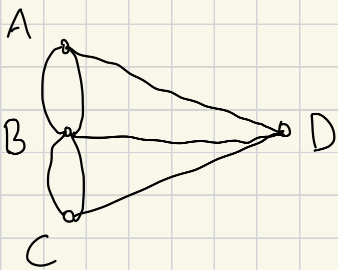
MAT 332

Introduction to Graph Theory

7 Bridges of Konigsburg



Want to use each bridge exactly once

Thm: this is impossible.

Pf (by contradiction) Sps such walk exists,

Represent the walk by a sequence of letters

i.e. ADBCB

Claim: if a city X is not start / end of a walk, then the walk crossed an even number of bridges touching X.

Pf easy

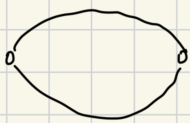
Notice that A, B, C, D all touch an odd # of bridges

Graph

Def (informal). A graph G is a finite list of nodes (called vertices) and the information of how they are connected (called edges)

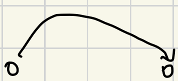
Def. A graph G is a pair (V, E) where V is a finite set (called the vertices) and $E \subseteq \{(v,w) \in V^2 : v \neq w\}$ (called the edges)

Forbidden:



(2 edges between vertices)

(self edge)



(directed edges)

- $V(G) / E(G)$: vertices / edges of G
- $|G| = |V(G)|$: order of G
- $e(G) = |E(G)|$: size of G

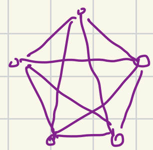
If $|G| = n$, then the possible values of $e(G)$ are $\in \left[0, \binom{n}{2}\right]$

order $= n$

size $= \binom{n}{2}$

$d_i = n-1$

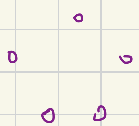
Complete graph K_n :

$K_5 =$ 

- $V = \{1, \dots, n\}$
- $E = \{ij : 1 \leq i < j \leq n\}$

$(n-1)$-regular

Complement $\overline{K_n}$: swap all edges and non-edges

$\overline{K_5} =$ 

order $= n$

size $= 0$

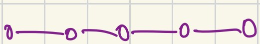
$d_i = 0$

0-regular

Neighbourhood $\Gamma(v) = N(v) = \{w \in V : vw \in E\}$

Degree of $v \in V$ is $d_v = |\Gamma(v)|$

Path graph P_n (length $n-1$ path)

$P_5 =$ 

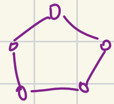
$d_1 = 1$ $d_2 = 2$ $\dots$ $d_5 = 1$

order $= n$

size $= n-1$

- $V = \{1, \dots, n\}$
- $E = \{i(i+1) : i = 1, \dots, n-1\}$

Cycle graph C_n (length n cycle)

$C_5 =$ 

order $= n$
size $= n$

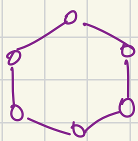
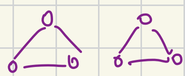
- $V = \{1, \dots, n\}$

2-regular

- $E = \{i(i+1) : i = 1, \dots, n-1\} \cup \{1n\}$

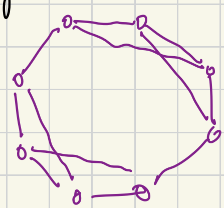
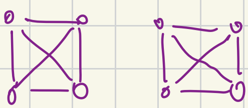
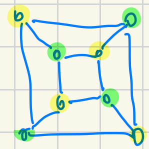
A k-regular graph is a graph w/ $d_v = k$ $\forall v \in V$

- 2-regular 6-vertex graph

C_6  

$C_3 \sqcup C_3$
↑
disjoint union

- 3-regular 8-vertex graph

- 3-regular 9-vertex graph

$e(G) = 3 \cdot 9/2 = 13.5$ impossible

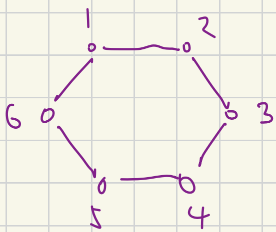
Lemma (Handshaking). $\sum_{v \in V} d_v = 2e(G)$

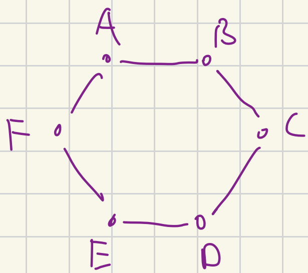
Isomorphism

Def. $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic (denoted $G_1 \cong G_2$) if $\exists$ bijective function $f: V_1 \to V_2$ s.t.

$vw \in E_1 \iff f(v)f(w) \in E_2$

Ex.





$f:$

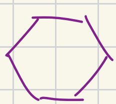
1	2	3	4	5	6
A	B	C	D	E	F

$f:$

1	2	3	4	5	6
F	E	D	C	B	A

(maps edge to edge, non-edge to non-edge)

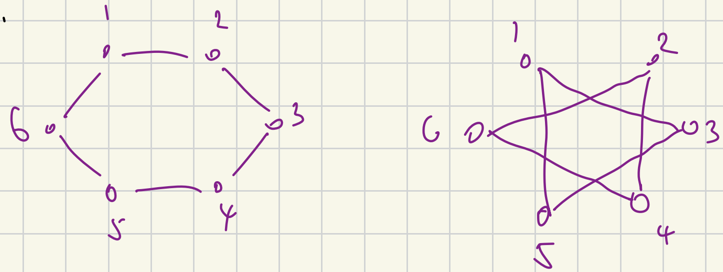
Or



(means graph of this shape labelled in some way)

$d_v = 2$

Ex.



i connected to $i-1$ & $i+1$ on left

Cannot happen for right no matter how we arrange vertices

also notice left has path of length 5

right does not

Thm. $G_1 \cong G_2 \iff \overline{G_1} \cong \overline{G_2}$

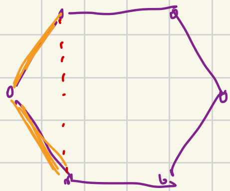
Def. H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

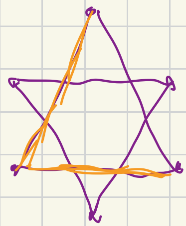
- H is an induced subgraph of G if $V(H) \subseteq V(G)$ and $E(H) = \{vw : v, w \in V(H), vw \in E(G)\}$

Def. G contains an H if H is isomorphic to a subgraph of G

- G contains an induced H if H is isomorphic to an induced subgraph of G

Ex.

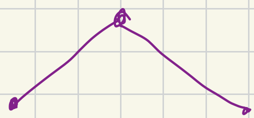




not induced b/c non-edge does not match

Contains an induced path of length 2

Contains a path of length 2

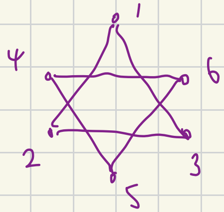


Ex. Every graph w/ ≤ 5 vertices is contained in K_5

Def. A clique in G is a complete (induced) subgraph of G

Def. An independent set in G is a subset $U \subseteq V$ s.t. there are no edges between pairs of vertices in U

Ex.



Cliques: $\emptyset$, 1, 2, 3, 4, 5, 6,

12, 23, 13, 45, 56, 46,

123, 456

Independent sets: $\emptyset$, 1, 2, 3, 4, 5, 6, 14, 15, 16, ...

Def. If $U \subseteq V$, we write $G[U]$ for the induced subgraph w/ vertex set U

- Indep. set of size $m \iff G[U] \cong \overline{K_m}$

Def. $G = (V, E)$. The complement $\overline{G} = (V, V^{(2)} \setminus E)$

- $V^{(2)}$ means all pairs in V

Def. A walk is a list of vertices $x_0, \dots, x_k$ of G s.t. $x_i x_{i+1} \in E$

- Allows for repeats
- A path is a walk with no repeated vertices

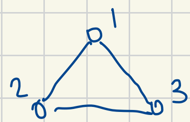
Def. $v, w \in V$ are linked, denoted $v \sim w$, if there is a walk $v = x_0, x_1, \dots, x_{k-1}, x_k = w$

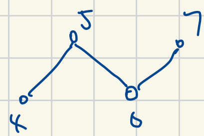
Thm. Being linked is an equivalence relation.

Thm. If $v \sim w$ then there is a path from v to w

Def. A connected component of G is the induced subgraph $G[U]$ where U is an equivalence class for $\sim$

Ex. Two linkage equivalent class 123, 4567

$G[123] =$ 

$G[4567] =$ 

Def. If G_1, G_2 are graphs on disjoint vertex sets, define $G_1 \amalg G_2 = (V(G_1) \amalg V(G_2), E(G_1) \amalg E(G_2))$

Thm. If $G_1, \dots, G_k$ are the connected component of G, then $G = G_1 \amalg \cdots \amalg G_k$

- This is the unique way of decomposing G into

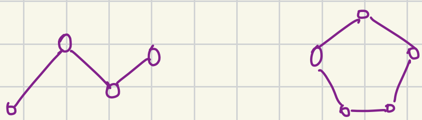
Connected graphs (up to reordering)

- $\emptyset$ is not connected

Lem. If H is a subgraph of G that is connected, then H is a subgraph of some connected comp. G'

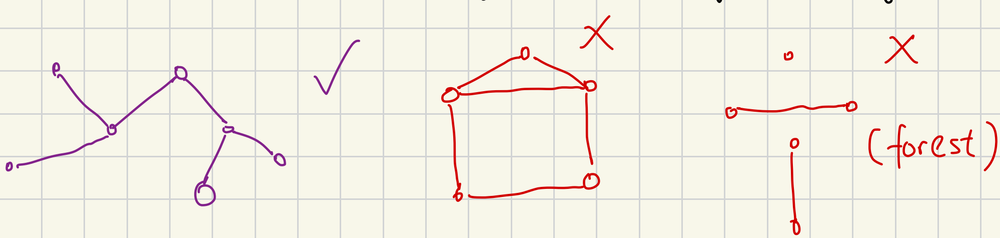
Def. $\Delta(G)$ = maximum degree of G = $\max_{v \in V(G)} d_v$

Thm. If $\Delta(G) \le 2$, then G is a disjoint union of paths and cycles



- To prove, break G into connected components
- <u>Technique of the longest path</u> : consider the longest path inside G

Def. A tree is a connected graph w/ no cycles (acyclic)



Def. A forest is an acyclic graph

- ϕ is a forest but not a tree (not connected)

Thm. Every forest is a disjoint union of trees

Def. A leaf is a vertex w/ degree 1 (usually in a tree)

- Degree 0 is not a leaf

Thm. Every tree (with at least 2 vertices) has at least 2 leaves

- Prove using longest path

Thm. The following are equivalent:

a) G is a tree

b) G is connected with $e(G) = |G| - 1$

c) G is minimally connected (i.e. G is connected and removing any edge results in disconnected graph)

d) G is maximally acyclic

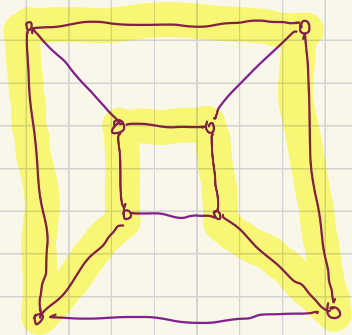
Thm. If G is a tree, then every pair of distinct vertices are connected by a unique path

- Prove by contradiction: assume $\exists$ paths P_1, P_2 connecting two vertices, must end up w/ a cycle

Thm. Every connected graph has a spanning tree (i.e. a subgraph which is a tree containing every vertex)

Def. A cycle $C \subseteq G$ is Hamiltonian if C uses all vertices of G, i.e. $V(C) = V(G)$

- G is Hamiltonian if it has a Hamiltonian cycle



Thm. (Bondy-Chvatal) Let G be a connected graph, $e = xy \in E(\overline{G})$, $|G| = n$. Sps $d_x + d_y \geq n$. Then G Hamiltonian $\iff$ $G \cup e$ Hamiltonian.

Cor. If every $e = xy \in E(\overline{G})$ has $d_x + d_y \geq n$, then G has a Hamiltonian cycle.

Proof to BCT:

Given $d_x + d_y \geq n$, WTS $\exists$ edges then

$$\Gamma(x) = A \subseteq \{2, 3, \ldots, n\}$$

$$\Gamma(y) = B \subseteq \{1, 2, \ldots, n-1\}$$

shift back by 1

Goal: find element common in $A-1$ and B

$|A-1| = d_x \qquad |B| = d_y$

$\Rightarrow A-1,\ B \subseteq \{1, \ldots, n-1\}$ PHP since

Def. A Eulerian circuit is a walk which uses every edge exactly once and returns to the start point

- Eulerian: every edge used
- Circuit: a walk that ends in the start point

Thm. If G has an Eulerian circuit, then all degrees are even.

- Prove using the fact that each edge is used exactly once

Thm. G has an Eulerian circuit iff G is connected and all degrees are even

- To prove, consider ~~longest~~ circuit (exists since all degrees ≥ 2
- Consider G - all edges from the circuit and decompose into connected components $G_1 \amalg \cdots \amalg G_c$
- Claim that G_i has all degrees even b/c $d_{v \in G}$ and $d_{v \in \text{circuit}}$ both even (same claim on smaller graphs), then use induction
- Notice that each G_i shares at least 1 vertex w/ circuit (otherwise G_i is disconnected from rest)
- Call original circuit C and other circuits C_i. First follow C then whenever we touch C_i, follow it $\Rightarrow$ obtain an Eulerian circuit

Def. An Eulerian walk is a walk in G where every edge is used exactly once

- Does not require start point = end point

Thm. A connected graph G has an Eulerian walk iff it has 0 or 2 vertices of odd degree

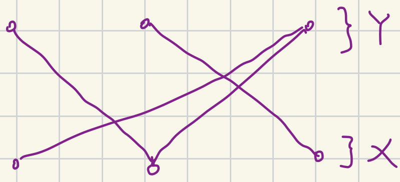
- Given a walk, if it's a circuit then 0 vertices of odd degree, if not then all vertices except start/end have even degree
- If all degrees are even, then Eulerian circuit.

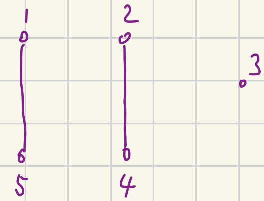
  If 2 vertices x, y have odd degree then
  - x, y not connected, then connect them to get Eulerian circuit, remove such edge to get Eulerian walk
  - $xy \in E$, then remove such edge. If this disconnect the graph, then both components have all vertices of even degree, then can find Eulerian circuit in each component, then can construct Eulerian walk with the edge.

    Otherwise, all vertices are of even degree. Start from x, follow circuit, then xy to get Eulerian walk.

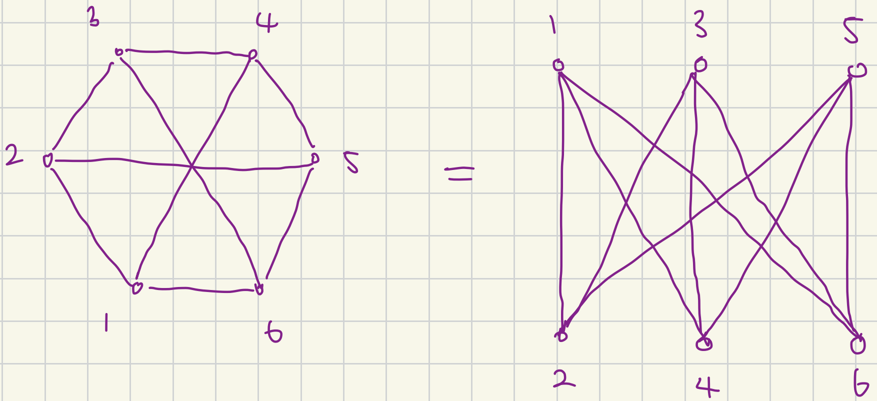
Def. A graph G is bipartite if we can divide $V = X \amalg Y$ w/ X, Y independent sets

- X, Y not necessarily unique





- A bipartition $V = X \amalg Y$ is a specific way of realizing a bipartite graph
- $K_{m,n}$ is the complete bipartite graph, where $|X| = m$, $|Y| = n$, with all possible edges
  - $K_{m,n} \cong K_{n,m}$



- $K_{m,n}$ has mn edges

Def. A proper bi-colouring of the vertices V of a graph G is an assignment of black/white to each vertex s.t. there is no BB/WW edges.

Thm. G bipartite $\Leftrightarrow$ $\exists$ proper bi-colouring

Thm. The following are equivalent:

① G is bipartite

② Every cycle has even length

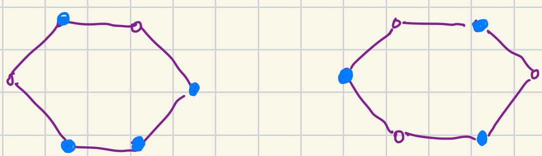
③ Every circuit has even length

Thm. If G is bipartite w/ bipartition $X \amalg Y$, then

$$e(G) = \sum_{x \in X} d_x = \sum_{y \in Y} d_y$$

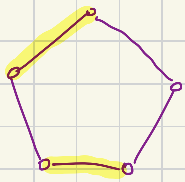
- Every edge is incident to 1 vertex in X and 1 vertex in Y

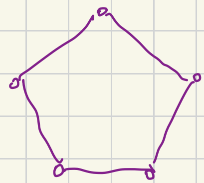
Def. A vertex cover $C \subseteq V$ is a set of vertices s.t. each edge touches at least 1 vertex from C



Lem. Complement of a vertex cover is an independent set (and vice versa)

Def. $M \subseteq E$ is a matching if no two edges share a vertex.





(empty)

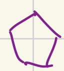
Thm. Let M be a matching and C be a vertex cover.

Then $|M| \leq |C|$

- Construct an injection $f: M \to C$
- For each $e \in M$, define $f(e)$ to be an endpoint covered by C
- If $f(e_1) = f(e_2) = v$, then v is an endpoint of both e_1, e_2. Since $e_1, e_2 \in M$, must have $e_1 = e_2$

Def. match(G) is the size of the largest matching in G

Def. cov(G) is the size of the smallest VC in G

match() = 2 cov() = 3

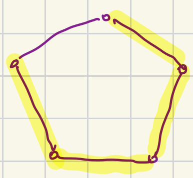
Cor. $\text{match}(G) \leq \text{cov}(G)$

Def. The independence number $\alpha(G)$ is the size of the largest independent set in G

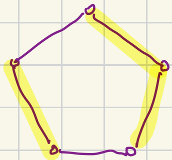
Thm. $\alpha(G) + \text{cov}(G) = |G|$

- Complement of largest IS is the smallest VC

Def. The edge covering number of a graph G (with no isolated vertices) is the smallest number of edges needed to touch every vertex of G, denoted $\text{Ecov}(G)$



edge cover



number $= 3$

Thm. If G has no isolated vertices, then

$\text{match}(G) + \text{Ecov}(G) = |G|$

- First show that $\text{Ecov}(G) \leq |G| - |M|$ for some maximum matching. Want to find EC w/ $|C| \leq |G| - |M|$.

Take $C = M \cup$ one edge touching each vertex not touching M. M accounts for $2|M|$ vertices in the graph. Remaining edges in $C \leq G - 2|M|$.

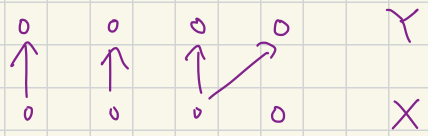
Therefore $|C| = |M| + |G| - 2|M| = |G| - |M|$.

- Then show a concrete example of $\text{match}(G) \geq |G| - \text{Ecov}(G)$. Given an extreme edge covering C, construct a matching M s.t. $|M| \geq |G| - |C|$. Start w/ no edges ($|G|$ connected comps), adding in edges of C one at a time decreases # of conn comps by at most 1 each time. So smallest # of conn comps possible is $|G| - |C|$, $|M|$ consisting of 1 edge from each connected component. This proves $|M| \geq |G| - |C|$.

Thm. (Konig's) For a bipartite graph G, $\text{match}(G) = \text{cov}(G)$.

- Proof is an algorithm, produce matching M, covering C s.t. $|M| \leq \text{match}(G) \leq \text{cov}(G) \leq |C| = |M|$
- Given matching M, start with $M = \emptyset$
- Put an $\uparrow$ on any edge $\notin M$

- Put a ↓ on any edge $\in M$
- Try and connect an unmatched vertex by following path (bottom vertices → top vertices)
- Take C as all vertices in Y reachable from U_X together w/ vertices in $X \setminus U_X$ on edges of matching not already covered
  - U: unmatched



(Ford-Fulkerson)

- This yields $|C| = |M|$
- C is a VC (proof by consider each kind of edge)

Def. An X-perfect matching is a matching using all vertices of X (allows nodes in Y to be unused)

Thm. (Hall's Marriage) Let G be bipartite. G has an X-perfect matching iff for each $A \subseteq X$, $|\Gamma(A)| \geq |A|$.

- Prove using G has X-perfect matching $\iff \text{match}(G) = |X|$ $\iff \text{cov}(G) = |X|$. Let C be such cover.

Ex. Consider a deck of 52 cards, deal out 13 piles of 4 cards each, show that can select one card from each pile s.t. all ranks $A \to K$ appear.

- Consider bipartite graph where $X = \{\text{piles}\}$ and $Y = \{A, \dots, K\}$. Connect each rank to each pile containing such rank. Consider $A \subseteq X$ a subset of piles. $\Gamma(A)$ is all ranks appearing in piles from A, # of cards in A is $4|A|$. If there are k ranks in A, there are at most $4k$ cards.

$$4|A| \leq 4k = 4|\Gamma(A)|$$

Def. An X-matching of defect d is a matching w/ $|X| - d$ edges

Thm. (Defect Hall's) $\exists X$-matching of defect d iff $\forall A \subseteq X,\ |\Gamma(A)| \geq |A| - d$

- Prove by adding d extra vertices to Y, fully connect X w/ added vertices. Call this graph G'. Then $|\Gamma_{G'}(A)| = |\Gamma_G(A) \cup d| \geq |A| - d + d = |A|$.

Apply Hall's marriage theorem in G', and an X-perfect matching would involve the extra vertices. Deleting those extra vertices defect the matching by at most d.

- Other direction: given matching, at most d unmatched vertices in A

Def. Given finite sets $S_1, \dots, S_k$, a transversal is a choice of elements $x_i \in S_i$ $\forall 1 \le i \le k$ s.t. all x_i are distinct

$\{1,2,3\}, \{3,4\}, \{4,5\}, \{5\}$

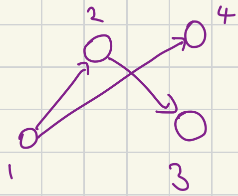
$1 \quad 3 \quad 4 \quad 5$

Thm. $\exists$ transversal iff $\forall A \subseteq \{1, \dots, k\}$, $\left|\bigcup_{i \in A} S_i\right| \ge |A|$

- Construct bipartite graph, $X = \{S_i\}$, $Y = \{e \in S_i\}$

Neighbourhood size is the size of union

Def. A directed graph is $G = (V, \vec{E})$ where $\vec{E} \subseteq V \times V$ is the ordered pairs with $(v,v) \notin \vec{E}$ and $(v,w) \in \vec{E} \Rightarrow (w,v) \notin \vec{E}$



Def. A flow network is a directed graph $G = (V, \vec{E})$ w/ 2 distinct vertices s (source) and t (sink). The capacity function $C : \vec{E} \to \mathbb{R}^{\geq 0}$ maps each edge to a positive real number.

- No edges of the form $\vec{vs}$ or $\vec{tv}$

Def. A flow within a flow network is an assignment $f : \vec{E} \to \mathbb{R}^{\geq 0}$ subject to 2 constraints:

① $0 \leq f(\vec{e}) \leq C(\vec{e})$

② $\forall v \in V \setminus \{s, t\}$, $\underbrace{\sum_{\vec{vw} \in \vec{E}} f(\vec{vw})}_{f^{out}(v)} = \underbrace{\sum_{\vec{wv} \in \vec{E}} f(\vec{wv})}_{f^{in}(v)}$

Thm. $\underbrace{\sum_{\vec{sv} \in \vec{E}} f(\vec{sv})}_{f^{out}(s)} = \underbrace{\sum_{\vec{vt} \in \vec{E}} f(\vec{vt})}_{f^{in}(t)}$

- Add up constraint 2 for all vertices, then cancel terms

Def. The value of a flow is $\sum_{\vec{sv} \in \vec{E}} f(\vec{sv}) = \sum_{\vec{vt} \in \vec{E}} f(\vec{vt})$

Def. A cut is any subset of vertices containing s but not t.

Def. The value of the cut X is $\sum_{\substack{\vec{vw} \in \vec{E} \\ v \in X,\, w \notin X}} C(\vec{vw})$

Thm. Value(flow) $\leq$ Value(cut) for all flows and cuts

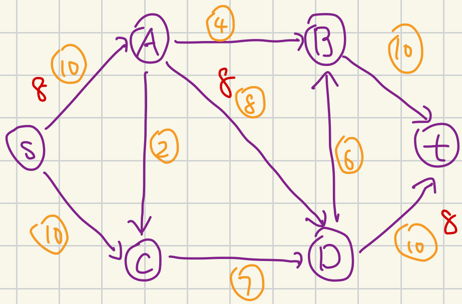
Thm. (max-flow min-cut) Value of max flow equals value of min cut

- Prove by creating a new directed graph $G' = (V, \vec{E}')$

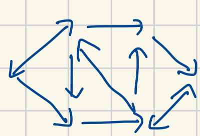
Fix some flow. Take the same unoriented edges as in G

but $\begin{cases} \text{same orientation if } 0 \text{ flow} \\ \text{reversed if capacity is reached by flow} \\ \text{bidirectional otherwise} \end{cases}$

E.g.



G'.



- Ford-Fulkerson algorithm
- Let X be all nodes reachable from s
- By end of algorithm, all edges between $V \setminus X$ and X are pointed to X (not bidirectionally)

Def. An augmenting path is a path from s to t $s = v_0 v_1 \dots v_k = t$ which has the property that $v_i v_{i+1}$ is always $\overrightarrow{v_i v_{i+1}}$ if edge has no flow currently, always $\overleftarrow{v_i v_{i+1}}$ if edge is at capacity, otherwise does not matter

Thm. If all capacities are $\in \mathbb{N}$, then there exists a max flow w/ all flow being integers.

Cor. (of max-flow min-cut) Can prove König's theorem by connecting s to X, t to Y, capacities all being 1 for sx, yt, and capacites very large (e.g. 10000000) for xy

- Value of cut is $|X \cap T| + |Y \cap S|$, which is the size of VC $|X \cap T| \cup |Y \cap S|$

Cor. (of max-flow min-cut) Can prove Hall's theorem using same construction as previous corollary

Connectivity

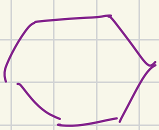
 (disconnected) $\Rightarrow$ connectivity $= 0$

 $\Rightarrow$ connectivity $= 1$

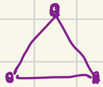
removing this breaks connectivity

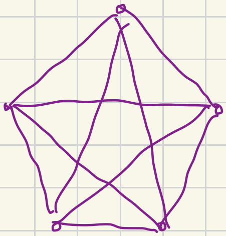
Trees have connectivity 1 b/c take any nonleaf vertex, remove it to disconnect the tree

Special case:  has connectivity 1

 $\Rightarrow$ connectivity $= 2$

remove 2 nonadjacent vertices to disconnect the graph

Special case:  has connectivity 2



Complete graph K_n, by convention has connectivity $n-1$

By convention • has connectivity 0

Def. The connectivity of G, $\kappa(G)$, is the smallest size of a subset $S \subseteq V$ s.t. $G - S$ is either disconnected or a single point

Def. G is k-connected if $\kappa(G) \geq k$

Lem. G is k-connected iff $|G| \geq k+1$ and no set of size $< k$ when removed disconnects G

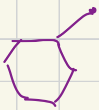
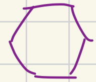
Def. x is a cut vertex if $G - x$ is disconnected.

Thm. $\kappa(G) \leq$ minimum degree.

- Consider minimum degree vertex x, w/ degree d

- Remove $\Gamma(x)$, may result in a disconnect graph or singleton x
- Connectivity $\leq |N(x)| = d$

Ex. removing vertex may increase connectivity

$\kappa\Big($  $\Big) = 1$ $\quad$ $\kappa\Big($  $\Big) = 2$

Thm. $\kappa(G-e) \leq \kappa(G)$

- Remove edge does not increase connectivity

Def. $S \subseteq V$ is a (vertex) separator if $G-S$ is a single point or disconnected

Lem. $\kappa(G)$ is the smallest size of vertex separator

Thm. $\kappa(G) - 1 \leq \kappa(G-e) \leq \kappa(G)$

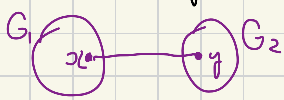
- Sps S is a min size vertex separator for G. Show $\kappa(G-e) \leq |S|$
- Since S is a vertex separator for $G-e$, have

$$G - e - S = \underbrace{(G-S)}_{\text{single point / disconnected}} - e$$

- Show $\kappa(G) - 1 \leq |S|$, $\kappa(G) \leq |S| + 1$

Goal: produce a v-sep of G of size $\leq |S| + 1$

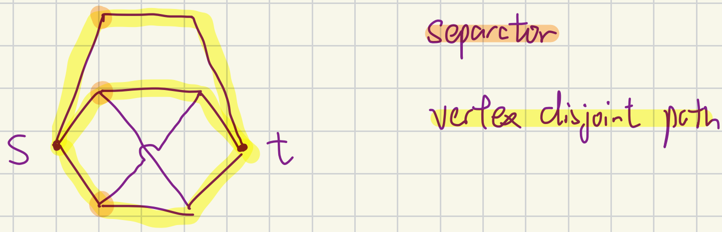
- $G - e - S$ is a single point $\Rightarrow$ $G - S$ is a single point
- $G - e - S$ is disconnected $\Rightarrow$ look at $G - S$
  - $G - S$ disconnected, then we're done
  - $G - S$ connected, then edge e connects between two components



  - Case $G_1 = \{x\}$, $G_2 = \{y\}$ then $S \cup \{x\}$ is a v-sep
  - Case $|G_1| \geq 2$, then $S \cup \{x\}$ is disconnected

Thm. (Vertex Menger's) If G is a connected graph and s, t are non-adjacent vertices. Then define an s-t separator to be $S \subseteq V$ where $s, t \notin S$. $s, t \in G - S$ and s, t lie in different connected components. The minimum size of s-t vertex separator

= max # of vertex disjoint path $s \to t$



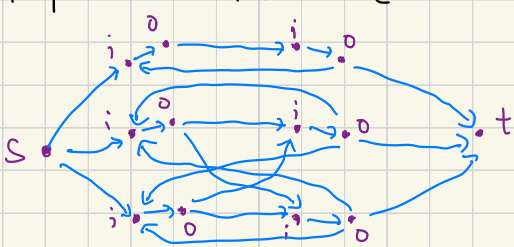
Def. An edge cut in a flow network is a subset of directed edges C s.t. in $G-C$, there is no directed path $s \to t$

Thm. Min value of cut = min value of edge cut (Sum of capacities)

- Given an edge cut of value V, Can produce a usual vertex cut of value v by taking C = all vertices explorable from s in G − edge cut

To prove Vertex Menger's Thm:

- Duplicate each vertex (in/out) besides s, t



i.e. makes undirected graph directed

- Vertex capacities are 1 for $i \to o$ and ∞ otherwise
- Each out vertex can only have 1 unit leaving
  〃 in 〃 〃 〃 entering
- Max flow is # of vertex disjoint paths
- Edge cut of flow network has to be $i \to o$, which corresponds to vertices in original graph
  $\Rightarrow$ edge cut $=$ s-t vertex separator

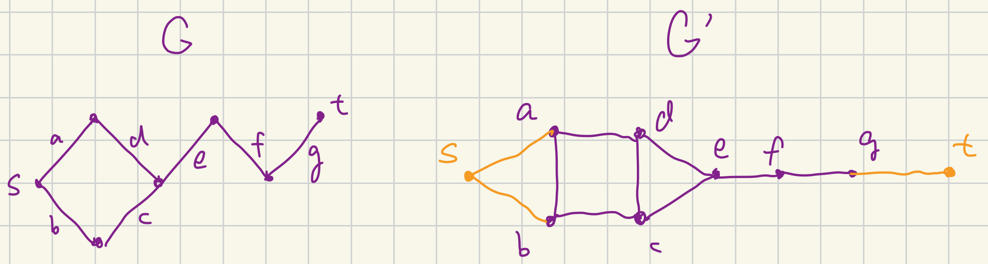
Cor. G is k-connected iff $\forall$ distinct vertices $s, t \in V$, there are $\geq k$ vertex disjoint paths $s \rightsquigarrow t$

Def. Let G be a connected graph where $|G| \geq 2$. The edge connectivity $\lambda(G)$ is the smallest size of a set $C \subseteq E$ s.t. $G - C$ is disconnected.

- G is k-edge-connected if $\lambda(G) \geq k$
- Every connected graph is 1-edge-connected
- A 2-edge-connected graph must not have a "bridge"

Thm. (Edge Menger's) The smallest # of edges needed to remove to disconnect $s, t \in G$ = largest number of edge-disjoint paths $s \rightsquigarrow t$

- Prove by making a line graph $\ell(G) = (V', E')$ where
  - $V' = E$
  - $uv \in E'$ if $u \in E$, $v \in E$ have a common vertex in V
- Add source and sink



- Claim: $\exists$ path $s \rightsquigarrow t$ in $G \iff \exists$ path $s \rightsquigarrow t$ in G'

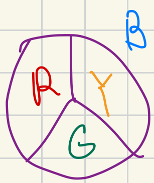
  b/c path in $G' \iff$ connected cluster of edges in G
- Use Vertex Menger's Thm

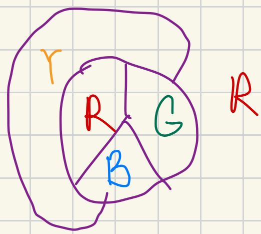
Ex. (4-colour question) Given a map, can you colour every region RYGB s.t. no 2 adjacent regions have same colour?

- 6-colour problem: easy ← polynomial time!
- 5-colour problem: solved by Kempe
- 4-colour problem: requires supercomputer + 5TB memory

Reduction to Graph: given a map, let

- V be all regions
- E connect all adjacent regions

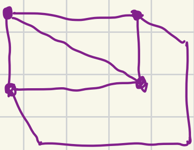




does not produce something like a complete graph!

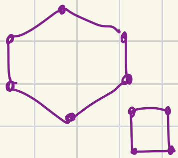
Def. A graph G is a planar graph if it can be drawn in the plane (using polygonal edge) with no edge crossings

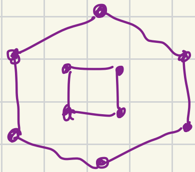
- A graph transformed from a map is planar
- E.g. K_4



- Show a graph is planar is easy (just draw it)

Def. A plane graph is a planar graph + choice of planar drawing





same planar graph

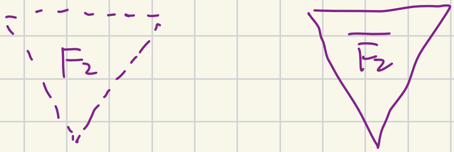
different plane graph

Def. A face of a plane graph is a "connected component" of $\mathbb{R}^2 - G$ (i.e. remove all edges + vertices)

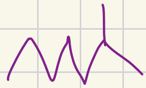
Def. $x, y \in \mathbb{R}^2 - G$ are linked by a polygonal arc if we can draw zigzag $x \to y$ avoiding G

- "Path-connectedness" in an analysis course

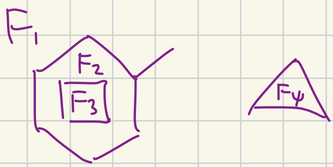
Def. Closure $\overline{F}$ = Face F + "limit points" in G

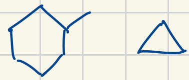


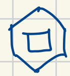
has 1 face

∂F = the tree 

Ex. List boundaries of all faces



$\partial F_1 =$ 

$\partial F_2 =$ 

$\partial F_3 =$ 

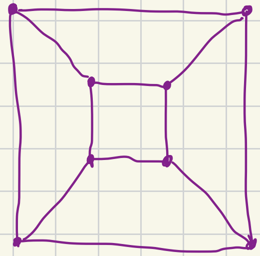
$\partial F_4 =$ 

Thm. Every tree is planar, and every plane drawing has exactly 1 face

- Prove by induction on $|V|$, choose a leaf v
- $G - v$ satisfies IH
- Induct on # of zigs on zigzag of v  G
- Adding a zig does not add a face, need algorithm to eliminate crossing when adding zig
- Can take facts like these for granted in proofs

Thm. (Euler's Formula) If a connected plane graph G has n vertices, m edges, and f faces, then

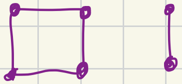
$n-m+f=2$



$n=8$

$m=12$

$f=6$



$n=6$, $m=5$, $f=2$

without connectedness $6-5+2=3$

- First simplify each zigzag where we add a vertex in each non-differentiable point, then n += 1, m += 1 each operation, which balances out in the formula
- Induct on # of edges (base case: tree)
- If G has no cycles, G is a tree

  $m=n-1$, $f=1 \Rightarrow n-(n-1)+1=2$
- Sps G contains a cycle, then the cycle bounds

a region in $\mathbb{R}^2$, so for $e \in C$, e touches 2 faces

- Because $e \in C$, have $G - e$ connected
- By IH $G - e$ satisfies
- Adding e back gives $m \mathrel{+}= 1$, $f \mathrel{+}= 1$, which balances out in the equation

Thm. If G has n vertices + m edges where $n \geq 3$, and G is a planar graph, then $m \leq 3n - 6$

- If G not connected, then can add edges to connect G, which implies the theorem for original graph
- Assume G connected, then can add edges so that all face boundaries have a cycle. Then Euler's formula implies $n + m - f = 2$. If we sum for each face F the # of edges in ∂F, then each edge is counted at most twice
- Every face contains ≥ 3 edges, so

$$3f = \sum_F 3 \leq \sum_F \#\text{ edges} \in \text{face boundary} \leq 2m$$

Therefore $f \leq 2/3\, m$

- $2 = n - m + f \leq n - m + \frac{2}{3}m = n - \frac{1}{3}m$

  $\Rightarrow \frac{1}{3}m \leq n - 2 \Rightarrow m \leq 3n - 6$

Cor. K_5 is not planar.

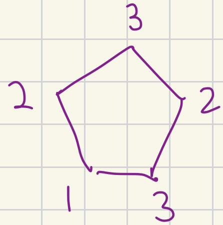
- $n = 5$, $m = 10$, inequality does not hold

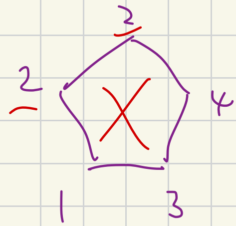
Lem. $K_{3,3}$ is not planar.

- Every face boundary of a connected graph which is not a tree contains a cycle
- Assume that $K_{3,3}$ is planar for contradiction, then there are faces $F_1, \dots, F_k$, each face boundary contains a cycle
- $K_{3,3}$ has no odd cycles, so those cycles have ≥ 4 edges
- $4f \leq \sum_{i=1}^{k} \#\text{ of edges in boundary of } F_i \leq 2m$

  $2 = n - m + f \leq n - m + \frac{1}{2}m = \overset{6}{n} - \frac{1}{2}\overset{9}{m}$

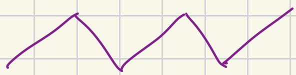
Def. A proper r-colouring of a graph G is a function $C : V \to \{1, \dots, r\}$ s.t. $C(x) \neq C(y)$ when $xy \in E$





- Bipartite $\Leftrightarrow \exists$ proper 2-colouring

Def. The chromatic number of a graph G, $\chi(G)$, is the smallest r s.t. G has a proper r-colouring



$\chi(G) = 2$

- $\chi(\overline{K_n}) = 1$
- $\chi(K_n) = n$
- $\chi(G) \leq |G|$

Def. A k-partite graph G is a graph s.t. we can partition $V = V_1 \amalg \cdots \amalg V_k$ s.t. $G[V_i]$ has no edges

- V_i could be empty

Thm. $\chi(G) \leq k \Leftrightarrow G$ is k-partite

Def. $\omega(G)$ is the largest r s.t. $K_r \subseteq G$

- aka clique number, max clique size

Thm. $\chi(G) \geq \omega(G)$

Thm. $\chi(G) \leq \Delta + 1$

- $\Delta = \max_{v \in V} d_v$ max degree
- Colour vertices one at a time, each vertex v sees at most Δ colours, use extra colour for v

Def. Graph G is k-degenerate if we can list the vertices $v_1, \dots, v_n$ s.t. v_i has degree $\leq k$ in $G - \{v_1, \dots, v_{i-1}\}$

Thm. G is k-degenerate $\Rightarrow \chi(G) \leq k+1$

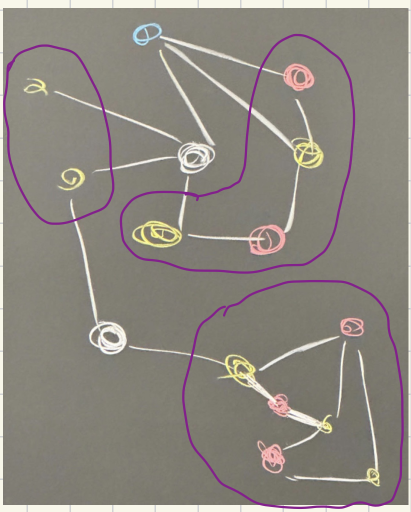
- Greedy colouring algorithm in the order $v_n, \dots, v_1$
- There are only k edges going forward, so worst case we use $k+1$ colours

Thm. (6-Colour) G is planar $\Rightarrow \chi(G) \leq 6$

- $\exists v_1$ where $d_{v_1} \leq 5$ (since K_5 not planar)
- $G - v_1$ planar, $\exists v_2$ where $d_{v_2} \leq 5$
- Iterate
- $v_1, \ldots, v_n$ shows G is 5-degenerate

Thm. (5-Colour) G is planar $\Rightarrow \chi(G) \leq 5$

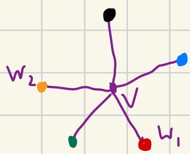
- A red-yellow Kempe chain is a connected component on the induced subgraph of G on red and yellow vertices
- Each Kempe chain is bipartite (2-colourable)
- Given a proper colouring of G, can swap colours inside a Kempe chain



Kempe chains

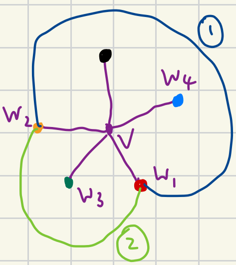
Can swap all reds $\leftrightarrow$ yellows here

- Induct on $|G|$, base case trivial
- Pick a vertex v, $d_v \leq 5$, which exists since graph is planar
- If $d_v \leq 4$, colour $G-v$ (IH), then colour v
- If $d_v = 5$, colour $G-v$ (IH), then look at v

  If 2 neighbours of v have the same colour, use the remaining colour available
- 

  Consider a red-yellow Kempe chain

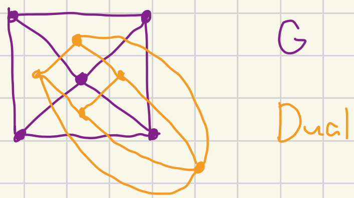
  If the Kempe chain of w_1 does not contain w_2, then swap colour
- If the Kempe chain of w_1 contains w_2, then



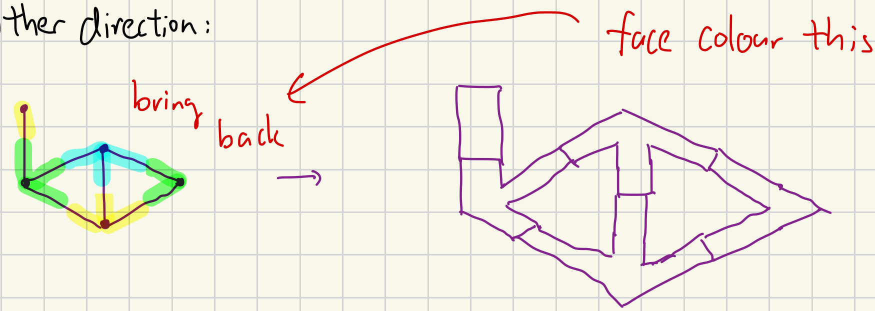
in either cases, repeat the same logic for green and blue. There cannot be a Kempe chain containing w_3 and w_4 b/c no edge crossings
- Then swap colour for Kempe chain of w_3

Thm. All planar graphs are k-colourable $\Leftrightarrow$ all planar graphs are k-face colourable (i.e. no two adjacent) faces have same colour

- Create a dual graph where the vertices are the faces of G and adjacent faces are connected by edges



- Dual is planar (from topology). Then vertex-colour dual.
- Other direction:

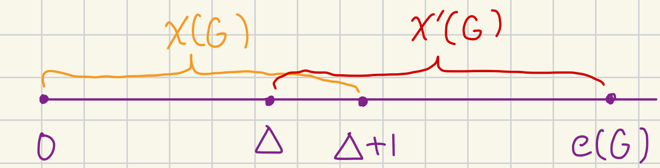


each vertex occupies half of each incident edge

Def. A (proper) r-edge colouring of G is a function $c: E \to \{1, \dots, r\}$ s.t. $c(e) \neq c(e')$ if e, e' share a vertex

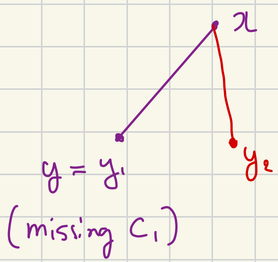
- $\chi'(G)$ = smallest # of colours for edge colouring called edge chromatic number

Thm. $\Delta \leq \chi'(G)$



Lem. $\chi'(G) = \chi(L(G))$ (line graph)

Thm. (Vizing's) $\Delta \leq \chi'(G) \leq \Delta + 1$

- Show G is $(\Delta+1)$-edge colourable
- Induct on $e(G)$, base case trivial
- Choose an edge $e = xy$, colour $G-e$ using IH
- Observe that having $\Delta+1$ colours, every vertex is "missing" some colour
- 

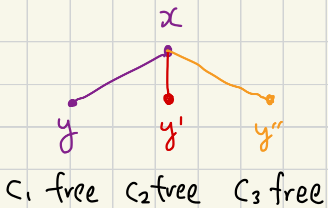
  If c_1 also free on x, then colour e with c_1

  otherwise x has an edge using c_1

- y_2 has some colour c_2 free. If c_2 also free at

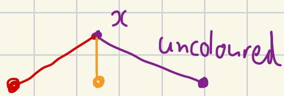
x, then can re-colour xy_2 with c_2 & e w/ c_1

- If c_2 not free at x, let xy_3 have colour c_2

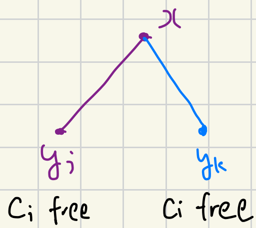


Keep doing this until we get a repeated colour $c_k = c_i$ for $k > i$

- Shift all colours up to but not including c_i back by 1



- Let c be colour not used at x and consider the



$c_i - c$ Kempe chain containing y_i

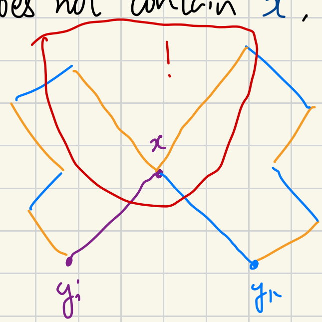
If it does not contain x, then recolour and we're done

- If it does contain x, then consider the $c_i - c$ Kempe chain containing y_k

- Do the same thing. If it does not contain x, then shift colours for y_k

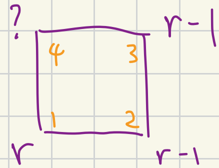
- It cannot contain x

b/c



Def. $P_G(r)$ = # of proper r-colourings of G

- $P_{\overline{K_n}}(r) = r^n$
- $P_{K_n}(r) = r(r-1)\cdots(r-n-1) = \dfrac{r!}{(r-n)!}$
- $P_{P_n}(r) = r(r-1)^{n-1}$
- $P_{C_4}(r)$:



$$\begin{cases} 1 \text{ \& } 3 & \text{same} \\ 1 \text{ \& } 3 & \text{diff} \end{cases}$$

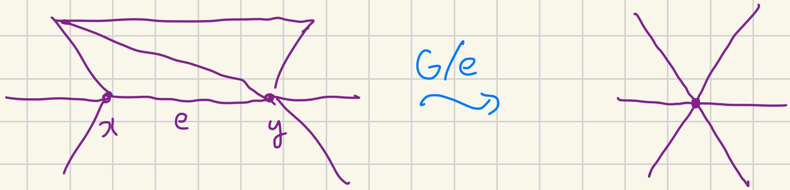
Cor. A planar graph G satisfies

① $P_G(5) > 0$ (5-Colour Theorem)

② $P_G(4) > 0$ (4-Colour Theorem)

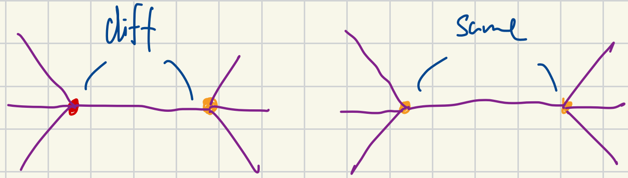
Thm. $P_G(r)$ is a polynomial for all graph G

Def. For $e = xy$ in G, G/e is the contraction where



Thm. $P_G(r) = P_{G \setminus e}(r) - P_{G/e}(r)$

- 2 cases

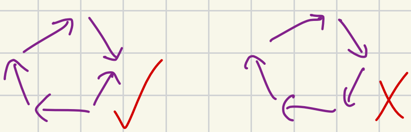


- $P_{G \setminus e}(r) - P_G(r)$ = colourings of G proper everywhere except $e = xy$ and $C(x) = C(y)$
- To prove polynomial, induct on $|G| + e(G)$
- Base case: $e(G) = 0$, then $P_G(r) = r^n$
- Know that $P_{G/e}(r)$, $P_{G \setminus e}(r)$ are polynomial by IH

Lem. $\deg P_G(r) = |V|$, and has leading coefficient 1

- Can prove by induction, or by matching upper/lower bounds
- Every monic polynomial of degree n is determined by its values at any n points
    - Monic polynomial: leading coefficient is 1

Def. An acyclic orientation of a graph G is a choice of direction for each edge so that no directed cycles occur



Thm. $P_G(r) = P_{G\setminus e}(r) - P_{G/e}(r) \quad \forall r \in \mathbb{R}$ (or even $\mathbb{C}$)

- If polynomials agree for $\geq r$ points, then they agree everywhere

Thm. $(-1)^{|G|} P_G(-1) = A_G$ = # of acyclic orientations

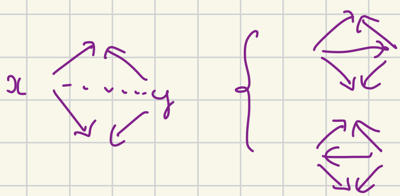
- Look at acyclic orientations of $G\setminus e$, G, G/e, $e = xy$ split them up according to there is a path $x \rightsquigarrow y$, $y \rightsquigarrow x$, or neither

not counting edge xy	$A_{G\setminus e}$	A_G	$A_{G/e}$
$x \rightsquigarrow y$	1	1	0
$y \rightsquigarrow x$	1	1	0
Neither	1	2	1

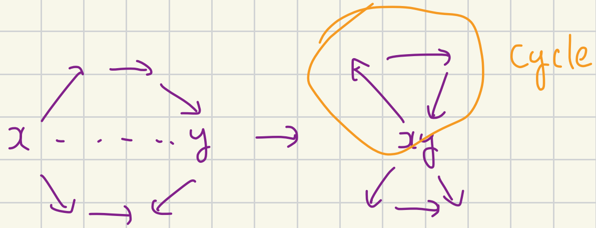
E.g $\text{Table}_{0i} = T_{1i}T_{10} + T_{2i}T_{20} + T_{3i}T_{30}$

$i = 1$ holds b/c three cases are disjoint, therefore adding them up (all by 1 time) gives $A_{G\setminus e}$

- T_{32} :



- T_{13}, T_{23}:



- T_{33}: above, but does not form cycle

- Prove by induction on $e(G)$
- Base case: $(-1)^{|G|} P_G(-1) = (-1)^{|G|}(-1)^{|G|} = 1$
  - $P_G(r) = r^{|G|}$
- Inductive step:

$$
\begin{aligned}
A_G &= A_{G/e} + A_{G \setminus e} \\
&= (-1)^{|G|-1} P_{G/e}(-1) + (-1)^{|G|} P_{G \setminus e}(-1) \qquad (IH) \\
&= (-1)^{|G|}\left(-P_{G/e}(-1) + P_{G \setminus e}(-1)\right) \\
&= (-1)^{|G|} P_G(-1)
\end{aligned}
$$

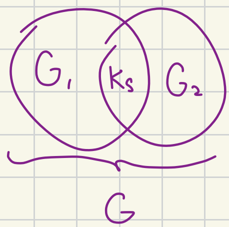
Example: $G = K_n$, $P_G(r) = r(r-1)\cdots(r-n+1)$

- $(-1)^n P_G(-1) = (-1)^n (-1)(-2)\cdots(-n) = n!$

Thm. For G_1, G_2 subgraph of G "overlap" in K_s and

$G_1 \cup G_2 = G$, we have

$$P_G(r) = \frac{P_{G_1}(r)\,P_{G_2}(r)}{P_{K_s}(r)} = \frac{P_{G_1}(r)\,P_{G_2}(r)}{r(r-1)\cdots(r-n+1)}$$



- To prove, first colour K_s, where # of ways is $P_{K_s}(r)$
- $P_G(r) = P_{K_s}(r)$ (# of ways to colour G_1 w/ K_s coloured)
  (〃 〃 〃 G_2 〃)
- $P_{G_1}(r) = P_{K_s}(r)$ (# of ways to colour G_1 w/ K_s coloured)
- $P_{G_1}(r)P_{G_2}(r)/P_{K_s}(r)$ cancels to $P_G(r)$

Example. $P(r)$

$$= \frac{P(r)\,P(r)}{P_{\bullet}(r)} = \frac{\left(r(r-1)\cdots(r-n)\right)^2}{r}$$

Thm. There is a unique way of writing $P_G(r) = \sum_{k=0}^{|G|} \binom{r}{k} f_G(k)$

where $f_G(k)$ = # of k-colourings of G w/ exactly k colours

- Uniqueness: $P_G(r) = r^{|G|} + \_\, r^{|G|-1} + \cdots$

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!} = \frac{1}{k!}r^k + \_\, r^{k-1} + \cdots + \_\, r^{k-r}$$

Only $\binom{r}{k}$ contains an r^k term

To match up $r^{|G|}$'s coefficient, $f_G(|G|)$ is determined uniquely. To match up $r^{|G|-1}$'s coeff, only $f_G(|G|)\binom{r}{|G|} + f_G(|G|-1)\binom{r}{|G|-1}$ contribute

so $f_G(|G|-1)$ uniquely determined

Then repeat.

- Consider the # of r-colourings of G, $P_G(r)$, split up those r-colouring according to which colours are actually used

For each subset $A \subseteq \{1, \ldots, c\}$, # of colourings of G which actually use each colour from A is $f_G(|A|)$

$$P_G(r) = \sum_{k=0}^{|G|} \underbrace{\binom{r}{k}}_{\text{\# of subsets of size } k} f_G(k)$$

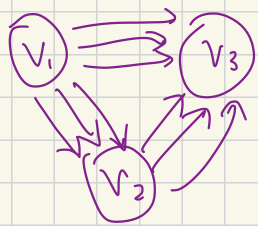
Thm. (Gallai–Hasse–Roy–Vitaver)

$$\chi(G) = \min_{\substack{\text{orientations } O \\ \text{of the graph} \\ \text{(not nec. acyclic)}}} \left(\max_{\substack{\text{directed} \\ \text{path } P}} \# \text{ of vertices in } P \right)$$

- Application: to prove $\chi(G) = x$, can show one that gives x, and none gives $x-1$
- To prove that $\chi(G) \geq$ RHS, find orientation O s.t. $\chi(G) \geq$ max # of vertices in directed path

  First consider $\chi(G)$ partitions $V = V_1 \amalg \cdots \amalg V_{\chi(G)}$

  Direct edges toward higher-number partitions

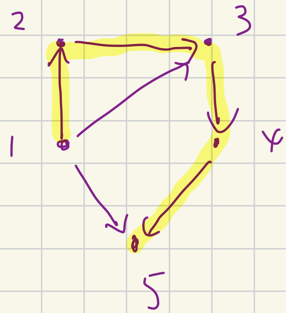
  

  completes this direction of proof
- To prove that $\chi(G) \leq$ RHS, WTS $\forall O$, $\chi(G) \leq$ max # of vertices in directed path in O

  Equivalently, can colour G using ↑ many colours

  Take a maximal directed acyclic subgraph of O

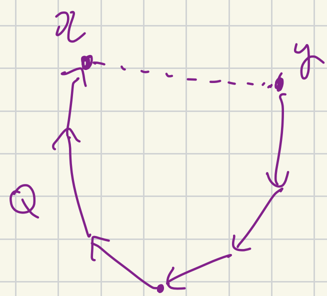
  Let $c(x) =$ # of vertices of the longest directed path ending at x in this subgraph be colour of x



Note that $1 \leq c(x) \leq$ max size of a directed path in this subgraph $\leq$ max size of a directed path for O in any subgraph

If we can show for $xy \in E(G)$ that $c(x) \neq c(y)$ then we are done

- Sps that $\overrightarrow{xy}$ is an edge of subgraph, then $\exists$ directed path P w/ vertices ending at x in the subgraph. Note $y \notin P$ otherwise we have a cycle. Add $x \to y$, then this is path of length $a+1$
- Sps that $\overrightarrow{xy}$ is not in Q in subgraph. Then adding $\overrightarrow{xy}$ results in a cycle (since the path is max acyclic)
- Look at longest directed path containing y, P_y.



P cannot have vertices in common w/ Q otherwise a cycle would exist.

If v is the last vertex Q and P_y have in common (other than y), then P_y followed by Q is a longer directed path from y to x $\lightning$

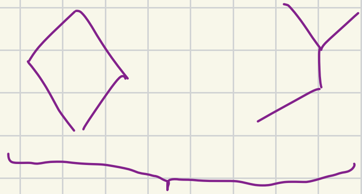
Cor. $\min_{O}\left(\max_{P} |P|\right)$ is achieved by an acyclic orientation

- Already know that $\min_{\text{acyclic } O}(\cdot) \geq \min_{O}(\cdot) = \chi(G)$
- Take orientation $x \to y$ if $x \in V_i$, $y \in V_j$, $i<j$, this is acyclic. With this orientation, max # of vertices in a directed path $\leq \chi(G)$
- By GHRV, max number $\geq \chi(G)$, so they are equal

Cor. If we orient the edges of K_n, there is always a directed path using all vertices

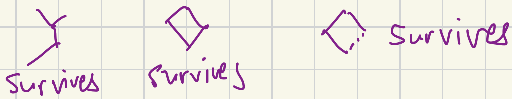
- Use GHRV, max # of vertices in directed path under O $\geq \chi(K_n) = n$

Def. For graph G, the reliability polynomial $R_G(p) =$ probability that after deleting every edge randomly w/ probability p, no connected component gets disconnected.



$R_G(0) = 1$

$$R_G(1/3) = \underbrace{\left(\frac{2}{3}\right)^4}\left[\underbrace{\left(\frac{2}{3}\right)^4} + \underbrace{4\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3}\right]$$



Thm. $R_G(p) = pR_{G\setminus e}(p) + (1-p)R_{G/e}(p)$ multigraph contraction

- First, either delete/don't delete edge e,

  then ↯ remaining edges

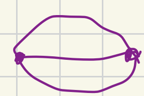
- If e is not removed, then it becomes undestroyable.

  Connects 2 vertices for sure, can contract this edge

  "merge the two cities"

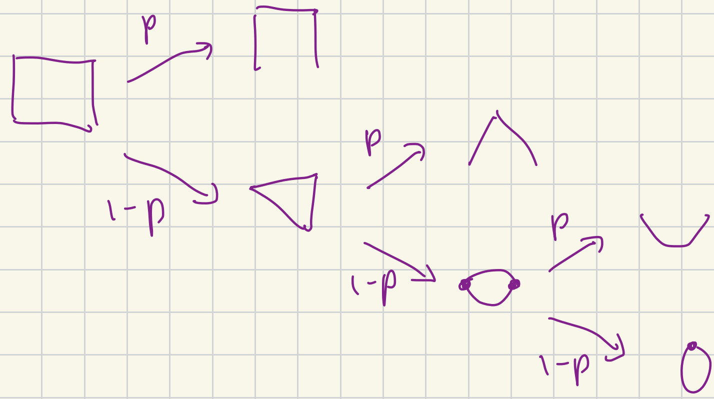
- If e is removed, then $G\setminus e$

Cor. $R_G(p)$ is a polynomial in p

Def A multigraph is a graph but multiple edges 

and loops  are allowed

- Can contract as a multigraph (i.e. Karger's algorithm)



Thm. For multigraph G:

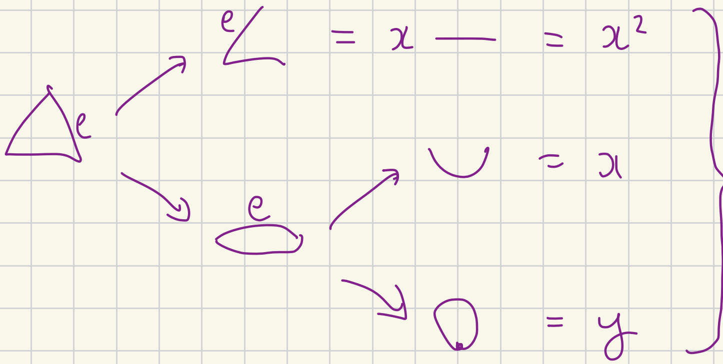
$$R_G(p) = \begin{cases} R_{G/e}(p) & \text{if } e \text{ is a bridge} \\ (1-p) R_{G/e}(p) & \text{if } e \text{ is a self edge} \\ p R_{G\setminus e}(p) + (1-p) R_{G/e}(p) & \text{elsewise} \end{cases}$$

Moreover $R_{G \amalg H}(p) = R_G(p) R_H(p)$

Thm. (Tutte polynomial) There is a unique polynomial $T(x,y)$ s.t. the following hold:

1. $T_{K_1}(x,y) = 1$
2. $T_{G \amalg H}(x,y) = T_G(x,y) T_H(x,y)$
3. If e is a bridge, then $T_G(x,y) = x T_{G/e}(x,y)$
4. If e is a loop, then $T_G(x,y) = y T_{G\setminus e}(x,y)$
5. If e is not a bridge or loop, then

$$T_G(x,y) = T_{G/e}(x,y) + T_{G\backslash e}(x,y)$$



x^2+x+y

$= T_{K_3}(x,y)$

- x, y do not represent meaningful things

Thm. Sps we have a function f : multigraphs $\to$ polynomials s.t.

① $f(G \sqcup H) = f(G)f(H)$

② $f(G) = \begin{cases} af(G\backslash e) + bf(G/e) & e \text{ not bridge/loop} \\ x_0 f(G/e) & e \text{ bridge, } G \neq K_2 \\ y_0 f(G\backslash e) & e \text{ loop, } |G| = 1 \end{cases}$

③ $f(\cdot) = z_0$ $\quad f(\bullet\!\!-\!\!\bullet) = x_0$ $\quad f(\bigcirc) = y_0$

Then for $m = e(G)$, $n = |G|$, k = # of conn. comps,

$$f(G) = a^{m-n-k} b^{n-k} z_0^k T_G\left(\frac{x_0}{b z_0}, \frac{y_0}{a z_0}\right)$$

called master Tutte recursion