Conjectures on Hidden Onsager Algebra Symmetries in Interacting Quantum Lattice Models

Yuan Miao*

Institute for Theoretical Physics, University of Amsterdam,
Postbus 94485, 1090 GL Amsterdam, The Netherlands
* y.miao@uva.nl

March 29, 2021

Abstract

We conjecture the existence of hidden Onsager algebra symmetries in two interacting quantum integrable lattice models, i.e. spin-1/2 XXZ model and spin-1 Zamolodchikov–Fateev model at *arbitrary* root of unity values of the anisotropy. The conjectures relate the Onsager generators to the conserved charges obtained from semi-cyclic transfer matrices. The conjectures are motivated by two examples which are spin-1/2 XX model and spin-1 $U(1)$-invariant clock model. A novel construction of the semi-cyclic transfer matrices of spin-1 Zamolodchikov–Fateev model at arbitrary root of unity value of the anisotropy is carried out via transfer matrix fusion procedure.

Contents

1 Introduction

Onsager algebra was used for the first time to solve two-dimensional Ising model with zero magnetic field by Lars Onsager in his seminal paper in 1944 [1], which is considered to be the herald of exactly solvable models in statistical mechanics.

Later Onsager has been used to study 2-dimensional chiral Potts model, a generalisation of Ising model and its quantum counterpart, $\mathbb{Z}_N$-symmetric spin chain [2, 3]. Those models studied via Onsager algebra are integrable and possess Kramers–Wannier duality [4]. Later, Dolan and Grady used Onsager algebra to show the self-duality of these models and constructed infinitely-many conserved charges without using integrability [5]. This approach has led to deeper understanding of the algebraic structure of Onsager algebra and its relation to self-duality and integrability [6–8]. A thorough and comprehensive summary of the mathematical structures of Onsager algebra is provided in [9]. Recently there have been several influential results using Onsager algebra to study the spectra of quantum lattice models [10], the out-of-equilibrium dynamics of quantum states [11], and the construction of quantum many-body scars [12], reigniting the interest on the applications of Onsager algebra in theoretical physics.

Meanwhile, the spectra of quantum integrable models at root of unity values of the anisotropy have been investigated using Bethe ansatz techniques. The definition of the parametrisation of root of unity value of the anisotropy is given in (3.5). Specifically, spin-1/2 XXZ model [13–16] at root of unity, a prototypical quantum integrable lattice model, has drawn lots of attention. Due to the underlying quantum group structure at root of unity [17], the spectra have exponentially many degeneracies. This phenomenon has been studied in [18–24]. More recently, the author and collaborators have constructed the Q operator for XXZ model at root of unity and studied the spectra in terms of descendant towers [24], which elucidated the origin of the exponentially many degeneracies due to the existence of eigenstates associated with exact (Fabricius–McCoy) strings. In particular, XX model with Hamltonian in the form of (4.1) possesses Onsager algebra symmetry [10], cf. Sec. 4.1, and its spectrum has similar descendant tower structure as the spectra at other roots of unity [24]. It is thus natural to consider whether XXZ models at arbitrary roots of unity value of the anisotropy would possess similar Onsager algebra symmetries. The difficulty of solving the problem is that the Onsager generators in XX model are expressed in terms of local operators, while the generators at other roots of unity are not if they were to exist, which are discussed in Secs. 3 and 4.2. Moreover, it has been shown that the spectra of higher spin generalisations of XX model, dubbed $U(1)$-invariant clock models, have similar exponentially many degeneracies in terms of descendant tower structure, which are demonstrated using Onsager algebra in [10]. This provides further motivation to the current work. This article is set to compose two conjectures about the explicit form of the Onsager generators of XXZ model and its spin-1 generalisation, i.e. Zamolodchikov–Fateev (ZF) model, at arbitrary root of unity values of the anisotropy and their relations to the conserved charges from semi-cyclic transfer matrices.

The structure of the article is as follows. First, we introduce the basic properties of

Onsager algebra in Sec. 2, which will be used extensively. Second, we focus on the spin-1/2 case, i.e. XXZ model at root of unity. We construct the semi-cyclic transfer matrix and the conserved charges in Sec. 3. Motivated by the example of XX model, the conjectures of hidden Onsager algebra symmetries in XXZ models at arbitrary root of unity values of the anisotropy are given in Sec. 4. For the spin-1 case, we construct the semi-cyclic transfer matrices for spin-1 ZF models at arbitrary root of unity values of the anisotropy via transfer matrix fusion procedure in Sec. 5, which has not been reported before. Another example of spin-1 $U(1)$-invariant clock model and similar conjectures of hidden Onsager algebra symmetries in spin-1 ZF models at arbitrary roots of unity are formulated in Sec. 6. We conclude the article with Sec. 7.

2 Onsager algebra

We use the notations in Onsager's original paper [1] to define Onsager algebra. Consider the following infinite-dimensional Lie algebra with basis $\{\mathbf{A}_m, \mathbf{G}_n | m, n \in \mathbb{Z}\}$. The *canonical* generators satisfy the following relations,

$$[\mathbf{A}_m, \mathbf{A}_n] = 4\mathbf{G}_{n-m}, \quad [\mathbf{G}_m, \mathbf{A}_n] = 2\left(\mathbf{A}_{n+m} - \mathbf{A}_{n-m}\right), [\mathbf{G}_m, \mathbf{G}_n] = 0. \tag{2.1}$$

From the definition, we easily find out that

$$\mathbf{G}_{-m} = -\mathbf{G}_m, \quad \forall m \in \mathbb{Z}. \tag{2.2}$$

As shown by Dolan and Grady [5] and later proven by Davies [6], Onsager algebra is equivalent to the *Dolan–Grady (DG) relation*, which imposes requirement only on the first two generators, i.e.

$$\Big[\mathbf{A}_0, \Big[\mathbf{A}_0, \big[\mathbf{A}_0, \mathbf{A}_1\big]\Big]\Big] = 16\big[\mathbf{A}_0, \mathbf{A}_1\big], \quad \Big[\mathbf{A}_1, \Big[\mathbf{A}_1, \big[\mathbf{A}_1, \mathbf{A}_0\big]\Big]\Big] = 16\big[\mathbf{A}_1, \mathbf{A}_0\big]. \tag{2.3}$$

We will mainly use the DG relation in this article as the defining property of the existence of Onsager algebra. Namely, once finding two operators that satisfy DG relation in certain physical systems, we can construct a family of operators fulfilling the definition (2.1), which can be considered as a representation of Onsager algebra. For example, in the case of one-dimensional transverse field Ising model, i.e. the quantum counterpart of two-dimensional Ising model considered by Onsager, we have

$$\mathbf{A}_0 = \sum_{j=1}^{N} \sigma_j^z, \quad \mathbf{A}_1 = \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x, \tag{2.4}$$

satisfying DG relation (2.3), thus being a representation of Onsager algebra (2.1).

As demonstrated in [10], it is natural to rewrite the operators as

$$\mathbf{A}_m = \mathbf{A}_m^0 + \mathbf{A}_m^+ + \mathbf{A}_m^-, \tag{2.5}$$

where $\mathbf{A}_m^0$ commute with total magnetisation $\mathbf{A}_0^0 = \sum_{j=1}^N \sigma_j^z$, and $\mathbf{A}_m^\pm$ satisfy

$$\big[\mathbf{A}_0^0, \mathbf{A}_m^\pm\big] = \pm 4\mathbf{A}_m^\pm. \tag{2.6}$$

From relation (2.1), we obtain the following relations,

$$\mathbf{A}^0_{-m} = \mathbf{A}^0_m, \quad \mathbf{A}^\pm_{-m} = -\mathbf{A}^\pm_m, \tag{2.7}$$

$$[\mathbf{A}^r_m, \mathbf{A}^r_n] = 0,\ r \in \{0, +, -\}; \quad \left[\mathbf{A}^-_m, \mathbf{A}^+_n\right] = \mathbf{A}^0_{n+m} - \mathbf{A}^0_{n-m}, \tag{2.8}$$

$$\left[\mathbf{A}^-_m, \mathbf{A}^0_n\right] = 2\left(\mathbf{A}^-_{n+m} - \mathbf{A}^-_{n-m}\right), \quad \left[\mathbf{A}^+_m, \mathbf{A}^0_n\right] = 2\left(\mathbf{A}^+_{n-m} - \mathbf{A}^+_{n+m}\right). \tag{2.9}$$

Self-duality DG relation (2.3) also implies Kramers–Wannier self-duality [4], which has been used to obtain the value of the phase transition point of models possessing Onsager algebra. Suppose that the operators $\mathbf{A}_0$ and $\mathbf{A}_1$ can be expressed in terms of local terms, i.e.

$$\mathbf{A}_0 = \sum_{j=1}^{N} \mathbf{a}_{0,j}, \quad \mathbf{A}_1 = \sum_{j=1}^{N} \mathbf{a}_{1,j}. \tag{2.10}$$

Kramers–Wannier self-duality implies that the mapping $\mathbf{a}_{0,j} \to \mathbf{a}_{1,j}$ (and conversely $\mathbf{a}_{1,j} \to \mathbf{a}_{0,j}$) leaves the algebraic structure intact, which is obvious from DG relation (2.3). Let us consider a Hamiltonian $\mathbf{H}$ that can be expressed in terms of $\mathbf{A}_0$ and $\mathbf{A}_1$,

$$\mathbf{H} = \mathbf{A}_0 + \lambda \mathbf{A}_1, \quad \lambda \in \mathbb{R}, \tag{2.11}$$

such as 1-dimensional transverse field Ising model and chiral Potts model. Using Kramers–Wannier self-duality, one could detect a phase transition at $\lambda = 1$ without solving the entire system [4].

$U(1)$**-invariant Hamiltonian** In the remaining part of the article, we will focus on a specific type of Hamiltonians that commute with both $\mathbf{A}_0$ and $\mathbf{A}_1$, i.e.

$$[\mathbf{H}, \mathbf{A}_0] = [\mathbf{H}, \mathbf{A}_1] = 0. \tag{2.12}$$

These models are referred as $U(1)$-invariant [10], because both operators $\mathbf{A}_0$ and $\mathbf{A}_1$ are considered as $U(1)$ charges. However, $\mathbf{A}_0$ and $\mathbf{A}_1$ do not commute with each other, cf. (2.1).

From (2.1), it is easy to observe that $\mathbf{H}$ commutes with all generators in Onsager algebra,

$$[\mathbf{H}, \mathbf{A}^r_m] = 0, \quad r \in \{0, +, -\},\ m \in \mathbb{Z}. \tag{2.13}$$

Two examples of $U(1)$-invariant Hamiltonians are spin-1/2 XX model and spin-1 ZF model with anisotropy parameter $\eta = \frac{\mathrm{i}\pi}{3}$ or $\frac{2\mathrm{i}\pi}{3}$. These examples are $U(1)$-invariant clock models defined in [10]. These two cases will be examined in details in Secs. 4.1 and 6.1, respectively. As explained in latter sections, we conjecture that spin-1/2 XXZ model and spin-1 ZF model at arbitrary root of unity values of the anisotropy belong to the class of $U(1)$-invariant Hamiltonians that possess hidden Onsager algebra symmetries.

3 Spin-1/2 case: semi-cyclic transfer matrices in XXZ model

We consider the N-site quasi-periodic (periodic with twist ϕ) spin-1/2 XXZ Hamiltonian which can be expressed as

$$\mathbf{H}(\phi) = \sum_{j=1}^{N} \left(\frac{1}{2} \big(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ \big) + \frac{\Delta}{4} \left(\sigma_j^z \sigma_{j+1}^z - 1 \right) \right). \tag{3.1}$$

Here $\sigma_j^{\pm} := (\sigma_j^x \pm \mathrm{i}\, \sigma_j^y)/2$, $\sigma_j^{\alpha} = \mathbb{1}^{\otimes(j-1)} \otimes \sigma^{\alpha} \otimes \mathbb{1}^{\otimes(N-j)}$ denotes the αth Pauli matrix acting at site j and $\sigma_{N+1}^{\pm} = e^{\pm \mathrm{i}\phi} \sigma_1^{\pm}$. The Hamiltonian is hermitian provided the twist ϕ and anisotropy parameter Δ are real. The latter can be parametrised as

$$\Delta = \frac{q + q^{-1}}{2} = \cosh \eta, \qquad q = e^{\eta}. \tag{3.2}$$

The spin-1/2 XXZ Hamiltonian is integrable, allowing us to construct transfer matrices that commute with the Hamiltonian. It can be considered as the Hamiltonian limit of the 6-vertex model [25]. We use transfer matrices as the generating functions for infinitely many conserved charges for the Hamiltonian. Moreover, transfer matrices can be written in terms of Lax operator with auxiliary space labelled by a,

$$\begin{aligned} \mathbf{L}_{aj}(u) = & \sinh u \left(\frac{\mathbf{K}_a + \mathbf{K}_a^{-1}}{2} \right) \otimes \mathbb{1}_j + \cosh u \left(\frac{\mathbf{K}_a - \mathbf{K}_a^{-1}}{2} \right) \otimes \sigma_j^z \\ & + \sinh \eta \left(\mathbf{S}_a^+ \otimes \sigma_j^- + \mathbf{S}_a^- \otimes \sigma_j^+ \right). \end{aligned} \tag{3.3}$$

The operators in the auxiliary space a satisfy $\mathcal{U}_q(\mathfrak{sl}_2)$ algebra,

$$\mathbf{K}_a^2 \mathbf{S}_a^{\pm} \mathbf{K}_a^{-2} = q^{\pm 2} \mathbf{S}_a^{\pm}, \quad \left[\mathbf{S}_a^+, \mathbf{S}_a^- \right] = \frac{\mathbf{K}_a^2 - \mathbf{K}_a^{-2}}{q - q^{-1}}. \tag{3.4}$$

In this paper, we only consider the root of unity case, i.e.

$$\eta = \mathrm{i}\pi \frac{\ell_1}{\ell_2}, \quad q = \exp\left(\mathrm{i}\pi \frac{\ell_1}{\ell_2} \right), \quad \varepsilon = q^{\ell_2} = \pm 1, \tag{3.5}$$

where ℓ_1 and ℓ_2 are coprimes. It is obvious that parameter q is a root of unity with $q^{2\ell_2} = 1$. In this case, we use the ℓ_2-dimensional semi-cyclic representation for the auxiliary space. The transfer matrix is therefore denoted as $\mathbf{L}^{\mathrm{sc}}$. As proven in [24], the semi-cyclic Lax operator $\mathbf{L}_{aj}^{\mathrm{sc}}(u, s, \beta)$ satisfies the following "RLL" relation,

$$\mathbf{R}_{am}^{\mathrm{sc}}(u - v, s, \varepsilon\beta) \mathbf{L}_{an}^{\mathrm{sc}}(u, s, \varepsilon^2\beta) \mathbf{L}_{mn}(v) = \mathbf{L}_{mn}(v) \mathbf{L}_{an}^{\mathrm{sc}}(u, s, \varepsilon\beta) \mathbf{R}_{am}^{\mathrm{sc}}(u - v, s, \varepsilon^2\beta), \tag{3.6}$$

with $\beta \in \mathbb{C}$. Hilbert spaces m and n both correspond to $\mathbb{C}^2$, and R matrix $\mathbf{R}_{am}^{\mathrm{sc}}(u) = \mathbf{L}_{am}^{\mathrm{sc}}(u + \eta/2)$. When $\beta = 0$, the ℓ_2-dimensional semi-cyclic representation becomes ℓ_2-dimensional highest weight representation, cf. Appendix A in [24].

Therefore, we define the monodromy matrix for a system with N sites and twist ϕ

$$\mathbf{M}_s^{\mathrm{sc}}(u, \beta, \phi) = \mathbf{L}_{aN}^{\mathrm{sc}}(u, s, \varepsilon^N \beta) \cdots \mathbf{L}_{aj}^{\mathrm{sc}}(u, s, \varepsilon^j \beta) \cdots \mathbf{L}_{a1}^{\mathrm{sc}}(u, s, \varepsilon\beta) \mathbf{E}_a(\phi), \tag{3.7}$$

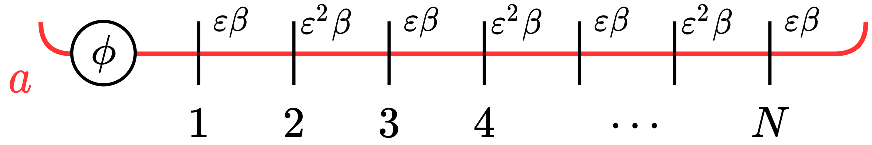

Figure 1: A pictorial illustration of semi-cyclic transfer matrix $\mathbf{T}^{\rm sc}$. The auxiliary space is denoted as a, while ϕ corresponds to the twist matrix. Different values of parameter β in semi-cyclic representation are written explicitly next to each physical sites.

with twist operator $\mathbf{E}_a(\phi) = \sum_{j=0}^{\ell_2-1} e^{{\rm i}\phi j}|j\rangle_a\langle j|_a$ acting in the auxiliary space. The twist ϕ considered here is *commensurate*, cf. (3.15). As shown in [24], commensurate value of the twist ϕ leads to the descendant tower structure and Onsager algebra due to the existence of eigenstates associated with exact (Fabricius-McCoy) strings.

The transfer matrices are defined as

$$\mathbf{T}_s^{\rm sc}(u,\beta,\phi) = {\rm tr}_a\mathbf{M}_s^{\rm sc}(u,\beta,\phi), \tag{3.8}$$

depicted in Fig. 1.

In addition to the generating functions for (quasi-)local charges [26–29] associated with $(2s+1)$-dimensional spin-s representation ($2s \in \mathbb{Z}_{>0}$), cf. the constructions in [24], we define two generating functions for quasilocal Z and Y charges,

$$\mathbf{Z}(u,\phi) = \frac{1}{2\eta}\,\partial_s \log \mathbf{T}_s^{\rm sc}(u,\beta,\phi)|_{s=(\ell_2-1)/2,\beta=0}\,, \tag{3.9}$$

and

$$\mathbf{Y}(u,\phi) = \frac{1}{2\sinh\eta}\,\partial_\beta \log \mathbf{T}_s^{\rm sc}(u,\beta,\phi)|_{s=(\ell_2-1)/2,\beta=0}\,, \tag{3.10}$$

where the prefactors are chosen for later convenience in Sec. 2. The significance of the quasilocality of conserved charges in the thermodynamic limit is beyond the scope of this work, and we refer the readers to [30–32] for detailed discussions.

From the "RLL" relation, we could show that these two operators are in involution with themselves respectively [24],

$$[\mathbf{Z}(u,\phi),\mathbf{Z}(v,\phi)] = [\mathbf{Y}(u,\phi),\mathbf{Y}(v,\phi)] = 0, \quad u,v\in\mathbb{C}, \tag{3.11}$$

and they commute with the Hamiltonian

$$[\mathbf{Z}(u,\phi),\mathbf{H}(\phi)] = [\mathbf{Y}(u,\phi),\mathbf{H}(\phi)] = 0, \quad u\in\mathbb{C}. \tag{3.12}$$

As for the commutation relations with transfer matrix with auxiliary space as $(2s+1)$-dimensional spin s representation, we have

$$[\mathbf{Z}(u,\phi),\mathbf{T}_s(v,\phi)] = [\mathbf{Y}(u,\phi),\mathbf{T}_s(v,\phi)] = 0, \quad \varepsilon=+1, \quad 2s\in\mathbb{Z}_{>0},\, u,v\in\mathbb{C}; \tag{3.13}$$

$$\begin{aligned} &[\mathbf{Z}(u,\phi),\mathbf{T}_s(v,\phi)] = [\mathbf{Y}(u,\phi),\mathbf{T}_s(v,\phi)] = 0,\, \varepsilon=-1,\, s\in\mathbb{Z}_{>0},\, u,v\in\mathbb{C},\\ &[\mathbf{Z}(u,\phi),\mathbf{T}_{s'}(v,\phi)] = \{\mathbf{Y}(u,\phi),\mathbf{T}_{s'}(v,\phi)\} = 0,\, \varepsilon=-1,\, s'\in\frac{\mathbb{Z}_{>0}}{2}\setminus\mathbb{Z}_{>0},\, u,v\in\mathbb{C}. \end{aligned} \tag{3.14}$$

The anti-commutation between $\mathbf{Y}(u,\phi)$ and $\mathbf{T}_{s'}(v,\phi)$ when s' is a half-integer might seem confusing. The reason is that $\mathbf{Y}(u,\phi)$ anti-commute with momentum operator when $\varepsilon=-1$, that is $\mathbf{Y}(u,\phi)$ is not translational invariant but 2-site translationally invariant which would be clear in Sec. 4.1. This means acting by $\mathbf{Y}(u,\phi)$ would change momentum of a state by π, resulting in the anti-commutation relation in (3.14). Moreover, operators $\mathbf{Z}(u,\phi)$ and $\mathbf{Y}(u,\phi)$ do not commute, which is closely related to the conjectured Onsager algebra, cf. Sec. 4.2.

Note that generating Y charges satisfy the properties (3.13) or (3.14) only when the twist ϕ is commensurate [24] which depends on the parameters ε and N,

$$\begin{aligned} \varepsilon^N &= +1 \;\Rightarrow\; \phi=\frac{(2n-2)\pi}{\ell_2}, \\ \varepsilon^N &= -1 \;\Rightarrow\; \phi=\frac{(2n-1)\pi}{\ell_2}, \end{aligned} \qquad 1\le n\le \ell_2,\; n\in\mathbb{N}. \tag{3.15}$$

It is worth emphasising that $\phi=0$ (with no twist) is commensurate when $\varepsilon^N=+1$, which has been considered previously in [18–20].

Quasilocal Z and Y charges [29,33] can be obtained by expanding the generating functions at $u=u_0$, i.e.

$$\mathbf{Z}(u,\phi)=\sum_{n=0}^{\infty}(u-u_0)^n\,\mathbf{Z}_n, \quad \mathbf{Y}(u,\phi)=\sum_{n=0}^{\infty}(u-u_0)^n\,\mathbf{Y}_n, \tag{3.16}$$

where

$$\varepsilon=-1 \;\Rightarrow\; u_0=\frac{\eta}{2}; \quad \varepsilon=+1 \;\Rightarrow\; u_0=\frac{\eta-\mathrm{i}\pi}{2}. \tag{3.17}$$

The reason for the different values of u_0 with respect to different ε values is given in Appendix A.

Generating function $\mathbf{Z}(u,\phi)$ is closely related to the Q operator and 2-parameter transfer matrix for XXZ model at root of unity [24], while generating function $\mathbf{Y}(u,\phi)$ is conjectured to be the creation operator for exact (Fabricius–McCoy) strings in [24].

From (3.11), all Z or Y charges are in involution with each other,

$$[\mathbf{Z}_m,\mathbf{Z}_n]=[\mathbf{Y}_m,\mathbf{Y}_n]=0, \quad m,n\in\mathbb{Z}_{\ge 0}, \tag{3.18}$$

and they can be expressed as

$$\mathbf{Z}_n=\frac{1}{n!}\partial_u^n\,\mathbf{Z}(u,\phi)|_{u=u_0}, \quad \mathbf{Y}_n=\frac{1}{n!}\partial_u^n\,\mathbf{Y}(u,\phi)|_{u=u_0}, \tag{3.19}$$

with the first terms

$$\mathbf{Z}_0=\mathbf{Z}(u_0,\phi), \quad \mathbf{Y}_0=\mathbf{Y}(u_0,\phi). \tag{3.20}$$

4 Onsager algebra symmetry in spin-$1/2$ XXZ model at root of unity

It is well-known that XX model (XXZ model with $\Delta=0$) is a $U(1)$-invariant Hamiltonian that possesses Onsager algebra symmetry, cf. (2.13). We start this section by discussing the known results in the XX case. Afterwards, we generalise the results by conjecturing the existence and properties of hidden Onsager algebra symmetries for XXZ model at arbitrary root of unity.

4.1 Example: XX model

We illustrate the relation between Onsager generators and semi-cyclic transfer matrix $\mathbf{T}_s^{\text{sc}}(u,\beta,\phi)$ using the example of spin-1/2 XX model. Twist ϕ considered here is always commensurate, satisfying (3.15). When system size N is even, $\phi \in \{0,\pi\}$, and $\phi \in \{\pi/2, 3\pi/2\}$ with system size N being odd.

The XX Hamiltonian is [1]

$$\mathbf{H}_{\text{XX}} = \sum_{j=1}^{N} \frac{1}{2}\big(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+\big), \tag{4.1}$$

where the Onsager generators are

$$\mathbf{Q}_0^0 = \frac{1}{2}\sum_{j=1}^{N} \sigma_j^z, \quad \mathbf{Q}_0^\pm = 0, \quad \mathbf{Q}_0 = \mathbf{Q}_0^0 + \mathbf{Q}_0^+ + \mathbf{Q}_0^-, \tag{4.2}$$

$$\begin{aligned} \mathbf{Q}_1^0 &= \frac{\mathrm{i}}{2}\sum_{j=1}^{N}\left(\sigma_j^+\sigma_{j+1}^- - \sigma_j^-\sigma_{j+1}^+\right), \quad \mathbf{Q}_1^+ = -\frac{\mathrm{i}}{2}\sum_{j=1}^{N}(-1)^j\sigma_j^+\sigma_{j+1}^+, \\ \mathbf{Q}_1^- &= \frac{\mathrm{i}}{2}\sum_{j=1}^{N}(-1)^j\sigma_j^-\sigma_{j+1}^-, \quad \mathbf{Q}_1 = \mathbf{Q}_1^0 + \mathbf{Q}_1^+ + \mathbf{Q}_1^-. \end{aligned} \tag{4.3}$$

They satisfy DG relation

$$\Big[\mathbf{Q}_0, \Big[\mathbf{Q}_0, \big[\mathbf{Q}_0, \mathbf{Q}_1\big]\Big]\Big] = \ell_2^2\big[\mathbf{Q}_0, \mathbf{Q}_1\big], \quad \Big[\mathbf{Q}_1, \Big[\mathbf{Q}_1, \big[\mathbf{Q}_1, \mathbf{Q}_0\big]\Big]\Big] = \ell_2^2\big[\mathbf{Q}_1, \mathbf{Q}_0\big], \tag{4.4}$$

with $\ell_2 = 2$. It is easy to check that $\mathbf{H}_{\text{XX}}$ is $U(1)$ invariant, i.e. commuting with all Onsager generators,

$$[\mathbf{H}_{\text{XX}}, \mathbf{Q}_m^r] = 0, \quad r \in \{0,+,-\}, \quad m \in \mathbb{Z}. \tag{4.5}$$

The relation to the canonical generators is

$$\mathbf{Q}_m^r = \frac{4}{\ell_2}\mathbf{A}_m^r, \quad m \in \mathbb{Z}, \quad r \in \{0,+,-\}. \tag{4.6}$$

The Onsager generators in terms of local spin operators are given in Appendix B.

What is truly striking here is that the Onsager generators $\mathbf{Q}_0^r$ can be identified with the Z and Y charges $\mathbf{Z}_0$ and $\mathbf{Y}_0$, i.e.

$$\mathbf{Q}_1^0 = \mathbf{Z}_0, \quad \mathbf{Q}_1^- = \mathbf{Y}_0, \quad \mathbf{Q}_1^+ = \mathbf{Y}_0^\dagger. \tag{4.7}$$

This relation has been pointed out in [10], and all operators $\mathbf{Q}_n^r$, $\mathbf{Z}_n$ and $\mathbf{Y}_n$ can be written in terms of bilinear fermion operators, reflecting the free fermion nature of XX model. One of the consequences is that there exists a closure condition for Onsager generators in XX model, namely

$$\mathbf{Q}_{n+2N}^r = \mathbf{Q}_n^r, \quad \forall n \in \mathbb{Z}. \tag{4.8}$$

[1] The twist ϕ is included in the Hamiltonian through the definition of $\sigma_{L+1}^\pm$, explained in the line below (3.1).

The discussion of the physical meaning of the closure condition is postponed to Sec. 4.3. Since there are infinitely many $\mathbf{Q}_n^r$, $\mathbf{Z}_n$ and $\mathbf{Y}_n$, one might wonder the relation between all of them. In fact, all operators can be obtained recursively, and the first few read

$$\begin{aligned} &\mathbf{Z}_1 = \frac{1}{1!}\left(\mathbf{Q}_2^0 - \mathbf{Q}_0^0\right), \quad \mathbf{Z}_2 = \frac{1}{2!}\left(2\mathbf{Q}_3^0 - \mathbf{Q}_1^0\right), \quad \mathbf{Z}_3 = \frac{1}{3!}\left(6\mathbf{Q}_4^0 - 8\mathbf{Q}_2^0 + 2\mathbf{Q}_0^0\right), \\ &\mathbf{Y}_1 = \frac{1}{1!}\left(\mathbf{Q}_2^- - \mathbf{Q}_0^-\right), \quad \mathbf{Y}_2 = \frac{1}{2!}\left(2\mathbf{Q}_3^- - \mathbf{Q}_1^-\right), \quad \mathbf{Y}_3 = \frac{1}{3!}\left(6\mathbf{Q}_4^- - 8\mathbf{Q}_2^- + 2\mathbf{Q}_0^-\right), \end{aligned} \tag{4.9}$$

revealing a deep connection between the Onsager generators $\mathbf{Q}_n^r$ and charges $\mathbf{Z}_n$, $\mathbf{Y}_n$. Relations between higher order terms can be obtained similarly.

4.2 Conjectures on hidden Onsager algebra symmetries in spin-1/2 XXZ models at root of unity

Motivated by the exact correspondence between the Onsager generators $\mathbf{Q}_0^r$ and Z and Y charges in the XX case ($\eta = \mathrm{i}\pi/2$), cf. (4.7) and (4.9), it is natural to generalise similar relations for the spin-1/2 XXZ model at arbitrary root of unity ($q = \exp(\mathrm{i}\pi\ell_1/\ell_2)$), despite that the generators are no longer able to be expressed in local densities but quasilocal ones [26–29]. Another motivation to the following conjectures is that the structure of descendant towers and exact (Fabricius–McCoy) strings are of no difference between XX model and XXZ model at other root of unity. After numerically verifying the relations, we are able to compose conjectures for the existence of hidden Onsager algebra symmetry in spin-1/2 XXZ spin chain at arbitrary root of unity, which are as follows:

Conjecture I:
There exists a hidden Onsager algebra symmetry in spin-1/2 XXZ spin chain at root of unity with commensurate twist (3.15). Spin-1/2 XXZ models at root of unity with commensurate twist (3.15) are $U(1)$ invariant, i.e.

$$[\mathbf{H}(\phi), \mathbf{Q}_m^r] = 0, \quad r \in \{0, +, -\}, \quad m \in \mathbb{Z}. \tag{4.10}$$

The generators of the hidden Onsager algebra are

$$\mathbf{Q}_0^0 = \frac{1}{2}\sum_{j=1}^{N} \sigma^z, \quad \mathbf{Q}_0^\pm = 0, \tag{4.11}$$

$$\begin{aligned} &\mathbf{Q}_1^0 = \mathbf{Z}_0 = \frac{1}{2\eta}\partial_s \log \mathbf{T}_s^{\mathrm{sc}}(u,\beta,\phi)\Big|_{s=(\ell_2-1)/2, u=u_0, \beta=0}, \\ &\mathbf{Q}_1^- = \mathbf{Y}_0 = \frac{1}{2\sinh\eta}\partial_\beta \log \mathbf{T}_s^{\mathrm{sc}}(u,\beta,\phi)\Big|_{s=(\ell_2-1)/2, u=u_0, \beta=0} = \left(\mathbf{Q}_1^+\right)^\dagger, \end{aligned} \tag{4.12}$$

which satisfy DG relation

$$\Big[\mathbf{Q}_0, \Big[\mathbf{Q}_0, \big[\mathbf{Q}_0, \mathbf{Q}_1\big]\Big]\Big] = \ell_2^2\big[\mathbf{Q}_0, \mathbf{Q}_1\big], \quad \Big[\mathbf{Q}_1, \Big[\mathbf{Q}_1, \big[\mathbf{Q}_1, \mathbf{Q}_0\big]\Big]\Big] = \ell_2^2\big[\mathbf{Q}_1, \mathbf{Q}_0\big], \tag{4.13}$$

with any $\ell_2 \in \mathbb{Z}_{>0}$. Notice that the definition of semi-cyclic transfer matrix $\mathbf{T}_s^{\mathrm{sc}}(u,\beta,\phi)$ depends on the root of unity through $\varepsilon = \pm 1$.

Similar to (4.9), the higher order Onsager generators are conjectured to be related to the higher order Z and Y charges.

Conjecture II:
In general, higher order Z and Y charges are functions of higher order Onsager generators such that

$$\begin{aligned}\mathbf{Z}_n &= \frac{1}{n!}\left(\frac{\ell_2}{2}\right)^n \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} c_j^n \mathbf{Q}^0_{(n+1)-2j},\\ \mathbf{Y}_n &= \frac{1}{n!}\left(\frac{\ell_2}{2}\right)^n \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} c_j^n \mathbf{Q}^-_{(n+1)-2j}, \quad n \in \mathbb{Z},\end{aligned} \tag{4.14}$$

where $c_j^n \in \mathbb{N}$. The first three terms $\mathbf{Z}_n$ and $\mathbf{Y}_n$, $n \in \{1,2,3\}$ can be expressed as

$$\begin{aligned}&\mathbf{Z}_1 = \frac{1}{1!}\frac{\ell_2}{2}\left(\mathbf{Q}^0_2 - \mathbf{Q}^0_0\right), \quad \mathbf{Z}_2 = \frac{1}{2!}\left(\frac{\ell_2}{2}\right)^2\left(2\mathbf{Q}^0_3 - \mathbf{Q}^0_1\right),\\ &\mathbf{Z}_3 = \frac{1}{3!}\left(\frac{\ell_2}{2}\right)^3\left(6\mathbf{Q}^0_4 - 8\mathbf{Q}^0_2 + 2\mathbf{Q}^0_0\right),\\ &\mathbf{Y}_1 = \frac{1}{1!}\frac{\ell_2}{2}\left(\mathbf{Q}^-_2 - \mathbf{Q}^-_0\right), \quad \mathbf{Y}_2 = \frac{1}{2!}\left(\frac{\ell_2}{2}\right)^2\left(2\mathbf{Q}^-_3 - \mathbf{Q}^-_1\right),\\ &\mathbf{Y}_3 = \frac{1}{3!}\left(\frac{\ell_2}{2}\right)^3\left(6\mathbf{Q}^-_4 - 8\mathbf{Q}^-_2 + 2\mathbf{Q}^-_0\right).\end{aligned} \tag{4.15}$$

Conjecture I (4.12) and part of Conjecture II (4.15) are proven for the case of XX model ($\ell_2 = 2$), illustrated in Sec. 4.1. (4.14) of Conjecture II can be considered as a generalisation of (4.15). Conjectures I and II, cf. (4.12) and (4.15) have been verified numerically for cases whose roots of unity satisfy $\ell_2 = 3, 4, 5$ and all permitted values of ℓ_1 with system size N up to 12. The numerical evidence is convincing that Conjectures I and II are true for arbitrary root of unity value of the anisotropy and system size.

4.3 Closure condition: free v.s. interacting

Let us assume that the conjectures above are true. One might wonder the question about the physical difference between XX model and XXZ model at root of unity other than $\exp(\mathrm{i}\pi/2)$. On the one hand, they all possess Onsager algebra symmetries, which are identical on the level of algebraic structure; on the other hand, XX model permits a free fermionic description [10], while XXZ models at other roots of unity do not, because of their intrinsically interacting nature [14].

The Onsager generators $\mathbf{Q}_m$ of different models in consideration can be regarded as different representations of Onsager algebra. Even though the algebraic structure of those generators is identical, they still might have different properties. All the Onsager generators are bilinear in fermionic operators after Jordan–Wigner transformation [10]. Hence, only the representation associated with $\eta = \mathrm{i}\pi/2$ has the free fermionic behaviour, guaranteed by the closure condition (4.8) [10, 11]. For other roots of unity $q = \exp\left(\mathrm{i}\pi\frac{\ell_1}{\ell_2}\right) \neq \exp\left(\mathrm{i}\pi/2\right)$, the closure condition is no longer satisfied, since these models are interacting. This observation means that we cannot transform all the nice properties and methods used for XX model to XXZ models at other roots of unity.

Remark. Historically speaking, Onsager solved the partition function of two-dimensional Ising model using both commutation relations between Onsager generators (2.1) and the closure condition (4.8) [1]. Both of them are crucial to the exact solution.

5 Spin-1 case: transfer matrix fusion

We can generalise the construction in the spin-1/2 case by considering the spin-1 generalisation of XXZ model (3.1), i.e. ZF model [34]. Spin-1 ZF model is the Hamiltonian limit to Izergin-Korepin 19-vertex model [35], permitting a transfer matrix formalism. The N-site quasi-periodic ZF model is

$$\begin{aligned}\mathsf{H}_{\mathrm{ZF}}(\eta,\phi) = \sum_{j=1}^{N}\Big(&\left[\mathsf{S}_j^x\mathsf{S}_{j+1}^x + \mathsf{S}_j^y\mathsf{S}_{j+1}^y + \cosh(2\eta)\mathsf{S}_j^z\mathsf{S}_{j+1}^z\right] \\ &+ 2\left[\left(\mathsf{S}_j^x\right)^2 + \left(\mathsf{S}_j^y\right)^2 + \cosh(2\eta)\left(\mathsf{S}_j^x\right)^2\right] - \sum_{a,b}\mathsf{A}_{ab}(\eta)\mathsf{S}_j^a\mathsf{S}_j^b\mathsf{S}_{j+1}^a\mathsf{S}_{j+1}^b\Big),\end{aligned} \tag{5.1}$$

where coefficients $\mathsf{A}_{ab} = \mathsf{A}_{ba}$ take the values of

$$\mathsf{A}_{xx} = \mathsf{A}_{yy} = 1,\ \mathsf{A}_{zz} = \cosh(2\eta),\ \mathsf{A}_{xy} = 1,\ \mathsf{A}_{xz} = \mathsf{A}_{yz} = 2\cosh\eta - 1. \tag{5.2}$$

The spin-1 operators are

$$\begin{aligned}&\mathsf{S}^x = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad \mathsf{S}^y = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -\mathrm{i} & 0\\ \mathrm{i} & 0 & -\mathrm{i}\\ 0 & \mathrm{i} & 0\end{pmatrix}, \\ &\mathsf{S}^z = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}, \quad \mathsf{S}^{\pm} = \mathsf{S}^x \pm \mathrm{i}\mathsf{S}^-,\end{aligned} \tag{5.3}$$

and $\mathsf{S}_j^\alpha = \mathbb{1}_3^{\otimes(j-1)} \otimes \mathsf{S}^\alpha \otimes \mathbb{1}_3^{\otimes(N-j)}$. The twist is encoded in the relation $\mathsf{S}_{N+1}^{\pm} = e^{\pm\mathrm{i}\phi}\mathsf{S}_1^{\pm}$.

The anisotropy parameter η can be re-parametrised in terms of $q = \exp\eta$. At root of unity value $q = \exp\left(\mathrm{i}\pi\frac{\ell_1}{\ell_2}\right)$ with ℓ_1 and ℓ_2 being coprimes, we define parameter $\varepsilon = q^{\ell_2} = \pm 1$.

When $\eta = 0$, i.e. the isotropic limit, $\mathsf{H}_{\mathrm{ZF}} \propto \sum_j \vec{\mathsf{S}}_j \cdot \vec{\mathsf{S}}_{j+1} - (\vec{\mathsf{S}}_j \cdot \vec{\mathsf{S}}_{j+1})^2$. This is known in the literature as spin-1 Takhtajan–Babujian model [36–38], which can be obtained through transfer matrix fusion of spin-1/2 XXX model.

Spin-1 ZF model is integrable in the same way as spin-1/2 XXZ model, and its Lax operator can be obtained through transfer matrix fusion relation [39,40]. What has not been obtained previously is the semi-cyclic transfer matrix for ZF model with $\varepsilon = -1$. Here we perform the transfer matrix fusion procedure for semi-cyclic transfer matrix with arbitrary $\varepsilon = \pm 1$ at root of unity.

The fusion procedure for Lax operator can be described pictorially in Fig. 2.

Equivalently, in terms of formulae, we express the semi-cyclic Lax operator $\mathsf{L}_{aj}^{\mathrm{sc}}$ as

$$\begin{aligned}\mathsf{L}_{aj}^{\mathrm{sc}}(u,s,\beta) = \frac{1}{\sinh^2\eta}(\mathbb{1}_a \otimes \mathsf{M}_j)(\mathbb{1}_a \otimes \mathsf{P}_{mn})\mathbf{L}_{am}^{\mathrm{sc}}(u-\eta/2,s,\varepsilon\beta)\\ \mathbf{L}_{an}^{\mathrm{sc}}(u+\eta/2,s,\varepsilon^2\beta)(\mathbb{1}_a \otimes \mathsf{M}_j^{-1})(\mathbb{1}_a \otimes \mathsf{P}_{mn}^{\mathrm{T}}),\end{aligned} \tag{5.4}$$

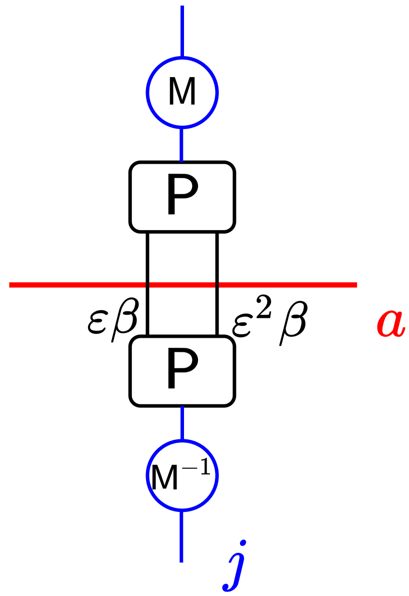

Figure 2: The semi-cyclic Lax operator of ZF model is expressed as a fusion of two Lax operators of spin-1/2 XXZ model, cf. (5.4). Blue lines correspond to physical Hilbert space with spin-1 representation of $\mathfrak{su}_2$ algebra, which are obtained through the fusion of two spin-1/2 representations denoted as black lines.

where P operator projects operators in Hilbert space mn (spin-$\frac{1}{2}\otimes\frac{1}{2}$) to operators in Hilbert space j (spin-1) and M operator fixes the normalisation of Lax operator, i.e.

$$\mathsf{P}_{mn} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{mn}, \quad \mathsf{M}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{[2]}}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}_j . \tag{5.5}$$

Here q-number is defined as $[x] = (q^x - q^{-x})/(q - q^{-1})$.

When $\varepsilon = +1$, (5.4) reproduces the known result [10],

$$\mathsf{L}^{\mathrm{sc}}_{aj}(u,s,\beta) = \begin{pmatrix} [\frac{u}{\eta} - \frac{1}{2} + \mathbf{S}^z_a][\frac{u}{\eta} + \frac{1}{2} + \mathbf{S}^z_a] & \mathbf{S}^-_a[\frac{u}{\eta} - \frac{1}{2} + \mathbf{S}^z_a] & (\mathbf{S}^-)^2 \\ \mathbf{S}^+_a[\frac{u}{\eta} + \frac{1}{2} + \mathbf{S}^z_a] & \mathbf{S}^+_a\mathbf{S}^-_a + [\frac{u}{\eta} + \frac{1}{2} + \mathbf{S}^z_a][\frac{u}{\eta} - \frac{1}{2} - \mathbf{S}^z_a] & \mathbf{S}^-_a[\frac{u}{\eta} - \frac{3}{2} + \mathbf{S}^z_a] \\ (\mathbf{S}^+_a)^2 & \mathbf{S}^+_a[\frac{u}{\eta} - \frac{1}{2} - \mathbf{S}^z_a] & [\frac{u}{\eta} + \frac{1}{2} - \mathbf{S}^z_a][\frac{u}{\eta} - \frac{1}{2} - \mathbf{S}^z_a] \end{pmatrix}_j . \tag{5.6}$$

However, when $\varepsilon = -1$, relation (5.6) no longer holds. One should use (5.4) instead.

It is straightforward to show the RLL relation for the semi-cyclic Lax operator $\mathsf{L}^{\mathrm{sc}}_{aj}(u,s,\beta)$,

$$\mathsf{R}^{\mathrm{sc}}_{aj}(u-v,s,\beta)\mathsf{L}^{\mathrm{sc}}_{ak}(u,s,\beta)\mathsf{L}_{jk}(v) = \mathsf{L}_{jk}(v)\mathsf{L}^{\mathrm{sc}}_{ak}(u,s,\beta)\mathsf{R}^{\mathrm{sc}}_{am}(u-v,s,\beta), \tag{5.7}$$

where $\beta \in \mathbb{C}$, and R matrix $\mathsf{R}^{\mathrm{sc}}_{aj}(u) = \mathsf{L}^{\mathrm{sc}}_{aj}(u+\eta/2)$. The proof is constructive and it is shown in Fig. 3.

We could consider spin-1 ZF model as a fused spin-1/2 XXZ model with even site. Hence, only the first condition of (3.15) remains for spin-1 ZF model. We have

$$\varepsilon = \pm 1 \ \Rightarrow \ \phi = \frac{(2n-2)\pi}{\ell_2}, \quad 1 \le n \le \ell_2, \ n \in \mathbb{N}. \tag{5.8}$$

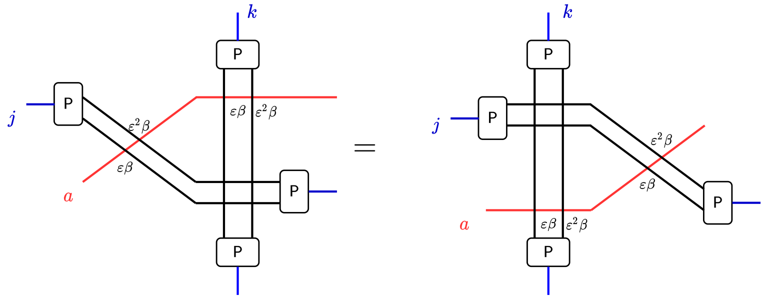

Figure 3: A pictorial proof of (5.7). We omit the M matrix parts in the Lax operator (5.4), which do not change the result.

We define the monodromy matrix accordingly

$$\mathsf{M}_s^{\mathrm{sc}}(u,\beta,\phi) = \mathsf{L}_{aN}^{\mathrm{sc}}(u,s,\beta)\cdots\mathsf{L}_{aj}^{\mathrm{sc}}(u,s,\beta)\cdots\mathsf{L}_{a1}^{\mathrm{sc}}(u,s,\beta)\mathbf{E}_a(\phi), \tag{5.9}$$

and semi-cyclic transfer matrix becomes

$$\mathsf{T}_s^{\mathrm{sc}}(u,\beta,\phi) = \mathrm{tr}_a \mathsf{M}_s^{\mathrm{sc}}(u,\beta,\phi). \tag{5.10}$$

Precisely the same as the spin-1/2 case, we define two generating functions for quasilocal Z and Y charges [40],

$$\mathsf{Z}(u,\phi) = \frac{1}{2\eta}\,\partial_s \log \mathsf{T}_s^{\mathrm{sc}}(u,\beta,\phi)|_{s=(\ell_2-1)/2,\beta=0}\,, \tag{5.11}$$

and

$$\mathsf{Y}(u,\phi) = \frac{1}{2\sinh\eta}\,\partial_\beta \log \mathsf{T}_s^{\mathrm{sc}}(u,\beta,\phi)|_{s=(\ell_2-1)/2,\beta=0}\,. \tag{5.12}$$

More importantly, from the RLL relation (5.7), $\mathsf{Z}(u,\phi)$ and $\mathsf{Y}(u,\phi)$ satisfy identical relations as their counterparts in spin-1/2 case, cf. Eqs. (3.11), (3.13) and (3.14). We shall not recite them again here.

We expand the generating functions at spectral parameter $u = \eta/2$, i.e.

$$\mathsf{Z}(u,\phi) = \sum_{n=0}^{\infty}\left(u-\frac{\eta}{2}\right)^n \mathsf{Z}_n, \quad \mathsf{Y}(u,\phi) = \sum_{n=0}^{\infty}\left(u-\frac{\eta}{2}\right)^n \mathsf{Y}_n. \tag{5.13}$$

Identical to the spin-1/2 counterparts, Z or Y charges in spin-1 case are in involution with each other respectively,

$$[\mathsf{Z}_m,\mathsf{Z}_n] = [\mathsf{Y}_m,\mathsf{Y}_n] = 0, \quad m,n\in\mathbb{Z}_{\geq 0}. \tag{5.14}$$

They are expressed as

$$\mathsf{Z}_n = \frac{1}{n!}\partial_u^n\,\mathsf{Z}(u,\phi)|_{u=\eta/2}\,, \quad \mathsf{Y}_n = \frac{1}{n!}\partial_u^n\,\mathsf{Y}(u,\phi)|_{u=\eta/2}\,. \tag{5.15}$$

The first terms are

$$\mathsf{Z}_0 = \mathsf{Z}\left(\frac{\eta}{2},\phi\right), \quad \mathsf{Y}_0 = \mathsf{Y}\left(\frac{\eta}{2},\phi\right), \tag{5.16}$$

which are important when constructing Onsager generators, cf. Sec. (6.1).

It has been shown in [40] that for arbitrary root of unity, $\mathsf{Z}(u,\phi)$ and $\mathsf{Y}(u,\phi)$ are quasilocal and thus Z_m and Y_m are quasilocal too. One special case is at $\eta = \frac{\mathrm{i}\pi}{3}$, when Z_m and Y_m can be written in terms of local operators [10], which we will exploit in Sec. 6.1.

6 Onsager algebra symmetry in spin-1 ZF model at root of unity

We proceed in analogue to the spin-1/2 case. We start with an example of spin-1 ZF model with $\eta = \mathrm{i}\pi/3$, which possesses Onsager algebra symmetry in terms of operators with local density in spite of its interacting nature. This model is also known as spin-1 $U(1)$-invariant clock model in [10]. Furthermore, we compose the conjectures for the hidden Onsager algebra symmetries for spin-1 ZF models at arbitrary root of unity values of the anisotropy, similar to the spin-1/2 case.

6.1 Example: spin-1 $U(1)$-invariant clock model

We concentrate on the case of spin-1 $U(1)$-invariant clock model in this section. We introduce three additional operators for the later convenience, i.e.

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \mathcal{S}^+ = \sum_{k=1}^{2} \mathbf{e}^{k,k+1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left(\mathcal{S}^-\right)^\dagger, \tag{6.1}$$

where $\omega = \exp(2\mathrm{i}\pi/3)$ and matrix $(\mathbf{e}^{ab})_{cd} = \delta^a_c \delta^b_d$.

We rewrite the Hamiltonian of spin-1 $U(1)$-invariant clock model, i.e. ZF Hamiltonian with $\eta = \frac{\mathrm{i}\pi}{3}$ in terms of operators defined above, i.e.

$$\begin{aligned} \mathsf{H}_{\mathrm{ZF}}(\phi) = -\sum_{j=1}^{N}\sum_{a=1}^{2} \Big[(-1)^a \left(\mathcal{S}_j^- \mathcal{S}_{j+1}^+\right)^a + (-1)^a \left(\mathcal{S}_j^+ \mathcal{S}_{j+1}^-\right)^a \\ +1 - \frac{2a}{3} e^{\mathrm{i}\pi a/3} \tau_j^a \Big] + \text{const}, \quad \eta = \frac{\mathrm{i}\pi}{3}, \end{aligned} \tag{6.2}$$

where $\mathcal{S}_j^\pm = \mathbb{1}_3^{\otimes(j-1)} \otimes \mathcal{S}^\pm \otimes \mathbb{1}_3^{\otimes(N-j)}$, $\tau_j = \mathbb{1}_3^{\otimes(j-1)} \otimes \tau \otimes \mathbb{1}_3^{\otimes(N-j)}$ and $\mathcal{S}^\pm_{L+1} = e^{\mathrm{i}\phi}\mathcal{S}^\pm_1$. The Hamiltonian (6.2) is the same as Eq. (2.9) in [10] up to a unitary gauge transformation, and it can be seen as a generalisation of the spin-1/2 XX model [10].

As explained in [10], the Hamiltonian (6.2) is a $U(1)$ invariant Hamiltonian that possesses Onsager algebra symmetry (2.13),

$$[\mathsf{H}_{\mathrm{ZF}}(\phi), \mathsf{Q}^r_m] = 0, \quad r \in \{0, +, -\}, \quad m \in \mathbf{Z}. \tag{6.3}$$

The generators in terms of operators (6.1) are

$$\mathsf{Q}^0_0 = \sum_{j=1}^{N} \mathsf{S}^z_j, \quad \mathsf{Q}^\pm_0 = 0, \quad \mathsf{Q}_0 = \mathsf{Q}^0_0 + \mathsf{Q}^+_0 + \mathbf{Q}^-_0, \tag{6.4}$$

$$\begin{aligned} \mathsf{Q}^0_1 &= \sum_{j=1}^{N}\sum_{a=1}^{2} \frac{\omega^a}{1-\omega^{-a}} \left[\left(\mathcal{S}_j^- \mathcal{S}_{j+1}^+\right)^a - \left(\mathcal{S}_j^+ \mathcal{S}_{j+1}^-\right)^a \right], \quad \mathsf{Q}^+_1 = \left(\mathsf{Q}^-_1\right)^\dagger, \\ \mathsf{Q}^-_1 &= \sum_{j=1}^{L-1}\sum_{a=1}^{2} \frac{\omega^a}{1-\omega^{-a}} \left(\mathcal{S}_j^-\right)^a \left(\mathcal{S}_{j+1}^-\right)^{3-a}, \quad \mathsf{Q}_1 = \mathsf{Q}^0_1 + \mathsf{Q}^+_1 + \mathsf{Q}^-_1, \quad \omega = e^{2\mathrm{i}\pi/3}. \end{aligned} \tag{6.5}$$

They satisfy the DG relation, i.e.

$$\Big[\mathsf{Q}_0, \Big[\mathsf{Q}_0, [\mathsf{Q}_0, \mathsf{Q}_1]\Big]\Big] = \ell_2^2 [\mathsf{Q}_0, \mathsf{Q}_1], \qquad \Big[\mathsf{Q}_1, \Big[\mathsf{Q}_1, [\mathsf{Q}_1, \mathsf{Q}_0]\Big]\Big] = \ell_2^2 [\mathsf{Q}_1, \mathsf{Q}_0], \tag{6.6}$$

with $\ell_2 = 3$ in this case. Higher order generators can be obtained through applying relations (2.8) and (2.9) recursively. This Hamiltonian (6.2) is special, since the Onsager generators are in fact local, instead of quasilocal in the generic cases.

Moreover, identical to the example in Sec. 4.1, the Onsager generators are expressed in terms of Z and Y charges, which are local with $\eta = \mathrm{i}\pi/3$,

$$\mathsf{Q}_1^0 = \mathsf{Z}_0, \quad \mathsf{Q}_1^- = \mathsf{Y}_0, \quad \mathsf{Q}_1^+ = \mathsf{Y}_0^\dagger. \tag{6.7}$$

This analogue are extended further. We find precisely the same relation as in spin-1/2 cases, cf. (4.15),

$$\begin{aligned}
&\mathsf{Z}_1 = \frac{\ell_2}{2}\frac{1}{1!}\left(\mathsf{Q}_2^0 - \mathsf{Q}_0^0\right), \quad \mathsf{Z}_2 = \left(\frac{\ell_2}{2}\right)^2 \frac{1}{2!}\left(2\mathsf{Q}_3^0 - \mathsf{Q}_1^0\right), \\
&\mathsf{Z}_3 = \left(\frac{\ell_2}{2}\right)^3 \frac{1}{3!}\left(6\mathsf{Q}_4^0 - 8\mathsf{Q}_2^0 + 2\mathsf{Q}_0^0\right), \\
&\mathsf{Y}_1 = \frac{\ell_2}{2}\frac{1}{1!}\left(\mathsf{Q}_2^- - \mathsf{Q}_0^-\right), \quad \mathsf{Y}_2 = \left(\frac{\ell_2}{2}\right)^2 \frac{1}{2!}\left(2\mathsf{Q}_3^- - \mathsf{Q}_1^-\right), \\
&\mathsf{Y}_3 = \left(\frac{\ell_2}{2}\right)^3 \frac{1}{3!}\left(6\mathsf{Q}_4^- - 8\mathsf{Q}_2^- + 2\mathsf{Q}_0^-\right),
\end{aligned} \tag{6.8}$$

with $\ell_2 = 3$.

In fact, the example in this section is an interacting system. As explained in Sec. 4.3, the closure condition (4.8) is absent for spin-1 ZF model with $\eta = \mathrm{i}\pi/3$, or equivalently spin-1 $U(1)$-invariant clock model [10]. Hence, the Onsager generators Q_m obtained from (6.4) and (6.5) can be considered as an explicit interacting representation of Onsager algebra.

6.2 Conjectures on hidden Onsager algebra symmetries in spin-1 ZF models at root of unity

Motivated by the spin-1/2 case in Sec. 4.2 and the example of spin-1 $U(1)$-invariant clock model in Sec. 6.1, we arrive at exactly the same conjectures: (4.12), (4.14) and (4.15) with operators acting on spin-1/2 physical Hilbert space replaced by the ones acting on spin-1 physical Hilbert space. We shall not repeat the same equations here.

In the spin-1 case, Conjecture I (4.12) and part of Conjecture II (4.15) are proven for $\ell_2 = 3$, as illustrated in Sec. 6.1. Conjectures I and II, cf. (4.12) and (4.15) have been verified numerically for cases whose roots of unity satisfy $\ell_2 = 3, 4$ and all permitted values of ℓ_1 with system size N up to 8. Similar to the spin-1/2 case, the numerical evidence is convincing that Conjectures I and II are true for arbitrary root of unity value of the anisotropy and system size.

7 Conclusion and outlook

In this article we focus on the hidden Onsager algebra symmetry structure in spin-1/2 XXZ model and its spin-1 generalisation, ZF model, at root of unity value of the anisotropy. By constructing the semi-cyclic transfer matrices and the generating functions for conserved charges, we propose two conjectures for the hidden Onsager algebra symmetries in the aforementioned models, motivated by two examples of spin-1/2 XX model and spin-1 $U(1)$-invariant clock model (ZF model with $\eta = \mathrm{i}\pi/3$). It is straightforward to observe that one can obtain similar results for higher spin generalisations of XXZ model at root of unity through transfer matrix fusion procedure, exemplified in Sec. 5. Despite the credibility of the conjectures, it would be interesting to prove them using quantum integrability. The conjectures also hint at the relation between the underlying quantum group structure of those models at root of unity and Onsager algebra symmetry. Future investigations in this direction would reveal possible connections between them.

For spin-1/2 XXZ model at root of unity, we have two sets of commuting charges $\mathbf{Z}_m$ and $\mathbf{Y}_n$, while they do not commute with operators in the other set. The non-commutability between $\mathbf{Z}_m$ and $\mathbf{Y}_n$ has consequences in the thermodynamic limit, leading to oscillatory behaviour of auto-correlation functions [41,42]. The relation between the oscillatory behaviour of correlation functions in the thermodynamic limit to the hidden Onsager algebra symmetries in those models still needs further consideration.

Moreover, there are recent works on Onsager algebra and its q-deformation in XXZ model with open and half-infinite boundary conditions [43, 44]. It would be of great interest to understand the relation to the results in this article concerning the same model with quasi-periodic boundary condition. At last, a generalisation of Onsager algebra has appeared in the mathematics literature [45], and it would be interesting to investigate its applications in theoretical physics too.

Acknowledgements

I am very grateful to Jules Lamers and Vincent Pasquier for the previous collaborations on a related topic and numerous discussions. I thank Oleksandr Gamayun, Enej Ilievski, Jiří Minář and Eric Vernier for valuable discussions and critical remarks on the manuscript. I thank Paul Fendley, Hosho Katsura, Marko Medenjak and Lenart Zadnik for valuable discussions. I acknowledge the support from the European Research Council under ERC Advanced grant 743032 DYNAMINT.

A Choice of u_0 in (3.17)

In (3.17) we expand the generating functions $\mathbf{Z}(u,\phi)$ and $\mathbf{Y}(u,\phi)$ at different spectral parameter values for cases with $\varepsilon = \pm 1$. We would like to provide some details in this appendix. As usual, the twist ϕ satisfies commensurate condition (3.15).

To begin with, we notice that when $\varepsilon = +1$,

$$\mathbf{T}^{\mathrm{sc}}_{(\ell_2-1)/2}\left(\frac{\eta}{2}, 0, \phi\right) \tag{A.1}$$

is not invertible (i.e. not full-ranked), while it is invertible when $\varepsilon = -1$. Meanwhile, transfer matrices with $\varepsilon = +1$ are related to transfer matrices with $\varepsilon = -1$. We can see that from the existence of a unitary gauge transformation $\mathbf{U}$ [24, 46]

$$\mathbf{U} = \exp\left(\mathrm{i}\pi \sum_{j=1}^{N} \frac{j}{2}\sigma_j^z\right), \tag{A.2}$$

such that

$$\mathbf{U}\mathbf{H}(\Delta,\phi)\mathbf{U}^\dagger = -\mathbf{H}(-\Delta,\phi'). \tag{A.3}$$

The twists are related as

$$\phi' = \begin{cases} \phi & N \text{ even}, \\ \phi+\pi & N \text{ odd}. \end{cases} \tag{A.4}$$

We consider 2 transfer matrices, one with η ($\varepsilon = \exp(\ell_2\eta) = -1$) and the other one with $\eta' = \mathrm{i}\pi - \eta$ ($\varepsilon' = \exp(\ell_2\eta') = +1$). This implies that ℓ_2 is odd. When $s = \frac{\ell_2-1}{2}$ and $\beta = 0$, we have

$$\mathbf{U}\mathbf{T}_s^{\mathrm{sc}}(u,0,\phi,\eta)\mathbf{U}^\dagger = \mathbf{T}_s^{\mathrm{sc}}(-u,0,\phi,\eta'). \tag{A.5}$$

The above equation is satisfied only when $s \in \mathbb{Z}_{>0}$. This implies that $\frac{\ell_2-1}{2} \in \mathbb{Z}_{>0}$, i.e. ℓ_2 is odd. Moreover, if ϕ with parameter η in (A.4) satisfies commensurate condition (3.15), ϕ' with parameter η' in (A.4) also satisfies commensurate condition (3.15).

Therefore, if we were to define

$$\mathbf{Q}_1^0(\eta,\phi) = \mathbf{Z}_0 = \frac{1}{2\eta}\,\partial_s \log \mathbf{T}_s^{\mathrm{sc}}(u,\beta,\phi,\eta)\big|_{s=(\ell_2-1)/2,\beta=0,u=\eta/2}\,, \tag{A.6}$$

it is natural to define

$$\begin{aligned}\mathbf{Q}_1^0(\eta',\phi') &= \mathbf{U}\mathbf{Q}_0(\eta,\phi)\mathbf{U}^\dagger \\ &= \frac{1}{2\eta'}\,\partial_s \log \mathbf{T}_s^{\mathrm{sc}}(u,\beta,\phi',\eta')\big|_{s=(\ell_2-1)/2,\beta=0,u=-\eta/2}\,,\end{aligned} \tag{A.7}$$

satisfying the same algebraic relations after applying the unitary gauge transformation $\mathbf{U}$. Similar relations for $\mathbf{Q}_1^\pm$ can be inferred.

In this case $-\frac{\eta}{2} = \frac{\eta'}{2} - \frac{\mathrm{i}\pi}{2}$, indicating (3.17). We have used the value of u_0 defined in (3.17) to numerically verify the conjectures in Sec. 4.2. For instance, for the cases of $\eta = 2\mathrm{i}\pi/3$, $2\mathrm{i}\pi/5$ and $4\mathrm{i}\pi/5$, the conjectures remain true with system size N up to 12.

B Onsager generators in XX case

In the case of XX model, we obtain analytically all the Onsager generators when the twist is commensurate, cf. (3.15) by calculating the recursion relation analytically. The results are as follows.

$$\mathbf{Q}_m^0 = \frac{\mathrm{i}}{2}\sum_{j=1}^{N}(-\mathrm{i})^{m-1}\sigma_j^+\sigma_{j+1}^z\cdots\sigma_{j+m-1}^z\sigma_{j+m}^- - \mathrm{i}^{m-1}\sigma_j^-\sigma_{j+1}^z\cdots\sigma_{j+m-1}^z\sigma_{j+m}^+, \tag{B.1}$$

$$\mathbf{Q}_m^- = \frac{\mathrm{i}}{2}\sum_{j=1}^{N}(-\mathrm{i})^{m-1}(-1)^j\sigma_j^-\sigma_{j+1}^z\cdots\sigma_{j+m-1}^z\sigma_{j+m}^- = \left(\mathbf{Q}_m^+\right)^\dagger, \tag{B.2}$$

where $\sigma_{N+k}^\pm = e^{\pm\mathrm{i}\phi/2}\sigma_k^\pm$ with $1 \le k < N$. All generators are bilinear in fermionic operators after Jordan–Wigner transformation [10].

References

[1] L. Onsager, *Crystal statistics. i. a two-dimensional model with an order-disorder transition*, Phys. Rev. **65**, 117 (1944), doi:10.1103/PhysRev.65.117.

[2] S. Howes, L. P. Kadanoff and M. Den Nijs, *Quantum model for commensurate-incommensurate transitions*, Nucl. Phys. B. **215**(2), 169 (1983), doi:10.1016/0550-3213(83)90212-2.

[3] G. von Gehlen and V. Rittenberg, *z_n-symmetric quantum chains with an infinite set of conserved charges and z_n zero modes*, Nucl. Phys. B. **257**, 351 (1985), doi:10.1016/0550-3213(85)90350-5.

[4] H. A. Kramers and G. H. Wannier, *Statistics of the two-dimensional ferromagnet. part i*, Phys. Rev. **60**, 252 (1941), doi:10.1103/PhysRev.60.252.

[5] L. Dolan and M. Grady, *Conserved charges from self-duality*, Phys. Rev. D **25**, 1587 (1982), doi:10.1103/PhysRevD.25.1587.

[6] B. Davies, *Onsager's algebra and superintegrability*, J. Phys. A **23**(12), 2245 (1990), doi:10.1088/0305-4470/23/12/010.

[7] E. Date and S. Roan, *The algebraic structure of the Onsager algebra*, Czechoslovak J. Phys. **50**, 37 (2000), doi:10.1023/A:1022812728907.

[8] E. Date and S. Roan, *The structure of quotients of the Onsager algebra by closed ideals*, J. Phys. A **33**(16), 3275 (2000), doi:10.1088/0305-4470/33/16/316.

[9] C. El-Chaâr, *The Onsager algebra*, arXiv preprint arXiv:1205.5989 (2012).

[10] E. Vernier, E. O'Brien and P. Fendley, *Onsager symmetries in $U(1)$ -invariant clock models*, J. Stat. Mech. **2019**(4), 043107 (2019), doi:10.1088/1742-5468/ab11c0.

[11] O. Lychkovskiy, *Closed hierarchy of Heisenberg equations in integrable models with Onsager algebra*, arXiv preprint arXiv:2012.00388 (2020).

[12] N. Shibata, N. Yoshioka and H. Katsura, *Onsager's scars in disordered spin chains*, Phys. Rev. Lett. **124**, 180604 (2020), doi:10.1103/PhysRevLett.124.180604.

[13] L. Hulthén, *Über das Austauschproblem eines Kristalles*, Arkiv Mat. Astron. Fysik. **26A**, 1 (1938).

[14] R. Orbach, *Linear Antiferromagnetic Chain with Anisotropic Coupling*, Phys. Rev. **112**, 309 (1958), doi:10.1103/PhysRev.112.309.

[15] C. N. Yang and C. P. Yang, *One-Dimensional Chain of Anisotropic Spin-Spin Interactions. I. Proof of Bethe's Hypothesis for Ground State in a Finite System*, Phys. Rev. **150**, 321 (1966), doi:10.1103/PhysRev.150.321.

[16] C. N. Yang and C. P. Yang, *One-Dimensional Chain of Anisotropic Spin-Spin Interactions. II. Properties of the Ground-State Energy Per Lattice Site for an Infinite System*, Phys. Rev. **150**, 327 (1966), doi:10.1103/PhysRev.150.327.

[17] V. Pasquier and H. Saleur, *Common structures between finite systems and conformal field theories through quantum groups*, Nucl. Phys. B. **330**(2-3), 523 (1990), doi:10.1016/0550-3213(90)90122-t.

[18] K. Fabricius and B. M. McCoy, *Bethe's Equation Is Incomplete for the XXZ Model at Roots of Unity*, J. Stat. Phys. **103**, 647 (2001), doi:10.1023/A:1010380116927.

[19] K. Fabricius and B. M. McCoy, *Completing Bethe's Equations at Roots of Unity*, J. Stat. Phys. **104**, 573 (2001), doi:10.1023/A:1010372504158.

[20] R. J. Baxter, *Completeness of the Bethe Ansatz for the Six and Eight-Vertex Models*, J. Stat. Phys. **108**, 1 (2002), doi:10.1023/A:1015437118218.

[21] C. Korff, *Auxiliary matrices for the six-vertex model at $q^N = 1$ and a geometric interpretation of its symmetries*, J. Phys. A **36**(19), 5229 (2003), doi:10.1088/0305-4470/36/19/305.

[22] C. Korff, *Auxiliary matrices for the six-vertex model at $q^N = 1$: II. Bethe roots, complete strings and the Drinfeld polynomial*, J. Phys. A **37**(2), 385 (2003), doi:10.1088/0305-4470/37/2/009.

[23] C. Korff, *A Q-operator identity for the correlation functions of the infinite XXZ spin-chain*, J. Phys. A **38**(30), 6641 (2005), doi:10.1088/0305-4470/38/30/002.

[24] Y. Miao, J. Lamers and V. Pasquier, *On the Q operator and the spectrum of the XXZ model at root of unity*, arXiv preprint arXiv:2012.10224 (2020).

[25] R. Baxter, *Exactly solved models in statistical mechanics*, ISBN 978-0-486-46271-4 (1982).

[26] E. Ilievski, M. Medenjak, T. Prosen and L. Zadnik, *Quasilocal charges in integrable lattice systems*, J. Stat. Mech. **2016**(6), 064008 (2016), doi:10.1088/1742-5468/2016/06/064008.

[27] T. Prosen, *Quasilocal conservation laws in XXZ spin-1/2 chains: Open, periodic and twisted boundary conditions*, Nucl. Phys. B **886**, 1177 (2014), doi:10.1016/j.nuclphysb.2014.07.024.

[28] R. G. Pereira, V. Pasquier, J. Sirker and I. Affleck, *Exactly conserved quasilocal operators for the XXZ spin chain*, J. Stat. Mech. **2014**(9), P09037 (2014), doi:10.1088/1742-5468/2014/09/p09037.

[29] L. Zadnik, M. Medenjak and T. Prosen, *Quasilocal conservation laws from semicyclic irreducible representations ofUq(sl2)in XXZ spin-1/2 chains*, Nucl. Phys. B **902**, 339 (2016), doi:10.1016/j.nuclphysb.2015.11.023.

[30] E. Ilievski and T. Prosen, *Thermodyamic Bounds on Drude Weights in Terms of Almost-conserved Quantities*, Commun. Math. Phys. **318**(3), 809 (2012), doi:10.1007/s00220-012-1599-4.

[31] B. Doyon, *Exact large-scale correlations in integrable systems out of equilibrium*, SciPost Phys. **5**, 54 (2018), doi:10.21468/SciPostPhys.5.5.054.

[32] B. Doyon, *Hydrodynamic projections and the emergence of linearised Euler equations in one-dimensional isolated systems*, arXiv preprint arXiv:2011.00611 (2020).

[33] A. De Luca, M. Collura and J. De Nardis, *Nonequilibrium spin transport in integrable spin chains: Persistent currents and emergence of magnetic domains*, Phys. Rev. B **96**, 020403 (2017), doi:10.1103/PhysRevB.96.020403.

[34] A. B. Zamolodchikov and V. A. Fateev, *Model factorized S Matrix and an integrable Heisenberg chain with spin 1*, Sov. J. Nucl. Phys. **32**, 298 (1980).

[35] A. G. Izergin and V. E. Korepin, *The inverse scattering method approach to the quantum Shabat-Mikhailov model*, Commun. Math. Phys. **79**(3), 303 (1981), doi:10.1007/bf01208496.

[36] L. A. Takhtajan, *The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins*, Phys. Lett. A **87**(9), 479 (1982), doi:10.1016/0375-9601(82)90764-2.

[37] H. M. Babujian, *Exact solution of the one-dimensional isotropic heisenberg chain with arbitrary spins s*, Phys. Lett. A **90**(9), 479 (1982), doi:10.1016/0375-9601(82)90403-0.

[38] H. M. Babujian, *Exact solution of the isotropic heisenberg chain with arbitrary spins: Thermodynamics of the model*, Nucl. Phys. B **215**(3), 317 (1983), doi:10.1016/0550-3213(83)90668-5.

[39] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, *Quantum Integrable Models and Discrete Classical Hirota Equations*, Commun. Math. Phys. **188**(2), 267 (1997), doi:10.1007/s002200050165.

[40] L. Piroli and E. Vernier, *Quasi-local conserved charges and spin transport in spin-1 integrable chains*, J. Stat. Mech. **2016**(5), 053106 (2016), doi:10.1088/1742-5468/2016/05/053106.

[41] M. Medenjak, B. Buča and D. Jaksch, *Isolated heisenberg magnet as a quantum time crystal*, Phys. Rev. B **102**, 041117 (2020), doi:10.1103/PhysRevB.102.041117.

[42] M. Medenjak, T. Prosen and L. Zadnik, *Rigorous bounds on dynamical response functions and time-translation symmetry breaking*, SciPost Phys. **9**, 3 (2020), doi:10.21468/SciPostPhys.9.1.003.

[43] P. Baseilhac and K. Koizumi, *A deformed analogue of Onsager's symmetry in the XXZ open spin chain*, J. Stat. Mech. **2005**(10), P10005 (2005), doi:10.1088/1742-5468/2005/10/p10005.

[44] P. Baseilhac and S. Belliard, *The half-infinite XXZ chain in Onsager's approach*, Nucl. Phys. B **873**(3), 550 (2013), doi:10.1016/j.nuclphysb.2013.05.003.

[45] J. V. Stokman, *Generalized Onsager Algebras*, Algebras Represent. Theory **23**(4), 1523 (2019), doi:10.1007/s10468-019-09903-6.

[46] M. Gaudin, *The Bethe Wavefunction*, Cambridge University Press (2014).