MAT334 Lecture

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Contents

1 Basic Operations of Complex Numbers

Def. A **complex number** is an expression of the form

$$z \triangleq x + iy$$

where $x, y \in \mathbb{R}$

- x is called the **real** part of z, i.e. $\operatorname{Re} z \triangleq x$
- y is called the **imaginary** part of z, i.e. $\operatorname{Im} z \triangleq y$
- Can be geometrically understood

Notation: $\mathbb{C} = \{\text{complex numbers}\}$

Def. Let $z = x + iy$ and $w = a + ib$ be complex numbers. Then

- $z + w \triangleq (x + a) + i(y + b)$
- $z - w \triangleq (x - a) + i(y - b)$
- $z \cdot w \triangleq (x + iy) \cdot (a + ib) = (xa - yb) + i(ya + xb)$

Def. Let $z = x + iy \in \mathbb{C}$. The **complex conjugate** $\overline{z}$ of z is the complex number $x - iy$

- Geometrically flipping around the real axis

by this definition

$$z \cdot \overline{z} = (x + iy) \cdot (x - iy) = x^2 + y^2 \triangleq |z|^2$$

where $|z|$ is the **absolute value** of z. Hence

$$(z \cdot \overline{z}) \cdot \frac{1}{x^2 + y^2} = 1$$

and so the multiplicative inverse of z is

$$\frac{1}{z} \triangleq \overline{z} \cdot \frac{1}{x^2 + y^2} = (x - iy) \cdot \frac{1}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$$

Therefore if $z = x + iy$ and $w = a + ib$, then

$$\frac{z}{w} \triangleq z \cdot \frac{1}{w} = (x + iy) \cdot \left(\frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \right) = \left(\frac{xa + yb}{a^2 + b^2} \right) + i \left(\frac{ya - xb}{a^2 + b^2} \right)$$

Lemma 1.1. *Let $z, w \in \mathbb{C}$. Then*

$$\begin{aligned} \overline{z + w} &= \overline{z} + \overline{w} \\ \overline{z - w} &= \overline{z} - \overline{w} \\ \overline{z \cdot w} &= \overline{z} \cdot \overline{w} \\ \overline{\frac{z}{w}} &= \frac{\overline{z}}{\overline{w}} \qquad \textit{For } w \neq 0 \\ |z \cdot w| &= |z| \cdot |w| \end{aligned}$$

Proof. Follows from the above definitions. □

2 Polar Coordinates

Let $z \in \mathbb{C}$ be geometrically represented. Then in the notion of polar coordinates, we have the length as $|z|$ and the angle as θ (from the positive real axis). Define the **argument** of z as

$$\arg z \triangleq \theta$$

- The argument of a nonzero complex number is only well-defined up to integral multiples of 2π
- The argument is not unique (can add/subtract 2π)

The polar coordinates of z is the pair

$$(|z|, \arg z)$$

Let $z, w \in \mathbb{C}$ where $z = x + iy$ and $w = a + ib$ with polar coordinates $z = (r_1, \theta_1)$ and $w = (r_2, \theta_2)$, we have

$$\begin{aligned} x &= r_1 \cos\theta_1 & y &= r_1 \sin\theta_1 \\ a &= r_2 \cos\theta_2 & b &= r_2 \sin\theta_2 \end{aligned}$$

Then

$$z \cdot w = (r_1 \cos\theta_1 + ir_1 \sin\theta_1) \cdot (r_2 \cos\theta_2 + ir_2 \sin\theta_2) = r_1 r_2 \cos(\theta_1 + \theta_2) + ir_1 r_2 \sin(\theta_1 + \theta_2)$$

Hence

$$|zw| = \sqrt{(r_1 r_2 \cos(\theta_1 + \theta_2))^2 + (r_1 r_2 \sin(\theta_1 + \theta_2))^2} = r_1 r_2 = |z| \cdot |w|$$

This implies that

$$\arg(zw) = \arg z + \arg w$$

up to multiples of 2π.

3 Geometry of Complex Numbers

Line

Equation of a line:

$$y = ax + b$$

where a is the slope and b is the y-intercept.

For the expression $(a + i)z + b$:

$$\mathrm{Re}((a + i)z + b) = \mathrm{Re}((a + i)(x + iy) + b) = ax - y + b$$

Then

$$y = ax + b \iff \mathrm{Re}((a + i)z + b) = 0$$

Generalizing,

$$\mathrm{Re}(Az + B) = 0, \qquad A, B \in \mathbb{C}$$

defines a line in the complex plane.

Circle

Equation of a circle:

$$(x - a)^2 + (y - b)^2 = r^2$$

where (a, b) is the center and r is the radius.

Let $z = x + iy$ and $z_0 = a + ib$. Then

$$|z - z_0| = |(x - a) + i(y - b)| = \sqrt{(x - a)^2 + (y - b)^2}$$

Hence

$$(x - a)^2 + (y - b)^2 = r^2 \iff |z - z_0| = r$$

4 Topology

Def. An **open disk** is a set of the form

$$\{z \in \mathbb{C} : |z - z_0| < r\}$$

where $z_0 \in \mathbb{C}, r \in \mathbb{R}^{>0}$.

Def. Let $S \subseteq \mathbb{C}$. A point z_0 of S is an **interior point** of S if there exists an open disk centered at z_0 which is contained in S.

Def. Let $S \subseteq \mathbb{C}$. The **interior** of S is the set of all interior points of S, denoted S^o.

- S is **open** if all points of S are interior

Def. Let $S \subseteq \mathbb{C}$. A point $z_0 \in \mathbb{C}$ is a **boundary point** of S if every open disk centered at z_0 contains both points of S and points of $\mathbb{C} \backslash S$.

- A boundary point of S may or may not be in S

Def. Let $S \subseteq \mathbb{C}$. The **boundary** of S is the set of all boundary points of S, denoted ∂S.

- S is **closed** if $S \supseteq \partial S$

Theorem 4.1. *Let $S \subseteq \mathbb{C}$. Then S is open iff $\mathbb{C} - S$ is closed. Moreover, S is open iff $S \cap \partial S = \varnothing$.*

Def. A **polygonal curve** in a subset S of $\mathbb{C}$ is a union of finitely many line segments in S of the form

$$\overline{z_0 z_1} \cup \overline{z_1 z_2} \cup \cdots \cup \overline{z_{n-1} z_n}$$

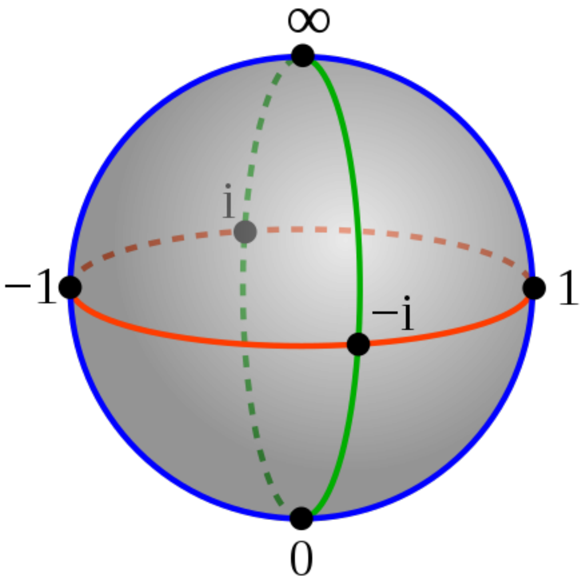
Def. Let $S \subseteq \mathbb{C}$. We say that S is **connected** if $\forall p, q \in S, \exists$ polygonal curve $\overline{z_0 z_1} \cup \overline{z_1 z_2} \cup \cdots \cup \overline{z_{n-1} z_n}$ such that $p = z_0$ and $q = z_n$.

- *Not* the actual definition of connectedness, but instead a stronger and simpler definition

Def. An open connected subset of $\mathbb{C}$ is called a **domain**.

5 Riemann Sphere

Def. The **Riemann sphere** is the set $\mathbb{C} \cup \{\infty\}$



- This provides a map from $\mathbb{C}$ to the unit sphere in $\mathbb{R}^3$
- This map is injective
- The image of the map is the unit sphere without the north pole
- By including ∞, we get a bijection

6 Function

Def. A **function** of a complex variable is a rule that assigns to each complex number within same subset S of $\mathbb{C}$ a complex number.

- S is called the **domain** of this function
- The collection of all possible values of the function is called its **range**

Def. Let $\{z_n\}$ be a sequence of complex numbers. L is a **limit** of $\{z_n\}$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n > N, |z_n - L| < \epsilon$$

- Notation:
$$\lim_{n\to\infty} z_n = L$$
- If $\{z_n\}$ has a limit, then it is **convergent**; otherwise, it is **divergent**

Theorem 6.1. *Let $z_n = x_n + iy_n$ where $x_n, y_n \in \mathbb{R}$. Then $\{z_n\}$ converges iff $\{x_n\}, \{y_n\}$ converge. Moreover, if $\{x_n\}$ and $\{y_n\}$ converge, then*
$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

Theorem 6.2. *Let $\{z_n\}$, $\{w_n\}$ be convergent sequnces. Then*

$$\begin{aligned}
\lim_{n\to\infty}(z_n + w_n) &= \lim_{n\to\infty} z_n + \lim_{n\to\infty} w_n \\
\lim_{n\to\infty}(z_n - w_n) &= \lim_{n\to\infty} z_n - \lim_{n\to\infty} w_n \\
\lim_{n\to\infty}(z_n w_n) &= \left(\lim_{n\to\infty} z_n\right)\left(\lim_{n\to\infty} w_n\right) \\
\lim_{n\to\infty}\frac{z_n}{w_n} &= \frac{\lim_{n\to\infty} z_n}{\lim_{n\to\infty} w_n} \qquad \text{if } w_n \neq 0\ \forall n \wedge \lim_{n\to\infty} w_n \neq 0
\end{aligned}$$

Def. Let $f : S \to \mathbb{C}$. Let $z_0 \in S \cup \partial S$. f has **limit** L as z approaches to z_0 if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall z \in S, |z - z_0| < \delta \implies |f(z) - L| < \epsilon$$

- Notation:
$$\lim_{z\to z_0} f(z) = L$$

Theorem 6.3. *Let $f, g : S \to \mathbb{C}$. Let $z_0 \in S \cup \partial S$ such that $\lim_{z\to z_0} = L$ and $\lim_{z\to z_0} g(z) = M$. Then*

$$\begin{aligned}
\lim_{z\to z_0}(f+g)(z) &= L + M \\
\lim_{z\to z_0}(f-g)(z) &= L - M \\
\lim_{z\to z_0}(fg)(z) &= LM \\
\lim_{z\to z_0}\frac{f}{g}(z) &= \frac{L}{M} \qquad \text{if } g(z) \neq 0\ \forall z \in S \wedge M \neq 0
\end{aligned}$$

Def. Let f be a function. f has **limit** L a z approaches ∞ if

$$\forall \epsilon > 0, \exists R > 0 \text{ such that } \forall z \in \mathbb{C}, |z| > R \implies |f(z) - L| < \epsilon$$

Def. Let $f : S \to \mathbb{C}$. f is **continuous at** $z_0 \in S$ if $\lim_{z\to z_0} f(z)$ exists and $\lim_{z\to z_0} f(z) = f(z_0)$.

- f is **continuous** if it is continuous at every $z_0 \in S$

Def. An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} z_n$$

The nth **partial sum** is

$$S_n \triangleq z_1 + \cdots + z_n$$

The infinite series **converges (diverges)** if the sequence $\{S_n\}$ converges (diverges).

Exponential Function

$$e^z \triangleq e^x(\cos y + i \sin y) \qquad \text{where } z = x + iy$$

- $e^z \cdot e^w = e^{z+w}$
- $|e^z| = e^x = e^{\operatorname{Re} z}$
- $\left|e^{iy}\right| = 1$
- $e^{z+2\pi i} = e^z$
- $\arg e^z = y + 2k\pi$ for some $k \in \mathbb{Z}$
- e^z in polar coordinates: $\left(e^{\operatorname{Re} z}, \operatorname{Im} z\right)$
- $e^z \cdot e^w$ in polar coordinates: $\left(e^{\operatorname{Re} z + \operatorname{Re} w}, \operatorname{Im} z + \operatorname{Im} w\right)$
- $e^z \neq 0$ for all $z \in \mathbb{C}$

Logarithm Function

- Inverse of the exponential function, i.e. find z satisfying $e^z = w$
- Solution: $z = \log|w| + i \arg w$
- The above definition is *not* well-defined
- Can define

  $$\operatorname{Log} : \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\} \to \mathbb{C} \quad \text{by} \quad \operatorname{Log} w = \log|w| + i \operatorname{Arg} w$$

  where

  $$\operatorname{Arg} : \mathbb{C} \to \{x \in \mathbb{R} : x \leq 0\} \to (-\pi, \pi)$$

- The property $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$ is **not** satisfied
  - E.g. let $z_1 = z_2 = -1 + i$. Then $\operatorname{Log}(z_1 z_2) = \operatorname{Log}(-2i) = \log|-2i| + i \operatorname{Arg}(-2i) = \log 2 + -\frac{\pi}{2} i$; however, $\operatorname{Log} z_1 + \operatorname{Log} z_2 = 2 \operatorname{Log}(-1+i) = 2\left(\log|-1+i| + i \operatorname{Arg}(-1+i)\right) = 2 \log\sqrt{2} + \frac{3\pi}{2} i$

Def. Let $a \in \mathbb{C} - \{x \in \mathbb{R} : x = 0\}$ and $z \in \mathbb{C}$. Define

$$a^z \triangleq e^{\operatorname{Log} a^z} = e^{z \operatorname{Log} a}$$

Def.

$$\cos z \triangleq \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$
$$\sin z \triangleq \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

7 Curves and Integrals

Def. A **curve** is a continuous function $\gamma : [a, b] \to \mathbb{C}$

Def. A curve $\gamma : [a, b] \to \mathbb{C}$ is

- **Closed** if $\gamma(a) = \gamma(b)$
- **Simple** if $\gamma(t_1) \neq \gamma(t_2)$ for all $a \leq t_1 < t_2 \leq b$

Theorem 7.1 (Jordan Curve Theorem)**.** *Let $\gamma : [a, b] \to \mathbb{C}$ be a simple closed curve. Then the complement of the range of γ is the disjoint union of two open connected subsets of $\mathbb{C}$. Moreover, one of the two subsets is bounded (called the* inside *of γ) and the other is unbounded (called the* outside *of γ).*

Def. Let $\gamma : [a, b] \to \mathbb{C}$. Let $\gamma(t) = x(t) = +iy(t)$ for all $t \in [a, b]$ where $x, y : [a, b] \to \mathbb{R}$. Then γ is

- **Differentiable** if both x and y are differentiable
- **Smooth** if x and y are differentiable and x' and y' are continuous
- **Piecewise smooth** if γ is composed of finite number of smooth curves, the ending point is coinciding with the starting point of the next

Def. A curve $\gamma : [a, b] \to \mathbb{C}$ is **oriented** by increasing t. We say that γ **starts** at $\gamma(a)$ and **ends** at $\gamma(b)$.

- The reverse orientation is given to γ by starting at $\gamma(b)$ and ending at $\gamma(a)$. Notation:

$$-\gamma : [a, b] \to \mathbb{C} \quad \text{where} \quad -\gamma(t) = \gamma(a + b - t)$$

Def. Let $\gamma : [a, b] \to \mathbb{C}$ be a simple closed curve (so the Jordan curve theorem applies). The **positive orientation** is if $\forall p \in$ inside of γ, the argument of $\gamma(t) - p$ increases by 2π as t increases from a to b.

Def. Let $f : [a, b] \to \mathbb{C}$ be continuous. Define

$$\int_a^b f(t)dt \triangleq \int_a^b \operatorname{Re} f(t)dt + i \int_a^b \operatorname{Im} f(t)dt$$

Def. Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve. Let f be a continuous function defined on the range of γ. Then the **integral** of f along γ is

$$\int_a^b f(\gamma(t))\gamma'(t)dt$$

- $f(\gamma(t))\gamma'(t)$ is a complex function, so the meaning of the integral is as in the previous definition
- γ is smooth implies that γ' is continuous, so the integral is well-defined

Lemma 7.2. *Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve. Let $g : [c, d] \to [a, b]$ be a bijection such that g and g^{-1} are C^1. Then $\gamma \circ g : [c, d] \to \mathbb{C}$ is also a smooth curve. Moreover,* $\operatorname{range}(\gamma) = \operatorname{range}(\gamma \circ g)$*. We can compute*

$$\int_c^d f(\gamma \circ g(t))(\gamma \circ g)'(t)dt = \int_c^d f(\gamma(g(t)))\gamma'(g(t))g'(t)dt$$

by Chain Rule.

Lemma 7.3. *Let $u = g(t)$ so that $du = g'(t)dt$. Then*

$$\int_a^b f(\gamma(u))\gamma'(t)du = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Therefore the integral of f along γ depends only on the range of γ. We use the notation

$$\int_\gamma f(z)dz$$

Lemma 7.4. *Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous curve. Let f_1, f_2 be continuous functions on the range of γ and $c_1, c_2 \in \mathbb{C}$. Then*

$$\int_\gamma (c_1 f_1 + c_2 f_2) dz = c_1 \int_\gamma f_1 dz + c_2 \int_\gamma f_2 dz$$

Def. Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve. Then $-\gamma : [a, b] \to \mathbb{C}$ is the reverse orientation satisfying

$$(-\gamma)(t) \triangleq \gamma(a + b - t)$$

Def. Let $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ be smooth curves so that $\gamma_1(b_1) = \gamma_2(a_2)$. Then

$$(\gamma_1 + \gamma_2)(t) \triangleq \begin{cases} \gamma_1(t), & \text{if } t \in [a_1, b_1] \\ \gamma_2(t + b_1 - a_2), & \text{if } t \in [b_1, b_1 + b_2 - a_2] \end{cases}$$

Lemma 7.5. *Let γ_1, γ_2 be defined as above. Then*

$$\int_{-\gamma} f(z) dz = -\int_\gamma f(z) dz$$
$$\int_{\gamma_1 + \gamma_2} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz$$

Lemma 7.6. *Let γ be a smooth curve. Then*

$$\left\| \int_\gamma f(z) dz \right\| \leq \int_\gamma \|f(z)\| \, dz$$

Def. The length $l(\gamma)$ of a smooth curve γ is

$$\int_a^b \|\gamma'(t)\| \, dt$$

Lemma 7.7.

$$\left\| \int_\gamma f(z) dz \right\| \leq l(\gamma) \max_{z \in \text{range}(\gamma)} \|f(z)\|$$

- Intuition: recall the integral definition of the average from single-variable calculus

Lemma 7.8. *Let $f : D(z_0; r) \to \mathbb{C}$ be continuous and $0 < \epsilon < r$. Then*

$$\lim_{\epsilon \to 0} \int_{\partial D(z_0, \epsilon)} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

Def. Let $f(z) = f(x + iy) = p(x + iy) + iq(x + iy)$. Then

$$\frac{\partial f}{\partial x} \triangleq \frac{\partial p}{\partial x} + i \frac{\partial q}{\partial x}$$
$$\frac{\partial f}{\partial y} \triangleq \frac{\partial f}{\partial y} + i \frac{\partial q}{\partial y}$$

Theorem 7.9 (Green's Theorem)**.** *Let Ω be a domain and let $\Gamma = \partial\Omega = \gamma_1 \cup \cdots \cup \gamma_n$ where $\gamma_1, \ldots, \gamma_n$ are piecewise smooth simple closed curves. Let $f : \Omega \to \mathbb{C}$ be C^1. Then*

$$\int_\Gamma f(z) dz = i \iint_\Omega \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) dx dy$$

8 Holomorphic Functions

Def. Let D be a domain and let $f : D \to \mathbb{C}$. Let $z_0 \in D$. f is **analytic/holomorphic** at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

- f is **analytic** if it is analytic at all $z_0 \in D$.
- An analytic function defined on all of $\mathbb{C}$ is called **entire**
- f is analytic at z_0 implies that f is continuous at z_0
- f being analytic at z_0 is stronger than the existence of partial derivatives of $\operatorname{Re} f$, $\operatorname{Im} f$ at z_0

Lemma 8.1 (Cauchy-Riemann Equations)**.** *Let $f = u + iv$ be analytic. Then*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Lemma 8.2 (Harmonic Functions)**.** *Let $f = u + iv$ be analytic. Then*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Def. If u and v are harmonic and they satisfy the Cauchy-Riemann equations, then they are called **harmonic conjugates**.

Lemma 8.3. *Let $f : D \to \mathbb{C}$ be analytic and denote $f = u + iv$. If u is a constant or $u^2 + v^2$ is a constant, then f is a constant.*

Theorem 8.4. *Let $f : D \to \mathbb{C}$ and denote $f = u + iv$. Assume that $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous. Then*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \wedge \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

implies that f is analytic.

9 Power Series

Def. A **power series** is a formal expression of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $z_0, a_0, a_1, a_2, \ldots$ are complex numbers.

Theorem 9.1. *Suppose that we have $z_1 \in \mathbb{C}$ such that $z_1 \neq z_0$ and*

$$\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$$

converges, then for any $z \in \mathbb{C}$ such that $\|z - z_0\| < \|z_1 - z_0\|$,

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

absolutely converges.

Radius of Convergence

- Within the radius, the power series absolutely converges
- On the boundary, we cannot conclude whether the power series converges
- Outside the radius, the power series diverges

Lemma 9.2. *Suppose $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ has radius of convergence R ($R \neq 0$). Then*

- *If $\lim_{n \to \infty} \left\| \frac{a_{n+1}}{a_n} \right\|$ exists, then $\lim_{n \to \infty} \left\| \frac{a_{n+1}}{a_n} \right\| = \frac{1}{R}$*
- *If $\lim_{n \to \infty} \sqrt[n]{\|a_n\|}$ exists, then $\lim_{n \to \infty} \sqrt[n]{\|a_n\|} = \frac{1}{R}$*

Lemma 9.3. *Suppose $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ has radius of convergence $R \neq 0$. Then $f : D(z_0; R) \to \mathbb{C}$ is analytic. Moreover $f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}$*

- $\sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}$ also converges on $D(z_0; R)$, so can again apply the lemma to this power series
- Can differentiate infinitely many times

10 Cauchy's Integral Formula

Theorem 10.1 (Cauchy's integral formula)**.** *Let D be a domain. Let $f : D \to \mathbb{C}$ be analytic. Let γ be a piecewise smooth simple closed curve in D. Assume that the inside Ω of γ is also in D. Then*

$$\int_\gamma f(z)dz = 0$$

- Assume that f' is continuous
- Green's Theorem states that $\int_\gamma f(z)dz = i \iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dxdy$
- Cauchy-Riemann equation states that for $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
- Hence

$$\begin{aligned} \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} &= \frac{\partial}{\partial x}(u+iv) + i\frac{\partial}{\partial y}(u+iv) \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \\ &= 0 \end{aligned}$$

Def. A domain D is **simply connected** if for any piecewise smooth simple closed curve γ in D, the inside Ω of γ is also in D.

Lemma 10.2. *Let D be simply connected, $f : D \to \mathbb{C}$ be analytic, γ be a piecewise smooth simple closed curve, in D. Then*

$$\int_\gamma f(z)dz = 0$$

Theorem 10.3. *Let D be simply connected, $f : D \to \mathbb{C}$ be analytic. Then there exists $F : D \to \mathbb{C}$ such that*

$$F'(z) = f(z) \quad \forall z \in D$$

Def. Define

$$F(z) \triangleq \int_{z_0}^{z} f(w)dw$$

This is well-defined (independent of the curve from z_0 to z) since for two curves γ_1, γ_2 from z_0 to z,

$$\int_{\gamma_1} f(w)dw - \int_{\gamma_2} f(w)dw = \int_{\gamma_1} f(w)dw + \int_{-\gamma_2} f(w)dw = \int_\gamma f(w)dw = 0$$

by Cauchy's integral formula.

Theorem 10.4 (Cauchy's theorem)**.** *Let D be a domain, $f : D \to \mathbb{C}$ be analytic, γ be a piecewise smooth simple closed curve in D whose inside Ω is also in D. Then $\forall z \in \Omega$, we have*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

- The integral is not defined at $\zeta = z$
- Let $g(\zeta) = \frac{f(\zeta)}{\zeta - z}$. Then g is defined on $D - \overline{D(z;\epsilon)}$ for some small ϵ
- We have that

$$\int_\gamma g(\zeta)d\zeta = \int_{\partial D(z;\epsilon)} g(\zeta)d\zeta$$

- Recall that

$$\lim_{\epsilon \to 0} \int_{\partial D(z;\epsilon)} \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i f(z)$$

- Plug in the terms to obtain Cauchy's theorem

Theorem 10.5. *Let D be a domain, $f : D \to \mathbb{C}$ an analytic function. Suppose that $D(z_0; R) \subseteq D$, then f has a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

valid on $D(z_0; R)$. Moreover, $\forall 0 < r < R$,

$$a_n = \frac{1}{2\pi i} \int_{\partial D(z_0;r)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

Corollary 10.5.1. *Let D be a domain and $f : D \to \mathbb{C}$ an analytic function. Then f' is also analytic on D. In particular, f is differentiable infinitely many times.*

Def.

$$\sinh(z) \triangleq \frac{e^z - e^{-z}}{2}$$
$$\cosh(z) \triangleq \frac{e^z + e^{-z}}{2}$$

Property:

$$\sinh^2 z - \cosh^2 z = \frac{e^{2z} - 2 + e^{-2z}}{4} - \frac{e^{2z} + 2 + e^{-2z}}{4} = -1$$

Corollary 10.5.2. *Let $f : D \to \mathbb{C}$ be analytic. Let $z_0 \in D$ such that $f^{(n)}(z_0) = 0$ for all n. Then $f = 0$ on D.*

Order of Zero

- Let D be a domain and $f : D \to \mathbb{C}$ be analytic
- Suppose that $z_0 \in D$ such that $f(z_0) = 0$
- Choose $r > 0$ such that $D(z_0; r) \subseteq D$ and expand $f(z)$ on $D(z_0; r)$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

- Let m be the smallest natural number such that $a_m \neq 0$, then

$$\begin{aligned} f(z) &= a_m (z - z_0)^m + a_{m+1} (z - z_0)^{m+1} + \cdots \\ &= (z - z_0)^m \underbrace{(a_m + a_{m+1}(z - z_0) + \cdots)}_{g(z)} \end{aligned}$$

- We have

$$f(z) = (z - z_0)^m g(z)$$
$$g(z_0) = a_m \neq 0$$

- Therefore g is also analytic

Def. We call m the **order of zero** of f at z_0

Theorem 10.6 (Morera)**.** *Assume $f : D \to \mathbb{C}$ is continuous. If*

$$\int_\gamma f(z)dz = 0$$

for all triangles γ that lies together with its inside in D, then f is analytic.

Proof. Fix $z_0 \in D$, WTS f is analytic at z_0. Choose $r > 0$ such that $D(z_0; r) \subseteq D$. Define $F : D(z_0; r) \to \mathbb{C}$ by $F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$ (i.e. integrtion along the radial curve from z_0 to z). We claim that F is analytic and $F' = f$, and we will show that

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{(z+h) - z}$$

exists and is equal to $f(z)$. By definition,

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(\zeta)d\zeta - \int_{z_0}^{z} f(\zeta)d\zeta$$

Notice that $z_0, z, z+h$ form a triangle on the complex plane. Let γ be such triangle. By assumption,

$$0 = \int_\gamma f(\zeta)d\zeta = \int_{z_0}^{z} f(\zeta)d\zeta + \int_{z}^{z+h} f(\zeta)d\zeta + \int_{z+h}^{z_0} f(\zeta)d\zeta$$

Hence,

$$F(z+h) - F(z) = \int_{z}^{z+h} f(\zeta)d\zeta$$

Since f is continuous at z, $\forall\epsilon > 0, \exists\delta > 0$ such that $\|f(\zeta) - f(z)\| < \epsilon$ satisfying $\|\zeta - z\| < \delta$. For $\|h\| < \epsilon$, we have

$$\begin{aligned}
\|F(z+h) - F(z) - hf(z)\| &= \left\| \int_{z}^{z+h} f(\zeta)d\zeta - hf(z) \right\| \\
&= \left\| \int_{z}^{z+h} f(\zeta)d\zeta - \int_{z}^{z+h} f(z)d\zeta \right\| \\
&= \left\| \int_{z}^{z+h} [f(\zeta) - f(z)]d\zeta \right\| \\
&\leq \epsilon \cdot \|h\|
\end{aligned}$$

Which implies that

$$\left\| \frac{F(z+h) - F(z)}{h} - f(z) \right\| \leq \epsilon$$

Therefore

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

$\square$

Theorem 10.7 (Liouville)**.** *Let f be an entire function. Assume that $\exists M$ such that $\|f(z)\| \leq M$ for all $z \in \mathbb{C}$. Then f is a constant.*

Proof. Consider $\tilde{f}(z) = f(z) - f(0)$. $\tilde{f}$ is an entire function such that $\tilde{f}(0) = 0$. Hence $\tilde{f}$ has order of zero at least 1 at $z_0 = 0$. So we can factorize $\tilde{f}(z) = (z-0)^1 g(z)$ for some entire function g. Then

$$g(z) = \frac{\tilde{f}(z)}{z} = \frac{f(z) - f(0)}{z}$$

By Cauchy's:

$$g(z) = \frac{1}{2\pi i}\int_{\partial D(0;R)} \frac{g(\zeta)}{\zeta - z} d\zeta$$

for any $R > \|z\|$. Then

$$\|g(\zeta)\| = \left\|\frac{f(\zeta) - f(0)}{\zeta}\right\| = \frac{\|f(\zeta) - f(0)\|}{\|\zeta\|} \leq \frac{\|f(\zeta)\| + \|f(0)\|}{R} \leq \frac{2M}{R}$$

Which implies that

$$\left\|\frac{1}{2\pi i}\int_{\partial D(0;R)} \frac{g(\zeta)}{\zeta - z} d\zeta\right\| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot \max_{\zeta \in \partial D(0;R)} \left\|\frac{g(\zeta)}{\zeta - z}\right\| \leq R \cdot \frac{2M/R}{R - \|z\|} = \frac{2M}{R - \|z\|}$$

Therefore, $\forall R > \|z\|$,

$$\|g(z)\| \leq \frac{2M}{R - \|z\|}$$

Which means that $\|g(z)\| = 0$ and so $g(z) = 0$. This implies that $f(z) = f(0)$, and so f is a constant. □

Log

- Let D be a simply connected domain
- Let $h : D \to \mathbb{C}$ be analytic and nowhere zero
- Then

  $$\frac{h'}{h} : D \to \mathbb{C}$$

  is analytic
- Define $\operatorname{Log} h : D \to \mathbb{C}$ by

  $$(\operatorname{Log} h)(z) \triangleq \int_{z_0}^{z} \frac{h'}{h}(\zeta) d\zeta$$

  where z_0 is some fixed point in D
- By Cauchy's, $\operatorname{Log} h$ is a well-defined, analytic function on D
- To compare $e^{\operatorname{Log} h}$ and h, observe that

$$\begin{aligned}
\left(e^{-\operatorname{Log} h} \cdot h\right)' &= \left(e^{-\operatorname{Log} h}\right)' \cdot h + \left(e^{-\operatorname{Log} h}\right) \cdot h' \\
&= e^{-\operatorname{Log} h}(-\operatorname{Log} h)' \cdot h + \left(e^{-\operatorname{Log} h}\right) \cdot h' \\
&= e^{-\operatorname{Log} h} \cdot \frac{h'}{h} \cdot h + \left(e^{-\operatorname{Log} h}\right) \cdot h' \\
&= 0
\end{aligned}$$

Theorem 10.8 (Fundamental Theorem of Algebra). *Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 1$. Then p has a root in $\mathbb{C}$*

Proof. (by contradiction) Assume that p has no roots. Then $\frac{1}{p}$ is an analytic function defined on $\mathbb{C}$ (i.e. $1/p$ is entire). Observe that for all z, we have

$$\begin{aligned} \|p(z)\| &= \left\|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0\right\| \\ &= ||a_n|| \cdot ||z||^n \cdot \left\| 1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \cdots + \frac{a_0}{a_n}\left(\frac{1}{z}\right)^n \right\| \\ &\geq ||a_n|| \cdot ||z||^n \cdot \left(1 - \left\|\frac{a_{n-1}}{a_n}\right\| \cdot \frac{1}{||z||} - \cdots - \left\|\frac{a_0}{a_n}\right\| \cdot \left(\frac{1}{||z||}\right)^n\right) \qquad \text{By Triangular Inequality} \end{aligned}$$

Since

$$\lim_{||z|| \to \infty} \left(1 - \left\|\frac{a_{n-1}}{a_n}\right\| \cdot \frac{1}{||z||} - \cdots - \left\|\frac{a_0}{a_n}\right\| \cdot \left(\frac{1}{||z||}\right)^n\right) = 1$$

by the $\epsilon - \delta$ definition of limit, $\exists M > 0$ such that

$$1 - \left\|\frac{a_{n-1}}{a_n}\right\| \cdot \frac{1}{||z||} - \cdots - \left\|\frac{a_0}{a_n}\right\| \cdot \left(\frac{1}{||z||}\right)^n > \frac{1}{2}$$

whenever $||z|| > M$. Therefore, for $||z|| > M$, we have

$$\begin{aligned} ||p(z)|| &\geq ||a_n|| \cdot ||z||^n \cdot \left(1 - \left\|\frac{a_{n-1}}{a_n}\right\| \cdot \frac{1}{||z||} - \cdots - \left\|\frac{a_0}{a_n}\right\| \cdot \left(\frac{1}{||z||}\right)^n\right) \\ &\geq ||a_n|| \cdot ||z||^n \cdot \frac{1}{2} \\ &> \frac{1}{2}||a_n|| \cdot M^n \\ &\iff \left\|\frac{1}{p(z)}\right\| < \frac{1}{\frac{1}{2}||a_n|| \cdot M^n} \end{aligned}$$

Notice that $\overline{D(0;M)}$ is compact and that $z \mapsto \left\|\frac{1}{p(z)}\right\|$ is continuous. Hence $\left\|\frac{1}{p(z)}\right\|$ is bounded on $\overline{D(0;M)}$. Therefore, $\left\|\frac{1}{p(z)}\right\|$ is bounded on $\mathbb{C}$. By Liouville, $\frac{1}{p(z)}$ is a constant function. Hence $p(z)$ is a constant function. However, we know that $p(z)$ is a polynomial of degree ≥ 1, which is a contradiction. $\square$

11 Singularity

Def. Let f be an analytic function defined on $D(z_0; r) - \{z_0\}$. Then f has an **isolated singularity** at z_0

- The function needs to be defined around the isolated singularity
- To create an isolated singularity:
  - Multiply by $\frac{1}{z-z_0}$
  - Multiply by $e^{\frac{1}{z-z_0}}$
- *Def.* z_0 is **removable** if $||f(z)||$ is bounded on $D(z_0; r) - \{z_0\}$
- *Def.* z_0 is a **pole** if $\lim\limits_{z \to z_0} ||f(z)|| = \infty$
- *Def.* z_0 is **essential** if it is neither removable nor a pole

Example: Let f be an analytic function on $D(z_0; r)$ and have a *removable singularity* on z_0. Let $g : D(z_0; r) \to \mathbb{C}$ defined by

$$z \longmapsto \begin{cases} (z - z_0)^2 f(z), & \text{if } z \neq z_0 \\ 0, & \text{if } z = z_0 \end{cases}$$

- g is analytic on $D(z_0; r) - \{z_0\}$
- g is analytic at z_0, since

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0)^2 f(z) - 0}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0$$

  and $f(z)$ is bounded on $D(z_0; r) - \{z_0\}$
- Power series expansion:

$$g(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

  - $a_0 = g(z_0) = 0$
  - $a = \frac{1}{1!} g'(z_0) = 0$ since the first derivative was computed to be zero
- So g has a zero of order at least 2, i.e. $g(z) = (z - z_0)^2 \hat{g}(z)$ for some analytic function $\hat{g}$ defined on $D(z_0; r)$
- Hence, on $D(z_0; r) - \{z_0\}$, we have $\hat{g}(z) = f(z)$
- Therefore, f can be extended to an analytic function on $D(z_0; r)$

Example: Let f be an analytic function on $D(z_0; r)$ and have a *pole* on z_0. Then $\lim\limits_{z \to z_0} ||f(z)|| = \infty$. Fix $M > 0$, by the $\epsilon - \delta$ definition of limit, $\exists \delta > 0$ such that $||f(z)|| > M$ for all $z \in D(z_0; \delta)$. So

$$\left\| \frac{1}{f(z)} \right\| < \frac{1}{M}$$

for all $z \in D(z_0; \delta) - \{z_0\}$. Define $\tilde{f} = \frac{1}{f}$, then $\tilde{f}$ is

- Analytic on $D(z_0; \delta) - \{z_0\}$
- $||\tilde{f}(z)||$ is bounded on $D(z_0; \delta) - \{z_0\}$
- Hence $\tilde{f}$ has an removable singularity at z_0

- Can extend $\tilde{f}$ to an analytic function $\tilde{\tilde{f}}$ on $D(z_0; r)$

Let $m \geq 0$ be the order of zero of $\tilde{\tilde{f}}$ at z_0. Then $\tilde{\tilde{f}}(z) = (z - z_0)^m \tilde{\tilde{\tilde{f}}}(z)$ for some analytic function $\tilde{\tilde{\tilde{f}}}$ on $D(z_0; \delta)$ such that $\tilde{\tilde{\tilde{f}}} \neq 0$. Therefore $\frac{1}{f(z)} = \tilde{f}(z) = \tilde{\tilde{f}}(z) = (z - z_0)^m \tilde{\tilde{\tilde{f}}}(z)$ on $D(z_0; \delta) - \{z_0\}$. Hence

$$f(z) = \frac{1}{(z - z_0)^m} \cdot \frac{1}{\tilde{\tilde{\tilde{f}}}(z)}$$

on $D(z_0; \delta) - \{z_0\}$. Since $\tilde{\tilde{\tilde{f}}}(z_0) \neq 0$, then $\exists \delta' > 0$ such that $\forall z \in D(z_0; \delta')$, $\tilde{\tilde{\tilde{f}}}(z) \neq 0$. So $\frac{1}{\tilde{\tilde{\tilde{f}}}(z)}$ is analytic on $D(z_0; \delta') \cap D(z_0; \delta)$. Therefore, $\exists \delta'' > 0$ such that

$$f(z) = \frac{1}{(z - z_0)^m} \cdot h(z)$$

on $D(z_0; \delta'')$ for some analytic function h defined on $D(z_0; \delta'')$ such tat $h(z_0) \neq 0$.

Def. From the above, m is the **order** of the pole of f at z_0.

12 Residue

Def. Let $f : D(z_0; r) - \{z_0\} \to \mathbb{C}$ be analytic. The **residue** $\operatorname{Res}(f; z_0)$ of f at z_0 is

$$\operatorname{Res}(f; z_0) = \frac{1}{2\pi i}\int_{\partial D(z_0;s)} f(z)dz \quad \forall 0 < s < r$$

- Notice that if $0 < s_1, s_2 < r$, then

$$\int_{\partial D(z_0;s_1)} f(z)dz = \int_{\partial D(z_0;s_2)} f(z)dz$$

Proof. (Well-definedness of residue) Consider an annulus $A(z_0; s_2, s_1) = \{z \in \mathbb{C} : s_2 < |z - z_0| < s_1\}$. The function f is analytic on $A(z_0; s_2, s_1)$. Let the annulus be parameterized by γ on the inside (with clockwise orientation) and Γ on the outside (with counterclockwise orientation). Then by Cauchy's Theorem, we have

$$0 = \int_{\Gamma\cup\gamma} f(z)dz = \int_{\Gamma} f(z)dz + \int_{\gamma} f(z)dz = \int_{\partial D(z_0;s_1)} f(z)dz - \int_{\partial D(z_0;s_2)} f(z)dz$$

Hence, the residue of f at z_0 is well-defined. $\square$

Assume that z_0 is a removable singularity or a pole of f. Then

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Define

$$g(z) = a_0 + a_1(z - z_0) + \cdots$$

Then g is analytic on $D(z_0; r)$. By Cauchy, $\int_{\partial D(z_0;s)} g(z)dz = 0$. Observe that

$$\int_{\partial D(z_0;s)} \frac{1}{z-z_0}dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it}dt = 2\pi i$$

Then for any $2 \le k \le m$,

$$\int_{\partial D(z_0;s)} \frac{1}{(z-z_0)^k}dz = \int_0^{2\pi} \frac{1}{(e^{it})^k} \cdot ie^{it}dt = \int_0^{2\pi} \frac{i}{(e^{it})^{k-1}}dt = \int_0^{2\pi} ie^{i(1-k)t}dt$$

Using Euler's formula, the above becomes

$$\int_0^{2\pi} i\left(\cos((1-k)t) + i\sin((1-k)t)\right)dt = -\int_0^{2\pi} \sin((1-k)t)dt + i\int_0^{2\pi} \cos((1-k)t)dt = 0$$

Therefore,

$$\begin{aligned}\operatorname{Res}(f; z_0) &= \frac{1}{2\pi i}\left(\int_{\partial D(z_0;s)} \frac{a_{-m}}{(z-z_0)^m}dz + \int_{\partial D(z_0;s)} \frac{a_{-m+1}}{(z-z_0)^{m-1}}dz + \cdots + \int_{\partial D(z_0;s)} \frac{a_{-1}}{(z-z_0)}dz + \int_{\partial D(z_0;s)} g(z)dz\right)\\ &= a_{-1}\end{aligned}$$

Let $0 < r < R$. Let $f : A(z_0; r, R) \to \mathbb{C}$ be analytic. Let $r < r' < R' < R$. Let γ parameterize the boundary of $D(z_0; r')$ with clockwise orientation and Γ parameterize the boundary of $D(z_0; R')$ with counterclockwise orientation. Then by Cauchy's Theorem, we have

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma\cup\gamma} \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{\partial D(z_0;R')} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{\partial D(z_0;r')} \frac{f(\zeta)}{\zeta - z}d\zeta$$

For the first term in the difference:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial D(z_0;R')}\frac{f(\zeta)}{\zeta - z}d\zeta &= \frac{1}{2\pi i}\int_{\partial D(z_0;R')}\frac{f(\zeta)}{(\zeta - z_0)-(z-z_0)}d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial D(z_0;R')}\frac{f(\zeta)}{(\zeta - z_0)\left(1-\frac{z-z_0}{\zeta-z_0}\right)}d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial D(z_0;R')}\frac{f(\zeta)}{\zeta - z_0}\cdot\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial D(z_0;R')}\frac{f(\zeta)}{\zeta - z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^k d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial D(z_0;R')}\sum_{k=0}^{\infty}\frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\cdot(z-z_0)^k d\zeta \\
&= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\int_{\partial D(z_0;R')}\frac{f(\zeta)}{(\zeta - z_0)^{k+1}}(z-z_0)^k d\zeta \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial D(z_0;R')}\frac{f(\zeta)}{(\zeta - z_0)^{k+1}}d\zeta\right)(z-z_0)^k
\end{aligned}$$

Doing the same for the second term:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial D(z_0;r')}\frac{f(w)}{w - z}dw &= \frac{1}{2\pi i}\int_{\partial D(z_0;r')}\frac{f(w)}{(w - z_0)-(z-z_0)}dw \\
&= \frac{1}{2\pi i}\int_{\partial D(z_0;r')}\frac{f(w)}{(z - z_0)\left(\frac{w-z_0}{z-z_0}-1\right)}dw \qquad \text{Notice the difference from above} \\
&= -\frac{1}{2\pi i}\int_{\partial D(z_0;r')}\frac{f(w)}{z - z_0}\cdot\frac{1}{1-\frac{w-z_0}{z-z_0}}dw \\
&= -\frac{1}{2\pi i}\int_{\partial D(z_0;r')}\frac{f(w)}{z - z_0}\sum_{k=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^k dw \\
&= -\frac{1}{2\pi i}\int_{\partial D(z_0;r')}\sum_{k=0}^{\infty}f(w)(w-z_0)^k\cdot\frac{1}{(z-z_0)^{k+1}}dw \\
&= -\frac{1}{2\pi i}\sum_{k=0}^{\infty}\left(\int_{\partial D(z_0;r')}f(w)(w-z_0)^k dw\right)\frac{1}{(z-z_0)}^{k+1} \\
&= -\frac{1}{2\pi i}\sum_{j=1}^{\infty}\left(\int_{\partial D(z_0;r')}f(w)(w-z_0)^{j-1} dw\right)(z-z_0)^{-j} \\
&= -\frac{1}{2\pi i}\sum_{j=-1}^{-\infty}\left(\int_{\partial D(z_0;r')}f(w)(w-z_0)^{-j-1} dw\right)(z-z_0)^{j} \\
&= \sum_{j=-1}^{-\infty}\left(-\frac{1}{2\pi i}\int_{\partial D(z_0;r')}\frac{f(w)}{(w-z_0)^{j+1}} dw\right)(z-z_0)^{j}
\end{aligned}$$

Therefore

$$f(z) = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial D(z_0;R')}\frac{f(\zeta)}{(\zeta - z_0)^{k+1}}d\zeta\right)(z-z_0)^k + \sum_{j=-1}^{-\infty}\left(\frac{1}{2\pi i}\int_{\partial D(z_0;r')}\frac{f(w)}{(w-z_0)^{j+1}} dw\right)(z-z_0)^{j}$$

Special case: $r = 0$, i.e. $f : D(z_0; R) - \{z_0\} \to \mathbb{C}$ is analytic, i.e. z_0 is an isolated singularity. Then

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{j=-1}^{-\infty} a_j (z - z_0)^j$$

is called the **Laurent series** of f

- In the case of a removable singularity, the Laurent series has *no negative powers*
- In the case of a pole (of order m), the Laurent series has *finitely many negative powers*
- In the case of an essential singularity, the Laurent series has *infinitely many negative powers*

Theorem 12.1 (Residue)**.** *Let D be a simply-connected domain and $z_1, \ldots, z_n \in D$. Let $f : D \backslash \{z_1, \ldots, z_n\} \to \mathbb{C}$ be analytic. Let γ be a positively oriented piecewise smooth simple closed curve in D which does not pass through $z_1, \ldots, z_n$. Then*

$$\frac{1}{2\pi i} \int_\gamma f(z) dz = \sum_{z_j \in \text{ inside of } \gamma} \operatorname{Res}(f; z_j)$$

Example. Let p, q be polynomials of degrees m, n resp. Assume $m \leq n - 2$ and $q(x) \neq 0$ for every $x \in \mathbb{R}$. Want to compute

$$\int_{-\infty}^{\infty} \frac{p(z)}{q(z)} dz$$

Visualize a semicircle centered at origin with radius $R \gg 0$ (upper half). By the Residue Theorem, we have

$$\frac{1}{2\pi i} \int_\gamma \frac{p(z)}{q(z)} dz = \sum_{q(w)=0,\ \operatorname{Re} w > 0} \operatorname{Res}\left(\frac{p}{q}; w\right)$$

Can compute RHS using Laurent series. LHS is equal to

$$\frac{1}{2\pi i} \int_\gamma \frac{p(z)}{q(z)} dz = \frac{1}{2\pi i} \int_{-R}^{R} \frac{p(z)}{q(z)} dz + \frac{1}{2\pi i} \int_{\gamma_R} \frac{p(z)}{q(z)} dz$$

where γ_R is the arc of the semicircle. Observe that

$$\lim_{R \to +\infty} \int_{-R}^{R} \frac{p(z)}{q(z)} dz = \int_{-\infty}^{\infty} \frac{p(z)}{q(z)} dz$$

Hence, sum of residues is equal to

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{p(z)}{q(z)} dz + \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\gamma_R} \frac{p(z)}{q(z)} dz$$

Write $p(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0$ and $q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0$. For $z \in \gamma_R$, we have that

$$\|p(z)\| = \left\| a_n z^n + \left(a_{n-1} z^{n-1} + \cdots + a_0\right) \right\| \leq 2 \|a_n\| \cdot R^n$$
$$\|q(z)\| = \left\| b_n z^n + \left(b_{n-1} z^{n-1} + \cdots + b_0\right) \right\| \geq \frac{1}{2} \|b_n\| \cdot R^n$$

Therefore

$$\left\| \frac{p(z)}{q(z)} \right\| \leq \frac{2||a_m||R^m}{\frac{1}{2}||b_n||R^n} = \frac{4||a_m||}{||b_n||} \cdot \frac{1}{R^{n-m}} \leq \frac{4||a_m||}{||b_n||} \cdot \frac{1}{R^2}$$

Hence

$$\left\| \int_{\gamma_R} \frac{p(z)}{q(z)} dz \right\| \leq \left(\max_{z \in \gamma_R} \left\| \frac{p(z)}{q(z)} \right\| \right) \cdot \operatorname{len}(\gamma_R) \leq \frac{4||a_m||}{||b_n||} \cdot \frac{1}{R^2} \cdot \pi R = \frac{4||a_m||\pi}{||b_n||} \frac{1}{R}$$

which tends to 0 as $R \to \infty$. Therefore

$$\lim_{R \to \infty} \int_{\gamma_R} \frac{p(z)}{q(z)} dz = 0$$

Lemma 12.2. *Let $f : D \to \mathbb{C}$ be analytic. Let $z_1, z_2, \ldots \in D$ such that $f(z_1) = f(z_2) = \cdots = 0$. Let $z_0 = \lim_{n\to\infty} z_n \in D$. Then $f = 0$.*

Proof. Taylor expand f around z_0 so that $f(z) = a_0 + a_1(z - z_0) + \cdots$. Then

$$a_0 = f(z_0) = f\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} 0 = 0$$

Now claim that $a_{n-1} = 0 \implies a_n = 0$. Assuming the claim proves the lemma.

To prove the claim, define

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^n}, & \text{if } z \neq z_0 \\ a_n, & \text{if } z = z_0 \end{cases}$$

g is analytic on $D \setminus \{z_0\}$. Moreover,

$$g(z) = \frac{f(z)}{(z - z_0)^n} = \frac{1}{(z - z_0)^n}\left(a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots\right) = a_n + a_{n+1}(z - z_0) + \cdots$$

This implies that

$$\lim_{z\to z_0} g(z) = a_n = g(z_0)$$

which means that g is continuous, and so g is bounded in a neighbourhood of z_0, which means that g has a removable singularity at z_0. Therefore for all $k \in \mathbb{N}$, $g(z_k) = 0$. ☐

Example: Let $f : D \to \mathbb{C}$ and $f(z_0) = 0$. Want to compute

$$\frac{1}{2\pi i}\int_{\partial D(z_0;r)} \frac{f'(z)}{f(z)} dz$$

Suppose z_0 is a zero of order n of f. Then $f(z) = (z - z_0)^n g(z)$, where g is analytic and that $g(z_0) \neq 0$. Hence,

$$\frac{f'(z)}{f(z)} = \frac{n(z - z_0)^{n-1}g(z) + (z - z_0)^n g'(z)}{(z - z_0)^n g(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}$$

Where the fraction involving g is analytic near z_0. Therefore

$$\text{Res}\left(\frac{f'}{f}; z_0\right) = \text{Res}\left(\frac{n}{z - z_0}; z_0\right) = n$$

Therefore

$$\frac{1}{2\pi i}\int_{\partial D(z_0;r)} \frac{f'(z)}{f(z)} dz = n$$

Now suppose z_0 is a pole of order n of f, then by a similar argument the integral evaluates to $-n$.

Theorem 12.3. *Let $f : D \to \mathbb{C}$ be analytic except at poles $z_1, \ldots, z_n$. Let γ be a positively oriented, smooth, simple closed curve which together with its inside is contained in D. Assume that γ does not pass through any zero or pole of f. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = \text{\# zeros in the inside of } \gamma - \text{\# poles in the inside of } \gamma$$

- Potential user case:

$$(\log(f(z)))' = \frac{f'(z)}{f(z)}$$

Corollary 12.3.1 (Argument Principle)**.**

$$\frac{1}{2\pi} \times \textit{(change in } \operatorname{Arg} f(z) \textit{ as } z \textit{ traverse } \gamma\textit{)}$$
$$= \textit{\# zeros of } f \textit{ in the inside of } \gamma - \textit{\# poles of } f \textit{ in the inside of } \gamma$$

Theorem 12.4 (Rouché)**.** *Let $f, g : D \to \mathbb{C}$ be analytic. Let γ be piecewise smooth, simple closed curve contained in D where the inside of γ is also contained in D. Suppose that*

$$\forall z \in \gamma, \|f(z) + g(z)\| < \|f(z)\|$$

Then f and g have the same number of zeros in the inside of γ.

Proof. Define $h(z) = g(z)/f(z)$. For every $z \in \gamma$, we have that

$$\|1 + h(z)\| = \left\|1 + \frac{g(z)}{f(z)}\right\| = \frac{1}{\|f(z)\|} \cdot \|f(z) + g(z)\| < \frac{1}{\|f(z)\|} \cdot \|f(z)\| = 1$$

Hence, $h(z) \in D(-1; 1)$ for every $z \in \gamma$ (which does not cross the origin). Therefore $\operatorname{Arg} h(z)$ does not change as z traverses γ. By the Argument Principle, h has the same number of zeros and poles enclosed by γ. □

13 Maximum Modulus and Mean Value

Theorem 13.1 (Open Mapping)**.** *Suppose that f is a nonconstant analytic function on a domain D. Then the range of f is an open set.*

Proof. Let $f : D \to \mathbb{C}$ be analytic. Suppose that f is not constant. Let $w_0 = f(z_0)$ be an arbitrary point in the range of f. Then $f(z) - w_0$ has a zero of order $m \geq 1$ at z_0. Choose a small enough r so that $f(z) - w_0$ has no zero in the region $0 < |z - z_0| \leq r$, which is possible since zeros of a nonconstant analytic function are isolated. Let

$$\delta = \min_z \{f(z) - w_0 : |z - z_0| = r\}$$

Let w be any point with $|w - w_0| < \delta$. Then on the circle $|z - z_0| = r$:

$$|[f(z) - w] - [f(z) - w_0]| = |w - w_0| < \delta \leq |f(z) - w_0|$$

By Rouché's Theorem, $f - w$ and $f - w_0$ have an equal number of zeros within the circle $|z - z_0| = r$. This shows that each point w_0 in the range of f lies at the center of a small disc, which is also within the range of f. Therefore the range of f is open. □

Corollary 13.1.1 (Maximum Modulus Principle)**.** *If f is a nonconstant analytic function on a domain D, then $|f|$ can have no local maximum on D.*

Proof. Suppose for a contradiction that $|f(z_0)| \geq |f(z)|$ for all z with $|z - z_0| < r$, then $f(z_0)$ lies on the boundary of the open set $W = \{f(z) : |z - z_0| < r\}$, which is a contradiction. □

- If f is analytic and nonconstant on a domain D, then $\operatorname{Re} f$ has no local maxima and no local minima on D.
- If f is analytic and on a bounded domain D and continuous on $D \cup \partial D$. Then each of $|f|, \operatorname{Re} f, -\operatorname{Re} f$ attains its maximum value on ∂D.

Lemma 13.2 (Schwarz)**.** *Suppose that f is analytic in the disc $|z| < 1$, that $f(0) = 0$, and that $|f(z)| \leq 1$ for all z in the disc. Then*

$$|f(z)| \leq |z|, \quad |z| < 1$$

- Equality can hold for some $z \leq 0$ only if $f(z) = \lambda z$, where λ is a constant of absolute value 1

Proof. Since $f(0) = 0$, we know that $g(z) = f(z)/z$ is also analytic on $|z| < 1$. For $|z| = r$,

$$|g(z)| = \frac{|f(z)|}{r} \leq \frac{1}{r}$$

By the maximum modulus principle, the inequality above is true for $|z| < r$ as well. Since r can be arbitrarily close to 1, we must have that $|g(z)| \leq 1$ if $|z| < 1$.

Furthermore, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $1 = |g(z_0)|$; consequently, $|g(z)$ has an interior maximum. This implies that g is a constant λ where $|\lambda| = 1$. This gives the conclusion that $f(z) = \lambda z$. □

Theorem 13.3 (Mean Value)**.** *Let f be analytic and z_0 in the domain of f. Then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt$$

Proof. Cauchy's Formula gives

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z_0} d\zeta$$

where γ is a circle and z_0 is the inside of γ. Taking z_0 to be the center of the circle, then $\zeta = z_0 + re^{it}, 0 \leq t \leq 2\pi, d\zeta = ire^{it} dt$. Plugging those values gives the result. □

14 Linear Fractional Transformations

Def. A **linear fractional transformation** T is a rational function of the special form

$$T(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$

- The restriction $ad - bc \neq 0$ is essential, since otherwise

  $$T'(z) = \frac{ad-bc}{(cz+d)^2} = 0$$

  for all z, so T is identically constant
- T is a *one-to-one* function
- T has a pole of order 1 at $-d/c$
- $\lim\limits_{|z|\to\infty} T(z) = a/c$
- There is a function T^{-1} that is the inverse of T such that $T^{-1}(T(z)) = z$, where

  $$T^{-1}(w) = z = \frac{-dw+b}{cw-a}$$
- T^{-1} is also a linear fractional transformation
- A linear fractional transformation is a one-to-one mmapping of the complex plane plus the point at ∞ onto itself
- A one-to-one analytic mapping of the complex plane plus ∞ onto itself is a linear fractional transformation

Def. A linear fractional transformation that is not identically equal to z has at most two distinct **fixed points**, i.e. points z for which $T(z) = z$

- z is the solution of the equation $T(z) = z$ when z is a root of the quadratic equation

  $$cz^2 + (d-a)z - b = 0$$

Lemma 14.1. *If T and S are two linear fractional transformations that are equal at 3 distinct points, then $T = S$*

Proof. The linear fractional transformation $S^{-1}(T(z))$ has three distinct fixed points, so it must be a constant. □

Lemma 14.2. *Given three distinct complex numbers z_1, z_2, z_3, and three other distinct complex numbers w_1, w_2, w_3, then there is a unique linear fractional transformation L with $L(z_j) = w_j$, $j = 1, 2, 3$.*

Proof. Set

$$T(z) = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}$$

Then $T(z_1) = 0, T(z_2) = 1, T(z_3) = \infty$. Let

$$S(w) = \frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1}$$

so that $S(w_1) = 0, S(w_2) = 1, S(w_3) = \infty$. Then L is given by

$$L(z) = S^{-1}(T(z))$$

□

- Can use this to find the linear fractional transformation that sends three points to three other points

Lemma 14.3. *A linear fractional transformation maps*

- *A circle onto another circle or a straight line*
- *A straight line onto another straight line or a circle*

Proof. If $T(z) = az + b, a \neq 0$, then T maps circles and straight lines to the same type

- The circle $\{Z : |z - z_0| = r\}$ is transformed to the circle $\{w : |w - (az_0 + b)| = |a|r\}$
- The straight line $\{z : \mathrm{Re}(Az + B) = 0\}$ is transformed to the straight line $\{w : \mathrm{Re}[(A/a)w + B - b(A/a)] = 0\}$

Let $T(z) = \frac{az+b}{cz+d}$ where $c \neq 0$. Now

$$T(z) = \frac{az + b}{cz + d} = \frac{1}{c}\left(\frac{bc - ad}{cz + d} + a\right)$$

and so T is the composition of the linear fractional transformations

$$T(z) = W(V(U(z)))$$

where

$$U(z) = cz + d \qquad V(w) = \frac{1}{w} \qquad W(\zeta) = \frac{1}{c}[(bc - ad)\zeta + a]$$

Knowing that U and W send circles to cirles and lines to lines, we need to show that V sends circles to circles and lines to lines. Define the equation

$$\alpha(x^2 + y^2) + \beta x + \gamma y = \delta$$

where $\alpha, \beta, \gamma, \delta$ are real and not all of α, β, γ are zero, represents either a circle (iff $\alpha \neq 0$ and $\beta^2 + \gamma^2 + 4\alpha\delta > 0$) or a straight line (iff $\alpha = 0$). Notice that

$$\frac{1}{z} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = u + iv$$

replacing z by $\frac{1}{z}$ yields

$$\delta(u^2 + v^2) - \beta u + \gamma v = \alpha$$

which is a line or a circle, completing this proof. $\square$

15 Conformal Mapping

Let $f : \mathbb{C} \to \mathbb{C}$ be anaytic. Let $z_0 \in \mathbb{C}$ and $w_0 = f(z_0)$. Let $\gamma : [a,b] \to \mathbb{C}$ such that $\exists t \in (a,b)$ such that $\gamma(t) = z_0$. Then $f \circ \gamma : [a,b] \to \mathbb{C}$ such that $f \circ \gamma(t) = w_0$. Assume that γ is smooth, then

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0) = f'(z_0)\gamma'(t_0)$$

In particular,

$$\arg(f \circ \gamma)'(t_0) = \arg(f'(z_0)) + \arg(\gamma'(t_0))$$

Now suppose there are two curves (with direction) $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ such that $\exists t_1 \in (a,b), \exists t_2 \in (c,d)$ such that $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ (i.e. the two curves intersect at z_0). We are interested in the angle between the two tangent lines (in the direction of the curves) at z_0.

Def. The **angle** from γ_1 to γ_2 is the angle θ, measured counterclockwise, from $\gamma_1'(t_1)$ to $\gamma_2'(t_2)$

Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and assume $f'(z_0) \neq 0$. Let $w_0 \triangleq f(z_0)$. Want to know that angle from $f \circ \gamma_1$ to $f \circ \gamma_2$ at w_0. We have the following:

$$\begin{aligned}\arg(f \circ \gamma_1)'(t_1) &= \arg f'(z_0) + \arg \gamma_1'(t_1)\\ \arg(f \circ \gamma_2)'(t_2) &= \arg f'(z_0) + \arg \gamma_2'(t_2)\end{aligned}$$

Hence,

$$\arg(f \circ \gamma_2)'(t_2) - \arg(f \circ \gamma_1)'(t_1) = \arg \gamma_2'(t_2) - \arg \gamma_1'(t_1)$$

Def. Let $\varphi : D(z_0; \epsilon) \to \mathbb{C}$ be a function. φ is **conformal** at z_0 if for any curves γ_1, γ_2 intersecting at z_0, the angle from γ_1 to γ_2 is equal to the angle from $\varphi \circ \gamma_1$ to $\varphi \circ \gamma_2$ at $\varphi(z_0)$.

Theorem 15.1. *Let $f : D(z_0; \epsilon) \to \mathbb{C}$ be analytic. If $f'(z_0) \neq 0$, then f is conformal at z_0.*

Corollary 15.1.1. *If f is analytic and injective on some domain D, then f is conformal on D.*

Examples of Conformal Mapping

- $f(z) = z$ is conformal on C
- $f(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$ is conformal on its domain of definition
- The Cayley Transform $C(z) = -\frac{z-i}{z+i}$ is conformal
  - The upper complex plane is mapped to the unit disc
  - Under the Cayley Transform, an imaginary number is mapped to the real number

$$C(iy) = -\frac{iy - i}{iy + i} = -\frac{i(y-1)}{i(y+1)} = -\frac{y-1}{y+1} = -\left(1 - \frac{2}{y+1}\right)$$

Def. Let $p : D \to \mathbb{R}$ where $D \subseteq \mathbb{C}$ is a domain. The **level curve** of p at level c is the set

$$\{(x,y) \in D : p(x,y) = c\}$$

Let $f : D \to \mathbb{C}, z_0 \in D, f'(z_0) \neq 0$. Denote $f(z) = u(z) + iv(z) = u(x,y) + iv(x,y)$. Assume for simplicity that f is injective on D. Let $\Omega = f(D)$. Notice that Ω is an open subset of $\mathbb{C}$. Let $g : \Omega \to \mathbb{C}$ be the (analytic) inverse function of f. Denote $g(z) = \sigma(z) + i\tau(z) = \sigma(x,y) + i\tau(x,y)$. Let

$$\begin{aligned}\gamma_1 &= \{z \in D : u(z) = u(z_0)\}\\ \gamma_2 &= \{z \in D : v(z) = v(z_0)\}\end{aligned}$$

- γ_1 is the level curve of u at level $u(z_0)$
- γ_2 is the level curve of v at level $v(z_0)$

Then

$$\begin{aligned}\gamma_1 &= \{z \in D : u(z) = u(z_0)\} \\ &= \{z \in D : \operatorname{Re} f(z) = u(z_0)\} \\ &= \{g(w) : \operatorname{Re} w = u(z_0)\} \qquad \text{Since } w = f(z) \\ &= g\left(\{w \in \Omega : \operatorname{Re} w = u(z_0)\}\right) \end{aligned} \tag{15.1}$$

and

$$\begin{aligned}\gamma_2 &= \{z \in D : v(z) = v(z_0)\} \\ &= \{z \in D : \operatorname{Im} f(z) = v(z_0)\} \\ &= \{g(w) : \operatorname{Im} w = v(z_0)\} \qquad \text{Since } w = f(z) \\ &= g\left(\{w \in \Omega : \operatorname{Im} w = v(z_0)\}\right) \end{aligned} \tag{15.2}$$

Notice that the set in (15.1) is a vertical line, and the set in (15.2) is a horizontal line. Therefore g maps orthogonal curves to orthogonal lines (which forms a grid). We can get "coordinate axes" by setting $u(z)$ and $v(z)$ to be constant.

16 Schwarz Christoffel Formula

Want to map the upper half plane to a polygon. Consider

$$f(z) = A(z - x_0)^{\beta} + B$$

where $A, B \in \mathbb{C}$, $x_0, \beta \in \mathbb{R}$, $\beta \in (0, 2)$. Choose arg such that $arg(z - x_0) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Let $x \in \mathbb{R}$. Want to compute

$$f'(x) = A\beta(x - x_0)^{\beta - 1}$$

In the case of $x > x_0$,

$$\begin{aligned}
\arg f'(x) &= \arg\left(A\beta(x - x_0)^{\beta-1}\right) \\
&= \arg A + \arg\beta + (\beta - 1)\arg(x - x_0) \\
&= \arg A + 0 + 0 \\
&= \arg A
\end{aligned}$$

In the case of $x < x_0$,

$$\begin{aligned}
\arg f'(x) &= \arg\left(A\beta(x - x_0)^{\beta-1}\right) \\
&= \arg A + \arg\beta + (\beta - 1)\arg(x - x_0) \\
&= \arg A + 0 + (\beta - 1)\pi \\
&= \arg A + (\beta - 1)\pi
\end{aligned}$$

Supose that $x_1 < x_2 < \cdots < x_N \in \mathbb{R}$. Let f be a function whose derivative is

$$A(z - x_1)^{\alpha_1}(z - x_2)^{\alpha_2} \cdots (z - x_N)^{\alpha_N}$$

where $A \in \mathbb{C}$ and $\alpha_1, \ldots, \alpha_N \in (-1, 1)$.

In the case of $x < x_1$,

$$\begin{aligned}
\arg f'(x) &= \arg\left(A(x - x_1)^{\alpha_1} \cdots (x - x_N)^{\alpha_N}\right) \\
&= \arg A + \alpha_1 \arg(x - x_1) + \cdots + \alpha_N \arg(x - x_N) \\
&= \arg A + \alpha_1\pi + \cdots + \alpha_N\pi
\end{aligned}$$

In the case of $x_1 < x < x_2$,

$$\begin{aligned}
\arg f'(x) &= \arg\left(A(x - x_1)^{\alpha_1} \cdots (x - x_N)^{\alpha_N}\right) \\
&= \arg A + \alpha_1 \arg(x - x_1) + \cdots + \alpha_N \arg(x - x_N) \\
&= \arg A + 0 + \alpha_2\pi + \cdots + \alpha_N\pi
\end{aligned}$$

In the general case of $x_j < x < x_{j+1}$ for $j = 1, \ldots, N - 1$,

$$\arg f'(x) = \arg A + \alpha_{j+1}\pi + \cdots + \alpha_N\pi$$

In the case of $x > x_N$,

$$\arg f'(x) = \arg A$$

Let P be the polygon with vertices $w_0, w_1, \ldots, w_N$. The exterior angles at $w_1, w_1, \ldots, w_N$ are $\theta_0, \theta_1, \ldots, \theta_N$. Write $\theta_i = \alpha_i\pi$, $\alpha_i \in (-1, 1)$. Note that $\alpha_0 + \alpha_1 + \cdots + \alpha_N = 2$.

Theorem 16.1 (Schwarz-Christoffel)**.** *$\exists x_1 < \cdots < x_N \in \mathbb{R}, A \in \mathbb{C}$ such that a function f whose derivative is $A(z - x_1)^{\alpha_1} \cdots (z - x_N)^{\alpha_N}$ gives a bijection from the upper half plane to P. Moreover, f maps $x_1, \ldots, x_N$ to $w_1, \ldots, w_N$, $\lim\limits_{x\to\infty} f(x) = w_0$.*