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|  |  |  |  |
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| 1 | Mathematical Institute of the Slovak Academy of Sciences | MISAS | Applicant/host organisation |

## 1. Excellence

### 1.1 PROJECT OBJECTIVES

*1.1.1 INTRODUCTION*

Many set-theoretic structures can be defined in terms of categorical constructions within the context of the categories **Set** and **Rel**. If some categorical construction that defines in such a way a structure also exists in another category **C**, then that allows us to transfer the definition of this structure to **C**, in which case we say that the structure can be *internalized* in **C**. For example, groups can be defined in terms of finite products in **Set,** and in the same way internal groups can be defined in any category **C** that also has all finite products, such as the category **Top** of topological spaces and continuous maps. Internal groups in the latter category are precisely the topological groups. The importance of the process of internalization lies in the fact that it allows for generalizations, transference of properties to other categories, and cross-disciplinary connections.

A program relevant to this project in which internalization plays an essential role is *categorical quantum mechanics[[1]](#footnote-2)[[2]](#footnote-3)[[3]](#footnote-4)*, in which one studies the foundations of quantum physics and quantum information theory in terms of dagger compact categories such as the category **FdHilb** of finite-dimensional Hilbert spaces. In particular, one tries to describe quantum mechanical phenomena and features as internal structures in **FdHilb**, for instance orthonormal bases, which can be characterized[[4]](#footnote-5) as commutative special Frobenius algebras in **FdHilb**. Also other dagger compact categories are studied as toy models (**Rel**) or as more elaborated models for quantum physics (such as the category **FdCStar** of finite-dimensional C\*-algebras and completely positive maps).  
  
Our primary motivation for exploring internalization lies in its connection with *quantization*, i.e., the process of finding noncommutative generalizations of mathematical structures: internalization in the order-enriched dagger category **WRel** of Neumann algebras and Weaver's quantum relations[[5]](#footnote-6) can be regarded as a form of quantization. However, since **WRel** is not compact, which is desirable for categorical quantum mechanics, we prefer to work instead with its full dagger compact subcategory **qRel** of *hereditarily atomic* von Neumann algebras, i.e., von Neumann algebras isomorphic to a (possibly infinite) sum of matrix algebras, which are less complicated to work with, while still having sufficient structure for most applications in quantum computing and quantum information theory. Following Kornell[[6]](#footnote-7), we call the process of quantization by internalization in **qRel** *discrete quantization*, since the objects in **qRel** have a discrete character, and are in particular useful to quantize discrete mathematics.

We give a brief summary of how quantize according to the scheme of discrete quantization. The starting point is often a mathematical structure for which we want to find a noncommutative version. A mathematical structure is a set equipped with some extra features, such as a partial order, a metric, a topology, (algebraic) operations, etcetera. These extra features can often be described as binary relations subject to constraints that can be rephrased in terms of the dagger compact structure of **Rel** and the order on the homsets of **Rel** (i.e., the inclusion order of binary relations). Next, we replace any occurrence of a set (so an object of **Rel**) in the definition of the structure by a quantum set (so an object of **qRel)**, and any occurrence of a binary relation between sets by an occurrence of a morphism in **qRel**. We still impose the same constraints in the language of order-enriched dagger compact categories.In this way, the structure is internalized in **qRel**, and in a similar way, we can internalize the structure-preserving maps. Next, we consider the category **C** of the structures we are interested in and their structure-preserving maps. We denote by **qC** the category of the internalized versions of these structures and their morphism in **qRel**. We proceed by investigating what categorical properties **qC** has in common with **C**. For instance, often **qC** is complete or cocomplete if **C** is complete or cocomplete, respectively. The biggest difference between **C** and **qC** is that if **C** is cartesian monoidal, **qC** will only be semicartesian monoidal due to the quantum character of **qC**. Nevertheless, there are sometimes several inequivalent generalizations of the same structure that leads to different categories **qC**.The guideline we adapt to choose which version of **qC** is the most desirable is the category that has the most categorical properties with **C** in common. This is of course not always measurable; hence it would be desirable to have a better understanding of the discrete quantization process.

Internalizations in other order-enriched dagger compact categories are also relevant for the study of quantum structures. Previously, I worked with Anna Jenčová and Gejza Jenča, with whom I developed an interest in another order-enriched dagger compact category, namely the category **RelPosInv** of involutive posets and monotone relations. This category is also of interest for the study of quantum structures, because effect algebras, which form an important class of quantum structures, can be described as Frobenius algebras in this category. The process of understanding **RelPosInv** and its related category **RelPos** of posets and monotone relations lead to the quantization of complete lattices[[7]](#footnote-8) via the discrete quantization process together with Jenča. With Jenčová, I plan also to work on the question whether higher-order maps between quantum channels can be understood in the framework of discrete quantization.

*1.1.2 OBJECTIVES*

The overarching theme of our objectives is internalizing structures in **qRel** and **RelPosInv** simultaneously and in order-enriched dagger compact categories in general. Firstly, we are interested in a categorical framework that describes our categories. Traditionally, *allegories[[8]](#footnote-9)* and *bicategories of relations[[9]](#footnote-10)*  were introduced as categorical generalizations of the category **Rel;** there is a rich theory of how to internalize structures in these kinds of categories. **RelPosInv** turns out to be both an allegory and a bicategory of relations. However, due to its quantum character, the category **qRel** fails to be either an allegory or a bicategory, essentially because its wide monoidal subcategory **qSet** of internal functions fails to be cartesian. It is hard to tweak the definitions of allegories or bicategories of relations in order to obtain a powerful concept for which the internal functions do not necessarily form a cartesian category. Hence, we propose to look for a categorical generalization of **Rel** that also includes **qRel** and **RelPosInv** as examples by looking at structure these categories have in common.

It will not be sufficient to just list a couple of properties that the categories **Rel**, **qRel**, and **RelPosInv** have in common. We will have to investigate how these properties relate to each other, which will result in verifiable theorems. In the end, whether a common generalization is interesting will depend on whether we can successfully apply the framework that the generalization yields, see also Objective 3.

**Objective O1** (Categorical structure): Create an effective categorical framework that allows for the internalization of many structures and that includes **Rel**, **qRel** and **RelPosInv** as examples.

Our second objective concerns a specific structure we want to quantize, namely topological spaces. Many mathematical structures have associated topologies that allow the study of these structures by topological methods. Hence, it would be desirable to have a notion of quantum topological spaces that allow to study quantized structures by means of topological methods. The standard approach to quantum locally compact Hausdorff spaces is via C\*-algebras. Our proposed quantization of topological spaces via discrete quantization is different and independent from that approach and includes spaces more general than the locally compact Hausdorff spaces.

**Objective O2** (Quantization): Apply our categorical framework to find a quantum version of topological

spaces that allows for the quantization of topological spaces beyond locally compact Hausdorff spaces.

There are several quantum generalizations of mathematical structures that are naturally endowed with a topology, and their quantum generalizations have associated structures that resemble these topologies. To verify whether our proposed quantization of topological spaces is successful, we will have to show that these associated structures are indeed examples of our notion of a quantum topological space.

Another guideline of deciding whether our notion of quantum topological spaces is suitable, is whether we can successfully use our notion to the applications we have in mind. One of these applications concerns the *denotational semantics* of quantum programming languages, i.e., the translation of any phrase in a programming language to a mathematical function in such a way that the function is the composition of the functions corresponding to the phrase’s subphrases. Since it is virtually impossible to debug quantum programs, it is pertinent to find tools for the verification of them. One possible tool is denotational semantics, and we plan to use discrete quantization to construct new denotational models and to refine existing models. Another application we have in mind lies in quantum information theory, where *higher-order* maps between q*uantum channels* are studied. A quantum channel is a communication channel which can transmit both quantum information and classical information. A *second-order* map is a map of quantum channels to quantum channels. In general, a higher-order map sends sets of maps of a lower order to quantum channels. Current research concerns the characterization of these higher-order maps, which we hope to characterize in terms of quantizations of known mathematical structures. Summarizing, our third objective is:

**Objective O3** (Applications): Describe higher-order maps between quantum channels in the framework of discrete quantum mathematics, and use discrete quantization to find denotational models of quantum programming languages.

For several quantum programming languages there is already an *abstract* denotational model. This means, there is a description of all the properties a category should satisfy in order to be a model for the specific language. If we use discrete quantization in order to construct a category that should be a denotational model for a certain quantum programming language, we can verify that our category is indeed a model by showing that it satisfies all the properties of an abstract model. Also, in order to show that a possible description of higher-order maps between quantum channels by means of discrete quantization, we have to verify that there is a one-to-one correspondence between higher-order maps as we describe them in our framework, and higher-order maps in their usual description.

1.2 RELEVANCE, QUALITY AND NOVELTY OF THE PROJECT

### *1.2.1 STATE OF THE ART*

Since locally compact Hausdorff spaces are dual to commutative C\*-algebras via Gelfand duality, the category of all C\*-algebras is traditionally regarded as the dual of a category of `formal' noncommutative locally compact Hausdorff spaces. The origin of quantum relations on von Neumann algebras lies in the work of Weaver and Kuperberg on the quantization of metric spaces[[10]](#footnote-11), which includes quantum generalizations of metric spaces that are not locally compact, and hence examples of quantum topological spaces beyond locally compact Hausdorff spaces. Quantum relations were also used to give a description of quantum graphs and to introduce quantum posets7. An alternative description of hereditarily atomic von Neumann algebras in terms of collections of finite-dimensional Hilbert spaces, called *quantum sets*, was given by Kornell[[11]](#footnote-12). In the same article, Kornell also showed that **qRel** is dagger compact, and explored the categorical properties of the wide subcategory **qSet** of internal functions in **qRel**, which he proved to be dual to the category of hereditarily atomic von Neumann algebras and normal unital \*-homomorphisms. The properties of the category **qPos** of quantum posets and monotone functions were investigated by Kornell, Mislove and me[[12]](#footnote-13) where we also showed the existence of a quantum power set monad on **qSet** for which it is essential that **qRel** is compact closed. This work was followed up by our quantization[[13]](#footnote-14) of complete partial orders (cpos), i.e., posets in which any monotonically ascending sequence has a supremum. These posets

are commonly used to construct denotational models for ordinary programming languages with recursive types. Using our notion of quantum cpos, we constructed a sound and computationally adequate model for the current state-of-the-art quantum programming language, Proto-Quipper-M extended with recursive types. Thus, discrete quantization has already proven its value because of this very important application to the denotational semantics of quantum programming languages.

Monotone relations between posets (or preorders) were studied under various names[[14]](#footnote-15). The category **RelPos** of posets and monotone relations is compact closed, but not a dagger category. However, its full subcategory **RelPosInv** of involutive posets is dagger compact. Recently, Jenča proved[[15]](#footnote-16) that effect algebras can be represented as certain commutative non-special dagger-Frobenius monoids in **RelPosInv** yielding an interesting link between the fields of algebraic quantum logic and categorical quantum mechanics. Using the methods of discrete quantization, Jenča and I found a quantum generalization of **RelPos**, which we used to construct a quantum version of lower set monad whose algebras we call *quantum suplattices*, since complete lattices (also called suplattices) are algebras of the ordinary lower set monad. We proved the quantum counterparts of several theorems on suplattices such as the Knaster-Tarski fixpoint theorem8.

Quantum channels are often represented by completely positive maps between operator algebras such as C\*-algebras and von Neumann algebras, whereas the ‘standard’ morphisms of C\*-algebras are \*-homomorphisms, and normal unital \*-homomorphisms in the case of von Neumann algebras. Nevertheless, normal completely positive maps between von Neumann algebras turn out be Kleisli morphisms of a certain monad T on the opposite of the category **WStar** of von Neumann algebras and normal unital \*-homomorphisms[[16]](#footnote-17). This monad T can be regarded as the noncommutative version of the Giry monad on the category **Meas** of measure spaces, whose Kleisli morphisms are probabilistic maps. A monad M on **qSet** (or equivalently, on opposite of the category of hereditarily atomic von Neumann algebras and normal unital \*-homomorphisms) similar to T also exists[[17]](#footnote-18); its Kleisli morphisms correspond to completely positive maps between hereditarily atomic von Neumann algebras. There exists also a description of completely positive maps between finite-dimensional operator algebras in the framework of categorical quantum mechanics; the category **FdCStar** of finite-dimensional C\*-algebras and completely positive maps turns out to be equivalent to CP\*(**FdHilb**), where CP\*(-) describes the *CP\*-construction[[18]](#footnote-19),* which to each dagger compact category **C** assigns a dagger compact category CP\*(**C**) whose objects can be regarded as abstract C\*-algebras (n the categorical sense) associated to **C**; its morphisms are abstract completely positive maps associated to **C**.

A second-order map between quantum channels transforms a quantum channel into another quantum channel. More generally, higher-order maps[[19]](#footnote-20), also called *superoperators*, send sets of maps of a lower order to quantum channels, and describe the most general quantum protocols. The idea behind higher-order maps resembles the idea of higher-order functions in programming languages, which can be modelled by the lambda calculus. Hence, it is not surprising that there are connections between higher-order quantum maps and higher-order quantum programming languages, such as the quantum lambda calculus or Proto-Quipper-M[[20]](#footnote-21). Higher-order maps also have been studied in the framework of categorical quantum mechanics[[21]](#footnote-22).

Finally, we should mention that there are links between our program and *fuzzification*, i.e., the process of generalizing mathematics to fuzzy set theory, where the membership relation takes values in the unit interval, or more generally, in a commutative unital quantale *V*, since fuzzification can be regarded as an internalization process in the order-enriched dagger compact category *V*-**Rel** of sets and *V*-valued relations. Related to *V*-**Rel** is the order-enriched dagger compact category *V-***Prof** of *V*-enriched categories and *V*-enriched profunctors, which generalizes **RelPos**.

This research is relevant for other research in the European Research Area, since it concerns quantum structures which are studied by many researchers in the EU, such as Anatolij Dvurečenskij and Silvia Pulmannová at the Host Institute, Gejza Jenča (Slovak University of Technology), Thomas Vetterlein (Johannes Kepler University Linz), Mirko Navara (Czech Technical University). Applications in the denotational semantics of quantum computing are relevant for Vladimir Zamdzhiev (Inria Saclay, Paris) and Benoit Valiron (Université Saclay Paris). Furthermore, this research aims for applications in quantum information theory, which are relevant for the proposed mentor at the host institute, Anna Jenčová, but also for Mario Ziman and Michal Sedlák (Physics Institute of the Slovak Academy of Sciences) and Teiko Heinosaari (University of Jyväskylä, Finland).

### *1.2.2 MAIN INNOVATIVE ASPECTS AND APPLICATIONS*

There are often several inequivalent ways to internalize the same structure in a category. However, for categories with sufficient structure, there are procedures to internalize several structures simultaneously and consistently. For instance, in topoi all constructive mathematics can be internalized. Every topos has an associated bicategory of relations, hence by studying the order-enriched dagger compact categories as generalizations of bicategories of relations, we obtain a better understanding of how to internalize structures in these categories, and thus how to consistently quantize several structures simultaneously. We expect to find connections with fuzzification (which has many applications in modern technology) because it can also be regarded as an internalization process in an order-enriched dagger compact category, namely the category *V*-**Rel** of *V*-valued relations for some unital commutative quantale *V*. Furthermore, our project concerns models of categorical quantum mechanics that have not been studied in this context before. In particular, the category **qRel** is not just a toy model of categorical mathematics, but a category with genuine applications in quantum computing (see Section 1.2.1 STATE OF THE ART). The addition of order enrichment to the program of categorical quantum mechanics is relevant, because **FdCStar** is order enriched, but there is not yet a framework that integrates order-enrichment and dagger compact categories.

Traditionally, C\*-algebras form the standard approach to quantum topology, but are only noncommutative generalizations of locally compact Hausdorff spaces. One of the project's most captivating objectives is to quantize topological spaces beyond locally compact Hausdorff spaces such that one can quantize specific topologies that are associated to structures that previously have been quantized, such as posets and graphs. This is particularly intriguing as many mathematical structures are illuminated by their associated topological spaces, which are not always locally compact Hausdorff—such as the Scott topology on a cpo.

Applications in the denotational semantics of quantum programming languages form one of the reasons why we are interested in the quantization of topological spaces. The quantum cpo model mentioned in Section 1.2.1 STATE OF THE ART lacks an important feature, namely support for probabilistic computation. Because of this, it is impossible to describe the preparation of states in the model. The problem would be resolved by finding a monad for probabilistic computation on the category **qCPO** of quantum cpos, but this problem turns out to be very difficult. Classically, valuations are used to construct such a monad, but valuations turn out to be very difficult to quantize. Topology offers a different route, but because the Scott topology on a cpo is not locally compact Hausdorff, the quantization of topological spaces that are not necessarily locally compact Hausdorff will be essential.

The second application to the denotational semantics of quantum programming languages we anticipate concerns differential programming languages, which are used in deep learning. The models of these languages are given by differential categories[[22]](#footnote-23), which include **Rel** and the category **Sup** of complete lattices and supremum-preserving maps as examples. Hence, we expect that the quantum counterparts of these categories, **qRel** and **qSup**, respectively, will be examples, too. Here, the latter category is precisely the category of quantum suplattices previously introduced by Jenča and me8. If these categories indeed are differential categories, this might lead to applications of categories of discrete quantum structures to quantum differential languages, hence to quantum deep learning.

Finally, we expect that a description of higher-order maps between quantum channels in the framework of discrete quantization would enlighten the structure of these maps, accentuating certain similarities with higher-order maps between classical channels, which itself has barely been explored, while at the same time explaining the differences due to the quantum setting. The approach of discrete quantization promises that categories of structures we want to quantize has similar properties with the categories of the original structures; the quantum character of the quantized structures is highlighted by replacing the occurrence of a cartesian product by a (semicartesian) monoidal product. We expect that this will apply to higher-order maps of channels as well.

1.3 METHODOLOGY

*1.3.1 OVERALL METHODOLOGY*

This proposal concerns an interdisciplinary project regarding research in mathematics, inspired by physics with applications in information theory and computer science. Within mathematics, this research employs elements and methods from various fields, in particular operator algebras, category theory, order theory, fuzzy mathematics, and monoidal topology. We aim to develop mathematics that should be applicable to the description of quantum-mechanical phenomena; direct applications are most likely to be found in quantum computing and quantum information theory. Our main structures of interest are order-enriched dagger compact categories. The main examples of such categories that we will consider are **Rel**, **qRel**, **RelPosInv**, **FdCStar**,*V***-Rel,** *V***-Prof**, in particular the former three.

*1.3.2 METHODOLOGY FOR OBJECTIVE O1*

There are several categorical generalizations of **Rel** such as *allegories* or *bicategories of relations*, for which there is a rich theory of how to internalize structure. However, **qRel** is neither an allegory nor a bicategory of relations due to the fact that **qSet**, which is equivalent to the category Map(**qRel**) of internal functions in **qRel**, is not cartesian monoidal. **Objective** **O1** concerns finding a different categorical generalization of **Rel** that does include **qRel** and **RelPosInv** as examples, and study the internalization process in this generalization. For our generalization, we will consider **C** that satisfy several properties that **Rel**, **qRel** and **RelPosInv** have in common. The main two properties that we will assume for are properties that are also satisfied by bicategories of relations:

1. order enriched;
2. dagger compact;

Possible extra conditions that we can require include:

1. the wide subcategory Map(**C**) of internal functions in **C** is symmetric monoidal closed;
2. The embedding of Map(**C**) into **C** has a right adjoint, or equivalently, there is a monad on Map(**C**) whose Kleisli category is equivalent to **C**;
3. The category **C** is a *quantaloid,* i.e., a category that is not just order enriched, but enriched over the category **Sup** of complete lattices and supremum-preserving maps;
4. Map(**C**) is semicartesian monoidal;
5. Map(**C**) is complete and cocomplete.

We note that (d) is a condition that is also enjoyed by *power allegories*, i.e., allegories **A** whose associated categories Map(**A**) of internal functions are topoi.

The categories that interest us the most, **Rel**, **qRel**, and **RelPosInv**,satisfy the conditions (a)-(g). The associated categories of internal functions of our categories are **Set**, **qSet** and **RelPos**. The associated right adjoint functors are given by the power set functor in case of **Rel**, the quantum power set functor in case of **qRel** and a functor resembling the power set functor in the case of **RelPosInv**.

We will investigate how the various requirements on **C** are related to each other, in order to simplify these requirements. Furthermore, we will investigate what kind of structures we can internalize in these kinds of categories, and what extra structure our categories satisfy further, and whether that leads to the internalization of more structures. A guide for this will be previous quantizations via the discrete quantization process, as well as the quantizations we aim to obtain in objective **O2**. In order to investigate what extra structure our categories possess, we will investigate our categories in terms of *double categories*[[23]](#footnote-24), which are large versions of internal categories in the category **Cat** of small categories and functions, and which can be used to combine a ‘category of functions’ and a ‘category of relations’ in a single framework. For instance, **Set** and **Rel** can be described together in a single double category, and we expect the same for **qSet** and **qRel**, and for **PosInv** and **RelPosInv**. A refinement of double categories are *framed bicategories*[[24]](#footnote-25)*.* An example of such a framed bicategory is the double category formed by **Set** and **Rel**, and we will verify whether **qRel** and **qSet** combine in a similar way into a framed bicategory, and similarly whether **PosInv** and **RelPosInv** combine into a framed bicategory. More generally, we will investigate whether and how order-enriched dagger categories **R** and their categories of internal functions Map(**R**) combine into a framed bicategory. As part of understanding the categorical structure of **qRel**, we will investigate double-categorical constructions such as double-categorical limits, adjunctions and monads in our associated double categories.

Since dagger compactness is a crucial element of our properties, we will also study our categories in the light of categorical quantum mechanics, which means we will investigate the following structures and constructions in our categories **qRel** and **RelPosInv**:

* internal monoids, comonoids and dagger Frobenius algebras
* dagger limits
* homsets

For monoids and comonoids, we will rely on Jenčová and Jenča’s previous work[[25]](#footnote-26) in the context of **Rel**. A connection between special dagger Frobenius algebras in **qRel** and quantum groupoids can be expected, since groupoids are the special dagger Frobenius algebras in **Rel**. The work of Heunen and Karvonen[[26]](#footnote-27) will be our main source for techniques to calculate dagger limits in our categories of interest. With respect to the homsets, we are interested in the identification of the categorical structure that assures that the homsets of categories satisfying (a) and (b) are complete orthomodular lattices, which is the case for **Rel**, **qRel** and **RelPosInv**. The homsets of *V-***Rel** and *V-***Cat** seem to be only ortholattices or orthomodular lattices if *V* is an ortholattice or an orthomodular lattice, respectively. Also, **FdCStar** satisfies (a) and (b), but its homsets do not have an orthocomplementation, and seem to be only bounded-directed complete, not complete. A road that we will follow to find conditions that assure the homsets are orthomodular lattices is by looking to our categoriesas compact-closed categories, hence as tracial categories. The existence of a trace yields a notion of orthogonality on the homsets. Moreover, since in contrast of **FdCStar** our categories **qRel** and **RelPosInv** are quantaloids, we can find a largest morphism in the homset that is orthogonal to our morphism. This largest orthogonal morphism should be the orthocomplement of our original morphism. A next step would be identifying the categorical structure that assures that the largest orthogonal morphism is indeed the orthocomplement, and that the orthomodular law holds. A reference point for this will be the work of Heunen and Kornell on the reconstruction of Hilbert spaces in terms of dagger categories[[27]](#footnote-28), which relies on reconstructing orthomodular lattices from the categorical structure of the category of Hilbert spaces.

*1.3.3 METHODOLOGY FOR OBJECTIVE O2*

**Objective** **O2** concerns the quantization of topological spaces and applications of quantization to the denotational semantics of quantum programming languages. The main strategy we will follow is to find categorical constructions in **Rel** that describe the structures that we want to quantize, lifting these constructions to **qRel**, and try to show that the quantized structures are subject to similar theorems. As quantization method we plan to use discrete quantization, which can be regarded as an internalization process in the order-enriched dagger compact category **qRel**. Since fuzzification can be regarded as an internalization process in the order-enriched dagger compact category *V*-**Rel** of *V*-valued relations for some commutative unital quantale *V*, we will use fuzzified versions of the structures we want to quantize as a source of possible constructions. The main structure we aim to quantize is that of topological spaces. Since topologies on sets are defined in terms of power sets on sets, a first step is identifying a quantum version of the power set construction. This was already achieved by Kornell, Mislove and me as an application of the quantization of posets via discrete quantization. The ordinary power set of an ordinary set can be defined in terms of functions into the two-element chain, and has ortholattice operations induced by the ortholattice operations on the two-element chain. The two-element chain is also used in the construction of the quantum power set of a quantum set, and its ortholattice operations also induce binary operations on the quantum power set that can be regarded as quantum versions of ortholattice operations.

More specifically, a topological space can be defined in terms of the intersection operator (one of the ortholattice operations on the power set) and the operation of taking arbitrary unions. The quantum version of the latter also exists, which follows from the previous quantization of suplattices by Jenča and me8: the quantum power set is an example of a quantum suplattice, and therefore has an operation that is the quantum version of the operation of taking arbitrary unions. Hence, the first possible definition of a quantum topology on a quantum set we propose is:

1. A subset of the quantum power of the quantum set that is closed under the quantum binary intersection operator, the quantum arbitrary unions operator, and that contains the least and largest element of the quantum power set.

The work on quantum suplattices also yields a notion of quantum Galois connections, and hence of quantum closure operators. This leads to a second possible definition of a quantum topology on a quantum set:

1. A closure operator on the quantum power set of the quantum set satisfying identities resembling the Kuratowski axioms.

A third definition of quantum topological spaces is inspired by the program of monoidal topology[[28]](#footnote-29), which was used for the fuzzification of topological spaces, which requires a quantum version of the ultrafilter monad. We identify two possible ways to obtain such a quantum ultrafilter monad, namely by quantizing the following constructions of the ordinary ultrafilter monad:

1. as the codensity monad of the embedding of the category **Setfin** of finite sets into **Set**;
2. as the monad induced by the adjunction between **Set** and **PosInv**op, where the left adjoint is given by the contravariant power set functor.

The first construction can be generalized to the quantum case. The existence of a quantum version of the monad as in the second construction is yet to be proven. If this monad exists as well, we will investigate whether the resulting monad coincides with the monad of the first construction. Given a satisfactory quantum ultrafilter monad on **qSet**, we will attempt to quantize the characterization of topological spaces as in the program of monoidal topology, yielding the third possible definition of a quantum topological space as:

1. lax algebras of the Barr extension of the quantum ultrafilter monad.

Hence, we have three possible definitions of a quantum topological space, possibly even four if the quantization of the ultrafilter monad bifurcates. We will investigate the relationships between the various definitions of quantum topological spaces, whether one definition is stronger than the others, or whether (some of) the definitions are equivalent. If the definitions are not equivalent, we will investigate which definition is the most suitable.

Previously quantized structures yield constructions that likely are examples of quantum topological spaces. We will verify whether our definitions of quantum topological spaces include these constructions as examples. If all our definitions of quantum topological spaces are equivalent, this will be useful for determining whether our definition make sense. If our definitions are not equivalent, this will be useful to determine what definition is the most useful. The candidate constructions include:

1. Quantum metric spaces in the sense of Kuperberg and Weaver;
2. The quantum lower sets of a quantum poset (generalizing the Alexandrov topology);
3. The quantum Scott topology on a quantum cpo;
4. Algebras of the quantum ultrafilter monad (generalizing compact Hausdorff spaces).

Unital C\*-algebras are traditionally seen as quantum generalizations of compact Hausdorff spaces. Due to the discrete character of our quantization process, it is unlikely that we will obtain a one-to-one correspondence between the algebras of the quantum ultrafilter monad and all unital C\*-algebras, but nevertheless we will investigate what exact relation exists between the two classes of objects.

*1.3.4 METHODOLOGY FOR OBJECTIVE O3*

**Objective** **O3** concerns applications of our research to quantum information theory and to the denotational semantics of quantum programming languages. We are interested in three specific applications, one in quantum information theory, and two in the denotational semantics of quantum programming languages. Both in the case of the application in quantum information theory and in one of the applications to quantum computing it concerns probability monads on a suitable category of quantized structures.

The application in quantum information theory concerns the characterization of higher-order maps of quantum channels in the framework of quantum sets, or more generally, in the framework of order-enriched dagger compact categories. Completely positive maps, which are often used to represent quantum channels, are well studied in the program of categorical quantum mechanics, since the category **FdCStar** of finite-dimensional C\*-algebras and completely positive maps is dagger compact. Since **FdCStar** is also order enriched, it forms a categorical setting for the description of completely positive maps that is relevant for this project. A different route via quantum sets follows from the following observation. As mentioned in Section 1.2.1 STATE OF THE ART**,** the idea behind higher-order maps resembles the idea of higher-order functions, which can be modelled by the lambda calculus, hence there exists connections with the quantum lambda calculus.

Related to the quantum lambda calculus is the quantum programming language Proto-Quipper-M, for which we recall that we already have a denotational model in terms of quantum sets, or more specifically in terms of quantum cpos14.These were obtained by applying the process of discrete quantization to ordinary cpos, which are the building bricks of denotational models of ordinary programming languages. In the same spirit, it is to be expected that higher-order maps of quantum channels can be obtained by applying the process of discrete quantization to higher-order maps of classical channels. In denotational semantics, higher-order functions of programming languages are often modelled by symmetric monoidal closed categories, since the internal hom in these categories models function spaces. However, as mentioned in Section 1.2.1 STATE OF THE ART, completely positive maps are modelled by Kleisli categories, which in general are not monoidal closed. Nevertheless, there is a weaker notion of *Kleisli exponentials[[29]](#footnote-30)* introduced by Moggi as *T-exponentials*[[30]](#footnote-31)*.* Generally, Kleisli categories are not monoidal closed, yet there is a weaker notion of Kleisli exponentials, which also allows to model higher-order functions in programming languages. The opposite of **WStar** has Kleisli exponentials with respect to the monad T17. The monad M on **qSet** whose Kleisli morphisms correspond to completely positive maps between hereditarily atomic von Neumann algebras is constructed in the same way as T; hence we also expect **qSet** to have Kleisli exponentials with respect to M as well. Hence, we aim to characterize higher-order maps between quantum channels in terms of these Kleisli exponentials. As is usual in discrete quantization, it will be imperative to understand the classical situation before moving to the quantum generalization. This means that we will study probabilistic functions first, which are the Kleisli morphisms of the distribution monad D on **Set**, the finitary version of the Giry monad. We expect that higher-order functions between probabilistic functions (which to our knowledge have not been described as well) can also be described by Kleisli exponentials of the distribution monad on **Set**. To our knowledge, higher-order functions between probabilistic functions has barely been the subject of research, and it will form the starting point of our research of higher-order functions between quantum channels, since as usual in discrete quantization, after understanding the classical case, we will try to lift results to the quantum setting, after which we expect that the (semicartesian) monoidal structure of our quantum categories reflect the quantum nature of higher-order maps between quantum channels. Since the category **Meas** of measureable spaces is a more fitting setting for probability theory than **Set**, we might also consider the Giry monad on **Meas** instead. The quantum version of the latter category is the opposite of the category **WStar** of all von Neumann algebras and normal unital \*-homomorphisms. Hence, if the framework of **qSet** turns out to be insufficient, we might move on to the setting of general von Neumann algebras and the monad M. The price to pay is that we won’t work in the framework of discrete quantization, but in the related framework of W\*-quantization (i.e., quantization by internalization in **WRel**). However, it also might be that the framework of quantum sets is actually more appropriate because **qRel** is dagger compact, and the description of higher-order maps might benefit of this feature.

The other applications concern the denotational semantics of quantum programming languages. Firstly, we recall the construction of a denotational model for Proto-Quipper-M extended with recursive types in terms of quantum cpos, obtained via the methods of discrete quantization14. This model lacks one important feature: the support for probabilistic computations, and as an effect, it is not possible to describe the execution of quantum circuits in the model. Traditionally, probabilistic computation is modelled by monads, for instance the distribution monad on **Set** or the monad M on **qSet** that was mentioned above. For the support of recursion, one needs to work with the category **CPO** of cpos instead of with **Set**. A monad for probabilistic computation on **CPO** is called a *probabilistic power domain*. Hence, a monad for probabilistic quantum computation, would be a similar monad on the category **qCPO** of quantum cpos. However, the construction of probabilistic power domains relies on the concept of valuations, which are very difficult to quantize. Hence, in order to find a monad for probabilistic quantum computation, we need to find different methods to construct monads for probabilistic computing. Classically, an alternative method is provided by *K-completions*[[31]](#footnote-32) of simple valuations*,* where the latter are much easier to quantize than arbitrary valuations. K-completions are essentially topological completions of topological spaces whose specialization orders are cpos, and are defined in terms of adjunctions between categories of topological spaces. Hence, we expect that the concept of a quantum topological space as in objective **O2** will be instrumental to find a similar construction in the quantum setting.

Our second application to quantum computing concerns investigating whether **qRel** and **qSup** form *differential categories[[32]](#footnote-33)*, just like their classical counterparts **Rel** and **Sup**. Differential categories form standard denotational models for the differential lambda-calculus, which is the blueprint for differential programming languages, which are used in deep learning. Hence, showing that categories of quantum objects form differential categories might yield applications to quantum deep learning. An important ingredient of differential categories is the monoidal structure, so we will have to find a suitable monoidal product on **qSup**. In fact, we suspect that the category is \*-autonomous, just like its classical counterpart **Sup**.A next step is showing that our categories **qRel** and **qSup** are equipped with a comonad corresponding with the exponential modality from linear logic. In order to prove the existence of such a comonad, we will have to investigate certain comonoids in our categories. For **qRel**, this is already part of objective **O1**. We will furthermore use the description of **Rel** and **Sup** as differential categories as a guidance.

1.3.5 DATA MANAGEMENT AND OPEN SCIENCE PRINCIPLES

The scientific output of this research will solely consist of research papers published in peer-reviewed journals and in conference proceedings. All research will be published as preprints on arXiv.org, which provides open access to our work. For peer-reviewed versions of our work, we will aim for publishing in journals that provide open access by default. If a publication is only possible in a journal that charges for open access, we will use part of the budget for research expenses for these charges. Furthermore, we will present our results in a weekly seminar, which will be accessible for anyone who is interested.

*1.3.5 GENDER EQUALITY/DIVERSITY ASPECTS*

Mathematical truth is independent of both gender or any other diversity aspect, and because of that, neither gender nor any diversity aspect will play a role in this research and its applications. Nevertheless, we are aware of the underrepresentation of women in mathematics, as well as the barriers women in mathematics are facing due to stereotypes, a lack of role models, bias and sexual intimidation. We are committed to encourage gender equality and inclusion; therefore we aim to create an inclusive and respectful research environment, and in case students are interested in writing a thesis in a subject related to this project, provide mentorship opportunities, especially for women and gender minorities.

1.4 EXCELLENCE OF THE RESEARCHER

This proposal requires a good understanding of several mathematical disciplines, namely operator algebras, category theory, orthomodular structures, logic, and domain theory, where the latter is the underlying mathematical theory for denotational semantics, which I obtained during my PhD project at Radboud University in Nijmegen with Klaas Landsman, and my postdocs at Tulane University in New Orleans with Michael Mislove, Johannes Kepler University in Linz with Thomas Vetterlein, and currently the Mathematical Institute of the Slovak Academy of Sciences in Bratislava with Anna Jenčová. During these years, I built a network that includes Andre Kornell (Dalhousie University, expertise on discrete quantization), Klaas Landsman, Michael Mislove (Tulane University) and Xiaodong Ji (Hunan University), both experts in domain theory and the semantics of programming languages, JS Lemay (Macquarie University, expertise on differential categories), Vladimir Zamdzhiev (Loria, expertise on quantum programming languages and their denotational semantics), and Isar Stubbe (Université du Littoral-Côte d'Opale), expertise on quantaloids, a kind of order-enriched category that includes both **qRel** and **RelPosInv**.  
During my initial postdoctoral position within the MURI project 'Semantics, Formal Reasoning, and Tool Support for Quantum Programming' at Tulane University Orleans, I collaborated with Andre Kornell and Michael Mislove. Our work centered on quantizing cpos to subsequently apply the thus-obtained quantum cpos to the denotational semantics of quantum programming languages14: in terms of quantum cpos, we constructed the first (and so far only) sound and computational adequate denotational model for extension of the state-of-the art quantum programming language, Proto-Quipper-M, with recursive types. It was during this collaboration that Kornell introduced me to the technique of discrete quantization, which also yielded a publication on the properties of the category **qPos** of quantum posets and monotone maps13. I am excited by the open problem of finding support for probabilistic computation in our denotational model. This concerns proving the existence of a quantum probabilistic power domain monad, a challenging task even in the classical scenario relying on topological tools. Moreover, I expect that a similar monad can be used in the description of higher-order functions between quantum channels in quantum information theory. Hence, because of my previous experiences in the denotational semantics of quantum programming languages, and because of my experience in the quantized structures, I expect to be of value in the research in quantum information theory by offering different perspectives.

Curriculum Vitae

**Personal information**

First and last name: Albertus Johannis Lindenhovius

Identifier: 0000-0001-5380-4705 (Orcid)

Date of birth:October 20, 1984

Nationality: Netherlands

Website (if relevant):

**Education**

07/2016 – PhD

Institute for Mathematics, Astrophysics and Particle Physics, Radboud University, the Netherlands

07/2011 – Master

Faculty of Science, University of Amsterdam, the Netherlands

**Current position/positions**

06/2023 – 12/2023

Institute of Mathematics, Slovak Academy of Sciences, Slovakia

**Previous positions**

01/2021 – 05/2023 – postdoc

Institute for Mathematical Methods in Medicine and Data Based Modeling, Johannes Kepler University, Austria

08/2016 – 12/2020 – postdoc

Department of Computer Science, Tulane University, United States of America

**Teaching activities**

2019 – Introduction to Discrete Math (both spring and fall), Tulane University, United States of America

2018 – Operator Algebras, Tulane University, United States of America

2017 – Calculus III, Tulane University, United States of America

**Organisation of scientific meetings**

2023 – co-organizer of the Dutch Mathematical Congress, the Netherlands, with 150 participants. I was in particular responsible for the organization around the PhD student prize.

**Major collaborations (if applicable)**

- John Harding, Bohrification (i.e., the research of reconstructing mathematical structures representing a quantum system from their posets of classical substructures), Department of Mathematical Sciences, New Mexico State University, United States of America

- Chris Heunen, Bohrification, School of Informatics, University of Edinburgh, United Kingdom

- Gejza Jenča, Quantization, Faculty of Civil Engineering, Department of Mathematics and Descriptive Geometry, Slovak University of Technology, Slovakia

- Andre Kornell, Quantization, Department of Mathematics and Statistics, Dalhousie University, Canada

- Michael Mislove, Denotational Semantics of Quantum Programming Languages/Quantization, Department of Computer Science, Tulane University, United States of America.

- Vladimir Zamdzhiev, Denotational Semantics of Quantum Programming Languages, Inria/Saclay, France

**Overview of the researcher’s most important projects in the last 5 years** (max. 5)

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Project name/identification** | **Source of funding** | **Budget (EUR)** | **Project period** | **The role of the researcher in the project** |
| Semantics, Formal Reasoning, and Tool Support for Quantum Programming/MURI grant number FA9550-16-1-0082 | AFOSR | 3.4 million euro ($3.67 million) | 2016-2020 | postdoc |
| The many facets of orthomodularity/Project I 4579/Project 20-09869L | FWF/GAČR | 228,000 euro (Austrian side)  350,000 euro (Czech side) | 2021-2023 | postdoc |

**Overview of the researcher’s most important outputs** (max. 5)

|  |  |  |  |
| --- | --- | --- | --- |
| **Output name/identification** | **Type of output** *(e.g., publication, dataset, software, patent, service, product, etc.)* | **Short description** | **The role of the researcher** |
| G. Jenča, B. Lindenhovius, *Quantum suplattices*, Proceedings of the 20th International Conference on  Quantum Physics and Logic, 58–74 (2023) | Conference paper | In this work, we present a noncommutative version of complete lattices, called *quantum suplattices*, and show that several classical theorems on complete lattices such as the Knaster-Tarski Fixpoint Theorem also hold for quantum suplattices. | author |
| A. Kornell, B. Lindenhovius, M. Mislove, *A category of quantum posets*, Indagationes Mathematicae, Vol. 33, Issue 6, Pages 1137-1171 (2022) | Paper | In this contribution, in the framework of quantum sets, we describe non-commutative generalizations  of partially ordered sets, which we call quantum posets. We define a category of quantum posets, and show that this category is complete, cocomplete and monoidal closed. Finally, we describe a quantum generalization of the power set, which can be equipped with a canonical quantum order that generalizes the inclusion order. | author |
| J. Harding, C. Heunen, B. Lindenhovius, M. Navara, *Boolean Subalgebras of Orthoalgebras*, Order, Vol. 36, Issue 3, pp 563–609, (2019) | Paper | We show that any quantum logic represented by an orthoalgebra can be reconstructed from its classical fragments represented by Boolean subalgebras of the orthoalgebra. Moreover, connections are made between orthoalgebras and hypergraphs similar to those arising in projective geometry. | author |
| A. Kornell, B. Lindenhovius, M. Mislove, *Quantum CPOs*, Proceedings of the 17th International Conference on Quantum Physics and Logic, 174–187 (2020) | Conference paper | Complete partial orders (cpos) are objects that can be used for the semantics of programming languages with recursion, but do not suffice for the semantics of quantum programming languages with recursion. Previous attempts to resolve this were either very complicated or were lacking a crucial property. In this contribution, we introduce a non-commutative generalization of these objects, which we call *quantum cpos*, and we show that the category of quantum cpos has all appropriate categorical properties for the semantics of quantum programming languages with recursion | author |
| B. Lindenhovius, M. Mislove, V. Zamdzhiev, *LNL-FPC: The Linear/Non-linear Fixpoint Calculus*, Logical  Methods in Computer Science, Vol. 17, Issue 2, Pages 9:1 – 9:61 (2021) | paper | We describe computationally adequate abstract models of the language LNL-FPC, which can be regarded as the extension of Plotkin’s Fixpoint Calculus (FPC) with linear types, or as the circuit-free fragment of the extension of Proto-Quipper-M with recursive types. | author |

My most important research is my work on the quantization of cpos, for two reasons: it shows that discrete quantization has strong applications, because we used the thus-obtained quantum cpos to construct denotational models for quantum programming languages. The second reason is that the work is a showcase of the construction of a category of quantum objects that bears many similarities with the category of the classical counterparts of these quantum objects, which allow applications to phenomena that are the quantum versions of phenomena described by these classical counterparts. My expectation is that for these reasons this work will be a blueprint for future denotational models of quantum programming languages.

1.5 EXCELLENCE OF THE APPLICANT/HOST ORGANISATION

Mathematical Institute of the Slovak Academy of Sciences is a scientific institute focused mainly on basic research in mathematics and theoretical informatics. The Institute has a long tradition in several important branches of pure and applied mathematics and participated in a number of successful projects in both basic and applied research, including projects of Frame Projects of EU, Structural projects of EU, and projects of domestic agencies APVV and VEGA. The researchers of the Institute belong to the top in their research, in a world-wide context, and are engaged in multiple collaborations with experts from internationally renowned institutions. In collaboration with the Comenius University, the Institute organises a PhD study program and many young scientists and students use the Slovak fellowship program SAIA for short term study stays at the institute.

There are several areas of research closely related to the project that are pursued strongly at our institute already for a long time. In particular, the institute is traditionally one of the world-wide most important centers of research in Quantum Structures. Of the current researchers, Anatolij Dvurečenskij and Silvia Pulmannová are leading experts in this field. They are both former presidents of the International Quantum Structures Association (IQSA), they co-authored a fundamental monograph in this field (New Trends in Quantum Structures) and contributed hundreds of papers in top journals with thousands of citations. Furthermore, the proposed mentor, Anna Jenčová, is an internationally renowned expert in quantum information theory, both in finite dimensional setting and in von Neumann algebras, and in quantum foundations. She authored and co-authored more than 50 papers, many of which were published in top mathematical physics and quantum information theory journals. She was awarded the Birkhoff-von Neumann prize of the IQSA in 2014. The field of fuzzy mathematics, also relevant to the project, is another strong research field at the Institute. The top researcher is Andrea Zemánková, who is a leading expert in fuzzy mathematics and aggregation functions. She authored and co-authored more than 60 papers in leading journals and was very recently awarded the price of the Slovak Academy of Sciences for an excellent publication. Fuzzy mathematics is also strong in the Košice branch of the Institute, with Jozef Pócs as the leading scientist. From the point of view of implementation of the present project, it is important that the Mathematical Institute is the only institution in Slovakia where all the three fields (quantum structures, quantum information theory and fuzzy mathematics) are simultaneously studied.

The researchers of the institute are well connected with world-wide experts in their respective branches. In the field of Quantum structures and quantum information theory, the Institute has a close connection to neighbouring institutions such as the Slovak Technical University in Bratislava, Palacký University in Olomouc or the Research Center for Quantum Information at the Institute of Physics of SAS, through extensive collaborations and numerous joint projects. The institute organizes several successful and long-term established seminars, such as the seminar on Quantum structures, with invited talks by distinguished scientists. Recently, a less formal seminar on Category theory and applications is organized on weekly basis, including Jenčová and me, but also with participants from Slovak Technical University (Gejza Jenča, Peter Sarkoci) and Palacký University (Dominik Lachman), which is an excellent opportunity for exchanging ideas and discussions.

The institute is well equipped with all the standard hardware and software needed for mathematics research, including quality equipment for online presentation and communication. The library of Mathematical Institute SAS belongs to the best mathematical libraries in Slovakia, with access to the most important scientific databases. Furthermore, the Slovak Academy of Sciences provides access to a supercomputer.

**Two-way transfer of knowledge**

To the two-way transfer of knowledge between host and researcher we allocate Work Package 4. We identify four interactions:

1. The interaction between the proposed mentor, Anna Jenčová, and me, in which we will exchange our knowledge on quantization and on quantum information theory. The latter subject is largely unfamiliar to me, on the other hand, my knowledge on quantization might yield a new perspective on quantum information theory, which will form part of the proposed research in Work Package 3, for which it will be essential that we ‘learn each other’s language’.
2. The interaction between me and the research group on fuzzy mathematics including Andrea Zemánková at the institute. The methodology of discrete quantization shares similarities with fuzzification as an internalization process, yet there might be conceptual differences, which are essential for the applications of fuzzification. I expect that my knowledge on internalization will yield a different perspective on fuzzification. On the other hand, I have not much knowledge on the standard approach to fuzzification, and on what applications are important, which I hope to learn during this project.
3. The interaction with other members of the institute, in particular Anatolij Dvurečenskij and Silvia Pulmannová, who are experts on quantum structures, which are of importance in most mathematical research on quantum phenomena, including the research of this project.
4. The exchange of networks. I will bring the work of my collaborators such as Andre Kornell and Isar Stubbe under the attention of the researchers at the Institute. On the other way, via the Institute I will be brought in contact with various other researchers, such as the researchers in quantum information at the Physics Institute of the Slovak Academy of Sciences, such as Mario Ziman and Michal Sedlák, and the fuzzy mathematics group at the Slovak University of Technology, including Radko Mesiar, Peter Sarkoci, Andrea Stupňanová and Martin Kalina. Moreover, a weekly seminar with Jenčová is already running in which also Gejza Jenča, Dominik Lachman and Peter Sarkoci participate.

## Impact

### 2.1 THE WIDER IMPACT OF THE PROJECT

**Scientific impact**: The biggest scientific impact of this project is short term: it opens the door to quantum topology for quantum structures. Many mathematical structures have associated topological spaces; the properties of the structures are often encoded in properties of these associated spaces. Hence, it is desirable to have a concept of quantum topological spaces, since it is to be expected that quantum structures do not have associated classical topological spaces, but a quantized version of these spaces. C\*-algebras form the traditional approach to quantum topology, but only generalize locally compact Hausdorff spaces. Our approach is independent of the C\*-algebra approach, but promises to include quantum versions of topological spaces that are not necessarily locally compact or Hausdorff.

As an impact on the middle term, we expect that the analysis of the discrete quantization process will increase the understanding of how to consistently quantize various structures simultaneously, which will stimulate the research on quantum structures by offering new perspectives and analytic methods. For example, we expect an impact on computer science by introducing new techniques that can be used in the denotational semantics of quantum programming languages. In the past, denotational models were either ad-hoc constructions or based on von Neumann algebras without extra quantized structures (and therefore lacking the necessary properties). Discrete quantization gives straightforward and structural methods of finding appropriate models. Similarly, we expect that a better understanding of higher-order maps between quantum channels will stimulate the design of new quantum protocols.

As a long-term impact, we anticipate that this project will contribute to the integration of several areas in the mathematics of quantum physics, namely the areas of quantum foundations, quantum information theory, and quantum programming. Moreover, this project might reveal deeper connections with fuzzy mathematics.

**Societal impact**: By participating in outreach events such as “the Night of researchers” and “the Day of open door”, I will increase the interest in and the awareness of mathematical physics and their possible applications in computer science. Furthermore, assisting Jenčová the supervision of students will give a direct contribution in that regard.

**Economic impact**: The most likely economic impact of the project is long term. The research direction we propose here might lead to a better understanding of quantum computing, which promises to be the next technological revolution.

**Impact on the career of the researcher**: My long-term goal is to secure a tenure-track position, which would allow me to continue my research in discrete quantization. This project would demonstrate my independence and organizational skills, as it concerns my own research and includes the organization of workshops. Additionally, discrete quantization is a new field of research, and this project would help to raise awareness of it by bringing it to the attention of the host institution’s network and the fuzzy mathematics community, which is represented by a strong research group at the host institution. Conversely, I expect that direct contact with Anna Jenčová will increase my understanding of quantum information theory. I have worked on the foundations of quantum theory, and on applications to quantum computing, but quantum information theory is still a subject with which I am not very familiar. The same can be said about fuzzy mathematics, and I expect that contact with the fuzzy mathematics groups at the host institution and in the rest of Bratislava will increase my knowledge and skills in fuzzy mathematics. We expect that this project will yield bridges between research in quantization at the one hand, and with quantum information theory and fuzzy mathematics at the other hand. Thus, this project will give me the opportunity to broaden my horizon, and to integrate my research with that of other research programs. Furthermore, the project would give me more opportunities to visit conferences, which would also contribute to the growth of my network, to the possibilities for new collaborations, and to my visibility within the science community.

Since this project would take place in Bratislava, it will also allow me to work with Gejza Jenča in a more structural way, which would benefit this research, because of his knowledge of how to describe mathematical structures by various categorical constructions. This already yielded a successful quantization of complete lattices8. The project would also allow research visits, for instance to Klaas Landsman (expert on noncommutative mathematics), Chris Heunen (expert on categorical quantum mechanics), Andre Kornell (expert on quantization), Isar Stubbe (expert on V-relations and quantaloids) and Vladimir Zamdzhiev (expert on the denotational semantics of quantum programming languages), which would enhance my research, hence ultimately my visibility. Moreover, most tenured positions require the ability to attract funding, which I will also enhance during the project by following workshops on grant writing, and by applying for local grants together with Jenčová.

Impact on the applicant/host organization: We expect that the research on discrete quantization will enrich the research of Anna Jenčová by giving a different perspective on one of her main interests: higher-order maps between quantum channels. We expect that these maps can be described in the language of discrete quantization, which will provide the different perspective. Similarly, we expect that describing fuzzification in terms of internalization will yield a different perspective on fuzzification, which will impact the research on fuzzy mathematics at the host organization.

Potential negative impact of the project: since this concerns mathematical research, which has no immediate technical application, there are no potential negative impact of this research. Its main potential future application concerns the quantum computer, whose biggest negative impact likely is that it will make the current password technology obsolete. However, the applications of this project concern understanding quantum computing better in general; there are no direct applications to the cryptography aspect of quantum computing, Moreover, quantum technology promises different cryptography techniques which will be unbreakable.

Potential obstacles to the planned impact of the project: This project has no potential legislative barrier, since it concerns mathematics. Competition might be a potential factor that impacts this project, but since discrete quantization is a relatively new area of research, we are aware of only one other researcher working on similar subjects, Andre Kornell, who actually has been a collaborator in the past, and we expect him to be a collaborator in the future as well. With respect to potential new competition, as a social impact of this project, we actually hope to make more researchers interested in the approach of discrete quantization, and we are looking forward to new collaborations.

**Monitored data:** we mainly expect that this project will yield several publications, and conference presentations. This project likely will increase my number of collaborators, since part of this project concerns the integration of my research with the quantum information and the fuzzy mathematics communities.

### 2.2 MEASURES TO MAXIMISE IMPACT – DISSEMINATION AND COMMUNICATION, EXPLOITATION OF RESULTS

To dissemination, exploitation and communication activities, we assign a Work Package 5 (WP5) .

**Dissemination**: Any significant result will be published immediately as preprints on arXiv, https://arxiv.org of the project, followed up, and validated by publications in a peer-reviewed journal. Furthermore, I will use the institute web server to set up a project homepage to inform about the progress and use it to centralize the links to arXiv preprints, conference abstracts, papers and other relevant information to a single hub. Further outreach to digital communities will be made by asking and answering questions on https://mathoverflow.net, but also via more general platforms such as X (Twitter) and Facebook.

With respect to nondigital communication, we plan to attend several conferences, such as QPL, LiCS, IQSA, BLAST, ACT, TACL, SSAOS, and FSTA in order to share our results, and increase our network, but also to discuss and exchange early ideas and partial results as is common practise in mathematics. Furthermore, we plan visits to K. Landsman, C. Heunen, J. Harding and A. Kornell. Finally, we plan to organize a workshop on quantization halfway the project to which we will invite the people involving quantum structures such as C. Heunen and A. Kornell, G. Kuperberg, and N. Weaver, but also people from the fuzzy mathematics community, from the host institution as well as J. Paseka, S. Solovjovs and I. Stubbe, hence connecting the quantum structures and the fuzzy mathematics communities.

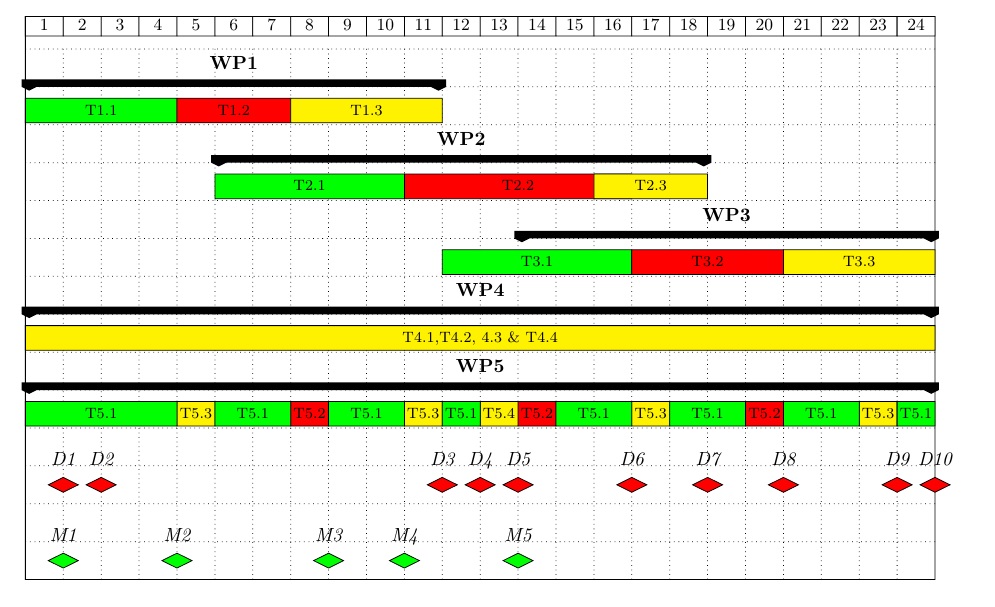
**Exploitation**: As is usual in mathematical physics, most applications are hard to predict, and appear in long time horizons, likely beyond the timeline of this project. However, there are two possible direct applications in the denotational semantics of quantum programming languages, and one possible direct application to quantum information theory. In particular, I expect the applications in the denotation semantics of quantum programming languages will be of importance in the future. *Domain theory* forms the mathematical field underneath denotational semantics. Discrete quantization already yielded a foundation for a possible quantum domain theory14, we expect to strengthen this foundation by constructing a monad for probabilistic quantum computing. Even if I will not be able to continue my scientific career after this project, the articles yielded by this project will still be available for other researchers who want to pick up the work.

**Communication**: I will annually participate in European researchers’ night by presenting the project outcomes to the general public. Any potential breakthrough result will be announced on the institute webpage, social media etc. The institute organizes a day of open door yearly, I will participate on this event as well. I am confident that the main aim of the project “extension of the mathematical notions to the quantum world” can be explained in layman terms to a talented high-school student, I will do this on the webpage of the project, and then try to contribute to popular science media in Slovakia like the Quark magazine or the https://vedator.space webpage.

## Implementation

3.1 PROJECT PLAN AND DELIVERABLES

We anticipate five work packages. The first three concern the scientific part of the proposal, the fourth concerns the two-way transfer of knowledge, and the last work package concerns the dissemination and communication. At any moment in the project, we allocate 50% of our time to the scientific part of the project, so on the first three work packages. To each of the remaining work packages we allocate 25% of our time. The timeline of the project is depicted schematically in the Gantt diagram below.



Here, WP refers to Work Package, T refers to Task, D refers to Deliveries and M refers to Milestones, all to be specified in the tables below.

3.1.1 Work packages

|  |  |
| --- | --- |
| Work package number | 1 |
| Title of the work package | Structure |
| **Start of implementation of the work package (Mx Month)** | M1 |
| End of implementation of the work package (Mx month) | M11 |
| **Involvement (expressed in Person Months)** | 4 person months. This is calculated as follows. We allocate 50% of our time to research (covered by WP1-3). The first 5 months of the project WP1 covers all research, which means 50% of 5 months equals 2,5 months. The remaining 6 months of this work package overlap with WP2, which means 25% of our time is spent on WP1 on these months. Hence 25% of 6 months is 1,5 month. |
| **Personnel costs (in EUR)** | 16,972 euro (4 person months times 4,243 euro of monthly salary costs). |
| Other eligible costs, excluding personnel costs (in EUR excluding VAT) | 2,120 euros indirect costs (2,544 euros including VAT: 636 euros times 4 person months) |
| Objectives | |
| We aim to investigate categorical generalization of Rel that contain qRel and RelPosInv as examples. | |
| Description of the work package | |
| We identify three tasks:  Task 1.1: Characterize the dagger limits, monoids, comonoids and dagger Frobenius algebras of qRel and RelPosInv and investigate what conditions on an order-enriched dagger compact category C assure that its homsets are complete orthomodular lattices such as in the case of Rel, qRel and RelPosInv.  Task 1.2: Investigate the relations between various proposed conditions on a category C that generalizes Rel, qRel, and RelPosInv, and simplify these conditions.  Task 1.3: Investigate our order-enriched dagger compact categories Rel, qRel and RelPosInv and their associated subcategories of internal functions in the light of double categories and framed bicategories. | |
| Deliverables | |
| We expect that this work package will yield one article. | |

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| Work package number | 2 |
| Title of the work package | Quantization |
| **Start of implementation of the work package (Mx Month)** | M6 |
| End of implementation of the work package (Mx month) | M18 |
| **Involvement (expressed in Person Months)** | 3,25 months. This is calculated as follows. We reserve 50% of our time to research, i.e., work on work packages 1-3. During the work on Work Package 2, there is always parallel work on either Work Package 1 or Work Package 3. Hence, per month, we spend 25% of our time to Work Package 2. The duration of the work package is 13 months, so we allocate 25% of 13 months of person months to this work package. |
| **Personnel costs (in EUR)** | 13,789.75 euros (3,25 person months times 4,243 euros of monthly salary costs. |
| Other eligible costs, excluding personnel costs (in EUR excluding VAT) | 1,722.50 euros indirect costs (2,067 euros including VAT: 636 euros times 3.25 person months) |
| Objectives | |
| We aim to quantize topological spaces that are not necessarily locally compact or Hausdorff. In particular, this quantization should allow to quantize specific topologies associated to structures that previously have been quantized, such as cpos and metric spaces. | |
| Description of the work package | |
| We identify three tasks:  **Task 2.1**: Investigate and compare two different possible definitions of the quantum ultrafilter monad:  **Task 2.2**: Investigate possible equivalences of the three proposed definitions of quantum topologies. If not equivalent, identify which version is the most suitable.  **Task 2.3**: Verify that the following previously-quantized structures have associated structures that yield  examples of quantum topological spaces. | |
| Deliverables | |
| We expect one article on quantum topological spaces. | |

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| Work package number | 3 |
| Title of the work package | Applications |
| **Start of implementation of the work package (Mx Month)** | M12 |
| End of implementation of the work package (Mx month) | M24 |
| **Involvement (expressed in Person Months)** | 4,75 person months. This is calculated as follows. We reserve 50% of our time to research (which is covered by WP1-3). The first 7 months of this work package overlap with Work Package 2, hence during these months, we spend 25% of our time to Work Package 3, which is 1,75 per son months. The last 6 months of Work Package 3 have no overlap with any other work package concerning research, hence 50% of 6 months is allocated to this work package during these months, corresponding to 3 person months. |
| **Personnel costs (in EUR)** | 20.154,25 euros (4,75 person months times 4.243 euros of monthly salary costs) |
| Other eligible costs, excluding personnel costs (in EUR excluding VAT) | 2,517.50 euros indirect costs (3,021 euros including VAT: 636 euros times 4.75 person months) |
| Objectives | |
| We investigate two applications of our research in discrete quantization to the denotational semantics of quantum programming languages. | |
| Description of the work package | |
| We identify three tasks:  **Task 3.1**: Describe higher-order maps of quantum channels in the framework of discrete quantization.  **Task 3.2**: Construct a monad for probabilistic computation on the category qCPO of quantum cpos.  **Task 3.3**: Show that **qRel** and the category **qSup** of quantum suplattices are differential categories. | |
| Deliverables | |
| We expect an article, or a conference presentation per task. | |

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| Work package number | 4 |
| Title of the work package | Knowledge transfer |
| **Start of implementation of the work package (Mx Month)** | M1 |
| End of implementation of the work package (Mx month) | M24 |
| **Involvement (expressed in Person Months)** | 6 person months. We allocate 25% of our time per month to this project, and we plan to work on this work package every month of the project, hence 25% of 24 months. |
| **Personnel costs (in EUR)** | 25,456 euro, which is 6 person months times 4,243 euro monthly salary costs. |
| Other eligible costs, excluding personnel costs (in EUR excluding VAT) | Total: 4,180 euros.  3,180 euros indirect costs (6 person months times 636 euros equals 3,816 euros inclusive VAT)  1,000 euros excluding VAT for a new laptop from the additional budget for research expenses |
| Objectives | |
| This work package concerns the two-way transfer of knowledge between the researcher and the host organization. | |
| Description of the work package | |
| **Task 4.1** concerns the most important transfer of knowledge and skills, namely by direct contact with the mentor, Anna Jenčová, which we plan at least twice a week during the whole duration of the project. During these interactions, we will work on the achieving the tasks in work packages 1-3, but Jenčová will also share her knowledge on quantum information theory, a subject which is largely unfamiliar to me. In particular, Jenčová’s knowledge on the role of von Neumann algebras in quantum information theory will be of importance, since this project concerns internalization in a category of von Neumann algebras.  **Task 4.2** concerns interactions with the other members of the institute, in particular Anatolij Dvurečenskij and Silvia Pulmannová, who are experts on quantum structures, and Andrea Zemánková whose expertise on fuzzy mathematics will be highly relevant for this project.  **Task 4.3** concerns interactions with Bratislava-based scientists outside the institute. In particular, Gejza Jenča (Slovak University of Technology) as a current collaborator will be of importance. A weekly seminar concerning subjects related to this project will be organized, in which Jenča, Jenčová, Dominik Lachman, Peter Sarkoci and I will participate. Furthermore, we expect interactions with the fuzzy set theory group at the Slovak University of Technology consisting amongst others of Rado Mesiar, Peter Sarkoci, Andrea Stupňanová and Martin Kalina. Furthermore, we anticipate interactions with members of the research center for quantum information at the Institute of Physics at the Slovak Academy of Sciences, in particular with Mario Ziman and Michal Sedlák. This center also organizes a weekly seminar, which I plan to attend.  **Task 4.4** concerns the training in soft skills, such as teaching, grant writing, work shop organizing, and science communication. For some of these skills, I plan to follow workshops at the Slovak Centre of Scientific and Technical Information (CVTI SR), which is accessible for employees of the Slovak Academy of Sciences. Improving communication/outreach to a broader public is addressed by activities outlined in Work Package 5. | |
| Deliverables | |
| A career development plan that Jenčová and I will write in the first months of this project. | |

|  |  |
| --- | --- |
| Work package number | 5 |
| Title of the work package | Dissemination |
| **Start of implementation of the work package (Mx Month)** | M1 |
| End of implementation of the work package (Mx month) | M24 |
| **Involvement (expressed in Person Months)** | 6 person months. We allocate 25% of our time per month to this project, and we plan to work on this work package every month of the project, hence 25% of 24 months. |
| **Personnel costs (in EUR)** | 25,456 euro, which is 6 person months times 4,243 euro monthly salary costs. |
| Other eligible costs, excluding personnel costs (in EUR excluding VAT) | Total 28,180 euros, which consists of:   * 3,180 euros indirect costs (6 person months times 636 euros equals 3,816 euros inclusive VAT) * 25,000 euros from the additional budget for research expenses (including VAT this is 30,000 euros from which we reserve 10,000 euros for possible charges for open access of our articles. (We estimate 5 articles, the average charges for open access are 2,000 euros per article). Furthermore, we reserve 15,000 euro for the costs of the organization of our workshop halfway the project, for which we will have to pay the travel costs of our invited speakers. Finally, we reserve 5,000 euros for our own travel costs). |
| Objectives | |
| We aim to maximise the expected outcomes and impacts of this project. | |
| Description of the work package | |
| We identify four tasks:  **Task 5.1**: all digital dissemination, i.e., the immediate output of preprints on arxiv.org, a homepage about the project, outreach to specialized communities such as mathoverflow.net, but also via more general platforms such as X (Twitter) and Facebook.  **Task 5.2**: The attendance of several conferences such as QPL, LiCS, IQSA, BLAST, ACT, TACL, SSAOS, and FSTA.  **Task 5.3**: Research visits  **Task 5.4**: the organization of a workshop on quantization halfway the project | |
| Deliverables | |
| The proceedings of the workshop we plan to organize in task 5.4. | |

3.1.2 List of work packages

|  |  |  |  |
| --- | --- | --- | --- |
| Work package number | Title of the work package | **Start of activities** | **End of activities** |
| 1 | Structure | M1 | M11 |
| 2 | Quantization | M6 | M18 |
| 3 | Applications | M12 | M24 |
| 4 | Knowledge transfer | M1 | M24 |
| 5 | Dissemination | M1 | M24 |

3.1.3 List of deliverables

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Deliverable number | Deliverable | Work package number | Type | Access and dissemination | Method of verification | Delivery (project implementation month) |
| D1 | Career development plan | WP4 | Work plan | N | Expert feedback | M1 |
| D2 | Website for the project | WP5 | website | P | Accessibility Check | M2 |
| D3 | Article on the categorical structure of **qRel** | WP1 | publication | P | peer-review | M11 |
| D4 | Interim Report of the project | WP5 | report | N | Progress Evaluation | M12 |
| D5 | Proceedings of the workshop | WP5 | publication | P | peer-review | M13 |
| D6 | Conference talk on higher-order maps between quantum channels in the framework of discrete quantization | WP3 | proceedings | P | peer-review | M16 |
| D7 | Article on the quantization of topological spaces | WP2 | publication | P | peer-review | M18 |
| D8 | Conference talk on a monad for probabilistic quantum computations | WP3 | proceedings | P | peer-review | M20 |
| D9 | Conference talk on differential categories of quantum objects | WP3 | proceedings | P | peer-review | M23 |
| D10 | Final report of the project | WP5 | report | N | Project Evaluation | M24 |

3.1.4 List of milestones

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Milestone number | Milestone | Work package number | Method of verification | Expected time to reach the milestone (project month) |
| M1 | Career development plan | WP4 | Peer review | M1 |
| M2 | Characterization of commutative monoids in **qRel** | WP1 | Peer review | M4 |
| M3 | The characterization of the categorical structure that includes **Rel**, **qRel** and **RelPosInv** | WP1 | Peer review | M8 |
| M4 | Analysis of possible equivalence between several proposed definitions of quantum topological spaces | WP2 | Peer review | M10 |
| M5 | Workshop | WP5 | Feedback | M13 |

3.2 IMPLEMENTATION RISKS AND PROPOSED MEASURES

In general, progress in mathematical research is hard to predict, hence our time line is tentative. It is also hard to predict whether a problem is solvable, until it is actually solved. Hence, the biggest risk is the stagnation of one of the objectives, which we mention as a specific risk in the table below, but we also mention several potential specific bottlenecks in the research. Overall, our general strategy to deal with stagnated objectives consists of the following steps. 1) We try to simplify the problem, and then we will try to solve the simplified problem. In the case of quantizations, this often means that we have to try to understand the classical situation better. We have to understand what steps in the classical proof can be rephrased in a categorical framework. Sometimes, we even have to design new proofs for the classical case, but in any case, the classical case will be easier. Once we solved the simplified version of our problem, we return to our original problem, for which we now hopefully have better intuitions and more techniques. If we are still stuck, then we will look at other fields in which similar problems have been solved. In our case, this concerns the fields of topos theory and especially fuzzification. If we still haven’t acquired the fitting techniques to solve our problems by looking at these similar problems, we will consult experts in our network. For problems in quantization, we will consult Gejza Jenča (Slovak University of Technology) and/or Andre Kornell (Dalhousie University Halifax), with whom I actually already have running collaborations. With Jenča I currently have the most intensive contact, thanks to the fact that he is also based in Bratislava. He has a great understanding of the categorical structure behind classical structures. Andre Kornell is the researcher who introduced me to discrete quantization and has a great intuition about what quantization of some structure is the ‘correct’ one if there are several inequivalent quantizations of the same structure are possible. To discuss possible similarities with fuzzification in terms of the category *V*-**Rel**, we can consult Jan Paseka (Masaryk University Brno), Sergejs Solovjovs (no longer in academia, but still in our network) and/or Isar Stubbe (Université du Littoral-Côte d'Opale, Calais). The latter is also an expert on quantaloids. For problems regarding categorical quantum mechanics, we will consult Chris Heunen (University of Edinburgh). With respect to denotational semantics of quantum programming languages, we can consult Vladimir Zamdzhiev (Inria Saclay, Paris). For differential categories, we will contact JS Lemay (Macquarie University Sydney).

3.2.1 Risks of implementation

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| --- | --- | --- | --- | --- |
| **Description of the risk of implementation** | **Probability of Risk** | **Severity of Risk** | **Work package** *(one or more)* | Proposed measures for risk mitigation or elimination |
| Delay due to relocation; not being able to find an apartment | low | Low | 1-5 | Rent an Airbnb apartment for a month in order to have time for finding a more permanent location to live. |
| Restrictions due to an outbreak of a new covid mutation | low | Medium | 1-5 | Work remote, in case the outbreak happens when we want to organize the workshop, organize the workshop online. |
| Stagnation in fulfilling one of  the objectives | high | Medium | 1-3 | Regular contact with Jenčová, and if necessary,  sharing and discussing preliminary results with external experts. |
| Failure to find a common generalization of the categories **Rel**, **qRel**, and **RelPosInv.** | low | Medium | 1 | Find additional conditions under which  the generalization works. |
| Fail to characterize the Frobenius  algebras in **qRel** and **RelPosInv** | medium | Medium | 1 | Try to understand Frobenius algebras in **Rel** better, then return to the original problem. |
| No single notion of quantum  topologies includes all our candidate  spaces as examples | medium | Medium | 2 | Describe and compare the various definitions  we obtain, in order to apply different  version to appropriate problems. |
| No notion of quantum topologies that allows for the quantization of K-spaces, hence of a probabilistic power domain | low | Medium | 2-3 | Investigate more possible categorical  constructions yielding topological spaces,  possibly inspired by the various constructions  of fuzzy topological spaces. |
| Quantum sets (or equivalently, hereditarily atomic von Neumann algebras) do not form a good setting for higher-order maps between quantum channels | medium | low | 3 | Work with all von Neumann algebras instead |

3.3 OPERATIONAL CAPACITY OF THE APPLICANT/HOST ORGANISATION

3.3.1 Description of the research/innovation infrastructure of the applicant/host organisation that is necessary for the implementation of the project

|  |  |
| --- | --- |
| Name of infrastructure or equipment | Short description |
| Office | A desk and a desktop computer, printer, quality hardware for video conferencing |
| DEVANA Supercomputer | The Slovak Academy of Sciences provides access to supercomputer DEVANA with available total performance about 800 Tflop/s (from 1.1. 2024) |
| Library and services | Access to the mathematical library of MISAS and library services |
| Access to databases | The Slovak Academy of Sciences provides access to scientific databases such as the Web of Science, Scopus (from 1.1. 2024), Springerlink, ScienceDirect, Mathematical Reviews etc. |

Research in mathematics doesn’t require much infrastructure. The most essential infrastructure concerns an office with a desk and a desktop computer, which the host institute has available during the whole project. Similarly, the host institute offers access to scientific journals and databases. The access to this infrastructure is generally necessary for any scientist, hence like any other scientific institution, the host institution will provide access to this infrastructure to any scientific employee during their employment.

3.3.2 List of the five most important projects of the applicant/host organisation and their relevance to the proposed project (in the last 5 years)

|  |  |  |
| --- | --- | --- |
| Project name/identification | Programme/scheme/grant provider | Short description |
| Mathematical support of quantum technologies/ NFP313011T683 | EU Operational Programme Research and Innovation/ ITMS-2014+ | The main goal of the project is the stabilisation of a quality research team and to realize independent research in the area of quantum technologies. The goals are reached by research of mathematical structures and functions relevant in the mathematical description of quantum mechanics. |
| Probabilistic, Algebraic and Quantum Mechanical Methods of Uncertainty Determination/ APVV-20-0069 | Slovak Research and Development Agency (APVV) | Investigation of uncertainty in quantum  structures and elsewhere, by combination of methods of algebra, probability theory, functional analysis, category theory and fuzzy mathematics. Joint with STU. |
| Mathematical models of non-classical events and uncertainty/ VEGA 2/0142/20 | Scientific Grant Agency of the Ministry of Education of the Slovak Republic and SAS (VEGA) | Mathematical models for quantum structures, quantum information theory and uncertainty. Joint with STU. |
| Designing quantum higher order structures/ APVV-22-0570 | Slovak Research and Development Agency (APVV) | Theoretical study and design of higher order maps – quantum networks. Potential applications of category theory methods proposed in the present project. Joint with Institute of Physics, SAS. |
| Probabilistic, Algebraic and Quantum-Mechanical Aspects of Uncertainty/ APVV-16-0073 | Slovak Research and Development Agency (APVV) | The aim of the project is to obtain original research results concerning description of uncertainty related to quantum structures. We focus on the study of total and partial algebraic structures derived from mathematical foundations of quantum mechanics, as well as on the description of quantum states, channels and more general processes and their estimation and discrimination procedures. |

3.3.3 List of maximum five most important outputs of the applicant/host organisation relevant to the submitted project

|  |  |  |
| --- | --- | --- |
| Output name/identification | **Type of output** | Short description |
| Pták, Pulmannová - *Orthomodular Structures as Quantum Logics: Intrinsic Properties, State Space and Probabilistic Topics, Kluwer Academic Publishers, Dordrecht, 1991* | Publication (Book) | A widely cited monograph dealing with orthomodular structures that form the basis of quantum logics. |
| Dvurecenskij, Pulmannová - *New Trends in Quantum Structures, Dordrecht : Kluwer Academic Publishers; Bratislava : Ister Science, 2000. 541+xvi pp. https://doi.org/10.1007/978-94-017-2422-7. ISBN 0-7923-6471-6* | Publication (Book) | A fundamental monograph describing structures arising in the mathematical description of quantum theory, fuzzy mathematics and related areas. |
| A. Jenčová - *A general theory of comparison of quantum channels (and beyond).* IEEE Transactions on Information Theory 67.6 (2021), 3945-3964, <https://doi.org/10.1109/TIT.2021.3070120> | Publication (Article) | Article on comparison of quantum channels dealing with the question of simulability or approximate simulability of a given channel by allowed transformations of another given channel. The paper introduces a general framework suitable for dealing with channels and higher-order structures. |
| A. Jenčová, G. Jenča – *On monoids in the category of sets and relations,* International Journal of Theoretical Physics, 56, pp 3757-3769, 2017, <https://doi.org/10.1007/s10773-017-3304-z> | Publiction (Article) | Article on internal monoids in the category **Rel.** |
| A. Jenčová – *Generalized channels: channels for convex subsets of the state space,* Journal of Mathematical Physics, 53(1), 012201, 2012, <https://doi.org/10.1063/1.3676294> | Publication (Article) | The paper introduces generalized channels as completely positive map transforming s subset of states of a finite dimensional C\*-algebra to states of another C\*-algebra. Properties of generalized channels are studied. Quantum channels and higher order maps are obtained as a special case. |

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