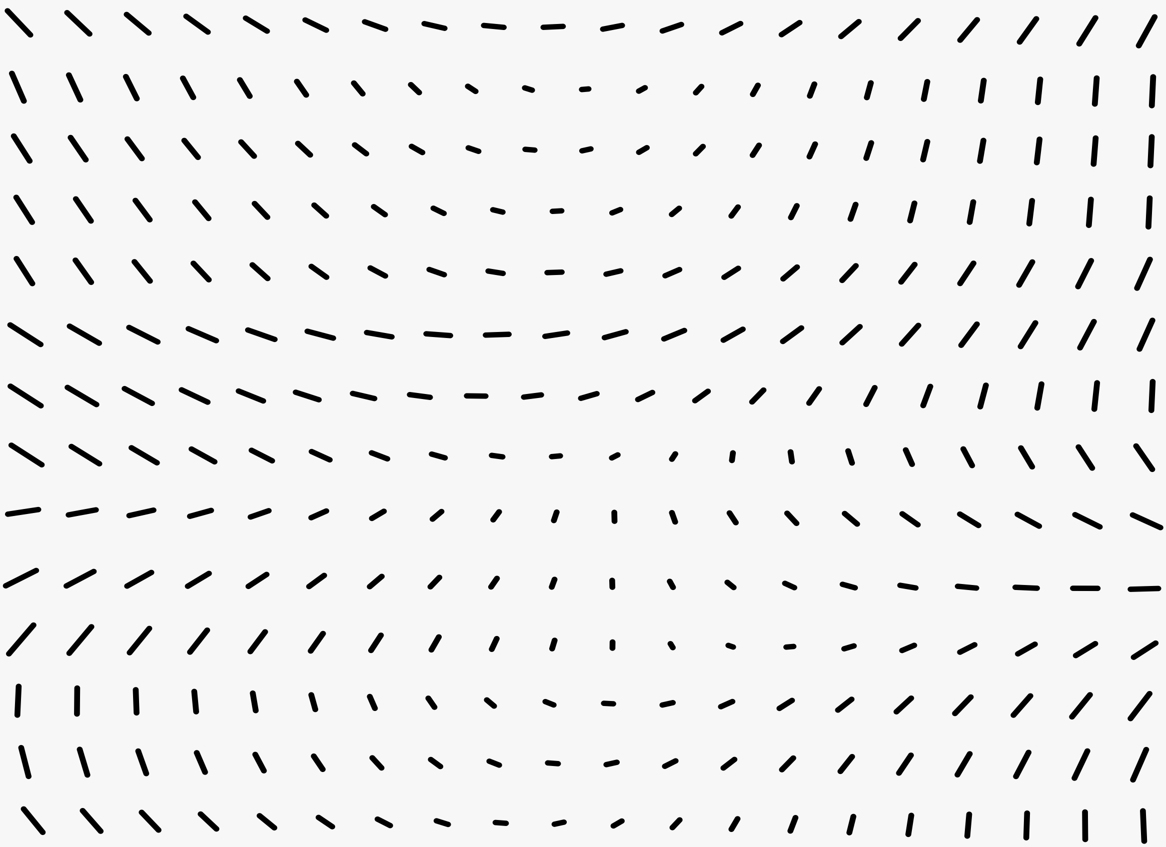
MAT137

Calculus with Proofs



1.1 Sets and Notation

Set

- a collection of things called "elements"
  - often numbers
- i.e. $\boxed{A} = \{\text{even integers}\}$

  $\boxed{B} = \{\underbrace{4, 5, 6}\}$ list of elements

  name of sets

Notation

- let $B = \{4, 5, 6\}$, $C = \{2, 4\}$, $D = \{4, 5\}$
- $\in$: "is an element of"
  - i.e. $4 \in B$
- $\notin$: "is not an element of"
  - i.e. $2 \notin B$
- $\subseteq$: "is a subset of"
  - i.e. $D \subseteq B$

-

	some books	other books
"is a subset of and may be equal"	$\subseteq$	$\subset$
"is a subset of and may not be equal"	$\subset$	$\subsetneq$

- use $\subseteq$ and $\subsetneq$ to avoid confusion

- $\cup$: "union of sets"
  - i.e. $C \cup D = \{2, 4, 5\}$
- $\cap$: "intersection of sets"
  - i.e. $C \cap D = \{4\}$
- $\emptyset$: "empty set"
  - i.e. $\emptyset = \{\}$

Important Sets of Numbers

- Naturals $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
- Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Rationals $\mathbb{Q} = \{$quotients of integers (fractions)$\}$
- Reals $\mathbb{R} = \{$numbers with decimal expansion$\}$
- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

1.2 Set-Building Notation

1.

$$\overbrace{A}^{\text{name of set}} = \{ \overbrace{x \in \mathbb{Z} \overbrace{:}^{\text{"such that"}} x^2 < 6}^{\text{description of the set}} \}$$

$$A = \{ \underbrace{x \in \mathbb{Z}}_{\text{I take elements from...}} \mid \underbrace{x^2 < 6}_{\text{extra constraints}} \}$$

$\rightarrow A = \{-2, -1, 0, 1, 2\}$

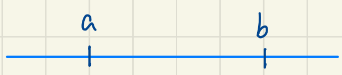
2.

$$B = \{ \underbrace{2x}_{\text{what elements in B look like}} \overbrace{:}^{\text{"such that"}} \underbrace{x \in A}_{\text{explains the notation}} \}$$

$\rightarrow B = \{-4, -2, 0, 2, 4\}$

Intervals

- let $a, b \in \mathbb{R}$



- $\underbrace{[a,b]}_{\text{name of set}} = \{x \in \mathbb{R} : a \leq x \leq b\}$

- $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

- $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$ 
- $(a,\infty) = \{x \in \mathbb{R} : a > x\}$ 
- $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$ 

1.3 Quantifiers

- $\forall$: "for all / every"
- $\exists$: "there exists / is" (at least one)

Examples

- $\forall x \in \mathbb{R}, x^2 \geq 0$
  - true
- $\forall x \in \mathbb{Z}, x > \pi$
  - false (i.e. $x = 1$)
- $\exists x \in \mathbb{R}$ such that $x^2 = 5$
  - true (i.e. $x = -\sqrt{5}$)
- $\exists x \in \mathbb{R}$ such that $x^2 = -1$
  - false
- $x^2 = 5$ (no quantifiers)
  - meaningless

1.4 Double Quantifiers

- s.t. : such that

1. $\exists y \in \mathbb{Z}$ s.t. $\boxed{\forall x \in \mathbb{Z},\ x < y}$
    - must be the same y for all values of x
    - "there is an integer (y) greater than all integers" → false

2. $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ s.t. $\boxed{x < y}$
    - for every x, there exists a different y
    - "Every integer is smaller than some other one" → true

- order matters for quantifiers

1.5 Simple Proofs With Quantifiers

1. Let $A = \{2,3,4\}$. $\forall x \in A,\ x > 0$

Pf: $2 > 0$ ✓

$3 > 0$ ✓

$4 > 0$ ✓ ■

- "■": end of proof

2. $\forall x \in \mathbb{Z},\ x > 0$
   - this is false. To show it, we need to prove

     $\exists\, x \in \mathbb{Z}$ s.t. $x \leq 0$

     negation

   - the negation of a statement is true when the original statement is false

Pf: take $x = -6$ $\quad -6 \in \mathbb{Z}$ ✓

$-6 \leq 0$ ✓ ■

3. $\forall x \in \mathbb{Z},\ \exists\, y \in \mathbb{Z}$ s.t. $x < y$

Pf: • Let $x \in \mathbb{Z}$

- "I fix an arbitrary $x \in \mathbb{Z}$"

- "arbitrary" : x can be any integer
- "fix" : I can always take $y = x + 1$

- Take $y = x+1$
- Verify $y \in \mathbb{Z}$ ✓

  $y > x$ ✓ ∎

1.6 Quantifiers and the Empty Set

1. $\forall x \in \emptyset, \boxed{x > 0}$ (*)
   - to prove ① is T, we need to verify all elements in $\emptyset$ satisfy (*) ✓
   - to prove ① is F, we need to find one element in $\emptyset$ doesn't satisfy (*) ✗
   - all elements in $\emptyset$ satisfy (*) since there's nothing to check
   - we cannot find any element in $\emptyset$ that doesn't satisfy (*) since we don't have any elements to begin with
   - therefore, the statement is true
2. $\exists x \in \emptyset$ s.t. $x > 0$ False

- "$\forall x \in \emptyset, \ldots$" is true
- "$\exists x \in \emptyset$ s.t. $\ldots$" is false

1.7 Conditionals

Conditional Statements

- If P, then Q → conditional statement
- $P \Rightarrow Q$ → P implies Q
- P implies Q
- Above three lines mean the same thing
- Whenever P is true, Q is true as well
- Whenever P is false, we don't care
- Ex.1 Let $x \in \mathbb{R}$. $x > 10 \Rightarrow x > 6$

	$x > 10$	$x > 6$
- $x = 12$	T	T
- $x = 8$	F	T
- $x = 3$	F	F

- Ex.2 Let $A \subseteq \mathbb{R}$, assume we know

  $x \in A \Rightarrow x > 0$, what can we conclude?

  - $x \notin A$ No conclusion (if part is false)
  - $x > 0$ No conclusion
  - $x \leq 0 \Rightarrow x \notin A$

- $\boxed{P \Rightarrow Q}$ and $\boxed{\text{not } Q \Rightarrow \text{not } P}$

  mean the same

- Ex. 3 Let $n \in \mathbb{Z}$

  - n is even $\Longleftarrow$ n is a multiple of 4

  - n is even $\underbrace{\Longleftrightarrow}_{}$ $n+1$ is odd

    "if and only if" = "iff"

- $P \Leftrightarrow Q$ $\rightarrow$ $P \Rightarrow Q$ and $Q \Rightarrow P$

  - P and Q are equivalent

  - P and Q are both true

    or both false

- Ex. 4 True or false ?

$$\boxed{\underbrace{0 > 1}_{\text{false}} \Rightarrow \underbrace{1035742189 \text{ is prime}}_{\text{we don't know}}}$$

True

  - the "if" part is false, we don't care

    about the "then" part

  - if the "if" part is false, the whole

    statement is true

1.8 How to Negate a Conditional

- Ex. Let $A \subseteq \mathbb{R}$

$$\underbrace{(\forall x \in \mathbb{R}),}_{\text{hidden quantifier for all conditionals}} \boxed{\text{if } x \in A, \text{ then } x > 0} \quad (*)$$

- $(*)$ means $\begin{cases} x \in A \text{ and } x > 0 \\ \quad \text{or} \\ x \notin A \text{ and } x > 0 \\ \quad \text{or} \\ x \notin A \text{ and } x \leq 0 \end{cases}$

- Negation of $(*)$: $\boxed{\begin{array}{l} \exists x \in \mathbb{R} \text{ s.t.} \\ (x \in A \text{ and } x \leq 0) \end{array}}$

- negation of an if-then is never another if-then
- negation of an if-then will always refer to the fourth case
  - i.e. if part is true, then part is false
- explicitly write the negation of the hidden quantifier

1.9 A Bad Proof

- Ex.

"<u>Thm</u>" : $\sqrt{xy} \le \dfrac{x+y}{2}$ (theorem)

"<u>Pf</u>" :

$$xy \le \left(\frac{x+y}{2}\right)^2$$

this step is assuming what we want to prove

what are x, y?
(statement is false when $x, y = -1$)

$$xy \le \frac{x^2+2xy+y^2}{4}$$

wrong

$$4xy \le x^2+2xy+y^2$$

no explanations

$$0 \le x^2-2xy+y^2 = (x-y)^2$$

- the above "proof" shows if the inequality on the top is true, $(x-y)^2$ is not negative, which doesn't tell us whether the inequality on the top is true or not
- we cannot assume what we want to prove
- we should go backwards to start from what we know is true and end up concluding the inequality on the top

- if the proof is correct, we should notice that the inequality does not work for neg. numbers
- a proof usually needs to explain what we are doing

- Ex.

Thm. Let $x, y \geq 0$, then $\sqrt{xy} \leq \frac{x+y}{2}$

Pf. since a square is always non-negative:

$$0 \leq (x-y)^2 = x^2 - 2xy + y^2$$

$$4xy \leq x^2 + 2xy + y^2$$

$$xy \leq \frac{x^2+2xy+y^2}{4} = \left(\frac{x+y}{2}\right)^2$$

since both sides are non-negative:

$$\boxed{\sqrt{xy}} \leq \sqrt{\left(\frac{x+y}{2}\right)^2} = \left|\frac{x+y}{2}\right| \underset{\substack{\text{because} \\ x,y \geq 0}}{=} \boxed{\frac{x+y}{2}}$$

■

- we need the "first proof" to come up with ideas to start our actual proof

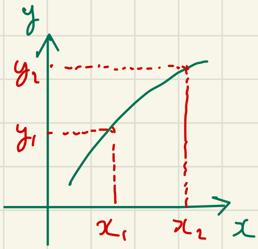
- the "first proof" is known as rough work
- if the rough work does not successfully reverse, we have to come up with something else

Summary

- statement of the theorem is not just an inequality, it tells us when the inequality is true
- proof begins w/ things we know that are true
  - then every step follows logically from the previous one
  - ends up concluding what we want to prove

1.10 How to Write a Rigorous Definition

- Ex. Goal: Define "increasing function"



$y = f(x)$

- ~~Definition: $f' > 0$~~ → bad definition

this definition would not work on the graph to the left

- $x_1 < x_2$, $f(x_1) < f(x_2)$
- it could be decreasing on some intervals
- all pairs of x_1, x_2 need to work ($\forall$)
- we need conditionals as some pairs cannot even satisfy $x_1 < x_2$
- $\boxed{\text{Def: a function } f \text{ is increasing on an interval } I \text{ when } \forall x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow f(x_1) < f(x_2)}$

1.11 Proofs: an Example

- Ex. Prove $f(x) = 3x+7$ is increasing on $\mathbb{R}$ directly from the definition
  - WTS $\forall x_1, x_2 \in \mathbb{R}$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
    - WTS - "want to show"

Pf: - let $x_1, x_2 \in \mathbb{R}$
- Assume $x_1 < x_2$

  $3x_1 < 3x_2$

  $f(x_1) = 3x_1 + 7 < 3x_2 + 7 = f(x_2)$
- I've shown $f(x_1) < f(x_2)$ ∎

1.12 Proofs: a Non-Example

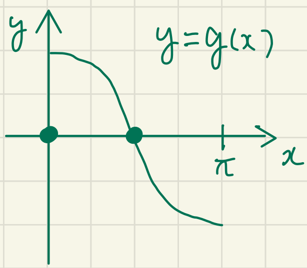
- Ex. Prove $g(x) = \cos x$ is not increasing on $[0, \pi]$ directly from the definition

  - "g isn't increasing on I"
  - NO $[\forall x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow g(x_1) < g(x_2)]$

  $\to \exists x_1, x_2 \in I$ s.t. $[\text{NO}(x_1 < x_2 \Rightarrow g(x_1) < g(x_2))]$

  $\to \boxed{\exists x_1, x_2 \in I \text{ s.t. } x_1 < x_2 \text{ and } g(x_1) \geq g(x_2)}$

<u>WTS</u> $\exists x_1, x_2 \in [0, \pi]$ s.t. $(x_1 < x_2$ and $g(x_1) \geq g(x_2))$



<u>Pf</u>: take $\begin{cases} x_1 = 0 \\ x_2 = \frac{\pi}{2} \end{cases}$

$0 < \frac{\pi}{2}$ ✓

$g(0) = 1 \geq 0 = g(\frac{\pi}{2})$ ✓ ■

1.13 Proofs: a Theorem

- Ex. Prove that the sum of increasing functions is increasing

Thm: Let f, g be functions on an interval I. Let $h = f + g$ $\Big\}$ set-up

If f, g increasing on I $\}$ hypothesis (I may assume this is true)

Then h increasing on I $\}$ conclusion (I need to prove this)

Pf: WTS $\forall x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow h(x_1) < h(x_2)$

- Let $x_1, x_2 \in I$. Assume $x_1 < x_2$

  WTS $h(x_1) < h(x_2)$

assume $\Big\{$
- Since f increasing on I, $f(x_1) < f(x_2)$
- Since g increasing on I, $g(x_1) < g(x_2)$

prove $\Big\{$
- add both inequalities:

  $h(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = h(x_2)$ ∎

1.14 Proof by Induction

- Ex. Prove that $\forall n \geq 4,\ n! > 2^n$ $\leftarrow S_n$
  - Proof by induction

    1) Base case: prove S_4

    2) Induction step: prove $\forall n \geq 4,\ S_n \Rightarrow S_{n+1}$

    $S_4 \Rightarrow S_5$

    $S_5 \Rightarrow S_6$

    $S_6 \Rightarrow S_7$

    $\dots$

  - Conclusion: $S_4, S_5, S_6, S_7, \dots$

- Pf: (by induction on n)
  - Base case ($n=4$) WTS $4! > 2^4$

    $4! = 24 \quad 2^4 = 16$ ✓

  - Induction step Let $n \geq 4$

    Assume $n! > 2^n$. $\boxed{\text{WTS } (n+1)! > 2^{n+1}}$

    $(n+1)! = (n+1)\,n! > (n+1)\cdot 2^n$

    by induction hypothesis

    $2^{n+1} = 2 \cdot 2^n < (n+1)\cdot 2^n$

    because $n \geq 4$ so $n+1 > 2$

$\blacksquare$

1.15 One Theorem, Two Proofs.

- Ex. Theorem: $\forall n \geq 1$, $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

- Proof #1. Let $n \geq 1$.

  Call $S_n = 1 + 2 + 3 + \ldots + n$

  $S_n = n + (n-1) + (n-2) + \ldots + 1$

  Add both lines:

  $2S_n = (n+1) + (n+1) + (n+1) + \ldots + (n+1)$

  $2S_n = n(n+1) \qquad S_n = \frac{n(n+1)}{2}$ $\blacksquare$

  - we did not assume anything

- Proof #2 (by induction on n)

  Base case $(n=1)$ WTS $1 = \frac{1(1+1)}{2}$ ✓

  Induction step Let $n \geq 1$

  Assume $1+2+3+\ldots+n = \frac{n(n+1)}{2}$

  WTS $1+2+3+\ldots+(n+1) = \frac{(n+1)(n+2)}{2}$

$$1+2+\dots+(n+1) = [1+2+\dots+n] + (n+1)$$

$$= \left[\frac{n(n+1)}{2}\right] + (n+1) = \frac{n(n+1)+2(n+1)}{2}$$

$$= \frac{(n+2)(n+1)}{2}$$

$\blacksquare$

2.1 The Idea of Limit

- Ex. 1 $f(x) = \dfrac{x^2-1}{x-1}$

  $f(1)$ is undefined

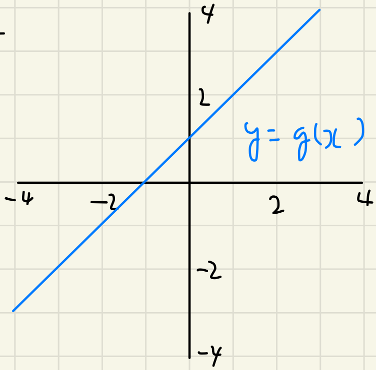
  $$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1 \quad \text{if } x \neq 1$$

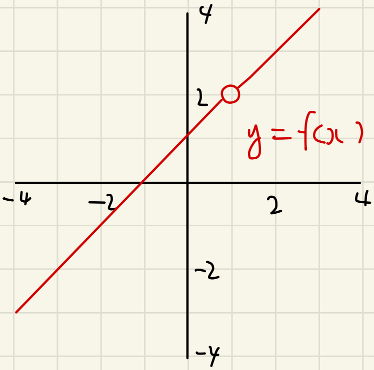
- $f(x) = \dfrac{x^2-1}{x-1}$ $\qquad$ $g(x) = x+1$

  $f(1)$ is undefined $\qquad$ $g(1) = 2$

  $f(x) = g(x)$ whenever $x \neq 1$

-





$g(x) = x+1$ $\qquad$ $f(x) = \dfrac{x^2-1}{x-1}$

$g(1) = 2$ $\qquad$ $f(1)$ undefined

- $\lim_{x \to 1} f(x) = 2$

  - "the limit as x approaches 1 of $f(x)$ is 2"

- idea: "if x is very close to 1 (but $x \neq 1$), then $f(x)$ is very close to 2"

- Ex. 2 $f(x) = \dfrac{1-\sqrt{1+x}}{x}$

- Domain $f = [-1, 0) \cup (0, \infty)$

- $f(0)$ is undefined

-

x	$f(x)$
1	-0.41421
0.1	-0.48809
0.01	-0.49876
0.001	-0.49988
0.000001	-0.4999999

- guess: when x is very close to 0 (but $x \neq 0$), then $f(x)$ is very close to -0.5

- Algebraic way:

$$\frac{1-\sqrt{1+x}}{x} = \frac{-x}{x(1+\sqrt{1+x})}$$

$$= \frac{-1}{1+\sqrt{1+x}} \quad (\text{when } x \neq 0)$$

$$\lim_{x \to 0} f(x) = \frac{-1}{2}$$

- Summary

  - $\lim_{x \to a} f(x) = L$ roughly means that

    "if x is close to a (but $x \neq a$),

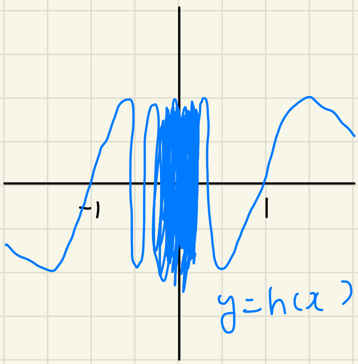
    then $f(x)$ is close to L"

2.2 Examples of Limits that Do Not Exist

- Ex. 3 $h(x) = \sin \frac{\pi}{2x}$

  $h(0)$ is undefined

x	$h(x)$
1	1
1/2	0
1/3	-1
1/4	0
1/5	1
1/6	0
1/7	-1
1/8	0

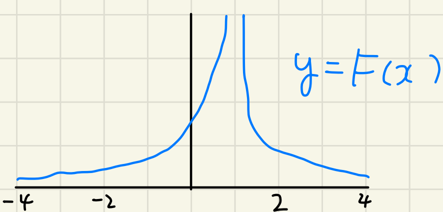


- if x is close to 0, then $h(x)$ is not close to one number

- $\lim_{x\to 0} h(x)$ DNE

- Ex. 4 $F(x) = \frac{1}{(x-1)^2}$

- if x is close to 1 (but $x \neq 1$), then $F(x)$ is very large



- $\lim_{x\to 1} F(x) = \infty$

- The limit does not exist
    - the graph does not approach any number (∞ is not a number, it can be as large as we want)
- the limit does not exist in a specific way
    - the number is arbitrarily large

2.3 Side Limits

- Ex. 5 $G(x) = \dfrac{x^2+x}{|x|}$

$G(0)$ undefined

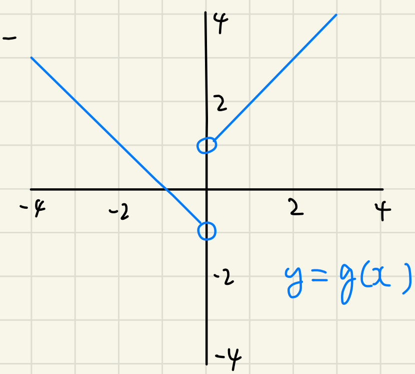
- if $x > 0$, then $|x| = x$,

$$G(x) = \frac{x(x+1)}{x} = x+1$$

- if $x < 0$, then $|x| = -x$,

$$G(x) = \frac{x(x+1)}{-x} = -x-1$$

- if $x > 0$, then $G(x) = x+1$
- if $x < 0$, then $G(x) = -x-1$
-



- $\lim_{x \to 0} G(x) = DNE$
    - when x is close to 0, $G(x)$ is sometimes close to 1 and sometimes close to -1
- $\lim_{x \to 0^+} G(x) = 1$
    - the limit as x approaches 0 on the right of $G(x)$ is 1
- $\lim_{x \to 0^-} G(x) = -1$
    - the limit as x approaches 0 on the left of $G(x)$ is -1
- idea:
    - "if x is very close to 0 (w/ $x > 0$), then $G(x)$ is very close to 1"
    - "if x is very close to 0 (w/ $x < 0$), then $G(x)$ is very close to -1"

<u>Summary</u>

- $\lim_{x \to a} f(x)$ exists – this means it is a single number

- The limit does not exist (DNE):
    - ∞
    - $-\infty$
    - different side limits
    - something else

2.4 Distance and Absolute Values

Absolute Value

- Algebraic definition
  - for every $x \in \mathbb{R}$, $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
- Geometric interpretation
  - $|x|$ is the distance between x and 0
  - $|x-a|$ is the distance between x and a
- Properties
  - For every $x, y \in \mathbb{R}$:
    1. $|xy| = |x||y|$
    2. $|x+y| \leq |x| + |y|$

Equivalent Expressions

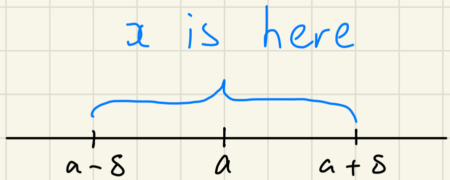
- $|x-a| < \delta$

  → "the distance between x and a is smaller than δ"

  → $-\delta < x-a < \delta$

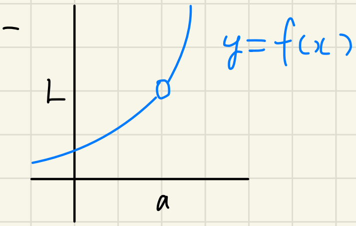
$\rightarrow a - \delta < x < a + \delta$

-



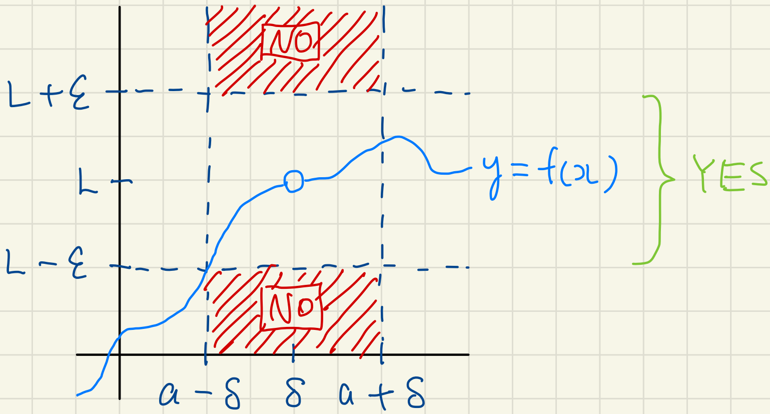
2.5 The Formal Definition of Limit

Towards a Formal Definition of $\lim_{x \to a} f(x) = L$

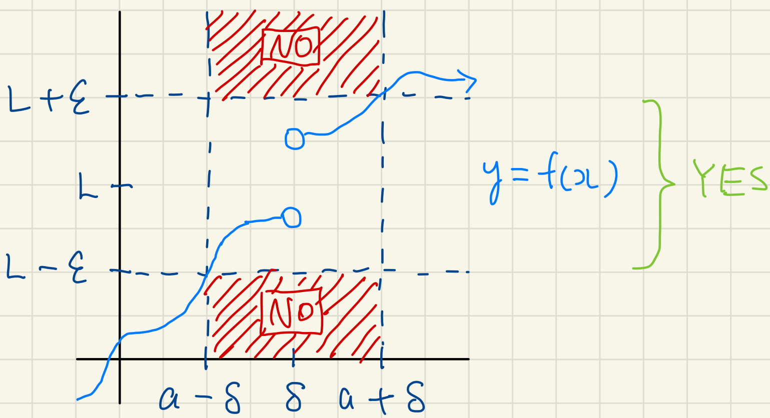
- 

$$\lim_{x \to a} f(x) = L$$

- "if x is close to a ($x \neq a$), then $f(x)$ is close to L"
- "x is close to a"
    - $|x-a|$ is "small"
    - $|x-a| < \delta$
- "x is close to a (but $x \neq a$)"
    - $0 < |x-a| < \delta$
- "if x is close to a, ($x \neq a$) then $f(x)$ is close to L"
    - $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
        - graph below



- $\lim_{x \to a} f(x)$ DNE

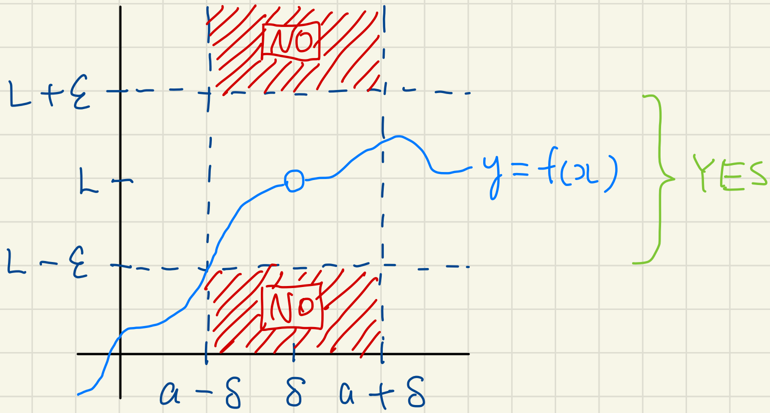


- narrowing δ makes this function easier to satisfy the implication
- if we narrow ε small enough, we will notice that the above graph does not have a limit

- $\forall \varepsilon > 0, \exists \delta > 0,$

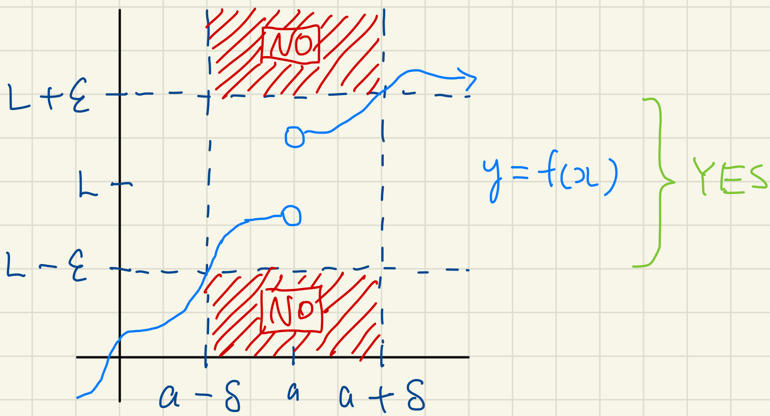
  $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

- for this graph:



  - no matter how small ε is, we can find a δ that satisfy the implication

- for this graph:



  - if we make ε small enough, no value of δ would work

The Formal Definition of Limit

- Let a, L be real numbers
- Let f be a function defined, at least, on an interval centered at a, except maybe at a

We say that

$$\lim_{x \to a} f(x) = L$$

when

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.}$$

$$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Additional Info

- since "$0 < |x-a| < \delta$" is a hypothesis, there is a hidden quantifier $\forall x \in \mathbb{R}$
  - it is there whether it's written or not

2.6 Limits at ∞

Intro

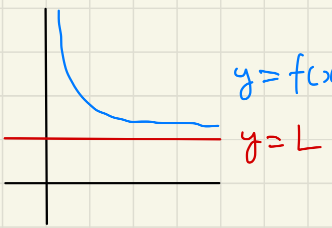
- $\lim_{x \to \infty} f(x)$
  - "when x is very large, what happens to $f(x)$?"

Non-Rigorous Example

- $f(x) = \dfrac{x+1}{x} = 1 + \dfrac{1}{x}$
- as $x \to \infty$, $\dfrac{1}{x} \approx 0$, $f(x) = 1 + \dfrac{1}{x} \approx 1$
  - $\lim_{x \to \infty} f(x) = 1$
- $\lim_{x \to \infty} f(x) = L$ means

  "if x is very large, then

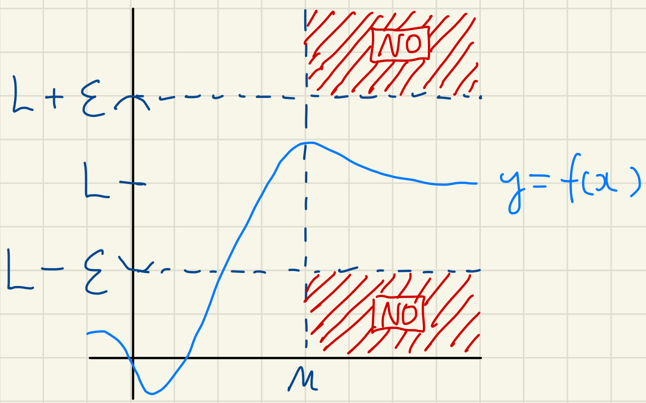
  $f(x)$ is close to L"



Towards a Formal Definition

- $\lim_{x \to \infty} f(x) = L$
- x lage $\Rightarrow$ $f(x)$ close to L

  $x > M \Rightarrow |f(x) - L| < \varepsilon$
- $\forall \varepsilon > 0, \exists M \in \mathbb{R}$ s.t. $x > M \Rightarrow$
  $|f(x) - L| < \varepsilon$



- no matter how small ε is, we can find an x to the right of M that works

Formal Definition

- Let $L \in \mathbb{R}$
- Let f be a function defined, at least, on an interval of the form (p, ∞) for some $p \in \mathbb{R}$

We say that

$$\lim_{x \to \infty} f(x) = L$$

when

$$\forall \varepsilon > 0, \exists M \in \mathbb{R} \text{ s.t.}$$

$$x > M \Rightarrow |f(x) - L| < \varepsilon$$

<u>Additional Info</u>

- if you tell me how close you want $f(x)$ and L to be (ε)
- then I can tell you how large x must be (M)

2.7 Prove from Definition I

Problem: Prove, directly from the formal definition of limit, that $\lim_{x \to 3} (2x+1) = 7$

- What we want to hs ow:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$0 < |x-3| < \delta \Rightarrow |(2x+1)-7| < \varepsilon$

Structure of the Proof

1. Let (or "Fix") $\varepsilon > 0$
2. Take (or "Set" or "Let") $\delta = ???$
3. Let $x \in \mathbb{R}$. Assume $0 < |x-3| < \delta$
4. We have concluded that $|(2x+1)-7| < \varepsilon$

Rough Work: What is δ?

- $|(2x+1)-7| = |2x-6| = 2|x-3| < 2\delta$
- if we take $2\delta = \varepsilon$, or $\delta = \frac{\varepsilon}{2}$, it will work
- any $\delta < \frac{\varepsilon}{2}$ will also work
- for simplicity, we will take $\delta = \frac{\varepsilon}{2}$

__Proof__

- Let $\varepsilon > 0$
- I take $\delta = \frac{\varepsilon}{2}$
- Let $x \in \mathbb{R}$. Assume that $0 < |x-3| < \delta$
- Then $|(2x+1)-7| = |2x-6| = 2|x-3| < 2\delta = \varepsilon$
- I have proven that $|(2x+1)-7| < \varepsilon$, as needed

$\blacksquare$

2.8 Prove from Definition II

Problem: Prove, directly from the formal definition of Limit, that $\lim_{x \to 4} (x^2+1) = 17$

- What we want to show:

  $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

  $0 < |x-4| < \delta \Rightarrow |(x^2+1)-17| < \varepsilon$

Structure of the Proof

1. Let (or "Fix") $\varepsilon > 0$
2. Take (or "Set" or "Let") $\delta = ???$
3. Let $x \in \mathbb{R}$. Assume $0 < |x-4| < \delta$
4. We have concluded that $|(x^2+1)-17| < \varepsilon$

Rough Work: What is δ?

- $|(x^2+1)-17| = |x^2-16| = |x+4||x-4| < |x+4|\delta$

- Bad idea: take $\delta = \dfrac{\varepsilon}{|x+4|}$ ✗

  - δ is only allowed to depend on ε, it cannot depend on x

- if we can make $|x+4| < C$ for some number C, then any $\delta \leq \frac{\varepsilon}{C}$ will work
    - we need to find out if $|x+4|$ can be bounded by a constant (i.e. 100)
- if we choose $\delta \leq 1$, then $|x+4| < 1$
    - we are allowed to do this as long as we show that the chosen value is valid later on
    - $3 < x < 5 \Rightarrow 7 < x+4 < 9 \Rightarrow |x+4| < 9$
- So we need $\delta \leq 1$ and $\delta \leq \frac{\varepsilon}{9}$ at the same time
    - take $\delta = \min\left\{1, \frac{\varepsilon}{9}\right\}$

Proof

- Let $\varepsilon > 0$
- I take $\delta = \min\left\{1, \frac{\varepsilon}{9}\right\}$
    - therefore $\delta \leq 1$ and $\delta \leq \frac{\varepsilon}{9}$
- Let $x \in \mathbb{R}$. Assume $0 < |x-4| < \delta$. This implies:
    - $|x-4| < \frac{\varepsilon}{9}$
    - $|x-4| < 1$

- Hence $3 < x < 5$, and $7 < x+4 < 9$

- Thus $|(x^2+1)-17| = |x^2-16|$

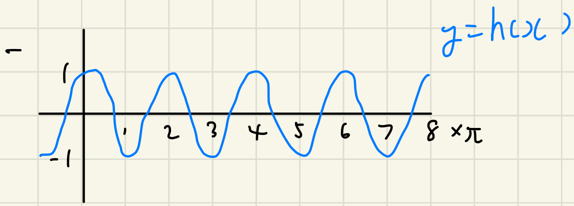
  $= |x+4||x-4| < 9 \cdot \frac{\varepsilon}{9} = \varepsilon$.

- We have proven $|(x^2+1)-17| < \varepsilon$, as needed. ∎

2.9 Prove a Limit DNE from Definition

Problem: Let $h(x) = \cos(\pi x)$. Prove, from the definition of limit, that

$\lim_{x \to \infty} h(x)$ does not exist



- if x is an even integer, then $h(x) = 1$
- if x is an odd integer, then $h(x) = -1$

Formal Definition of "Limit DNE"

- $\lim_{x \to \infty} h(x) = L$ means: $\forall \varepsilon > 0, \exists M \in \mathbb{R}$ s.t.

  $(\forall x \in \mathbb{R},) \quad x > M \Rightarrow |h(x) - L| < \varepsilon$

- $\lim_{x \to \infty} h(x) \neq L$ means: $\exists \varepsilon > 0$ s.t. $\forall M \in \mathbb{R}$,

  $\exists x \in \mathbb{R}$ s.t. $[x > M$ and $|h(x) - L| > \varepsilon]$

- $\lim_{x \to \infty} h(x)$ DNE means

  $\forall L \in \mathbb{R}, \lim_{x \to \infty} h(x) \neq L$

- expanding the definition of $\lim_{x \to \infty} h(x) \neq L$,

$\forall L \in \mathbb{R}, \exists \varepsilon > 0$ s.t. $\forall M \in \mathbb{R}, \exists x \in \mathbb{R}$ s.t.

$x > M$ and $|h(x) - L| \geq \varepsilon$

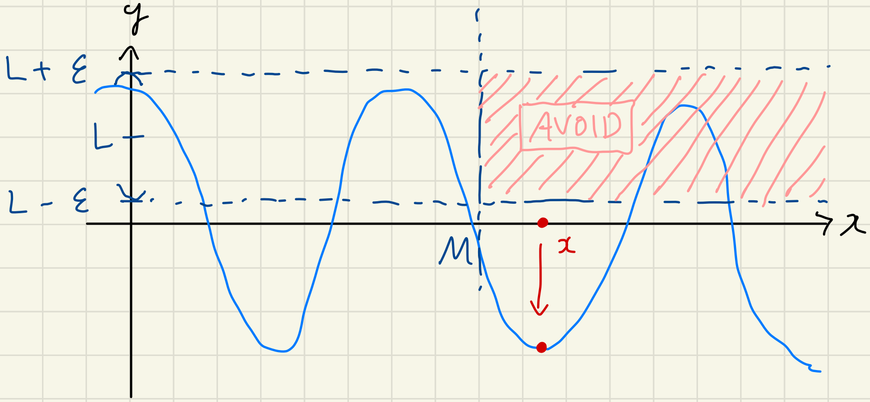
Structure of the Proof

- WTS $\forall L \in \mathbb{R}, \exists \varepsilon > 0$ s.t. $\forall M \in \mathbb{R}, \exists x \in \mathbb{R}$ s.t.

  $x > M$ and $|h(x) - L| \geq \varepsilon$

1. Let $L \in \mathbb{R}$
2. Take $\varepsilon = ???$
3. Let $M \in \mathbb{R}$
4. Take $x = ???$
5. Verify that $x > M$, and $|h(x) - L| \geq \varepsilon$

Graph



- pick ε as not too large
- pick $\varepsilon \leq 1$. Then 1 or $-1 \notin (L - \varepsilon, L + \varepsilon)$

- pick $x \in \mathbb{Z}$. Then $h(x) = 1$ or -1

Proof

- Let $L \in \mathbb{R}$. Take $\varepsilon = \frac{1}{2}$. Let $M \in \mathbb{R}$
- At least one of the following must be true:
    - Case A: $1 \notin (L - \varepsilon, L + \varepsilon)$

      I choose any $x \in \mathbb{Z}$, even,

      satisfying $x > M$. Then $h(x) = 1$
    - Case B: $-1 \notin (L - \varepsilon, L + \varepsilon)$

      I choose any $x \in \mathbb{Z}$, odd,

      satisfying $x > M$. Then $h(x) = -1$
- Either way, if satisfies $x > M$ and

  $|h(x) - L| \geq \varepsilon$ ■

Additional Info

- the words in red follow the structure

2.10 The Limit Laws

Why Do We Need Limit Laws?

- I want to prove rigorously that, for every polynomial P and every $a \in \mathbb{R}$:

$$\lim_{x \to a} P(x) = P(a)$$

The Plan

- Prove "basic" limits
  - $\lim_{x \to a} x = a$
  - $\lim_{x \to a} c = c$
- Prove Limit Laws
  - Assume $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$
  - Then:
    - $\lim_{x \to a} [f(x) + g(x)] = L + M$
    - $\lim_{x \to a} [f(x) g(x)] = LM$
    - $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{M}$ (assuming $M \neq 0$)

Example

Prove that $\lim_{x \to 2} (x^4 - 3x) = 10$

Pf: $\lim_{x \to 2} (x^4 - 3x)$

$= (\lim_{x \to 2} x^4) + (\lim_{x \to 2} [-3x])$

- Limit law for sum

$= (\lim_{x \to 2} x)^4 + (\lim_{x \to 2} (-3))(\lim_{x \to 2} x)$

- Limit law for product

$= (2)^4 + (-3)(2) = 10$

- Basic limits $\blacksquare$

- The same works for any polynomial

Warning: Limit Laws Only Apply if the Initial Limits Exist

- Ex. $\lim_{x \to 0} x = 0$, $\lim_{x \to 0} \frac{3}{x}$ D.N.E.

but $\lim_{x \to 0} \left[x \cdot \frac{3}{x} \right] = 3$

- Ex. $\lim_{x \to 0} \frac{1}{x}$ and $\lim_{x \to 0} \frac{2x-1}{x}$ D.N.E.

but $\lim_{x \to 0} \left[\frac{1}{x} + \frac{2x-1}{x} \right] = 2$

2.11 Proof of the Limit Law for Sums

The Limit Law for Sums, Carefully

- Theorem
    - Let $a, L, M \in \mathbb{R}$
    - Let f and g be functions defined at least on an interval centered at a, except maybe be at a
    - if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$
    - then $\lim_{x\to a} [f(x) + g(x)] = L + M$
- call $h(x) = f(x) + g(x)$

Structure of the Proof

WTS $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$0 < |x-a| < \delta \Rightarrow |h(x) - (L+M)| < \varepsilon$$

1. Let (or "fix") $\varepsilon > 0$
2. Take (or "set" or "let") $\delta = ???$
3. Let $x \in \mathbb{R}$. Assume $0 < |x-a| < \delta$
4. We have concluded that $|h(x) - (L+M)| < \varepsilon$

- We want to show:
  - $\lim_{x \to a} h(x) = L + M \qquad \forall \varepsilon, \exists \delta, \ldots$
  - I fix an arbitrary ε
  - I need to find a value of δ that works for this ε
- We know:
  - $\lim_{x \to a} f(x) = L \qquad \forall \varepsilon, \exists \delta, \ldots$
  - I can choose a value of ε
  - There exists a value of δ that works for that ε

<u>Rough Work</u>

- We need:

$$0 < |x-a| < \delta \Rightarrow |h(x) - (L+M)| < \varepsilon$$

- $$\begin{aligned} |h(x) - (L+M)| &= |f(x) + g(x) - L - M| \\ &= |[f(x) - L] + [g(x) - M]| \\ &\leq |f(x) - L| + |g(x) - M| \end{aligned}$$
- Make $|f(x) - L| < \frac{\varepsilon}{2}$. We know $\lim_{x \to a} f(x) = L$
- For the value $\frac{\varepsilon}{2}$, $\exists \delta_1 > 0$ s.t.

$$0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$$

- By doing the same thing for $|g(x) - M|$, $\exists\, \delta_2 > 0$ s.t.

$$0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$$

- Take $\delta = \min\{\delta_1, \delta_2\}$

Proof

- Let $\varepsilon > 0$
- I use $\frac{\varepsilon}{2}$ in the definition of $\lim_{x \to a} f(x) = L$. $\exists\, \delta_1 > 0$ s.t.

$$0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$$

- I use $\frac{\varepsilon}{2}$ in the definition of $\lim_{x \to a} g(x) = M$. $\exists\, \delta_2 > 0$ s.t.

$$0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$$

- Take $\delta = \min\{\delta_1, \delta_2\}$
- Let $x \in \mathbb{R}$. Assume $0 < |x-a| < \delta$. This implies:
    - $0 < |x-a| < \delta_1$. Thus $|f(x) - L| < \frac{\varepsilon}{2}$
    - $0 < |x-a| < \delta_2$. Thus $|g(x) - M| < \frac{\varepsilon}{2}$
- Then

$$\begin{aligned} |h(x) - (L+M)| &= |[f(x) - L] + [g(x) - M]| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

- I have shown that $|h(x) - (L+M)| < \varepsilon$,

  as needed $\square$

<u>Additional Info</u>

- words in blue corresponds to each step in the structure of the proof
- fix ε before using it

2.12 The Squeeze Theorem

- Ex. Compute $\lim_{x \to 0} x^2 \sin \frac{1}{x}$
  - we cannot use the limit laws here, because $\lim_{x\to 0} \sin \frac{1}{x}$ DNE.

Limit of Products, Carefully

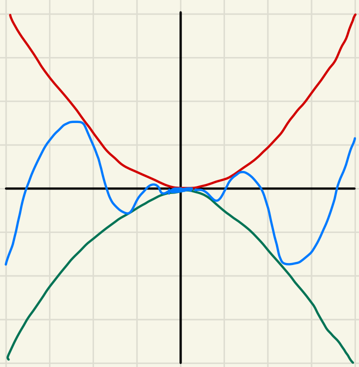
- $\lim_{x\to 0} x = 0$, $\lim_{x \to 0} \frac{3}{x}$ DNE, $\lim_{x\to 0} \left[x \cdot \frac{3}{x}\right] = 3$
- $\lim_{x\to 0} x = 0$, $\lim_{x \to 0} \frac{3}{x^2}$ DNE, $\lim_{x\to 0} \left[x \cdot \frac{3}{x^2}\right]$ DNE
- $\lim_{x\to 0} x = 0$, $\lim_{x \to 0} \sin \frac{1}{x}$ DNE, $\lim_{x\to 0} \left[x^2 \cdot \sin \frac{1}{x}\right] = ???$

Original Problem

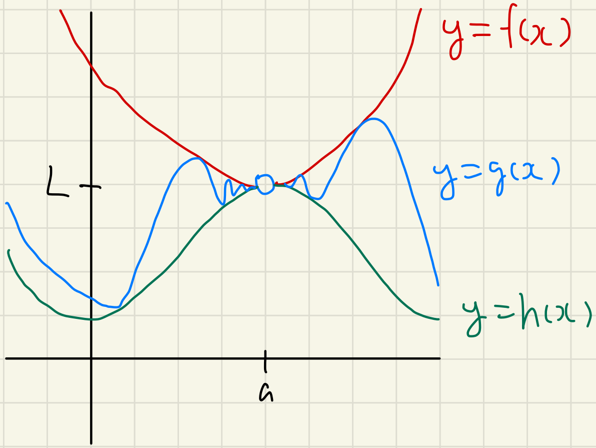
- For every $x \neq 0$:

$$-1 \leq \sin \frac{1}{x} \leq 1$$

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$



- generalizing the above graph:



The Squeeze Theorem

- Let $a, L \in \mathbb{R}$
- Let f, g, and h be functions defined near a, except possibly at a

If:

- For x close to a but not a, $h(x) \leq g(x) \leq f(x)$
- $\lim_{x \to a} f(x) = L$
- $\lim_{x \to a} h(x) = L$

Then:

- $\lim_{x \to a} g(x) = L$

Solution to the Original Problem

- Ex. Compute $\lim_{x\to 0} x^2 \sin\frac{1}{x}$
    - We know:
        - For every $x \neq 0$: $-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$
        - $\lim_{x\to 0} x^2 = 0$
        - $\lim_{x\to 0} (-x^2) = 0$
    - By the squeeze theorem, $\lim_{x\to 0} \left[x^2 \sin\frac{1}{x}\right] = 0$

2.13 Proof of the Squeeze Theorem

The Squeeze Theorem

- Let $a, L \in \mathbb{R}$
- Let f, g, and h be functions defined near a, except possibly at a

If:

- $\exists p > 0$, s.t. $0 < |x-a| < p \Rightarrow h(x) \leq g(x) \leq f(x)$
- $\lim_{x \to a} f(x) = L$
- $\lim_{x \to a} h(x) = L$

Then:

- $\boxed{\lim_{x \to a} g(x) = L}$

Structure of the Proof

WTS $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$0 < |x-a| < \delta \Rightarrow |g(x) - L| < \varepsilon$$

1. Let $\varepsilon < 0$
2. Take $\delta = ???$
3. Let $x \in \mathbb{R}$, assume $0 < |x-a| < \delta$

4. We have concluded that $|g(x) - L| < \varepsilon$

<u>We Want to Show</u>

- $\lim_{x \to a} g(x) = L$ $\quad \forall \varepsilon, \exists \delta, \ldots$
- I fix an arbitrary ε
- I need to find a value of δ that works for this ε

<u>We know</u>

- $\lim_{x \to a} f(x) = L$ $\quad \forall \varepsilon, \exists \delta, \ldots$
- I can choose a value of ε
- There exists a value of δ that works for that ε

<u>Rough Work: What to Take as δ?</u>

We need: $0 < |x-a| < \delta \implies |g(x) - L| < \varepsilon$

- $|g(x) - L| < \varepsilon$ is equivalent to $\boxed{L - \varepsilon < g(x) < L + \varepsilon}$
- We know $\boxed{0 < |x-a| < p} \implies \boxed{h(x) \leq g(x) \leq f(x)}$
- We know $\lim_{x \to a} f(x) = L$. $\exists \delta_1 > 0$ s.t.

  $\boxed{0 < |x-a| < \delta_1} \implies L - \varepsilon < \boxed{f(x) < L + \varepsilon}$
- We know $\lim_{x \to a} h(x) = L$. $\exists \delta_2 > 0$ s.t.

  $\boxed{0 < |x-a| < \delta_2} \implies \boxed{L - \varepsilon < h(x)} < L + \varepsilon$
- Take $\delta = \min\{\delta_1, \delta_2, p\}$

Proof

- Let $\varepsilon > 0$
- Use the same ε in the definition of $\lim_{x \to a} f(x) = L$.

  $\exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$

  $\Rightarrow f(x) < L + \varepsilon$
- Use the same ε in the definition of $\lim_{x \to a} h(x) = L$.

  $\exists \delta_2 > 0$ s.t. $0 < |x-a| < \delta_2 \Rightarrow |h(x) - L| < \varepsilon$

  $\Rightarrow L - \varepsilon < h(x)$
- take $\delta = \min\{\delta_1, \delta_2, p\}$
- Let $x \in \mathbb{R}$. Assume $0 < |x-a| < \delta$. This implies:
  - $0 < |x-a| < \delta_2 \Rightarrow L - \varepsilon < h(x)$
  - $0 < |x-a| < p \Rightarrow h(x) \leq g(x) \leq f(x)$
  - $0 < |x-a| < \delta_1 \Rightarrow f(x) < L + \varepsilon$
- Therefore $L - \varepsilon < g(x) < L + \varepsilon$
- Equivalently, we have proven that $|g(x) - L| < \varepsilon$,

  as needed. ■

Additional Info

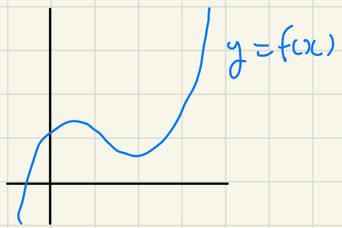
- steps in blue corresponds to each step in the structure

- all the variables are introduced in the right order
- every step follows logically from the previous ones
- for every step, we are only using things that are established to be true

2.14 The Definition of Continuity

Idea

- A function is continuous means we can sketch its graph w/o lifting the pen from the paper



Definition

- Let $a \in \mathbb{R}$
- Let f be a function defined, at least, on an interval centered at a

We say that f is continuous at a when

$$\lim_{x \to a} f(x) = f(a)$$

This means:

- $\lim_{x \to a} f(x)$ exists (is a number)
- $f(a)$ is defined
- $\lim_{x \to a} f(x) = f(a)$

An Equivalent Definition

- $\lim_{x \to a} f(x) = L$ means: $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

  $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

- $\lim_{x \to a} f(x) = f(a)$ means: $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

  $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Alternate (equivalent) Definition

- Let $a \in \mathbb{R}$
- Let f be a function defined, at least, on an interval centered at a

We say that f is continuous at a when

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t.}$$

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

Not Very Rigorously

- $\lim_{x \to a} f(x) = L$ means:

$$\left\{ \begin{array}{c} x \text{ close to } a \\ x \neq a \end{array} \right\} \Longrightarrow f(x) \text{ close to } L$$

- f continuous at a means.

$$x \text{ close to } a \Rightarrow f(x) \text{ close to } f(a)$$

Continuity

- Continuous at a point:
  - f continuous at c means

    $\lim_{x \to c} f(x) = f(c)$

- Continuous on an open interval
  - f continuous on the interval (a,b) means

    $\forall c \in (a,b)$, f is continuous at c

- Continuous on a closed interval
  - f continuous on the interval $[a,b]$ means

    $\lim_{x \to a^+} f(x) = f(a)$

    $\forall c \in (a,b)$, f is continuous at c

    $\lim_{x \to b^-} f(x) = f(b)$

- Just "continuous"
  - f is continuous means

    f is continuous on its domain

  -  This function is continuous on its domain. (domain : $x \neq 0$)

    It is not continuous at 0.

2.15 The Main Continuity Theorem

The Main Continuity Theorem

- A function f is continuous at c when $\lim_{x \to c} f(x) = f(c)$
- Any function we can construct w/ sum, product, quotient, and composition of polynomials, roots, trigonometric functions, exponentials, logarithms, and absolute values is continuous (on its domain).

- Ex. $\lim_{x \to 1} \frac{x^2 - \sin x}{e^x + \ln(\sqrt{x} + 3)} = \frac{1 - \sin 1}{e + \ln 4}$

- $f(x) = \frac{\sin x}{x}$ is not continuous at $x = 0$
    - $x = 0$ is not a domain for the function
- $g(x) = \tan x$ is not continuous at $x = \frac{\pi}{2}$
    - $x = \frac{\pi}{2}$ is not a domain for the function

How to Prove It

1. Prove that the "basic" functions are continuous
    - $f(x) = c$
    - $f(x) = x$

- $f(x) = e^x$
- $f(x) = \ln x$
- $f(x) = \sin x$
- $f(x) = |x|$
- The above is enough. i.e.
  - $\cos x = \sin\left(\frac{\pi}{2} - x\right)$
  - $\tan x = \frac{\sin x}{\cos x}$
  - $2^x = e^{x \ln 2}$
  - $\sqrt{x} = e^{\frac{1}{2}\ln x}$

2. Then prove that sum, product, quotient, and compositions of continuous functions is continuous

- step 1 is easy

<u>Step 2: Use Limit Laws</u>

Claim: The sum of continuous functions is continuous

Proof:

Assume f and g are continuous at a

This means $\lim_{x \to a} f(x) = f(a)$, $\lim_{x \to a} g(x) = g(a)$

Use limit law for sums:

$$\lim_{x \to a} [f(x) + g(x)] = f(a) + g(a)$$

So $f+g$ is continuous at a. ■

What Is Left to Prove?

- Prove that the composition of continuous functions is continuous
- Problem: There is no limit law for composition

2.16 Limits and Composition

False Theorem

- if $\lim_{x\to a} f(x) = L$, and

  $\lim_{y\to L} g(y) = M$

  ~~then~~ $\cancel{\lim_{x\to a} g(f(x)) = M}$ False!

Counterexample to False Theorem

$$f(x) = \begin{cases} x\sin\frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad \lim_{x\to 0} f(x) = 0$$

$$g(y) = \begin{cases} 2 & \text{if } y = 0 \\ 1 & \text{if } y \neq 0 \end{cases} \qquad \lim_{y\to 0} g(y) = 1$$

$$g(f(x)) = \begin{cases} 2 & \text{if } x\sin\frac{\pi}{x} = 0 \\ 1 & \text{if } x\sin\frac{\pi}{x} \neq 0 \end{cases} \qquad \lim_{x\to 0} g(f(x)) \text{ DNE}$$

(oscillates between 1 and 2)

The Core of the Problem

- $\lim_{x\to a} f(x) = L$ means:

$$\left\{ \begin{matrix} x \text{ close to } a \\ x \neq a \end{matrix} \right\} \Longrightarrow f(x) \text{ close to } L$$

- f continuous at a means:

$$x \text{ close to } a \implies f(x) \text{ close to } f(a)$$

- the first definition lacks symmetry
  - therefore it leads to problems with composition

Theorem 1

- if ① f continuous at a, and

  ② g continuous at $f(a)$

  ③ $g \circ f$ continuous at a

- ① means x close to $a \implies f(x)$ close to $f(a)$
- ② means y close to $f(a) \implies g(y)$ close to $g(f(a))$

Concatenate the two implications:

- ③ x close to $a \implies g(f(x))$ close to $g(f(a))$

False Theorem Explained

- if ① $\lim_{x \to a} f(x) = L$, and

  ② $\lim_{y \to L} g(y) = M$

  then ③ $\lim_{x \to a} g(f(x)) = ???$

- ① means $\left\{ \begin{array}{l} x \text{ close to } a \\ x \neq a \end{array} \right\} \implies f(x)$ close to L

- ② means $\left\{\begin{array}{l} y \text{ close to } L \\ y \neq L \end{array}\right] \Rightarrow g(y)$ close to M

- We cannot concatenate these two implications
    - The "then" part of ① does not match the "if" part of ②
- but there are ways to fix this theorem

<u>Theorem 2</u>

- if ① $\lim_{x \to a} f(x) = L$, and

    $f(x) \neq L$ for x on an interval centered at a, except maybe at a

    ② $\lim_{y \to L} g(y) = M$

    then ③ $\lim_{x \to a} g(f(x)) = M$

<u>Theorem 3</u>

- if ① $\lim_{x \to a} f(x) = L$

    ② g is continuous at L

    then ③ $\lim_{x \to a} g(f(x)) = g(L)$

2.17 Discontinuities

How May A Function Fail To Be Continuous?

- f is continuous at a means $\lim_{x \to a} f(x) = f(a)$
- Otherwise, f is discontinuous at a

1. Removable discontinuity
   - $\lim_{x \to a} f(x)$ exists, but $\lim_{x \to a} f(x) \neq f(a)$
2. Non-removable discontinuity
   - $\lim_{x \to a} f(x)$ DNE

Removable Discontinuity: It Can Be "Fixed"

- Example: $h(x) = \dfrac{\sin x}{x}$. $\lim_{x \to 0} h(x) = 1$, $h(0)$ undefined
- Define a new function H:

$$H(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

  - Then H is continuous at 0

Non-removable Discontinuity

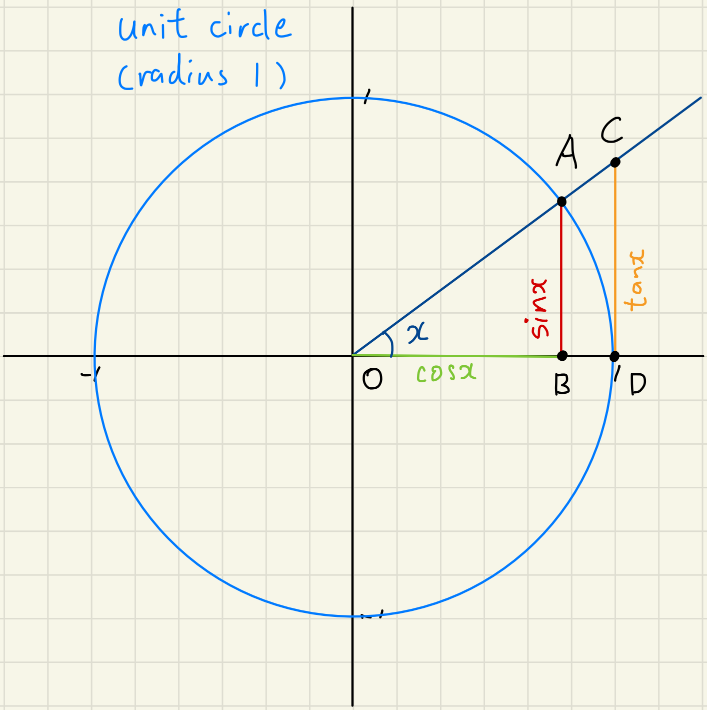
- jump
- vertical asymptote

- Oscillation

- None of the above can be fixed

2.18 A Geometric Proof for a Trig Limit

- Prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$



- Area of OAB $\leq$ Area of OAD $\leq$ Area of OCD
- Area of OAB $= \frac{1}{2} \cdot \cos x \cdot \sin x$
- Area of OCD $= \frac{1}{2} \cdot 1 \cdot \tan x$
- Area of OAD $= \frac{1}{2} \cdot x \cdot 1^2$
- Multiply the inequality by 2,

$$\cos x \cdot \sin x \leq x \leq \tan x = \frac{\sin x}{\cos x}$$

- From $\cos x \cdot \sin x \leq x$, we can conclude that

$$\frac{\sin x}{x} \leq \frac{1}{\cos x}$$

- From $x \leq \frac{\sin x}{\cos x}$, we can conclude that

$$\cos x \leq \frac{\sin x}{x}$$

- Therefore, $\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}$ for $0 < |x| < \frac{\pi}{2}$
- $\lim_{x \to 0} \cos x = 1$, $\lim_{x \to 0} \frac{1}{\cos x} = 1$
- By Squeeze Theorem, $\lim_{x \to 0} \frac{\sin x}{x} = 1$ as well

2.19 Computing the Limit of a Function at a Point

Method 1

- Is the function defined and continuous
  - if so, evaluate

- Ex: $\lim_{x \to 2} \frac{x \sin x + e^x}{\sqrt{x^2+7}}$

$$= \lim_{x \to 2} \frac{2\sin 2 + e^2}{\sqrt{2^2+7}} = \frac{2\sin 2 + e^2}{\sqrt{11}}$$

Method 2

- Algebraic manipulations
  - transform the function into a continuous one

- Ex. $\lim_{x \to 2} \frac{x^2-4}{x^2-3x+2}$

$$= \lim_{x \to 2} \frac{(x+2)(x-2)}{(x-1)(x-2)} = \lim_{x \to 2} \frac{x+2}{x-1} = \frac{2+2}{2-1} = 4$$

- Ex. $\lim_{x \to 0} \frac{1-\sqrt{1+x}}{x}$

$$= \lim_{x \to 0} \frac{[1-\sqrt{1+x}] \cdot [1+\sqrt{1+x}]}{x \cdot [1+\sqrt{1+x}]} = \lim_{x \to 0} \frac{1-(1+x)}{x[1+\sqrt{1+x}]}$$

$$= \lim_{x\to 0} \frac{-x}{x[1+\sqrt{1+x}]} = \lim_{x\to 0} \frac{-1}{1+\sqrt{1+x}}$$

$$= \frac{-1}{1+\sqrt{1}} = \frac{-1}{2}$$

Method 3

- Reduce to a problem we have already solved

- Ex. $\lim_{x\to 0} \frac{\sin(2x)}{x}$

  - $\lim_{x\to 0} \frac{\sin x}{x} = 1 \rightarrow \lim_{x\to 0} \frac{\sin(2x)}{2x} = 1$

$$\lim_{x\to 0} \left[2 \cdot \frac{\sin(2x)}{2x} \right] = 2 \cdot 1 = 2$$

- Ex. $\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$

  - $\lim_{x\to 0} \frac{\sin(ax)}{ax} = a$ for any $a \neq 0$

$$\lim_{x\to 0} \frac{2\cancel{x} \cdot \cancel{\frac{\sin(2x)}{2x}}}{3\cancel{x} \cdot \cancel{\frac{\sin(3x)}{3x}}} = \frac{2}{3}$$

- Ex. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

$$= \lim_{x\to 0} \frac{[1-\cos x][1+\cos x]}{x^2[1+\cos x]} = \lim_{x\to 0} \frac{1-\cos^2 x}{x^2[1+\cos x]}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2[1+\cos x]} = \lim_{x \to 0} \left[\cancel{\left(\frac{\sin x}{x}\right)^2} \frac{1}{1+\cos x} \right]$$

$$= \frac{1}{1+\cos 0} = \frac{1}{2}$$

Summary

1. Is the function defined and continuous?
2. Algebraic manipulations
3. Reduce to a problem we have already solved
4. Squeeze Theorem
5. L'Hôpital's rule (requires derivatives)
6. Taylor series (requires power series)

2.20 Compute the Limit as $x \to \pm\infty$

- $\lim_{x\to\infty} x = \infty$

- $\lim_{x\to\infty} \frac{1}{x} = 0$

- Ex. $\lim_{x\to\infty} \left[2x^3 - 3x^2 + 7\right]$

$$= \lim_{x\to\infty} x^3 \left[2 - \underbrace{3\cdot\frac{1}{x}}_{0} + \underbrace{7\cdot\frac{1}{x^3}}_{0}\right]$$

$$= \infty$$

- Ex. $\lim_{x\to\infty} \frac{2x^2+x-1}{3x^2+10x+e}$

$$= \lim_{x\to\infty} \frac{x^2\left[2+\frac{1}{x}-\frac{1}{x^2}\right]}{x^2\left[3+\frac{10}{x}+\frac{e}{x^2}\right]} = \lim_{x\to\infty} \frac{2+\overbrace{\frac{1}{x}}^{0}-\overbrace{\frac{1}{x^2}}^{0}}{3+\underbrace{\frac{10}{x}}_{0}-\underbrace{\frac{e}{x^2}}_{0}} = \frac{2}{3}$$

- Ex. $\lim_{x\to\infty} \frac{\sqrt{x^4+1}+x^2}{\sqrt{2x^4+1}+2x}$

$$\lim_{x\to\infty} \frac{\sqrt{x^4\left(1+\frac{1}{x^4}\right)}+x^2}{\sqrt{x^4\left(2+\frac{1}{x^4}\right)}+2x} = \lim_{x\to\infty} \frac{x^2\sqrt{1+\frac{1}{x^4}}+x^2}{x^2\sqrt{2+\frac{1}{x^4}}+2x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left[\sqrt{1+\frac{1}{x^4}}+1\right]}{\cancel{x^2}\left[\sqrt{2+\frac{1}{x^4}}+\frac{2}{x}\right]} = \frac{1+1}{\sqrt{2}} = \sqrt{2}$$

2.21 The Extreme Value Theorem

- Definition. We say that a function f has a maximum on a set I when:

  $\exists c \in I$, s.t. $\forall x \in I$, $f(x) \leq f(c)$

  - The maximum is $f(c)$
  - f has a maximum at c
  - If we do not specify, "the maximum of f" means "the maximum of f on its domain"

Extreme Value Theorem

- if f is a continuous function on an interval $[a,b]$ then f has a maximum and a minimum on $[a,b]$

2.22 The Intermediate Value Theorem

Problem

- Solve $x^5 + 9x - 6 = 0$
    - It is impossible to solve it exactly
    - Call $f(x) = x^5 + 9x - 6$
    - $f(0) = -6$, $f(1) = 4$
        - So there must exist $c \in (0, 1)$ s.t. $f(c) = 0$ because f is continuous
    - $f(0.5) < 0$
        - so there is $0.5 < c < 1$ s.t. $f(c) = 0$
    - With enough patience, we can get the first few digits of the solution

The Intermediate Value Theorem

- Let f be the function defined on the interval $[a, b]$
- If f is continous on $[a, b]$

  Then f takes all the values between $f(a)$ and $f(b)$

3.1 Derivative as Slope

Idea

- Let f be a function w/ domain I
- Let $a \in I$

$f'(a)$ = slope of line tangent to the graph of

$y = f(x)$ at the point w/ $x = a$

- We call the number $f'(a)$ the

  derivative of f at a

Towards a Definition

- Derivative

  ↓ define with

- Slope of a curve

  ↓ define with

- Tangent line to a curve

  ↓ define with

- Linear approximation / Instantaneous rate of change

  ↓ define with

- Derivatives (Oh, no!)
- We need to define one of the above from scratch, and then we can use it to define the rest
    - The only way: using limits

Definition

- Let $a \in \mathbb{R}$
- Let f be a function defined, at least, on an interval centered at a

The derivative of f at a is the number

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

- "f is differentiable at a" when this limit exists
- "f is differentiable" means at all points on its domain

Alternative Definition

- Change of variable: $h = x - a$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 0} \frac{f(a+h) - f(a)}{h}$$

Tangent Line

- the line tangent to the graph of $y=f(x)$ at the point w/ x-coordinate a is the line
    - through the point $(a, f(a))$
    - w/ slope $f'(a)$
- $y = f(a) + f'(a)(x-a)$

3.2 Computing a Derivative from the Definition

- Let $f(x) = 4x - x^2$

  Compute $f'(1)$ directly from the definition

$$- f(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{[4 \cdot (1+h) - (1+h)^2] - [4 \cdot 1 - 1^2]}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h - (1 + 2h + h^2) - 3}{h}$$

$$= \lim_{h \to 0} \frac{2h - h^2}{h} = \lim_{h \to 0} [2 - h] = 2$$

<u>Differentiation Rules</u>

- Much faster way to compute derivatives
- Will need to obtain and prove them (just once) from definition

3.3 Derivative as Rate of Change

What is Velocity?

- " Right now, my velocity is 180 km/h"
  - "~~In the next 1 h, I will drive 180 km~~"
    - Untrue
  - " If I continue driving at the same velocity, in the next 1 h I will drive 180 km"
    - Cheating
    - Defines "velocity" using "velocity"
  - "~~In the next minute, I will drive 180/60 = 3 km~~"
    - False, but better than the first one
  - " In the next 0.001 second, I will drive 50 mm"
    - Good approximation
  - " If Δt is a very small time interval, in the next Δt I will drive approximately a distance of $(180\,km/h \cdot \Delta t)$."
    - Need to use limits

Average vs. Instantaneous Velocity

- t = time, x = position

- $\begin{pmatrix} \text{average velocity} \\ \text{between } t_1 \text{ and } t_2 \end{pmatrix} = \dfrac{\Delta x}{\Delta t}$

- $\begin{pmatrix} \text{instantaneous} \\ \text{velocity at } t_1 \end{pmatrix} = \lim\limits_{t_2 \to t_1} \begin{pmatrix} \text{average velocity} \\ \text{between } t_1 \text{ and } t_2 \end{pmatrix}$

$$= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

- dx and dt are not numbers

- $\dfrac{dx}{dt} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta x}{\Delta t}$

    - " derivative of x with respect to t "

More Generally...

- physical quantities Q, x

- Q depend on x

- $\begin{pmatrix} \text{average} \\ \text{rate of change} \end{pmatrix} = \dfrac{\Delta Q}{\Delta x}$

- $\begin{pmatrix} \text{instantaneous} \\ \text{rate of change} \end{pmatrix} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta Q}{\Delta x} = \dfrac{dQ}{dx}$

$\frac{dx}{dt}$ vs. f'

- t = time, x = position, $x = f(t)$

$$\frac{dx}{dt} = \lim_{\Delta t = 0} \frac{\Delta x}{\Delta t} = \lim_{t_2 \to t_1} \frac{f(t_2) - f(t_1)}{t_2 - t_1} = f'(t_1)$$

3.4 Differentiation Rules

Differentiation Rules (Lazy Version)

- $\frac{d}{dx}[c] = 0$

- $\frac{d}{dx}[x^c] = cx^{c-1}$

- $(f+g)' = f' + g'$

- $(cf)' = cf'$

- $(f \cdot g)' = f'g + fg'$

- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

- f, g are functions

- $c \in \mathbb{R}$ is a constant

- Ex. $f(x) = x^7 - 5x^3 + 2x + 11$

  - $f'(x) = \frac{d}{dx}(x^7) - 5\frac{d}{dx}(x^3) + 2\frac{d}{dx}(x) + \frac{d}{dx}(11)$

    $= 7x^6 - 5 \cdot 3x^2 + 2 \cdot 1 + 0$

$$= 7x^6 - 15x^2 + 2$$

- Ex. $f(x) = \dfrac{x^2+1}{x^2+x}$

- $$f'(x) = \frac{\frac{d}{dx}(x^2+1)(x^2+x) - (x^2+1)\frac{d}{dx}(x^2+x)}{[x^2+x]^2}$$

$$= \frac{(2x)(x^2+x) - (x^2+1)(2x+1)}{(x^2+x)^2}$$

$$= \frac{x^2-2x-1}{(x^2+x)^2}$$

- Ex. $\dfrac{d}{dx}\left[\dfrac{1}{x^3}\right]$

$$= \frac{d}{dx}[x^{-3}] = -3x^{-4} = \frac{-3}{x^4}$$

- Ex. $\dfrac{d}{dx}[\sqrt{x}]$

$$= \frac{d}{dx}[x^{-1/2}] = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

<u>The Product Rule</u> (Formal Version)

- Let $a \in \mathbb{R}$

- Let f and g be functions defined at and near a

- We define the function h by $h(x) = f(x) \cdot g(x)$

If f and g are differentiable at a

Then h is differentiable at a, and

$$h'(a) = f'(a)g(a) + f(a)g'(a)$$

3.5 Differentiable Implies Continuous

Theorem

- Let $c \in \mathbb{R}$
- Let f be a function defined at and near c

If f is differentiable at c

Then f is continuous at c

- Note that its converse is not necessarily true

- f is differentiable at c means $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists

- f is continuous at c means $\lim_{x \to c} f(x) = f(c)$
- $\lim_{x \to c} [f(x)] = f(c)$ is equivalent to

  $\lim_{x \to c} [f(x) - f(c)] = 0$
- If f continuous at c, then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \to \frac{0}{0}$$

  - $f'(c)$ may or may not exist

- If f not continuous at c, then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \to \frac{\text{"not 0"}}{0}$$

- $f'(c)$ does not exist

<u>Proof</u>

- Assume f is differentiable at c
- Then

$$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \cdot (x - c) \right]$$

$$= \left[\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \right] \left[\lim_{x \to c} (x - c) \right]$$

$$= f'(c) \cdot 0 = 0$$

- So f is continuous at c $\square$

3.6 Proof of the Product Rule

- $(fg)' = f'g + fg'$

<u>Theorem</u>

- Let $a \in \mathbb{R}$
- Let f and g be functions defined at and near a
- We define the function h by $h(x) = f(x)g(x)$

If f and g are differentiable at a,

Then h is differentiable at a, and

$$h'(a) = f'(a)g(a) + f(a)g'(a)$$

<u>Proof</u>

- $h(x) = f(x)g(x)$

- $$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{[f(x)g(x) - f(a)g(x)] + [f(a)g(x) - f(a)g(a)]}{x - a}$$

$$= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} g(x) + f(a) \frac{g(x) - g(a)}{x - a} \right]$$

$$= \underbrace{\left[\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right]}_{f'(a)} \underbrace{\left[\lim_{x\to a} g(x)\right]}_{g(a)} + f(a)\cdot \underbrace{\left[\lim_{x\to a} \frac{g(x)-g(a)}{x-a}\right]}_{g'(a)}$$

f differentiable at a

g differentiable at a

g continuous at a

$$h'(a) = f'(a)g(a) + f(a)g'(a) \qquad \square$$

Additional Info

- must assume that f and g are differentiable at a
- explain how we know that g is continuous at a

3.7 Proof of the Power Rule

- $\frac{d}{dx}[x^c] = cx^{c-1}$
  - True for all $c \in \mathbb{R}$
  - For now, proof for $c = 1, 2, 3, \ldots$

<u>Proof</u> (by Induction on c)

- Base case: $c = 1$

WTS $\frac{d}{dx}[x] = 1 \cdot x^0 = 1$

Call $f(x) = x$

Then $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{h}$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

- Induction step

Fix $c \geq 1$ $\begin{cases} \text{Assume } \frac{d}{dx} x^c = cx^{c-1} \\ \text{WTS } \frac{d}{dx} x^{c+1} = (c+1)x^c \end{cases}$

$\frac{d}{dx} x^{c+1} = \frac{d}{dx}(x^c \cdot x)$ (by product rule)

$= \left(\frac{d}{dx} x^c\right) \cdot x + x^c \cdot \left(\frac{d}{dx} x\right)$ (by induction hypothesis)

$$= (cx^{c-1}) \cdot x + x^c \cdot 1$$

$$= (c+1)x^c \qquad \square$$

3.8 Higher-Order Derivatives

Lagrange Notation

- function f
- (first) derivative f'
- second derivative f''
- n^{th} derivative $f^{(n)}$ (parentheses are important!)
- Example: $f(x) = x^7$
  - $f'(x) = 7x^6$
  - $f''(x) = 42x^5$
  - $f'''(x) = 210x^4$
  - $f^{(8)}(x) = 0$

Leibniz Notation

- x, y are physical quantities, $y = f(x)$
- (1^{st}) derivative $\frac{dy}{dx} = \frac{d}{dx}(y)$
- 2^{nd} derivative $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}(y)\right)$

- n^{th} derivative $\qquad \dfrac{d^n y}{dx^n} = \dfrac{d}{dx}\left[\dfrac{d}{dx}\left[\cdots \dfrac{d}{dx}(y) \cdots\right]\right]$

<u>Units</u>

- position	x	(m)
- time	t	(s)
- velocity	$\dfrac{dx}{dt}$	(m/s)
- acceleration	$\dfrac{d^2x}{dt^2}$	(m/s^2)

3.9 Continuous but Not Differentiable Functions

How May a Function be Non-Differentiable?

- f is differentiable $\Longrightarrow$ f is continuous
- f can be continuous and not differentiable
- f is continuous at a means

  $\lim_{x \to a} f(x) = f(a)$
- f is not differentiable at a means

  $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{DNE}$$

  - because the side limits are different (corner)
  - because the limit is $\pm\infty$ (vertical tangent)

Example: Corner

- Let $f(x) = |x|$

  - $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$
  - $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$
  - $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$

- The limit $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$

does not exist b/c the side limits are different

- The graph has a sharp edge - a corner

- f has a corner at $x = a$
    - f is continuous at a
    - the limit in $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ DNE

      b/c the side limits are different

Example: Vertical Tangent Line

- Let $g(x) = x^{\frac{1}{3}}$

- $g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x^{\frac{1}{3}}}{x} = \lim_{x \to 0} \frac{1}{x^{\frac{2}{3}}} = \infty$

- g has a vertical tangent line at $x = a$
    - g is continuous at a
    - $g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \pm \infty$

- Moreover, if $x \neq 0$, $g'(x) = \frac{1}{3x^{\frac{2}{3}}}$ and

$$\lim_{x \to 0} g'(x) = \lim_{x \to 0} \frac{1}{3x^{\frac{2}{3}}} = \infty$$

- if g is continuous at a,

$$\lim_{x \to a} g'(x) = \pm\infty \Rightarrow \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \pm\infty$$

- a stronger definition of vertical tangent line
    - g is continuous at a
    - $\lim_{x \to a} g'(x) = \pm\infty$

Caution

- Vertical tangent line $\neq$ vertical asymptote

3.10 The Chain Rule - Examples

- Ex. $\frac{d}{dx}\sqrt{x^3+1}$

$$= \frac{1}{2\sqrt{x^3+1}} \cdot \frac{d}{dx}(x^3+1) = \frac{3x^2}{2\sqrt{x^3+1}}$$

- Ex. $\frac{d}{dx}[\sin(6x)]$

$$= \cos(6x) \cdot \frac{d}{dx}(6x) = 6\cos(6x)$$

- Ex. $\frac{d}{dx}(x^2+1)^{100}$

$$= 100(x^2+1)^{99} \cdot \frac{d}{dx}(x^2+1) = 200x(x^2+1)^{99}$$

3.11 Chain Rule - The Theorem

- $(f \circ g)'(x) = f'(g(x))\, g'(x)$

- $(f \circ g)(x) = f(g(x)) = \sqrt{x^3+1}$ $\begin{cases} f(x) = \sqrt{x} \\ g(x) = x^3+1 \end{cases}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$ $\quad f'(g(x)) = \dfrac{1}{2\sqrt{x^3+1}}$ $\quad g'(x) = 3x^2$

By chain rule, $(f \circ g)'(x) = \dfrac{1}{2\sqrt{x^3+1}}(3x^2)$

Theorem (Chain Rule)

- Let $a \in \mathbb{R}$. Let f, g be functions

- If $\begin{cases} g \text{ is differentiable at } a \\ f \text{ is differentiable at } g(a) \end{cases}$

- Then $f \circ g$ is differentiable at a, and

$$(f \circ g)'(a) = f'(g(a))\, g'(a)$$

~~Proof~~. $(f \circ g)'(a) = \lim_{x \to a} \dfrac{f(g(x)) - f(g(a))}{x-a}$

$$= \lim_{x\to a} \left[\frac{f(g(x)) - f(g(a))}{\underbrace{g(x) - g(a)}_{0???}} \cdot \frac{g(x) - g(a)}{x-a} \right]$$

I need an extra condition

$\left\{ \begin{matrix} x \text{ close to } a \\ x \neq a \end{matrix} \right\} \Rightarrow g(x) \neq g(a)$

$$= \underbrace{\left[\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x)-g(a)}\right]}_{f'(g(a))?} \underbrace{\left[\lim_{x\to a} \frac{g(x)-g(a)}{x-a}\right]}_{g'(a)}$$

however, this is NOT always true.

So this proof is invalid.

- $f'(g(a)) = \lim_{u\to g(a)} \frac{f(u) - f(g(a))}{u-g(a)}$

Chain Rule in Leibnitz Notation

- $(f \circ g)'(x) = f'(g(x))\, g'(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$u = g(x)$, $y = f(u) = f(g(x))$

3.12 Derivatives of Trigonometric Functions

Derivatives of Trig Functions

- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = \sec^2 x$
- $\frac{d}{dx} \cot x = -\csc^2 x$
- $\frac{d}{dx} \sec x = \sec x \tan x$
- $\frac{d}{dx} \csc x = -\csc x \cot x$

Plan for Proofs

1. Obtain $\frac{d}{dx} \sin x = \ldots$ from the definition
2. Use $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ to obtain $\frac{d}{dx} \cos x = \ldots$
3. Obtain other derivatives from

$$\tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{etc.}$$

Step 1

$f(x) = \sin x$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{[\sin x \cos h + \cos x \sin h] - \sin x}{h} \rightarrow \text{constants}$$

$$= \lim_{h \to 0} \left[\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right]$$

$$= (\sin x) \Big[\underbrace{\lim_{h \to 0} \frac{\cosh - 1}{h}}_{0} \Big] + (\cos x) \Big[\underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{1} \Big]$$

$$= \cos x$$

Caution

- $\lim_{h \to 0} \frac{\sin h}{h} = 1$

  ↓ allows us to use

- $\frac{d}{dx}(\sin x) = \cos x$

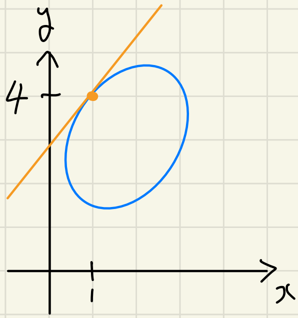
  ↓ allows us to use

- L'Hôpital's Rule, therefore

3.13 Implicit Differentiation

- Ex. Find the slope of the line tangent to the graph of

$3x^2+2y^2-4xy+2x-8y+11=0$ at $(1,4)$



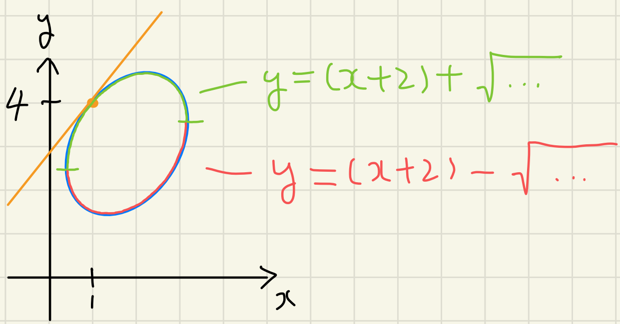
Method 1: Solve For y

$$3x^2+2y^2-4xy+2x-8y+11=0$$

$$2y^2+(-4x-8)y+(3x^2+2x+11)=0$$

$$y=\frac{(4x+8)\pm\sqrt{(-4x-8)^2-4\cdot 2\cdot(3x^2+2x+11)}}{2\cdot 2}$$

$$y=x+2\pm\sqrt{\frac{-x^2+6x-3}{2}}$$



Solution will be $f'(1)$ where $f(x) = x + 2 + \sqrt{\dfrac{-x^2+6x-3}{2}}$

...

<u>Method 2:</u> Think of $y = f(x)$ implicitly

$$\frac{d}{dx}\left[3x^2 + 2y^2 - \underbrace{4xy} + 2x - 8y + 11\right] = \frac{d}{dx}[0]$$

Note: $y' = \dfrac{dy}{dx}$

$$6x + \underbrace{4y \cdot y'} - \overbrace{4 \cdot 1 \cdot y - 4xy'} + 2 - 8y' = 0$$

think of y as a function and apply the chain rule

When $x = 1$, $y = 4$:

$$6 \cdot 1 + 4 \cdot 4y' - 4 \cdot 4 - 4 \cdot 1y' + 2 - 8y' = 0$$

$$-8 + 4y' = 0$$

$$y' = 2$$

4.1 What Is a Function?

Image | Invertible | $f(x) = x^2$

Domain | Range

Surjective | $\boxed{\text{FUNCTION}}$ | Inverse function

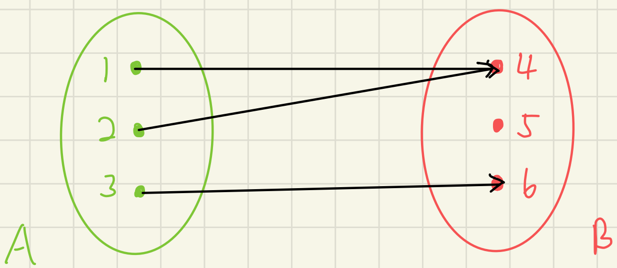
$f: A \to B$ | Codomain

Onto | One-to-one | Injective

Definition

A function f consists of

- a domain: set of inputs (A) $\{1, 2, 3\}$
- a codomain: set of potential outputs (B) $\{4, 5, 6\}$
- a rule that matches each input $(x \in A)$ to exactly one output $(f(x) \in B)$



- Name of function: f
- $x \in A$, $f(x) \in B$
- Codomain: set of potential outputs $\{4,5,6\}$
- Range: set of actual outputs $\{4,6\}$

Notation

$f: A \to B$ means

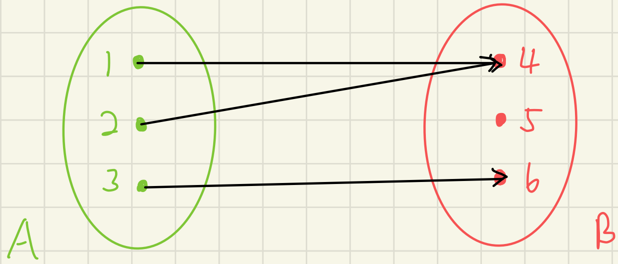
- f is the name of the function
- A is the domain
- B is the codomain
- Ex. $f: \mathbb{R} \to \mathbb{R}$

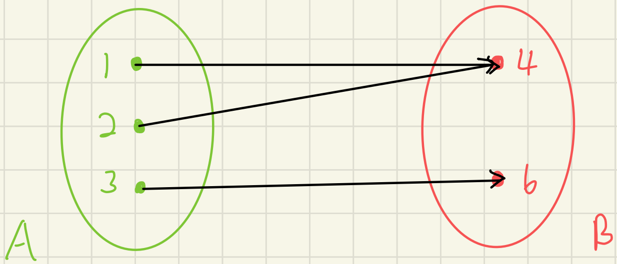
$$f(x) = 3 + x^2 + \sin x$$

Convention (in single-variable calculus)

- We may define a function with just the rule
- The domain is the largest subset of $\mathbb{R}$ possible
- The codomain is always $\mathbb{R}$
- Ex. $g(x) = \frac{1}{x^2}$
  - Domain $g = (-\infty, 0) \cup (0, \infty)$
  - Codomain $g = \mathbb{R}$

Are These Two the Same Function?





- Same domain, range, rule
- Different codomain
- Strictly speaking, they are different
- But in single-variable calculus, we can treat them as if they are the same

Warning

Mathmatics	Computer Science
Domain	Domain
Codomain	Range
Range	Image

4.2 Inverse Functions: The Theory

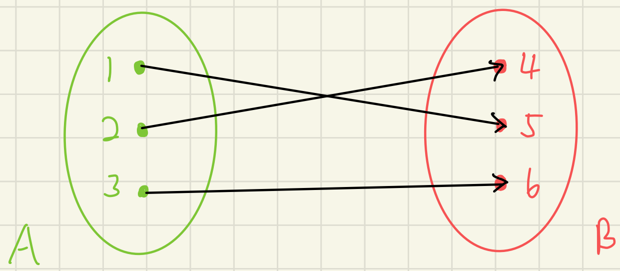
Intro

- A function f has an inverse if and only if...
  - Calc: f is injective
  - Lin. Alg.: f is injective and surjective

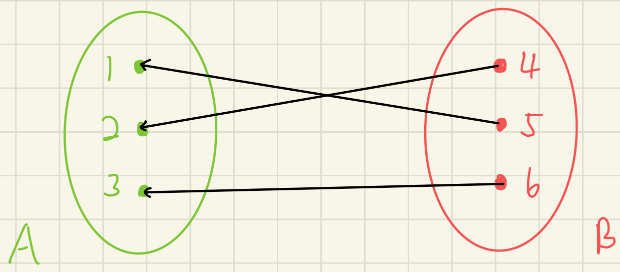
Idea

- Swap inputs and outputs

A function f



its inverse f^{-1}



Definition

Let $f : A \to B$ be a function.

The inverse of f is another function $f^{-1} : B \to A$

defined by $\forall x \in A, \forall y \in B, x = f^{-1}(y) \iff y = f(x)$

(assuming this is a function)

Definitions

Let $f : A \to B$ be a function.

- f is surjective or onto when $\text{Range } f = B$
- f is injective or one-to-one when

  $\forall x_1, x_2 \in A, \; x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

  Or, equivalently

  $\forall x_1, x_2 \in A, \; f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Theorem

$$f \text{ has an inverse} \iff \begin{cases} f \text{ is injective} \\ f \text{ is surjective} \end{cases}$$

Shortcut

- In calculus, we can forget about the parts of the codomain that are not in the range

- They're not doing anything

- We would not care about the codomain; we only care about the range

<u>Inverse Functions in Calculus</u>

Let f be a one-to-one function. Let

Domain $f = A$, Range $f = C$

Then the inverse f^{-1} is another function satisfying

Domain $f^{-1} = C$, Range $f^{-1} = A$ and defined by

$\forall x \in A, \forall y \in C, \ x = f^{-1}(y) \Leftrightarrow y = f(x)$

Or, equivalently

$$\begin{cases} \forall x \in A, \ f^{-1}(f(x)) = x \\ \forall y \in C, \ f(f^{-1}(y)) = y \end{cases}$$

4.3 Inverse Functions: Examples

- Ex. $f(x) = 2x+1$
  - Domain $f = \mathbb{R}$, Range $f = \mathbb{R}$
  - $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \ x = f^{-1}(y) \Leftrightarrow y = f(x)$
  - $y = 2x+1 \Leftrightarrow x = \frac{y-1}{2}$
  - $f^{-1}(y) = \frac{y-1}{2}$
  - $f^{-1}(f(x)) = \frac{f(x)-1}{2} = \frac{(2x+1)-1}{2} = x$
  - $f(f^{-1}(y)) = 2f^{-1}(y)+1 = 2\frac{y-1}{2}+1 = y$
- Ex. $E(x) = e^x$; $L(x) = \ln(x)$
  - For all $x \in \mathbb{R}$, $y > 0$

    $x = \ln y \Leftrightarrow y = e^x$
  - For all $x \in \mathbb{R}$, $\ln(e^x) = x$
  - For all $y > 0$, $e^{\ln y} = y$
- Ex. $f(x) = x^2$
  - $f(x) = x^2$ is not one-to-one

- i.e. $f(3) = f(-3)$

- Therefore, f does not have an inverse function
- We can try to construct something close to an inverse
    - We can restrict the function's domain while maintaining it's range so that the function becomes one-to-one
- $f(x) = x^2$ $\quad$ $g(x) = \sqrt{x}$
- g is not the inverse function of f
- g is the inverse function of the restriction of f to $[0, \infty)$
- $(-3)^2 = (3)^2 = 9$

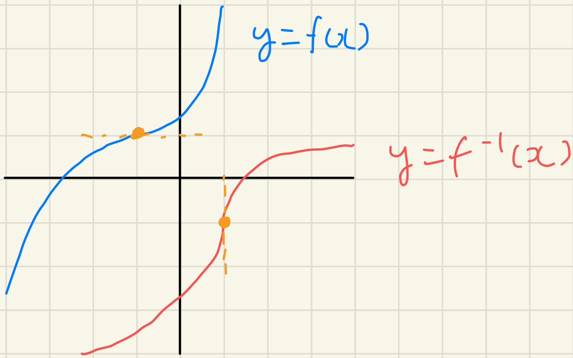
  $\sqrt{9} = 3$
- $g(x) = \sqrt{x}$ is the inverse function of $f(x) = x^2,\ x \geq 0$
- $$a = \sqrt{b} \iff \begin{cases} b = a^2 \\ a \geq 0 \end{cases}$$

$$-\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ |x| & \text{for all } x \in \mathbb{R} \end{cases}$$

4.4 Derivative of the Inverse of a Function

Questions

- If f is differentiable and has an inverse, is f^{-1} differentiable?
  - Not necessarily



- Theorem. Let f be a function defined on an interval I. If
  - f has an inverse
  - f is differentiable
  - for all $x \in I$, $f'(x) \neq 0$

  Then f^{-1} is differentiable
  - This is not the Inverse Function Theorem

- Can I write $(f^{-1})'$ in terms of f'?

  - Assume f, f^{-1} are differentiable

$$\frac{d}{dy}\left[f(f^{-1}(y))\right] = \frac{d}{dy}[y]$$

$$f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$

Call $f^{-1}(y) = x$ $\qquad y = f(x)$

$$f'(x) \cdot (f^{-1})'(y) = 1$$

$$(f^{-1})'(\underline{y}) = \frac{1}{f'(\underline{x})}$$

evaluated at different points

4.5 Derivative of Exponentials and the Number e

- $f(x) = a^x$

- $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{a^{x+h} - a^x}{h}$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x (a^h - 1)}{h}$$

$$= a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

- Call $L_a = \lim\limits_{h \to 0} \dfrac{a^h - 1}{h}$. $\dfrac{d}{dx} a^x = L_a a^x$

- Def. e is the only number s.t $L_a = 1$.

Equivalently, $\dfrac{d}{dx} e^x = e^x$

4.6 The Definition of Exponentials and Logarithms

Definition of Logarithms

- $\log_a x = y \iff a^y = x$
- i.e. $\log_2 8 = 3 \iff 2^3 = 8$

Question Regarding Exponentials

- How do we define a^c? $c \in \mathbb{R}$, $a > 0$

How do we define a^c for $c \in \mathbb{Q}$?

1. For $n \in \mathbb{Z}^+$, $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$

2. $a^{1/n} = \sqrt[n]{a}$ is defined by $(\sqrt[n]{a})^n = a$

3. For $n, m \in \mathbb{Z}^+$, $a^{n/m} = [a^n]^{1/m}$

4. $a^0 = 1$, $a^{-c} = \dfrac{1}{a^c}$

The Classical Analysis Solution to Define 2^π

- $\pi = 3.14159265358979323846264338 3\ldots$
- Define 2^π as the limit of the sequence:

$$2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, \ldots$$

- Problems

    - Does this limit exist?

    - What if I use a different sequence?

The Modern Analysis Solution

1. Define from scratch one of these two functions:

    - $E(x) = e^x$

    - $L(x) = \ln x$

2. Define the other function as its inverse:

    - $y = e^x \Leftrightarrow x = \ln y$

3. Define other exponentials as $a^c = e^{c \ln a}$

Option A: Differential Equations

- Define $E(x) = e^x$ as the only function that satisfies

$$\begin{cases} E'(x) = E(x) \\ E(0) = 1 \end{cases}$$

- Use Picard-Lindelöf Theorem

Option B: Power Series

- Define

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Option C: Using Integrals

- Define $\ln x = \int_1^x \frac{1}{t}\,dt$

- Use Fundamental Theorem of Calculus

4.7 Derivative of Logarithm

- We know $\frac{d}{dx} e^x = e^x$ $\qquad \frac{d}{dx} \ln x = ?$

- $\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x \quad$ for $x > 0$

$$e^{\ln x} \cdot \frac{d}{dx} \ln x = 1$$

$$x \cdot \frac{d}{dx} \ln x = 1$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

4.8 Derivative of Other Exponentials

- We know $\frac{d}{dx} e^x = e^x$. $\frac{d}{dx} a^x = ?$ $(a > 0)$

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}[x \ln a]$$

$$= a^x \cdot \ln a$$

- We know $\frac{d}{dx} \ln x = \frac{1}{x}$. $\frac{d}{dx} \log_a x = ?$ $(a > 0)$

$$\log_a x = \frac{\ln x}{\ln a} = \frac{1}{\ln a} \cdot \ln x$$

$$\frac{d}{dx} \log_a x = \frac{1}{\ln a} \cdot \frac{d}{dx} \ln x = \frac{1}{\ln a} \cdot \frac{1}{x}$$

$$= \frac{1}{x \ln a}$$

4.9 Logarithmic Differentiation

- Ex. $f(x) = (\cos x)^{\sin x}$ $\quad f'(x) = ?$

- Warning: f is not just a power or exponential
    - Power:
        - $\frac{d}{dx} x^3 = 3x^2$
        - $\frac{d}{dx} \cos^3 x = 3\cos^2 x(-\sin x)$
    - Exponential:
        - $\frac{d}{dx} 2^x = 2^x \cdot \ln 2$
        - $\frac{d}{dx} 2^{\sin x} = 2^{\sin x} \cdot \ln 2 \cdot \cos x$

- Method 1: Use $a^b = e^{b \ln a}$
    - $f(x) = (\cos x)^{\sin x} = e^{\sin x \cdot \ln(\cos x)}$
    - $f'(x) = e^{\sin x \cdot \ln(\cos x)} \cdot \frac{d}{dx}[\sin x \cdot \ln(\cos x)]$

$$= (\cos x)^{\sin x} \cdot \left[\cos x \cdot \ln(\cos x) + \sin x \cdot \frac{-\sin x}{\cos x}\right]$$

- Method 2: Logarithmic differentiation
    - $\ln f(x) = \ln (\cos x)^{\sin x}$
    - $\frac{d}{dx}[\ln f(x)] = \frac{d}{dx}[(\sin x) \ln(\cos x)]$

$$\frac{1}{f(x)} \cdot f'(x) = (\cos x)\ln(\cos x) + \sin x \frac{-\sin x}{\cos x}$$

$$f'(x) = f(x)\left[\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right]$$

4.10 Proof of the Power Rule

$$\frac{d}{dx}[x^c] = \frac{d}{dx}\left[e^{c\ln x}\right]$$

$$= e^{c\ln x}\frac{d}{dx}[c\ln x]$$

$$= x^c\left[\frac{c}{x}\right]$$

$$= cx^{c-1}$$

4.11 LN or LOG ?

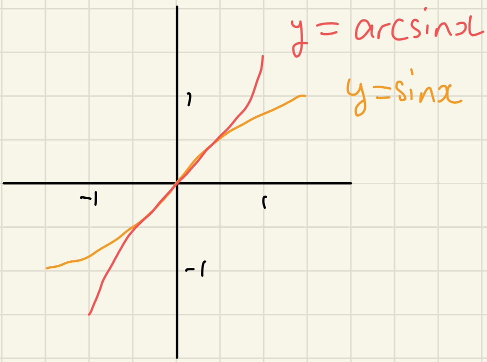
- $\log_e x = \ln x$
  - Natural logarithm
- $\log x = \log_{10} x$
  - Common logarithm
  - Decimal logarithm
- Current situation
  - $\ln x = \log_e x$
    - Used in science, engineering, calculus
  - $\log x = \log_e x$
    - Used in math

4.12 The Arcsin Function

- Arcsin is not the inverse function of sin
- Sin is not one-to-one, therefore, it does not have an inverse function

Definition

arcsin is the inverse function of the restriction of sin to $[-\pi/2, \pi/2]$



domain(sin) $= [-\pi/2, \pi/2]$

range(sin) $= [-1, 1]$

domain(arcsin) $= [-1, 1]$

range(arcsin) $= [-\pi/2, \pi/2]$

- $x = \arcsin y \iff y = \sin x$

  for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $-1 \le y \le 1$

- Ex. What is $\arcsin \frac{1}{2}$?
  - $\arcsin \frac{1}{2} = t$ means

$$\begin{cases} \sin t = \frac{1}{2} \\ -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \end{cases}$$

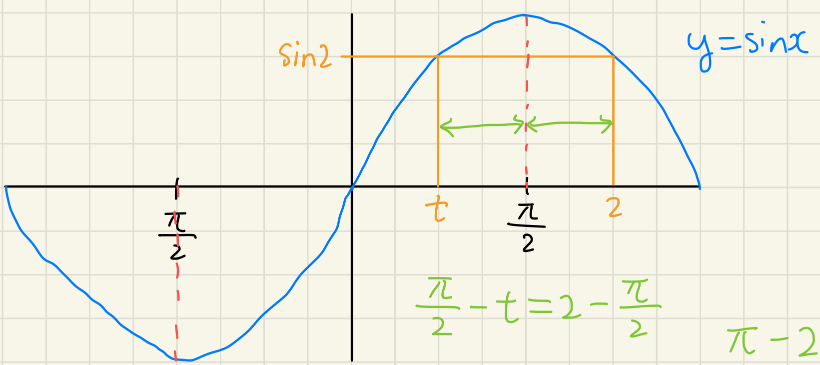
- $\arcsin \frac{1}{2} = \frac{\pi}{6}$

Composition of sin and arcsin

- $\arcsin(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
- $\sin(\arcsin y) = y$ for $-1 \leq y \leq 1$
- The above is not true for other numbers
    - $\sin(\arcsin 2)$ is undefined
    - $\arcsin(\sin 2) = ???$

Evaluate $\arcsin(\sin 2)$

$$\arcsin(\sin 2) = t \iff \begin{cases} \sin t = \sin 2 \\ -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \end{cases}$$



$$\frac{\pi}{2} - t = 2 - \frac{\pi}{2}$$

$$\pi - 2 = t$$

$$\arcsin(\sin 2) = \pi - 2$$

4.13 The Derivative of Arcsin

$-1 \leq x \leq 1$, $\frac{d}{dx}[\sin(\arcsin x)] = \frac{d}{dx}[x]$

$[\cos(\arcsin x)] \cdot \frac{d}{dx}(\arcsin x) = 1$

$\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin x)}$

- What is $\cos(\arcsin x)$? $\quad -1 \leq x \leq 1$
  - Call $\theta = \arcsin x$ $\quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

    $\begin{cases} \text{Know: } \sin\theta = x \\ \text{Want: } \cos\theta \end{cases}$ $\quad \cos\theta \geq 0$

  - $\cos^2\theta = 1 - \sin^2\theta$

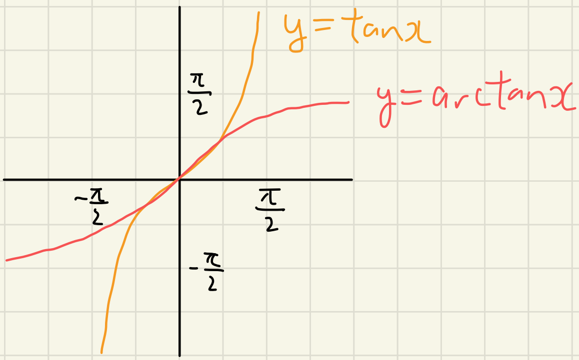
    $\cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\sqrt{1-x^2}$

$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

4.14 Arctan and Arccos

Definition of arctan

arctan is the inverse function of the restriction of tan to $(-\pi/2, \pi/2)$

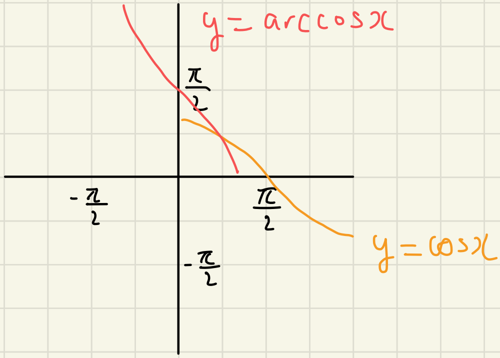


- $x = \arctan y \iff y = \tan x$

  for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $y \in \mathbb{R}$

Definition of arccos

arccos is the inverse function of the restriction of cos to $[0, \pi]$



- $x = \arccos y \iff y = \cos x$

  for $0 \leq x \leq \pi$, $-1 \leq y \leq 1$

<u>Derivatives</u>

- $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

5.1 The Mean Value Theorem

Goal of the MVT

- Use f' to obtain information about f
- Theorem. Let I be an open interval.

  Let f be a function defined on I.

  If $\forall x \in I$, $f'(x) = 0$ Then f is constant

  - Without the MVT, we do not know how to prove this simple theorem (or any other theorem about application of derivative)

Some Applications of the MVT

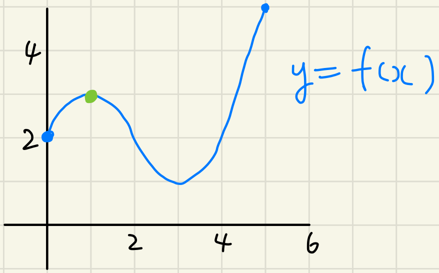
- A function w/ zero derivative must be constant
- A function w/ positive derivative must be increasing
- All integration methods
- L'Hôpital's Rule
- The Fundamental Theorem of Calculus
- Taylor's theorem
- $\partial_x \partial_y f(x,y) = \partial_y \partial_x f(x,y)$

- etc.

Steps Toward the MVT

- Extreme Value Theorem
- Local Extreme Value Theorem
- Rolle's Theorem
- Mean Value Theorem

5.2 The Local Extreme Value Theorem



Domain $f = [0, 5]$

- f has a.
  - minimum of 1 (at $x = 3$)
  - maximum of 5 (at $x = 5$)
  - local maximum at $x = 1$

Definitions

- Let f be a function w/ domain I
- Let $c \in I$
- We say that f has a maximum at c when

  $\forall x \in I, \ f(x) \leq f(c)$
- We say that f has a local maximum at c when

  $\exists \delta > 0$ s.t. $|x - c| < \delta \Rightarrow f(x) \leq f(c)$

Notes

- "Extremum" means "maximum or minimum"
- "Global extremum" means "extremum"
- The plural of extremum / maximum / minimum is

  extrema / maxima / minima.
- According to this definition,

  endpoints do not count as local extrema.

<u>The Local Extreme Value Theorem</u>

- Let f be a function w/ domain an interval I.

  Let $c \in I$.

If

- f has a local extremum at c
- c is an interior point to I (not an end-point)

Then

- $f'(c) = 0$ or DNE

<u>Critical Point</u>

C is called a critical point of f when

- c is an interior point of the domain of f, and
- $f'(c) = 0$ or DNE

5.3 Proof of the Local Extreme Value Theorem

Proof. Assume f has a local maximum at c.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

WTS this limit is 0 or DNE.

I will assume this limit exists, and I will prove it is 0.

- As $x \to c^+$,

$x - c > 0$

$f(x) - f(c) \leq 0$

$$\frac{f(x) - f(c)}{x - c} \leq 0$$

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

- As $x \to c^-$

$x - c < 0$

$f(x) - f(c) \leq 0$

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0$$

For the two side limits to exist and be equal, they must be 0. $\square$

5.4. Finding the Local Max. and Min. of a Function

Ex. Find the extrema (and local extrema) of

$f(x) = x^3 - 3x^2 - 9x + 35$ on the interval $[4, 4]$

1. f is continuous on $[-4, 4]$.

   By EVT, it has a max and a min.

2. It could happen...

   - at an end point
   - at an interior point
     - Then it's a local extremum
     - By Local EVT, it is a critical point

<u>Plan to Find Extrema</u> (f continuous at $[a, b]$)

1. Find endpoints and critical points

   ($f'(c) = 0$ or DNE)

2. Compare f at these points

3. One of them is the max,

   one of them is the min

<u>Calculation</u>

- Endpoints: $x=-4$ and $x=4$

$$f'(x) = 3x^2 - 6x - 9$$
$$= 3(x^2 - 2x - 3)$$
$$= 3(x+1)(x-3)$$

- Critical points: $x=-1$ and $x=3$

- Candidate points

x	$f(x)$		
-4	-41	min	
-1	40	max	local max
3	8		local min
4	15		

5.5 Rolle's Theorem

Rolle's Theorem

Let $a < b$. Let f be a function defined on $[a,b]$.

If

1. f is continuous on $[a,b]$
2. f is differentiable on (a,b)
3. $f(a) = f(b)$

Then

$\exists c \in (a,b)$ s.t. $f'(c) = 0$

Proof

- f is continuous on $[a,b]$
- By EVT, f must have a maximum and a minimum on $[a,b]$
- Case 1: If f has a maximum or minimum at some point $c \in (a,b)$, then it is a local extremum
    - By Local EVT, $f'(c) = 0$ or DNE

- Since f is differentiable at c, $f'(c) = 0$

- Case 2: If f has a maximum and minimum at endpoints, since $f(a) = f(b)$, f must be constant
  - Then $\forall x \in (a,b)$, $f'(x) = 0$ $\square$

Additional Info

- We used all hypotheses
- We used EVT and Local EVT

5.6 An Application of Rolle's Theorem

Definition. A number $c \in \mathbb{R}$ is a zero of a function f when $f(c) = 0$

- zero of f = solution of $f(x) = 0$

- How many zeroes does a function have?
  1. Use IVT to prove it has at least n
  2. Use Rolle's Theorem to prove it has at most n

- Ex. How many zeroes does the function $g(x) = x^6 + x^2 + x - 2$ have?
  - g is continuous on $\mathbb{R}$
  - $g(-2) = 64$, $g(0) = -2$, $g(1) = 1$
  - By IVT, g has at least 2 zeroes
    - $-2 < x_1 < 0$ s.t. $g(x_1) = 0$
    - $0 < x_2 < 1$ s.t. $g(x_2) = 0$
  - Side note: assume f is continuous and differentiable everywhere
    - Assume $f(x_1) = f(x_2) = 0$

- Use Rolle's theorem for f on $[x_1, x_2]$
- There exists $x_1 < a < x_2$ s.t. $f'(a) = 0$

- Conclusion 1: Between any 2 zeroes of f, there must be at least 1 zero of f'
  - Assume $x_1 < x_2 < x_3$ are zeroes of f
  - Then there exists $x_1 < a < x_2$ s.t. $f'(a) = 0$
  - and there exists $x_2 < b < x_3$ s.t. $f'(b) = 0$

- Conclusion 2:

  \# of zeroes of f' $\geq$ \# of zeroes of $f - 1$

- Theorem. (*) If f is continuous and differentiable on an interval,

  \# of zeroes of f $\leq$ \# of zeroes of $f' + 1$

- $g'(x) = 6x^5 + 2x + 1$
- $g''(x) = 30x^4 + 2$. It is always positive.
- g'' has no zeroes.
- Using Theorem (*):

$\#$ of zeroes of $g' \leq \#$ of zeroes of $g'' + 1 = 1$

- Using Theorem $(*)$:

  $\#$ of zeroes of $g \leq \#$ of zeroes of $g' + 1 = 2$

Summary

- IVT: g has at least 2 zeroes
- Rolle's: g has at most 2 zeroes
- Therefore, g has exactly 2 zeroes

5.7 The Mean Value Theorem

The Mean Value Theorem

Let $a < b$. Let f be a function defined on $[a,b]$.

If

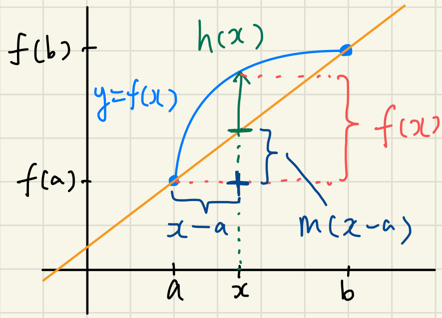
- f is continuous on $[a,b]$
- f is differentiable on (a,b)

Then

- $\exists c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

5.8 Proof of the Mean Value Theorem

Idea



$$m = \frac{f(b) - f(a)}{b - a}$$

Use Rolle's on $h(x) = f(x) - f(a) - m(x-a)$

The Mean Value Theorem

Let $a < b$. Let f be a function defined on $[a,b]$.

If

- f is continuous on $[a,b]$
- f is differentiable on (a,b)

Then

- $\exists c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Proof

- Let $m = \dfrac{f(b) - f(a)}{b - a}$. Define a new function h on $[a,b]$

by $h(x) = f(x) - f(a) - m(x-a)$

- Apply Rolle's Theorem to h on $[a,b]$. First, verify the hypotheses:
  - h is continuous on $[a,b]$, because so is f.
  - h is differentiable on (a,b), because so is f.
  - $h(a) = 0$ and $h(b) = 0$
- Therefore, by Rolle's, $\exists c \in (a,b)$ s.t. $h'(c) = 0$.
- $h'(x) = f'(x) - m$. Therefore,

$$f'(c) = m = \frac{f(b) - f(a)}{b - a}$$

$\square$

5.9 Zero Derivative Implies Constant

<u>Theorem</u>

Let $a < b$. Let f be a function defined on $[a, b]$.

If

- $\forall x \in (a,b)$, $f'(x) = 0$
- f is continuous on $[a, b]$

Then f is constant on $[a, b]$

<u>Proof</u>

- WTS $\forall x_1, x_2 \in [a, b]$, $f(x_1) = f(x_2)$
- Fix $x_1, x_2 \in [a, b]$. Assume $x_1 < x_2$.

  I want to use MVT for f on $[x_1, x_2]$.
- First, verify the hypotheses:
  - f continuous on $[a, b]$, so

    continuous on $[x_1, x_2] \subseteq [a, b]$.
  - f differentiable on (a, b), so

    differentiable on $(x_1, x_2) \subseteq (a, b)$
- Thus, by MVT, $\exists c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- $f'(c) = 0$. So $f(x_1) = f(x_2)$, as we wanted. $\square$

Variations

Theorem. Let $a < b$. Let f be a function defined on (a,b).

If $\forall x \in (a,b)$, $f'(x) = 0$

Then f is constant on (a,b).

Exercise

Prove that there exists a constant C such that

$$\arctan\sqrt{\frac{1-x}{1+x}} = C - \frac{1}{2}\arcsin x.$$

What is C? For which values of x is this true?

- Let $F(x) = \arctan\sqrt{\frac{1-x}{1+x}} + \frac{1}{2}\arcsin x$

- Domain $F = (-1, 1]$

- $\forall x \in (-1,1)$, $F'(x) = 0$

- Therefore, $\exists C \in \mathbb{R}$ s.t. for every $x \in (-1,1)$, $F(x) = C$

- F is continuous on $(-1,1]$, so for every $x \in (-1,1]$, $F(x) = C$

- $C = F(0) = \arctan 1 + \frac{1}{2}\arcsin 0 = \frac{\pi}{4}$

- Conclusion: $\forall x \in (-1,1]$, $\arctan\sqrt{\frac{1-x}{1+x}} = \frac{\pi}{4} - \frac{1}{2}\arcsin x$.

5.10 Why Integration is Possible ?

Ex. Find all functions f s.t. $\forall x \in \mathbb{R}$, $f'(x) = x^2$

- By guessing, $f(x) = \frac{1}{3}x^3$ is a solution
- For any constant C, $f(x) = \frac{1}{3}x^3 + C$ is also a solution
- There can be no other solutions

<u>Claim</u>

If f satisfies $\forall x \in \mathbb{R}$, $f'(x) = x^2$

Then there exists $C \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}$, $f(x) = \frac{1}{3}x^3 + C$

1. Theorem
   - If h has zero derivative on an open interval I, then h is constant on I
2. Corollary
   - If f and g have same derivative on an open interval I

     Then $f-g$ is constant on I

     Proof : $f-g$ has zero derivative

3. Proof of the claim

- Assume $f'(x) = x^2$ for all $x \in \mathbb{R}$

  Then $\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[\frac{1}{3}x^3\right]$

- Therefore $f(x) - \frac{1}{3}x^3 = C$

$\square$

<u>Conclusion</u>

- W/o the MVT, we could only say that

  $f(x) = \frac{1}{3}x^3 + C$ are some solutions to $f'(x) = x^2$

- W/ the MVT, we can say that

  $f(x) = \frac{1}{3}x^3 + C$ are all the solutions to $f'(x) = x^2$

- All integration methods are based on this

5.11 Monotonicity of Functions

Definition

Let f be a function defined on an interval I

- f is increasing on I when

  $\forall x_1, x_2 \in I, \ x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$

- f is non-decreasing on I when

  $\forall x_1, x_2 \in I, \ x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2)$

- Warning

Some books	Other books
increasing	strictly increasing
non-decreasing	increasing

Theorem

Let $a < b$. Let f be a function defined on (a, b)

- If $\forall x \in (a, b)$, $f'(x) > 0$
- Then f is increasing on (a, b)

Theorem

Let $a < b$. Let f be a function defined on $[a, b]$

- If
  - $\forall x \in (a,b)$, $f'(x) > 0$, and
  - f is continuous on $[a,b]$
- Then f is increasing on $[a,b]$

Summary

- On an open interval: $f' = 0 \Rightarrow f$ constant
- On an open interval: $f' > 0 \Rightarrow f$ increasing
- On an open interval: $f' < 0 \Rightarrow f$ decreasing
- At a point: if $f'(x) = 0$ or DNE...
  it could be anything

5.12 Finding the Intervals of Increasing / Decreasing

Ex. Find the intervals where $f(x) = 8x^5 + 5x^4 - 20x^3$ is increasing / decreasing

- $f'(x) = 40x^4 + 20x^3 - 60x^2 = 20x^2(2x^2 + x - 3)$

  $= 20x^2(2x+3)(x-1)$

- $f'(x) = 0 \iff x = 0$ or $-\frac{3}{2}$ or 1

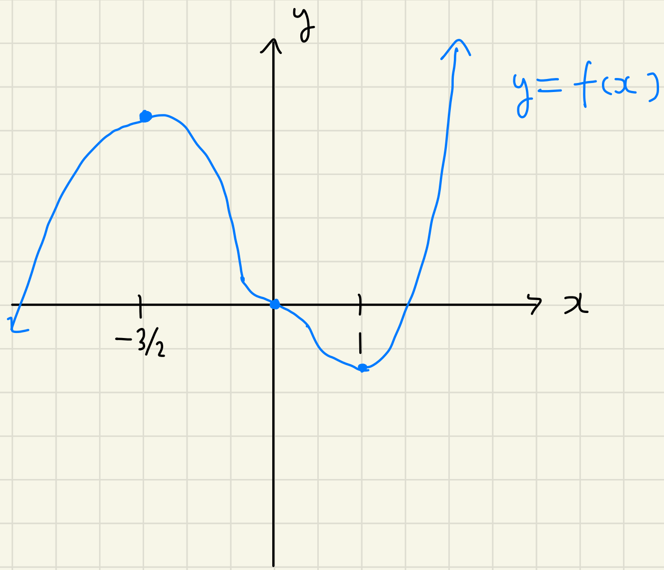
-

x		$-3/2$		0		1	
$20x^2$	$+$		$+$	0	$+$		$+$
$2x+3$	$-$	0	$+$		$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$-$	0	$+$
f	$\nearrow$	$\frown$	$\searrow$	$\hookrightarrow$	$\searrow$	$\smile$	$\nearrow$
		Local Max				Local Min	

- f increasing on $(-\infty, -3/2]$
  - Continuous on $-3/2$
- f decreasing on $[-3/2, 1]$
- f increasing on $[1, \infty)$

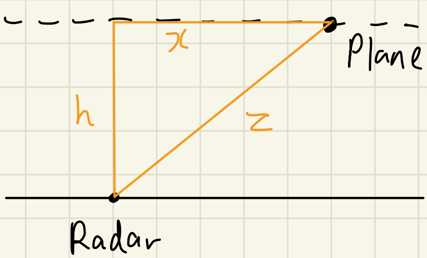
- 1 is in both $[-3/2, 1]$ and $[1, \infty]$, and this is fine

- increasing / decreasing is for intervals



6.1 Related Rates: Example 1

Modelling



- $h = 10\,\text{km}$ (constant)
- x, z depend on t

- Want $\dfrac{dx}{dt}$ when $\begin{cases} z = 20\,\text{km} \\ \dfrac{dz}{dt} = 1000\,\text{km/h} \end{cases}$

- $z^2 = x^2 + h^2$

Calculus

$$\frac{d}{dt} z^2 = \frac{d}{dt}\left[h^2 + x^2\right]$$

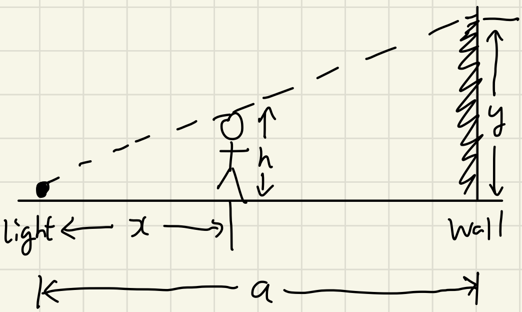
$$2z \frac{dz}{dt} = 0 + 2x \frac{dx}{dt} \qquad x = \sqrt{z^2 - h^2}$$

$$\frac{dx}{dt} = \frac{z}{x} \frac{dz}{dt} = \frac{20\,\text{km}}{\sqrt{20^2 - 10^2}\,\text{km}} \cdot 100\,\frac{\text{km}}{\text{h}}$$

$$= \frac{2000}{\sqrt{3}}\,\frac{\text{km}}{\text{h}}$$

6.2 Related Rates: Example 2

Modelling



- $a = 30m$ } constant
- $h = 1.8m$ }
- x, y depend on t

- want $\frac{dy}{dt}$ when $\begin{cases} x = 10m \\ \frac{dx}{dt} = 8 m/s \end{cases}$

- $\frac{h}{x} = \frac{y}{a}$ (by similar triangles)

Calculus

$$\frac{d}{dt}\frac{h}{x} = \frac{d}{dt}\frac{y}{a}$$

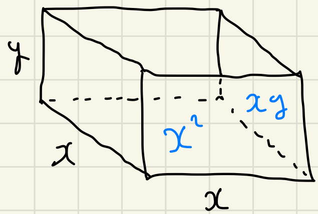
$$h\left(\frac{-1}{x^2}\right)\frac{dx}{dt} = \frac{1}{a}\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{-ah}{x^2}\frac{dx}{dt} = -\frac{(30m)(1.8m)}{(10m)^2}\cdot\left(8\frac{m}{s}\right)$$

$$= -4.32\ \frac{m}{s}$$

6.3 Applied Optimization: Example 1

Modelling



Want to minimize total area

$$S = x^2 + 4xy$$

Subject to $V = 500 = x^2 y$, $\begin{cases} x > 0 \\ y > 0 \end{cases}$ $\rightarrow y = \dfrac{500}{x^2}$

Goal: find the minimum of

$$S = x^2 + 4x\frac{500}{x^2} = x^2 + \frac{2000}{x} \quad \text{for } x > 0$$

Calculus

$$\frac{dS}{dx} = 2x - \frac{2000}{x^2} = \frac{2(x^3 - 1000)}{x^2}$$

$$\frac{dS}{dx} = 0 \iff x = 10\,\text{cm}$$

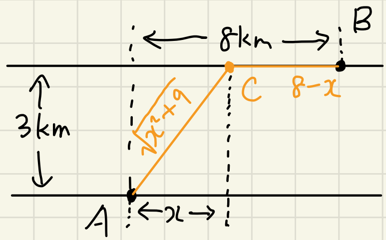
x	0		10	
$\frac{dS}{dx}$		$-$	0	$+$

$S \quad \searrow \; \cup \; \nearrow$

$$\begin{cases} x = 10\,\text{cm} \\ y = 5\,\text{cm} \end{cases}$$

6.4 Applied Optimization : Example 2

Modelling



$V_{row} = 6\,km/h$

$V_{run} = 8\,km/h$

total time = $(\text{time}_{A\to C}) + (\text{time}_{C\to B})$

$$T = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}, \quad 0 \le x \le 8$$

Need to find minimum of T

Calculus

By EVT, T must have a minimum

- It must be $\begin{cases} \text{a critical point} \\ \text{or an endpoint} \end{cases}$

$$\frac{dT}{dx} = \frac{1}{6}\frac{1}{2\sqrt{x^2+9}} \cdot 2x - \frac{1}{8} \qquad \text{(some algebra)}$$

$$\frac{dT}{dx} = 0 \iff x = \frac{9}{\sqrt{7}}\ \text{km}$$

	x	T
(end pt)	0 km	1.5 h
(crit. pt)	$9/\sqrt{7}$ km	1.33 h
(end pt)	8 km	1.42 h

6.5 Indeterminate Forms

Motivation

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x} = \frac{0}{0} \; ?$$

If $\begin{cases} \lim_{x\to a} f(x) = 2 \\ \lim_{x\to a} g(x) = 3 \end{cases}$ Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{2}{3}$

If $\begin{cases} \lim_{x\to a} f(x) = 1 \\ \lim_{x\to a} g(x) = \infty \end{cases}$ Then $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$

If $\begin{cases} \lim_{x\to a} f(x) = 0 \\ \lim_{x\to a} g(x) = 0 \end{cases}$ Then Can't draw conclusions about $\lim_{x\to a} \frac{f(x)}{g(x)}$

Examples

$$\lim_{x\to 0} \frac{2x}{x} = 2, \quad \lim_{x\to 0} \frac{x^2}{x} = 0, \quad \lim_{x\to 0} \frac{x}{x^3} = \infty$$

We say that $\frac{0}{0}$ is a (limit) indeterminate form

Remarks

1. We are only talking about limits. We are not actually dividing 0 by 0

2. This does not mean $\lim_{x \to a} \frac{f(x)}{g(x)}$ is undefined or DNE

3. The indeterminate form arises because of a tension:

   - f wants to make the limit of the quotient 0
   - g wants to make the limit of the quotient $\pm\infty$

Common Indeterminate Forms

$\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, $1^{\pm\infty}$

Useful tool for (some) indeterminate forms:

L'Hôpital's Rule

What About 1/0?

Assume $\begin{cases} \lim_{x \to a} f(x) = 1 \\ \lim_{x \to a} g(x) = 0 \end{cases}$

- We can conclude $\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = \infty$
- If $g(x) > 0$ as $x \to a$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \infty$
- If $g(x) < 0$ as $x \to a$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = -\infty$

$$\lim_{x \to 3^+} \frac{1}{\underbrace{\underbrace{(x-3)}_{\text{pos.}}\underbrace{(2-x)}_{\text{neg.}}}_{\text{neg.}}} = -\infty$$

6.6 L'Hôpital's Rule: The Theorem

L'Hôpital's Theorem

Let f, g be functions. Let $a \in \mathbb{R}$. I want to compute $\lim_{x \to a} \frac{f(x)}{g(x)}$.

- The theorem also works for $a = \infty$, $a = -\infty$, or side limits

If

1. f and g are differentiable as $x \to a$
2. g and g' are never 0 as $x \to a$
3. The limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an interdeterminate form of type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$
4. The limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or is ∞ or $-\infty$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

- Condition (4) means

  - if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$
  - if $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \infty$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \infty$ as well
  - if $\lim_{x \to a} \frac{f'(x)}{g'(x)} = -\infty$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = -\infty$ as well
  - if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ DNE for a different reason,

    then $\lim_{x \to a} \frac{f(x)}{g(x)}$ may or may not exist

6.7 L'Hôpital's Rule: Examples

Ex. $\lim_{x \to \infty} \frac{x}{\ln x}$ $\quad \left(\frac{\infty}{\infty}\right)$

$$\overset{(L'H)}{=} \lim_{x \to \infty} \frac{1}{1/x} = \lim_{x \to \infty} x = \infty$$

Ex $\lim_{x \to 0} \frac{\cos x - \cos(2x)}{xe^x - x}$ $\quad \left(\frac{0}{0}\right)$

$$\overset{(L'H)}{=} \lim_{x \to 0} \frac{-\sin x + 2\sin(2x)}{e^x + xe^x + 1} \quad \left(\frac{0}{0}\right)$$

$$\overset{(L'H)}{=} \lim_{x \to 0} \frac{-\cos x + 4\cos(2x)}{e^x + e^x + xe^x} = \frac{3}{2}$$

Ex. $\lim_{x \to 1} \frac{x^3 - 2x + 1}{x^2 + 3x + 2} = \frac{0}{6} = 0$

- Since we don't have an indeterminate form, we cannot use L'Hôpital's rule

6.8 When L'Hôpital's Goes Wrong

Ex. $\lim_{x\to\infty} \frac{x+\sin x}{2x+\cos x}$ $\quad \left(\frac{\infty}{\infty}\right)$

$$\left.\begin{aligned} x+\sin x &\geq x-1 \\ \lim_{x\to\infty}(x-1) &= \infty \end{aligned}\right\} \lim_{x\to\infty}(x+\sin x) = \infty$$

Same goes for the denominator

$$(\text{L'H}) = \lim_{x\to\infty} \frac{1+\cos x}{2-\sin x} \quad \text{DNE}$$

(may not accurately reflect the function)

(Periodic and non-constant)

- Recall the 4th assumption of L'Hôpital's Rule

4. The limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists or is ∞ or $-\infty$

And so L'Hôpital's Rule does not apply in this case.

Ex. $\lim_{x\to\infty} \dfrac{x+\sin x}{2x+\cos x}$ (without using L'Hôpital's Rule)

$$= \lim_{x\to\infty} \frac{1 + \frac{\sin x}{x} \to 0}{2 + \frac{\cos x}{x} \to 0} = \frac{1}{2}$$

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$-\frac{1}{x} \to 0$, $\frac{1}{x} \to 0$

By Squeeze Theorem

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

Same goes for $\dfrac{\cos x}{x}$

6.9 Indeterminate Form: Zero Times Infinity

Ex. $\lim_{x\to\infty} x \cdot [1 - \underbrace{e^{2/x}}_{1}]$ $\quad (\infty \cdot 0)$

$$= \lim_{x\to\infty} \frac{1 - e^{2/x}}{1/x} \qquad \left(\frac{0}{0}\right)$$

$$(\text{L'H}) = \lim_{x\to\infty} \frac{-e^{2/x} \cdot [-2/x^2]}{-1/x^2}$$

$$= \lim_{x\to\infty} [-2e^{2/x}] = -2$$

6.10 Indeterminate Form: $\infty - \infty$

Ex. $\lim_{x\to\infty}\left[\sqrt{x^2-x}-x\right]$ $\quad(\infty-\infty)$

Method 1

$$= \lim_{x\to\infty}\left[\sqrt{x^2}\sqrt{1-1/x}-x\right] \qquad \sqrt{x^2}=|x|=x$$

$$= \lim_{x\to\infty} x\left[\sqrt{1-1/x}-1\right] \qquad (\infty\cdot 0)$$

$$= \lim_{x\to\infty}\frac{\sqrt{1-1/x}-1}{1/x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{1-1/x}}\cdot\frac{1}{x^2}}{-1/x^2} = \lim_{x\to\infty}\frac{-1}{2\sqrt{1-1/x}} = \frac{-1}{2}$$

Method 2

$$= \lim_{x\to\infty}\frac{\left[\sqrt{x^2-x}-x\right]\left[\sqrt{x^2-x}+x\right]}{\sqrt{x^2-x}+x}$$

$$= \lim_{x\to\infty}\frac{x^2-x-x^2}{\sqrt{x^2-x}+x} = \lim_{x\to\infty}\frac{-x}{\sqrt{x^2-x}+x}$$

$$= \lim_{x\to\infty}\frac{-x}{x\left[\sqrt{1-1/x}+1\right]} = \lim_{x\to\infty}\frac{-1}{\sqrt{1-1/x}+1} = \frac{-1}{2}$$

6.11 Why Is 1^∞ an Indeterminate Form?

Intro

"1^∞ is an indeterminate form" means

- If $\begin{cases} \lim_{x \to a} f(x) = 1 \\ \lim_{x \to a} g(x) = \infty \end{cases}$

- Then we cannot draw a conclusion about

$$\lim_{x \to a} f(x)^{g(x)}$$

- We are raising numbers close to 1 to ∞

Example

Let $c \in \mathbb{R}$

$$\lim_{x \to 0^+} \left(\overset{0}{e^x}\right)^{\overset{\infty}{\frac{c}{x}}} \qquad (1^{\pm\infty})$$

$$= \lim_{x \to 0^+} \left[e^{x \cdot \frac{c}{x}} \right] = \lim_{x \to 0^+} e^c = e^c$$

c could be any number (other than 0), which justifies that $1^{\pm\infty}$ is an indeterminate form

6.12 Indeterminate Forms: Exponential Type

$$0^0, \quad \infty^0, \quad 1^{\pm\infty}$$

Ex. $\lim_{x \to 0^+} (1-x)^{1/x}$ $\qquad (1^\infty)$

Call $f(x) = (1-x)^{1/x}$

$\ln f(x) = \ln (1-x)^{1/x} = \frac{1}{x} \ln(1-x)$

1) $\lim_{x \to 0^+} [\ln f(x)]$

$$= \lim_{x \to 0^+} \frac{\ln(1-x)}{x} \qquad \left(\frac{0}{0}\right)$$

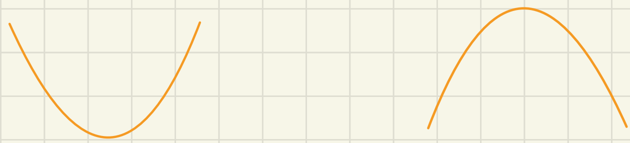
$$(L'H) = \lim_{x \to 0^+} \frac{\frac{1}{1-x} \cdot (-1)}{1} = \lim_{x \to 0^+} \frac{-1}{1-x} = -1$$

2) $\lim_{x \to 0^+} f(x)$

$$= \lim_{x \to 0^+} e^{\overbrace{\ln f(x)}^{-1}} = e^{-1} = \frac{1}{e}$$

- we know that the exponential function is continuous

6.13 The Definition(s) of Concavity



Concave up　　　　Concave down

Idea 1

- Concave up: "secant segments are above the graph"
- Concave down: "secant segments are below the graph"

Idea 2

- Concave up: "tangent lines are below the graph"
- Concave down: "tangent lines are above the graph"

Idea 3

- Concave up: "slope is increasing"
- Concave down: "slope is decreasing"

3 Ways to Define Concavity

- All three are equivalent for differentiable functions.

  The proof relies on MVT.

- 3rd way is the simplest (we'll use it for now)

- 1^{st} way is the best because it works for not differentiable functions

Definitions

Let f be a differentiable function defined on interval I

- f is concave-up on I when f' is increasing on I
- f is concave-down on I when f' is decreasing on I
- Let c be an interior point to I.

  f has an inflection point at c when

  "f changes concavity at c"
- "f changes concavity at c" means $\exists \delta > 0$ s.t.
  - f is concave-up on $[c-\delta, c]$
  - f is concave-down on $[c, c+\delta]$

  or vice versa

Theorem 1

Let I be an open interval.

Let f be a twice-differentiable function defined on I.

- If $\forall x \in I$, $f''(x) > 0$, then f is concave-up on I
- If $\forall x \in I$, $f''(x) < 0$, then f is concave-down on I

Theorem 2

Let I be an open interval. Let $c \in I$.

Let f be a differentiable function defined on I.

If f has an inflection point at c

Then $f''(c) = 0$ or DNE

6.14 Example: Monotonicity and Concavity of a Function

Ex. Let $f(x) = x^6(x+3)^3$. Find

- The intervals where f is increasing or decreasing
- Local extrema
- The intervals f is concave-up or concave-down
- Inflection points

Use this information to sketch the graph of f.

<u>Note</u>

- On an open interval
  - $f' > 0 \Rightarrow f \nearrow$
  - $f' < 0 \Rightarrow f \searrow$
  - $f'' > 0 \Rightarrow f \quad \cup$
  - $f'' < 0 \Rightarrow f \quad \cap$
- On a closed interval
  - In the interior, require same conditions
  - At endpoints, require only continuity
- Candidate points

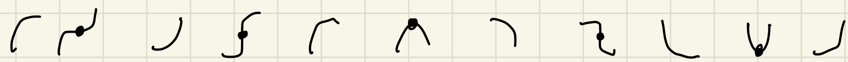
- Local extremum at $c \Rightarrow f'(c) = 0$ or DNE
- Inflection point at $c \Rightarrow f''(c) = 0$ or DNE

<u>Problem</u>

$f'(x) = 9x^5 (x+3)^2 (x+2)$

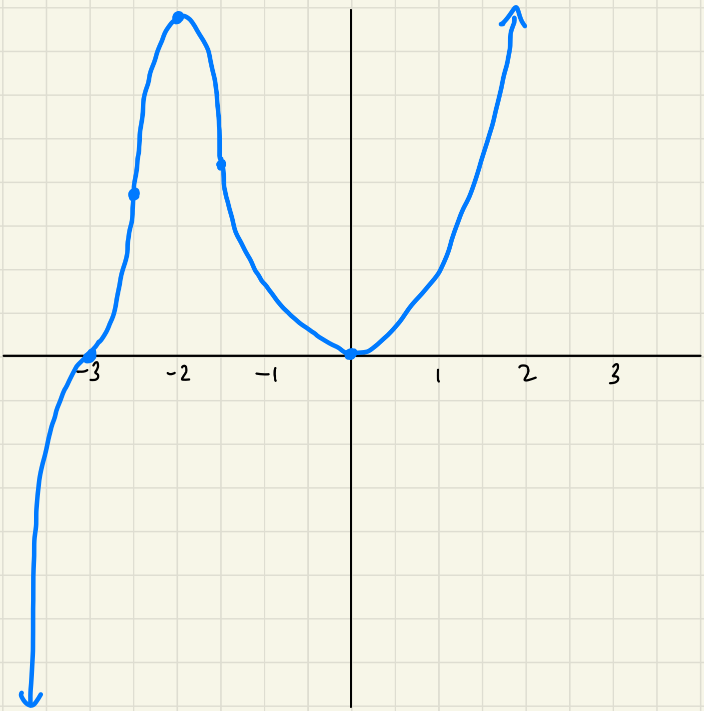
$f''(x) = 18x^4 (x+3)(2x+3)(2x+5)$

x		-3		$-5/2$		-2		$-3/2$		0	
f'	$+$	0	$+$		$+$	0	$-$		$-$	0	$+$
f''	$-$	0	$+$	0	$-$		$-$	0	$+$	0	$+$

f 

- f increasing on $(-\infty, -2]$
- f decreasing on $[-2, 0]$
- f increasing on $[0, \infty)$
- Local max at $x = -2$. $f(-2) = 64$
- Local min at $x = 0$. $f(0) = 0$
- f concave down on $(-\infty, -3]$
- f concave up on $[-3, -5/2]$
- f concave down on $[-5/2, -3/2]$

- f concave up on $[-3/2, \infty)$
- Inflection points at $x = -3, -5/2, -3/2$
  - $f(-3) = 0$, $f(-5/2) \approx 30.5$, $f(-3/2) \approx 38.4$



6.15 Asymptotes

Idea

Let L be a line and C be a curve in the plane.

L is an asymptote for C when "L and C become arbitrarily close as we move away from the origin in one direction"

Vertical Asymptotes

Let f be a function. Let $a \in \mathbb{R}$.

The vertical line $x=a$ is an asymptote of f when

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty$$

Horizontal Asymptotes

Let f be a function. Let $L \in \mathbb{R}$.

The horizontal line $y=L$ is an asymptote of f when

$$\lim_{x \to \pm\infty} f(x) = L$$

Slant Asymptotes

Let f be a function. Let $m, b \in \mathbb{R}$.

The line $y = mx+b$ is an asymptote of f when

$$\lim_{x \to \pm\infty} \left[f(x) - (mx+b) \right] = 0$$

6.16 Asymptotes: Example 1

Ex. Find the asymptotes of the function

$$f(x) = \frac{x^3+1}{x^3-2x^2}.$$

Vertical Asymptotes: Need $\lim_{x\to a^{\pm}} f(x) = \pm\infty$

$$f(x) = \frac{x^3+1}{x^2(x-2)}$$

$$\lim_{x\to 0^+} \frac{\overbrace{x^3+1}^{1}}{\underbrace{x^2}_{0}\underbrace{(x-2)}_{-2}} = -\infty$$

$$\lim_{x\to 0^-} \frac{x^3+1}{\underbrace{x^2}_{0}(x-2)} = -\infty$$

$$\lim_{x\to 2^+} \frac{\overbrace{x^3+1}^{1}}{\underbrace{x^2}_{4}\underbrace{(x-2)}_{0}} = \infty$$

$$\lim_{x\to 2^-} \frac{x^3+1}{x^2\underbrace{(x-2)}_{0}} = -\infty$$

V.A. $\begin{cases} x = 0 \\ x = 2 \end{cases}$

Horizontal Asymptotes: Need $\lim_{x \to \pm\infty} f(x) = L$

$$\lim_{x \to \pm\infty} \frac{x^3+1}{x^3-2x^2} = \lim_{x \to \pm\infty} \frac{x^3\left[1 - 1/x^3\right]}{x^3\left[1 - 2/x\right]} = 1$$

H.A. $y = 1$

6.17 Asymptotes: Example 2

Ex. Find the asymptotes of the function

$$g(x) = x + 2 + \frac{1}{e^{x^2} + 1}.$$

Continuous on $\mathbb{R}$ (no V.A.'s)

$\lim_{x \to \pm\infty} g(x) = \pm\infty$ (no H.A.'s)

$\lim_{x \to \pm\infty} [g(x) - (x+2)] = 0$

$y = x+2$ is an asymptote as $x \to \pm\infty$

6.18 Asymptotes: Example 3

Ex. Find two different slant asymptotes for

$h(x) = x \arctan x$

$$\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$$

As $x \to \infty$, $x\arctan x \sim \frac{\pi}{2}x$

- Is $y = \frac{\pi}{2}x$ an asymptote? NO
- Is $\lim_{x\to\infty} \left[x\arctan x - \frac{\pi}{2}x\right] = 0$?

$$\lim_{x\to\infty} \left[x\arctan x - \frac{\pi}{2}x\right] \qquad (\infty - \infty)$$

$$= \lim_{x\to\infty} x\left(\arctan x - \frac{\pi}{2}\right) \qquad (\infty \cdot 0)$$

$$= \lim_{x\to\infty} \frac{\arctan x - \frac{\pi}{2}}{1/x} \qquad \left(\frac{0}{0}\right)$$

$$(\text{L'H}) = \lim_{x\to\infty} \frac{1/1+x^2}{-1/x^2} = \lim_{x\to\infty} \frac{-x^2}{1+x^2} = -1$$

However, we know that

$$\lim_{x\to\infty} \left[x\arctan x - \left(\frac{\pi}{2}x - 1\right)\right] = 0$$

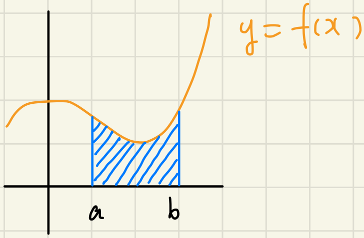
And so $y = \frac{\pi}{2}x - 1$ is an asymptote as $x \to \infty$

7.1 Preview of the Definition of Integral

Definite Integral

$$\int_a^b f(x)\,dx$$

- Integral from a to b of $f(x)$ with respect to x
- Integral from a to b of f



$\int_a^b f(x)\,dx$ = area of shaded region

Plan

1. Cut the region into slices
2. Under- and over-estimate each slice w/ a rectangle
3. Some sort of limit

We need:

- $\sum$ notation for sums
- A better concept than maximum and minimum: supremum and infimum

7.2 Sigma Notation for Sums

- $\displaystyle\sum_{i=3}^{7} \frac{1}{i} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$

  - "sum of $\frac{1}{i}$ from $i = 3$ to 7"

- $\displaystyle\sum_{i=1}^{N} a_i = a_1 + a_2 + a_3 + \ldots + a_N$

  - "sum of a_i from $i = 1$ to N"

- The summation index is a "dummy index"

  - $\displaystyle\sum_{i=1}^{N} a_i = \sum_{k=1}^{N} a_k = \sum_{\mu=1}^{N} a_\mu = \sum_{\heartsuit=1}^{N} a_\heartsuit$

- $\displaystyle\sum_{i=1}^{3} \frac{i}{k} = \frac{1}{k} + \frac{2}{k} + \frac{3}{k} = \frac{6}{k}$

- $\displaystyle\sum_{k=1}^{3} \frac{i}{k} = i + \frac{i}{2} + \frac{i}{3} = \frac{11i}{6}$

- $\displaystyle\sum_{\mu=1}^{3} \frac{i}{k} = \frac{i}{k} + \frac{i}{k} + \frac{i}{k} = \frac{3i}{k}$

Important Properties

- $$\sum_{i=1}^{N} (c \cdot a_i) = c \left(\sum_{i=1}^{N} a_i \right)$$

- $$\sum_{i=1}^{N} (a_i + b_i) = \sum_{i=1}^{N} a_i + \sum_{i=1}^{N} b_i$$

7.3 The Supremum and Infimum of a Set

Goal: Improve the Concept of "Maximum"

- 2 is the maximum of the set $[0,2]$
- 2 is the ? of the set $(0,2)$
- A number $c \in \mathbb{R}$ is the maximum of a set A when:
  - $c \in A$
  - $\forall x \in A,\ x \leq c$

Definitions (supremum)

Let $A \subseteq \mathbb{R}$. Let $c \in \mathbb{R}$.

- c is an upper bound of A means
  - $\forall x \in A,\ x \leq c$
- c is the least upper bound (lub) or supremum (sup) of A means:
  - c is an upper bound of A, and
  - If b is an upper bound of A, then $c \leq b$

- If the supremum of A is in A, it is called **maximum**
- A is **bounded above** means it has (at least) one upper bound

Examples (supremum)

set	upper bounds	supremum	maximum	bounded above?
$[0,2]$	$2, 2.1, \pi, 100, \ldots$	2	2	yes
$(0,2)$	$2, 2.1, \pi, 100, \ldots$	2	—	yes
$\mathbb{Z}$	—	—	—	no

Definition (infimum)

Let $A \subseteq \mathbb{R}$. Let $c \in \mathbb{R}$.

- c is a **lower bound** of A means
  - $\forall x \in A,\ x \geq c$
- c is the **greatest lower bound (glb)** or **infimum (inf)** of A means:
  - c is a lower bound of A, and
  - If b is a lower bound of A, then

$c \geq b$

- If the infimum of A is in A, it is called minimum
- A is bounded below means it has (at least) one lower bound

Examples (infimum)

set	lower bounds	infimum	minimum	bounded below?
$[0,2]$	$-100, -1, 0$	0	0	yes
$(0,2)$	$-100, -1, 0$	0	—	yes
$\mathbb{Z}$	—	—	—	no

A set is bounded means it is both bounded above and bounded below

Theorem (The L.U.B. Principle)

Let $A \subseteq \mathbb{R}$. If

- A is bounded above
- A is not empty

Then A has a least upper bound

Theorem (The G.L.B. Principle)

Let $A \subseteq \mathbb{R}$. If

- A is bounded below
- A is not empty

Then A has a greatest lower bound

7.4 The Supremum and Infimum of a Function

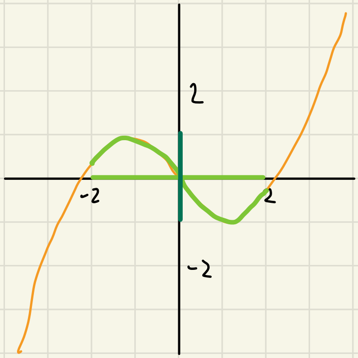
The Supremum of a Function

- When we say "supremum of a function", we mean "supremum of its range"
- Supremum of a function f (on the domain I)

  $= \sup \{ f(x) \mid x \in I \}$

  $= \sup\limits_{x \in I} f(x)$
- The same applies to upper bound of a function, maximum of a function, infimum of a function, . . .

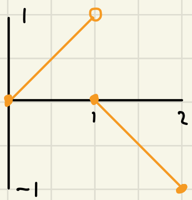
Ex. find sup of f on $[-2, 2]$



$y = f(x)$

$\sup [-1, 1] = 1$

Ex.



$y = g(x)$

- g is bounded above on $[0,2]$
- the supremum of g on $[0,2]$ is 1
- g has no maximum on $[0,2]$
- g is bounded below on $[0,2]$
- the infimum of g on $[0,2]$ is -1
- the minimum of g on $[0,2]$ is -1

<u>Theorem</u> (A consequence of the LUB Principle)

Let f be a function defined on a domain $I \neq \emptyset$

- If f is bounded above on I
- Then f has a supremum on I

<u>Theorem</u> (EVT)

Let $a < b$ Let f be a function defined on $[a,b]$.

- If f is continuous on $[a,b]$
- Then f has a maximum and minimum on $[a,b]$

7.5 The Definition of an Integral

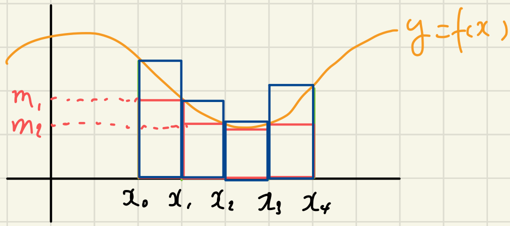
Let $a < b$

Let f be a bounded function on $[a,b]$

Step 1: Cut the Region Into Slices

- Definition. A partition of the interval $[a,b]$ is a set P such that
  - P is finite
  - $P \subseteq [a,b]$
  - $a \in P$ and $b \in P$
- Partitions of $[0,1]$:
  - $P_1 = \{0, 0.2, 0.5, 0.9, 1\}$
  - $P_2 = \{0, 1\}$
  - etc.
- Convention: when we write $P = \{x_0, x_1, x_2, \dots, x_N\}$, we implicitly mean $a = x_0 < x_1 < x_2 < \dots < x_N = b$

Step 2: Under- and Over-estimate the Area with Rectangles



- m_1 = infimum of f on $[x_0, x_1]$
- m_2 = infimum of f on $[x_1, x_2]$
- area of red rectangles

$$= m_1(x_1 - x_0) + m_2(x_2 - x_1) + \dots$$

$$= \sum_{i=1}^{4} m_i (x_i - x_{i-1})$$

- area of blue rectangles $= \sum_{i=1}^{4} M_i (x_i - x_{i-1})$
- M_i = supremum of f on $[x_{i-1}, x_i]$
- Definition.
  - Let f be a bounded function on $[a, b]$
  - Let $P = \{x_0, x_1, \dots, x_N\}$ be a partition of $[a, b]$
  - For each $i = 1, \dots, N$, let

- m_i be the infimum of f on $[x_{i-1}, x_i]$
- M_i be the supremum of f on $[x_{i-1}, x_i]$
- $\Delta x_i = x_i - x_{i-1}$

- The P-lower sum of f is the number

$$L_P(f) = \sum_{i=1}^{N} m_i \cdot \Delta x_i$$

- The P-upper sum of f is the number

$$U_P(f) = \sum_{i=1}^{N} M_i \cdot \Delta x_i$$

Step 3: Define Integral

- I want to define the number $\int_a^b f(x)\,dx$
  - It needs to satisfy

$$L_P(f) \leq \int_a^b f(x)\,dx \leq U_P(f)$$

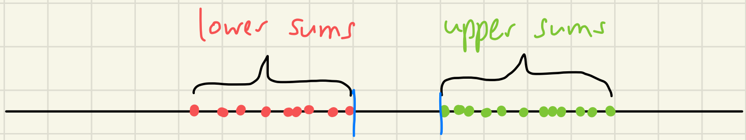
for every partition P

- Lemma. Every lower sum is less than or equal to every upper sum

- Definition. Let f be a bounded function on $[a,b]$.
  - Lower integral of f from a to b

    $= \underline{I}_a^b(f) = \sup\{\text{lower sums of } f\}$
  - Upper integral of f from a to b

    $= \overline{I}_a^b(f) = \inf\{\text{upper sums of } f\}$



$$\underline{I}_a^b(f) \leq \overline{I}_a^b(f)$$

- Definition. Let f be a bounded function on $[a,b]$. When $\underline{I}_a^b(f) = \overline{I}_a^b(f)$, we say that f is integrable on $[a,b]$, and that

$$\int_a^b f(x)\,dx = \underline{I}_a^b(f) = \overline{I}_a^b(f)$$

- Definition. Let f be a bounded function on $[a,b]$. When $\underline{I}_a^b(f) < \overline{I}_a^b(f)$, f is non-integrable on $[a,b]$, and

$$\int_a^b f(x)\,dx \text{ is undefined}$$

- Theorem. If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

7.6 Properties of Lower and Upper Sums

Let $a < b$

Let f be a bounded function on $[a,b]$

Property 1

For every partition P of $[a,b]$, $L_P(f) \leq U_P(f)$

Definition

Let P and Q be partitions of the interval $[a,b]$.

We say that Q is finer than P when $P \subseteq Q$.

- $Q = \{0, 0.2, 0.3, 0.5, 0.64, 0.8, 0.91, 1\}$

is finer than

$P = \{0, 0.2, 0.5, 0.64, 0.8, 1\}$

Property 2

Let P and Q be two partitions of $[a,b]$.

If $P \subseteq Q$, then $L_P(f) \leq L_Q(f)$ and $U_Q(f) \leq U_P(f)$

Property 3

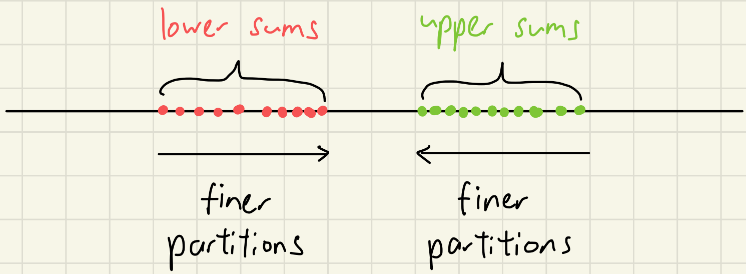
Let P and Q be any two partitions of $[a,b]$.

Then $L_P(f) \leq U_Q(f)$

Proof. Call $R = P \cup Q$. Then $P \subseteq R$ and $Q \subseteq R$.

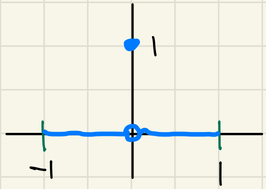
$$L_P(f) \underset{\text{(property 2)}}{\leq} L_R(f) \underset{\text{(property 1)}}{\leq} U_R(f) \underset{\text{(property 2)}}{\leq} U_Q(f)$$

$\square$



7.7 Example: An Integrable Function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$



Is f integrable on $[-1,1]$?

$\underline{I}_{-1}^{1}(f) = \sup\{\text{lower sums of } f\}$

$\overline{I}_{-1}^{1}(f) = \inf\{\text{upper sums of } f\}$

f is integrable on $[-1,1] \iff \underline{I}_{-1}^{1}(f) = \overline{I}_{-1}^{1}(f)$

- consider any partition, on every subinterval, the infimum is $m_i = 0$. For every partition P,

  $L_P(f) = 0$

  $\underline{I}_{-1}^{1}(f) = \sup\{0\} = 0$

- $\overline{I}_{-1}^{1}(f) = \inf(0,2] = 0$

Thus, f is integrable on $[-1,1]$ and $\int_{-1}^{1} f(x)\,dx = 0$

7.8 Example: A Non-Integrable Function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

g is called the Dirichlet function

Is g integrable on $[0,1]$?

$\underline{I}_0^1(g) = \sup\{\text{lower sums of } g\}$

$\overline{I}_0^1(g) = \inf\{\text{upper sums of } g\}$

g is integrable on $[0,1] \iff \underline{I}_0^1(g) = \overline{I}_0^1(g)$

- Let $a, b \in \mathbb{R}$ s.t. $a < b$, then
  - $\exists t \in [a,b]$ s.t. $t \in \mathbb{Q}$
  - $\exists s \in [a,b]$ s.t. $s \notin \mathbb{Q}$
- Let $P = \{x_0, x_1, x_2, \dots, x_N\}$ be any partition of $[0,1]$
  - inf of g on every subinterval $[x_{i-1}, x_i]$ is 0
  - sup of g on every subinterval $[x_{i-1}, x_i]$ is 1

- $L_P(g) = 0$, $U_P(g) = 1$

$\underline{I}_0^1(g) = \sup\{0\} = 0$

$\overline{I}_0^1(g) = \inf\{1\} = 1$

Since $\underline{I}_0^1(g) \neq \overline{I}_0^1(g)$, g is not integrable on $[0,1]$

$\int_0^1 g(x)dx$ is undefined

<u>Alternative Theories of Integration</u>

- Darboux integral (the one we used)
- Riemann integral
- Lebesgue integral

7.9 Integrals as Limits

Let $a < b$

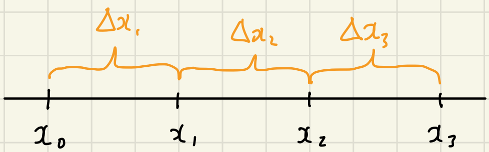
Let f be a bounded function on $[a,b]$

<u>Definition</u>

Let $P = \{x_0, x_1, x_2, \dots, x_N\}$ be a partition of $[a,b]$.

For each i, let $\Delta x_i = x_i - x_{i-1}$

The norm of P is $\|P\| = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_N\}$



We want to compute limit as $\|P\| \to 0$

<u>Definition</u>

$\underline{I}_a^b(f) = \sup\{L_P(f) \mid P \text{ is a partition of } [a,b]\}$

<u>Theorem 1</u>

$$\underline{I}_a^b(f) = \lim_{\|P\| \to 0} L_P(f)$$

$\forall \varepsilon > 0$, $\exists \delta > 0$. $\forall$ partition P of $[a,b]$,

$$\|P\| < \delta \Rightarrow |\underline{I}_a^b(f) - L_P(f)| < \varepsilon$$

Theorem 2

Pick a sequence of partitions $P_1, P_2, P_3, \ldots$

satisfying $\lim_{n \to \infty} \| P_n \| = 0$

(e.g. P_n : break the interval $[a,b]$ into

n subintervals of equal length)

Then

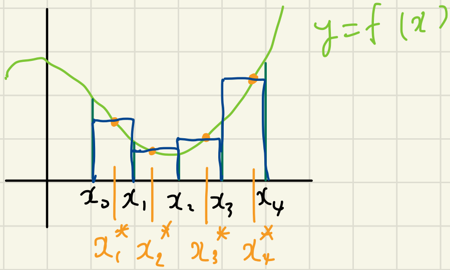
$$\underline{I}_a^b (f) = \lim_{x \to \infty} L_{P_n}(f)$$

$$\overline{I}_a^b (f) = \lim_{x \to \infty} U_{P_n}(f)$$

7.10 Riemann Sums

Let $a < b$

Let f be a bounded and integrable function on $[a,b]$.



Area of dark blue rectangles

$$= \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i$$

<u>Definition</u>

- Let f be a bounded function on the interval $[a,b]$
- Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$
- For each $i = 1, 2, \ldots, n$:
    - Let $\Delta x_i = x_i - x_{i-1}$
    - Choose a number $x_i^* \in [x_{i-1}, x_i]$

Then $S_P^*(f) = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i$

is called a Riemann-sum for f and P

Theorem

- Let f be a bounded function on the interval $[a,b]$.

  Assume f is integrable on $[a,b]$.

- Pick a sequence of partitions $P_1, P_2, P_3, \ldots$

  of $[a,b]$ s.t. $\lim_{n\to\infty} \|P_n\| = 0$

  - e.g. break the interval $[a,b]$ into

    n subintervals of equal length

- On each subinterval of each partition, pick $x_i^* \in [x_{i-1}, x_i]$

  - e.g. $x_i^* = x_i$

Then $\int_a^b f(x)\,dx = \lim_{N\to\infty} S_{P_n}^*(f)$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\cdot \Delta x_i\right]$$

Proof

- $\lim_{n\to\infty} L_{P_n}(f) = \underline{I}_a^b(f) = \int_a^b f(x)\,dx$

- $\lim_{n\to\infty} U_{P_n}(f) = \overline{I}_a^b(f) = \int_a^b f(x)\,dx$

because f is integrable on $[a, b]$.

$$L_{P_n}(f) \leq S^*_{P_n}(f) \leq U_{P_n}(f)$$

By Squeeze Theorem.

$$\lim_{n \to \infty} S^*_{P_n}(f) = \int_a^b f(x)\,dx$$

$\square$

We can only use Riemann sums on integrable functions!

Example

Calculate $\int_0^1 x\,dx$ using Riemann sums

- $f(x) = x$ is continuous on $[0, 1]$.

$$\int_0^1 x\,dx = \lim_{n \to \infty} S^*_{P_n}(f)$$

- Choose P_n: break $[0, 1]$ into n equal subintervals

$$P_n = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, 1 \right\}$$

- On each subinterval, choose the right endpoint

$$x_i^* = \frac{i}{n}$$

$$\int_0^1 x\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{i}{n}\cdot\frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$$

7.11 Five Properties of Definite Integrals

Definite Integrals are Linear

1. $$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Theorem 1. Let $a<b$. Let f and g be bounded functions on $[a,b]$.

If f and g are integrable on $[a,b]$

Then $f+g$ is integrable on $[a,b]$ and

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2. $$\int_a^b [C f(x)]\,dx = C\int_a^b f(x)\,dx$$

Theorem 2. Let $a<b$. Let f be a bounded function on $[a,b]$. Let $C \in \mathbb{R}$.

If f is integrable on $[a,b]$

Then $C\cdot f$ is integrable on $[a,b]$ and

$$\int_a^b [Cf(x)]\,dx = C\int_a^b f(x)\,dx$$

Definite Integrals Over Different Intervals

3. $$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Theorem 3. Let $a < b < c$. Let f be a bounded function on on $[a, c]$.

If f is integrable on $[a, b]$ and

f is integrable on $[b, c]$

Then f is integrable on $[a, c]$ and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Definition.

- $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$
- $$\int_a^a f(x)\,dx = 0$$

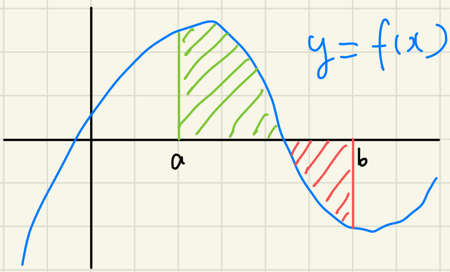
Then

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$ is true

no matter the order of a, b, c

Geometric Interpretation as Area (Property 4)

$$\int_a^b f(x)dx = \begin{pmatrix} \text{area above} \\ x\text{-axis} \end{pmatrix} - \begin{pmatrix} \text{area below} \\ x\text{-axis} \end{pmatrix}$$



Inequalities and Definite Integrals

5. If $\forall x \in [a,b]$, $f(x) \leq g(x)$

Then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

Theorem 5. Let $a<b$. Let f and g be bounded functions on $[a,b]$.

If f, g are integrable on $[a,b]$ and

for all $x \in [a,b]$, $f(x) \leq g(x)$

Then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

8.1 Antiderivatives

Two Different "Integrals"

1. Definite integral : $\int_a^b f(x)dx$

    - A number

    - Measures area

2. Indefinite integral : $\int f(x)\, dx$

- Definitions. Let f be a function defined on an interval.

    - An antiderivative of f is any function F such that $F' = f$

    - The collection of all antiderivatives of f is denoted $\int f(x) dx$

Example : $\int x^2 dx$

- Guess: $F(x) = \frac{1}{3}x^3$ is one antiderivative. $\frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2$

- For any constant C, $F(x) = \frac{1}{3}x^3 + C$ is also one antiderivative

- By MVT, if two functions have the same derivative on an interval, then they differ by a constant

- Conclusion: $\int x^2 \, dx = \frac{1}{3}x^3 + C$

<u>The Main Integration Technique: Guess and Check</u>

Ex. Compute $\int (2x+7)^{11} \, dx$

$$\frac{d}{dx}\left[???\right] = (2x+7)^{11}$$

$$\frac{d}{dx}\left[(2x+7)^{12}\right] = 12 \cdot (2x+7)^{11} \cdot 2 = 24 \cdot (2x+7)^{11}$$

$$\frac{d}{dx}\left[\frac{1}{24}(2x+7)^{12}\right] = \frac{1}{24} \cdot 24 \cdot (2x+7)^{11} = (2x+7)^{11}$$

Conclusion: $\int (2x+7)^{11} \, dx = \frac{1}{24}(2x+7)^{12} + C$

8.2 Functions Defined as Integrals

- We can use an integral to define a function

  $F(x) = \int_a^x f(t)\,dt$

- Let I be an interval. Let $a \in I$.

  Let f be a function integrable on I.

  Then for each value of $x \in I$,

  $F(x) = \int_a^x f(t)\,dt$ is a number

- Avoid writing $F(x) = \int_a^{x} f(x)\,dx$ bad notation

8.3 Fundamental Theorem of Calculus, Part 1

The Fundamental Theorem of Calculus

- FTC: Connections between integrals and antiderivatives
- FTC Part 1: What can we say about the function

  $F(x) = \int_a^x f(t)\,dt$?
- FTC Part 2: How do we compute $\int_a^b f(x)\,dx$ quickly?

Theorem (FTC 1)

Let I be an interval. Let $a \in I$.

Let f be a function on I.

We define $F(x) = \int_a^x f(t)\,dt$.

If f is continuous

Then F is differentiable and $F' = f$

- "A function defined as an integral is an antiderivative."

  $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Ex. 1

Let $F(x) = \int_1^x e^{-t^2}\,dt$. Calculate $F'(2)$.

$F'(x) = e^{-x^2}$ $\quad F'(2) = e^{-4}$

Ex. 2

Construct a function g such that

- For every $x \in \mathbb{R}$, $g'(x) = \frac{1}{1+x^2+x^{10}}$, and
- $g(2) = 5$

$$g(x) = 5 + \underbrace{\int_2^x \frac{1}{1+t^2+t^{10}}\,dt}_{0 \text{ when } x=2}$$

Ex. 3

Let $G(x) = \int_{-4}^{x^2} \frac{\sin t}{t}\,dt$ Calculate $G'(x)$

Define $F(x) = \int_{-4}^{x} \frac{\sin t}{t}\,dt$. We know that $F'(x) = \frac{\sin x}{x}$.

$G(x) = F(x^2)$, and so $G'(x) = 2x F'(x^2)$

$$= 2x \cdot \frac{\sin(x^2)}{(x^2)} = \frac{2\sin(x^2)}{x}$$

Ex. 4

Let $H(x) = \int_{x^3+1}^{x^2+2x} e^{-t^2}\,dt$. Calculate $H'(x)$.

$$H(x) = \int_0^{x^2+2x} e^{-t^2}\,dt - \int_0^{x^3+1} e^{-t^2}\,dt$$

Define $F(x) = \int_0^x e^{-t^2}\,dt$. $F'(x) = e^{-x^2}$

$$H(x) = F(x^2+2x) - F(x^3+1)$$

$$H'(x) = (2x+2)F'(x^2+2x) - 3x^2 F'(x^3+1)$$

$$= (2x+2)e^{-(x^2+2x)^2} - 3x^2 e^{-(x^3+1)^2}$$

8.4 Proof of Part 1 of FTC

Theorem (FTC 1)

Let I be an interval. Let $a \in I$.

Let f be a function on I.

We define $F(x) = \int_a^x f(t)\,dt$.

If f is continuous

Then F is differentiable and $F' = f$

Proof

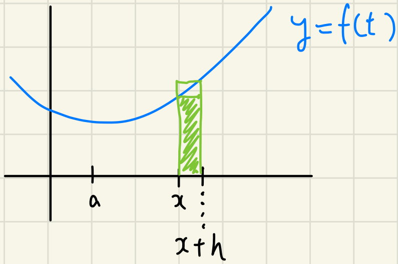
Fix $x \in I$. WTS $F'(x) = f(x)$

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\int_x^{x+h} f(t)\,dt \right]$$

WTS $\lim_{h \to 0} \frac{1}{h} \left[\int_x^{x+h} f(t)\,dt \right] = f(x)$



height width

When h is small, $\int_x^{x+h} f(t)\,dt \approx f(x)h$

$$\Rightarrow \frac{1}{h}\int_x^{x+h} f(t)\,dt \approx f(x)$$

Case 1: Assume $h>0$.

Call $\begin{cases} M_h = \sup \text{ of } f \text{ on } [x, x+h] \\ m_h = \inf \text{ of } f \text{ on } [x, x+h] \end{cases}$

Then $m_h \cdot h \leq \int_x^{x+h} f(t)\,dt \leq M_h \cdot h$

$$m_h \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq M_h$$

f is continuous. By EVT, it has a maximum on $[x, x+h]$.

$M_h = \max$ of f on $[x, x+h]$

$\forall h>0,\ \exists c_h \in [x, x+h]$ s.t. $M_h = f(c_h)$

$x \leq c_h \leq x+h$

As $h \to 0$, $c_h \to x$, and so

$M_n = f(c_n) = f(x)$ because f is continuous

We have proven that $\lim\limits_{h \to 0} m_h = \lim\limits_{h \to 0} M_h = f(x)$

By Squeeze Theorem,

$\lim\limits_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = f(x)$ as we wanted.

$\square$

8.5 Fundamental Theorem of Calculus, Part 2

Theorem (FTC 2)

Let $a < b$. Let f be a continuous function on $[a,b]$.

Let G be any antiderivative of f

Then $\int_a^b f(x)\,dx = G(b) - G(a)$

Notation: $G(b) - G(a) = G(x)\Big|_{x=a}^{x=b} = G(x)\Big|_a^b$

Example

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_{x=1}^{x=2} = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$$

8.6 Proof of Part 2 of FTC

Theorem (FTC 2)

Let $a < b$. Let f be a continuous function on $[a,b]$.

Let G be any derivative G of f

Then $\int_a^b f(x)\,dx = G(b) - G(a)$

Proof

- We know $G' = f$. WTS $\int_a^b f(x)\,dx = G(b) - G(a)$
- Define $F(x) = \int_a^x f(t)\,dt$. WTS $F(b) = G(b) - G(a)$
- Since f is continuous, by FTC 1, $F' = f$
- $F' = G'$, so $F - G$ is a constant
- $\exists c \in \mathbb{R}$ s.t. $F(x) = G(x) + C$
- Evaluate at $x = a$ to figure out the constant:

$$0 = F(a) = G(a) + C$$

Thus $C = -G(a)$

- Thus, for every $x \in [a,b]$, $F(x) = G(x) - G(a)$
- In particular $F(b) = G(b) - G(a)$, which is what we wanted. $\square$

8.7 The Three Notions of Integral

Three Concepts

1. Definite integral: $\int_a^b f(x)\,dx$
   - This is a number
2. Indefinite integral: $\int f(x)\,dx$
   - This is a collection of functions
3. Functions defined by an integral: $F(x) = \int_a^x f(t)\,dt$
   - This is one function

Definite Integral

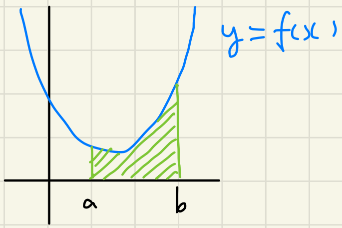
- Formal definition:

$\overline{I}_a^b(f) = \inf\{U_P(f) \mid P \text{ is a partition of } [a,b]\}$

$\underline{I}_a^b(f) = \sup\{L_P(f) \mid P \text{ is a partition of } [a,b]\}$

If they are equal, then $\int_a^b f(x)\,dx = \overline{I}_a^b(f) = \underline{I}_a^b(f)$

- Geometric interpretation: area under the curve



- To compute: (by FTC 2)
  1. Find one antiderivative G of f
  2. Then $\int_a^b f(x)\,dx = G(b) - G(a)$

<u>Indefinite Integral</u>

- Definition: the collection of all antiderivatives of f
- To compute: (by MVT)
  1. Find one antiderivative G of f
  2. Then $\int f(x)\,dx = G(x) + C$

<u>Function Defined By an Integral</u>

- If f is integrable, then $F(x) = \int_a^x f(t)\,dt$ is a valid way to define a new function F.
  - By FTC 1, if f is continuous, then F is an antiderivative of f
- Ex. $F(x) = \int_3^x e^{-t^2}\,dt$ satisfies $\begin{cases} \forall x \in \mathbb{R}, & F'(x) = e^{-x^2} \\ & F(3) = 0 \end{cases}$

<u>Notation: dx or dt</u>

- Definite integral: (number)

$$\int_1^2 x^2\,dx = \int_1^2 t^2\,dt = \int_1^2 ☺^2\,d☺$$

- Indefinite integral:

$$\int x^2 dx = \frac{x^3}{3} + C \text{ is different from } \int t^2 dt = \frac{t^3}{3} + C$$

- Functions defined by an integral

$$F(x) = \int_3^x e^{-t^2} dt = \int_3^x e^{-u^2} du = \int_3^x e^{-\text{☺}^2} d\text{☺}$$

but we cannot write $F(x) = \int_3^x e^{-x^2} dx$

9.1 Integration by Substitution, the Theory

- Ex. $\int x^4 \, dx = \frac{1}{5}x^5 + C$

- Ex. $\int (\ln x)^4 \, dx$
  - Guess: $\frac{1}{5}(\ln x)^5$
  - $\frac{d}{dx}\left[\frac{1}{5}(\ln x)^5\right] = (\ln x)^4 \cdot \frac{1}{x}$

  $\Rightarrow \int (\ln x)^4 \frac{1}{x} = \frac{1}{5}(\ln x)^5 + C$

- Ex. $\int g(x)^4 g'(x) \, dx = \frac{1}{5}g(x)^5 + C$
  - $\frac{d}{dx}\left[\frac{1}{5}g(x)^5\right] = g(x)^4 \cdot g'(x)$

<u>Theorem</u> (Substitution Rule or Reversed Chain Rule)

Let f, g, F be functions.

Assume F and g are differentiable.

If $\int f(x) dx = F(x) + C$ $\qquad$ (1)

Then $\int f(g(x)) \, g'(x) \, dx = F(g(x)) + C$ $\qquad$ (2)

- (1) means $F'(x) = f(x)$
- (2) means $(F \circ g)'(x) = f(g(x)) \, g'(x)$

The Substitution Rule in Practice

- Assume $\int f(x)\,dx = F(x) + C$
- $\int f(\underbrace{g(x)}_{u})\underbrace{g'(x)\,dx}_{du}$

Substitution:

$$\begin{cases} u = g(x) \\ du = g'(x)\,dx \end{cases}$$

$$= \int f(u)\,du$$

$$= F(u) + C = F(g(x)) + C$$

<u>Example</u>

$$\int (\ln x)^4 \frac{1}{x}\,dx \qquad \left(\text{Substitution: } \begin{cases} u = \ln x \\ du = \frac{1}{x}\,dx \end{cases} \right)$$

$$= \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}(\ln x)^5 + C$$

9.2 Integration by Substitution, Examples

- Ex. $\int x^2 \sin(x^3+7)\,dx$

Substitution: $\begin{cases} u = x^3+7 \\ du = 3x^2\,dx \end{cases}$

$$= \int \sin(u)\cdot\frac{1}{3}\,du = -\frac{1}{3}\cos(u) + C$$

$$= -\frac{1}{3}\cos(x^3+7) + C$$

- Ex. $\int \tan x\,dx$

$$= \int \frac{\sin x}{\cos x}\,dx$$

Substitution: $\begin{cases} u = \cos x \\ du = -\sin x\,dx \end{cases}$

$$= \int -\frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

- Ex. $\int x^3\sqrt{x^2+1}\,dx$

Substitution: $\begin{cases} u = x^2+1 \\ du = 2x\,dx \end{cases}$

$$= \int \frac{1}{2} x^2 \sqrt{x^2+1} \cdot 2x\,dx = \frac{1}{2} \int (u-1)\sqrt{u}\ du$$

$$= \frac{1}{2} \int \left(u^{3/2} - u^{1/2} \right) du$$

$$= \frac{1}{2} \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right] + C$$

$$= \frac{1}{5} u^{5/2} - \frac{1}{3} u^{3/2} + C$$

$$= \frac{1}{5} (x^2+1)^{5/2} - \frac{1}{3} (x^2+1)^{3/2} + C$$

9.3 Substitution for Definite Integrals

- Ex. $\int_1^2 x(x^2+1)^{100}\,dx$
- Method 1: find an antiderivative first

  i.e. $\int x(x^2+1)^{100}\,dx = \dots$

  then $\int_1^2 x(x^2+1)^{100}\,dx = \left[\dots\right]_{x=1}^{x=2}$
- Method 2: use substitution in the definite integral

  i.e. $\int_1^2 x(x^2+1)^{100}\,dx = \dots$

  (faster)

Example

$$\int_1^2 x(x^2+1)^{100}\,dx$$

Substitution: $\begin{cases} u = x^2+1 \\ du = 2x\,dx \end{cases}$

$x=1 \rightarrow u = 1^2+1 = 2$

$x=2 \rightarrow u = 2^2+1 = 5$

$$= \frac{1}{2}\int_1^2 (x^2+1)^{100}(2x\,dx) = \frac{1}{2}\int_2^5 u^{100}\,du$$

$$= \frac{1}{2}\cdot\frac{u^{101}}{101}\Bigg|_{u=2}^{u=5} = \frac{5^{101}-2^{101}}{202}$$

no need to undo substitution

Theorem (Change of variable for definite integrals)

- Let $a < b$
- Let g be a function with continuous derivative on $[a, b]$
- Let f be a continuous function. Assume the range of g on $[a, b]$ is contained in the domain of f.

Then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du$

9.4 Integration by Parts, the Theory

- Ex. $\int xe^x dx$
  - $\frac{d}{dx}(xe^x) = e^x + xe^x$
  - $\frac{d}{dx}(e^x) = e^x$

    $\frac{d}{dx}(xe^x - e^x) = xe^x$

  $= xe^x - e^x + C$

- $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$

$$\int [f'(x)g(x) + f(x)g'(x)]\,dx = f(x)g(x) + C$$

<u>Theorem</u> (Integration by Parts or Reversed Product Rule)

Let f and g be functions. Then:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

<u>Alternative Notation</u>

$$\begin{cases} u = f(x) & du = f'(x)\,dx \\ v = g(x) & dv = g'(x)\,dx \end{cases}$$

$$\int u\,dv = uv - \int v\,du$$

- Ex. $\int x e^{x} \, dx$

By Parts $\begin{cases} u = x & du = dx \\ dv = e^{x} dx & v = e^{x} \end{cases}$

$$= x e^{x} + \int e^{x} dx$$

$$= x e^{x} + e^{x} + C$$

9.5 Integration by Parts, Examples

- Ex. $\int x\cos x\,dx$

By Parts $\begin{cases} u = x & du = dx \\ dv = \cos x\,dx & v = \sin x \end{cases}$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

- Ex. $\int x^2 e^{-x}\,dx$

By Parts $\begin{cases} u = x^2 & du = 2x\,dx \\ dv = e^{-x}\,dx & v = -e^{-x} \end{cases}$

$$= -x^2e^{-x} - \int -2xe^{-x}\,dx$$

$$= 2\int xe^{-x}\,dx - x^2e^{-x}$$

By Parts $\begin{cases} u = x & du = dx \\ dv = e^{-x}\,dx & v = -e^{-x} \end{cases}$

$$= 2\left[-xe^{-x} - \int -e^{-x}\,dx\right] - x^2e^{-x}$$

$$= 2\int e^{-x}\,dx - 2xe^{-x} - x^2e^{-x}$$

$$= -2e^{-x} - 2xe^{-x} - x^2e^{-x} + C$$

- Ex. $\int \arctan x \, dx$

By Parts $\begin{cases} u = \arctan x & du = \dfrac{1}{x^2+1} dx \\ dv = 1 dx & v = x \end{cases}$

$$= x\arctan x - \int \frac{x}{x^2+1} dx$$

Substitution $\begin{cases} u = x^2+1 \\ du = 2x dx \Leftrightarrow x dx = \frac{1}{2} du \end{cases}$

$$= x\arctan x - \int \frac{1}{2} \cdot \frac{1}{u} du$$

$$= x\arctan x - \frac{1}{2} \ln|u| + C$$

$$= x\arctan x - \frac{1}{2} \ln|x^2+1| + C$$

9.6 Integration by Parts, One More Example

- Ex. $\int e^x \sin x \, dx = I$

By Parts $\begin{cases} u = e^x & du = e^x dx \\ dv = \sin x \, dx & v = -\cos x \end{cases}$

$$= -e^x \cos x + \int e^x \cos x \, dx$$

By Parts $\begin{cases} u = e^x & du = e^x dx \\ dv = \cos x \, dx & v = \sin x \end{cases}$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$I = -e^x \cos x + e^x \sin x - I$$

$$I = \frac{1}{2} e^x (\sin x - \cos x) + C$$

9.7 Integration of Products of Trig Functions, Part 1

- Ex. $\int \sin^5 x \cos^2 x \, dx$

  - First attempt: Substitution $\begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$

$$= \int (\sin^5 x \cos x)(\cos x \, dx)$$

$$= \int \sin^5 x \left[\pm\sqrt{1 - \sin^2 x} \right] (\cos x \, dx)$$

$$= \pm \int u^5 \sqrt{1 - u^2} \, du$$ Not helpful

  - Second attempt. Substitution $\begin{cases} u = \cos x \\ du = -\sin x \, dx \end{cases}$

$$= -\int \sin^4 x \cos^2 x \, (-\sin x \, dx)$$

$$= -\int (\sin^2 x)^2 \cos^2 x \, (-\sin x \, dx)$$

$$= -\int (1 - \cos^2 x)^2 \cos^2 x \, (-\sin x \, dx)$$

$$= -\int (1 - u^2)^2 u^2 \, du$$ Polynomial

Summary

To compute $\int \sin^n x \cos^m x \, dx$:

- If m odd, then try $\begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$

- If n odd, then try $\begin{cases} u = \cos x \\ du = -\sin x \, dx \end{cases}$

9.8 Integration of Products of Trig Functions, Part 2

- Ex. $\int \sin^2 x \, dx$
  - Method 1: Half-angle identities

    $$\sin^2 x = \frac{1-\cos(2x)}{2}, \quad \cos^2 x = \frac{1+\cos(2x)}{2}$$

    $$\int \sin^2 x \, dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\, dx$$

    - Works for higher powers of sin and cos
  - Method 2: Integration by parts

    $$\begin{cases} u = \sin x & du = \cos x \, dx \\ dv = \sin x \, dx & v = -\cos x \end{cases}$$

    $$\left.\begin{aligned} &\int \sin x \sin x \, dx = -\sin x \cos x + \int \cos x \cos x \, dx \\ &\int \sin^2 x \, dx + \int \cos^2 x \, dx = \int 1 \, dx \end{aligned}\right]$$

    System of two equations where we can solve for $\int \sin^2 x \, dx$

9.9 The Integral of Secant

Method 1

$$\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$$

this step is unnatural

how could anyone think of this?

Substitution $\begin{cases} u = \sec x + \tan x \\ du = (\sec x \tan x + \sec^2 x) \, dx \end{cases}$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Method 2

$$\int \sec x \, dx = \int \frac{1}{\cos x} dx$$

Power for cos is odd: Substitution $\begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$

$$= \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x \, dx}{1 - \sin^2 x} = \int \frac{du}{1 - u^2} = \ldots$$

- Natural progression

9.10 Integration of Rational Functions, Example 1

Ex. $\displaystyle\int \frac{x+3}{(x-1)(x+1)}\,dx$

I know: $\begin{cases} \displaystyle\int \frac{1}{x-1}\,dx = \ln|x-1| + C \\ \displaystyle\int \frac{1}{x+1}\,dx = \ln|x+1| + C \end{cases}$

Goal: $\displaystyle\frac{x+3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$

I want to find

$$\frac{x+3}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

$$x+3 = A(x+1) + B(x-1)$$

Method 1:

$$= (A+B)x + (A-B)$$

$$\begin{cases} A+B = 1 \\ A-B = 3 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = -1 \end{cases}$$

Method 2:

$$\begin{cases} \text{When } x=1, & 4 = 2A \\ \text{When } x=-1, & 2 = -2B \end{cases}$$

$$\int \frac{x+3}{(x-1)(x+1)}\,dx = \int \left[\frac{2}{x-1} - \frac{1}{x+1} \right] dx$$

$$= 2\ln|x-1| - \ln|x+1| + C$$

$$= \ln\left|\frac{(x-1)^2}{x+1}\right| + C$$

Ex $\displaystyle\int \frac{P(x)}{x^3-3x^2+2x}\,dx$ polynomial

$$\frac{P(x)}{x^3-3x^2+2x} = q(x) + \frac{ax^2+bx+c}{x^3-3x^2+2x}$$

$$= q(x) + \frac{ax^2+bx+c}{\underbrace{x(x-1)(x-2)}_{\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-2}}}$$

9.11 Integration of Rational Functions, Example 2

Ex. $\displaystyle\int \frac{x^4+1}{x^3+2x^2+x}\,dx$

$$\frac{x^4+1}{x^3+2x^2+x} = (x-2) + \frac{3x^2+2x+1}{x^3+2x+x}$$

$$= (x-2) + \frac{3x^2+2x+1}{x(x+1)^2}$$

Decompose into partial fractions

$$\frac{3x^2+2x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\begin{cases} \displaystyle\int \frac{1}{x}\,dx = \ln|x| \\ \displaystyle\int \frac{1}{x+1}\,dx = \ln|x+1| + C \end{cases}$$

$$\int \frac{1}{(x+1)^2}\,dx = \frac{-1}{x+1} + C$$

$A = 1,\ B = 2,\ C = -2$

$$\int \frac{x^4+1}{x^3+2x^2+x}\,dx = \int \left[x-2 + \frac{3x^2+2x+1}{x(x+1)^2} \right] dx$$

$$= \int \left[x-2 + \frac{1}{x} + \frac{2}{x+1} - \frac{2}{(x+1)^2} \right] dx$$

$$= \frac{x^2}{2} - 2x + \ln|x| + 2\ln|x+1| + \frac{2}{x+1} + C$$

9.12 Integration of Rational Functions, Example 3

Ex. $\int \frac{3x+7}{x^2+4}\,dx = 3\int \frac{x}{x^2+4}\,dx + 7\int \frac{1}{x^2+4}\,dx$

- $\int \frac{x}{x^2+4}\,dx$ Substitution $\begin{cases} u = x^2+4 \\ du = 2x\,dx \end{cases}$

$= \int \frac{1/2\,du}{u} = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln(x^2+4) + C$

- $\int \frac{1}{x^2+4}\,dx$

We know that $\int \frac{1}{x^2+1}\,dx = \arctan x + C$

$= \int \frac{1}{4}\frac{1}{\frac{x^2}{4}+1}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$

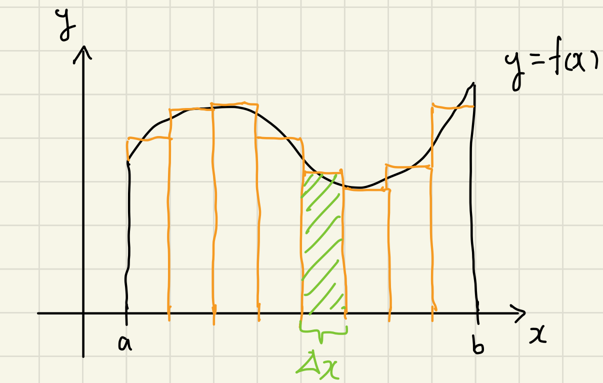
Substitution $\begin{cases} u = x/2 \\ du = \frac{1}{2}\,dx \end{cases}$

$= \frac{1}{4}\int \frac{2\,du}{u^2+1} = \frac{1}{2}\arctan u + C$

$= \frac{1}{2}\arctan\frac{x}{2} + C$

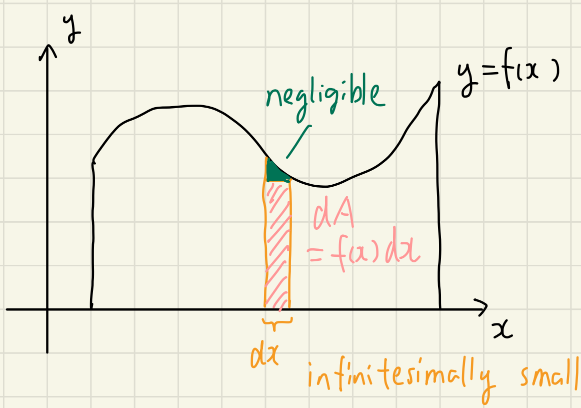
10.1 Volumes as Integrals

Riemann Integral



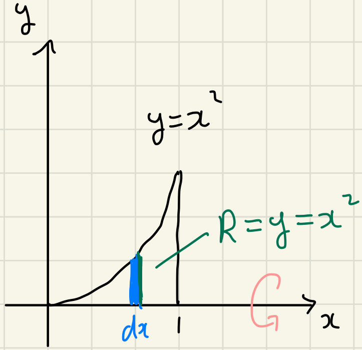
"infinitesimally small"

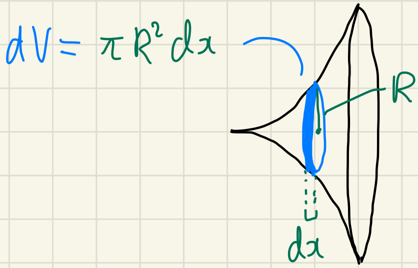
$$\lim_{\Delta x \to 0} \sum_i f(x)\,\Delta x = \int_a^b f(x)\,dx$$



$$A = \int_a^b dA = \int_a^b f(x)\,dx$$

Ex. Let R be the region on the first quadrant between the graph of $y = x^2$ and the x-axis, from $x = 0$ to $x = 1$. Rotate R around the x-axis. Compute the volume of the solid this creates.





$$V = \int_0^1 \left(\pi (x^2)^2\right) dx$$

$$= \int_0^1 \pi x^4 \, dx = \pi \cdot \frac{1}{5} x^5 \Big|_0^1$$

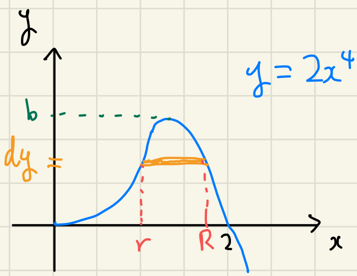
$$= \frac{\pi}{5}$$

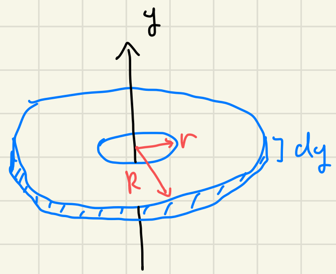
10.2 More Volumes as Integrals

Ex.

- Let R be the region on the first quadrant bounded by the graph of $y = 2x^4 - x^5$ and the x-axis.
- Rotate R around the y-axis.
- Compute the volume of the solid this creates.

Method 1: Use y as a variable





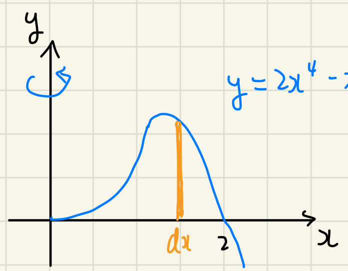
$$dV = [\pi R^2 - \pi r^2]\, dy$$

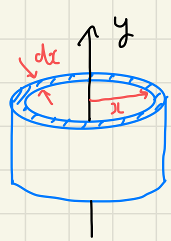
$$V = \int_0^b [\pi R^2 - \pi r^2]$$

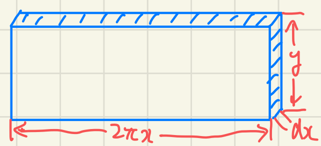
solve for x: $y = 2x^4 - x^5$

infeasible

Method 2: Use x as the variable







$$dV = 2\pi x \cdot y \cdot dx$$

$$V = \int_0^2 2\pi x \cdot (2x^4 - x^5)\, dx$$

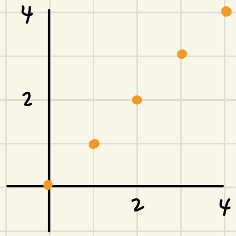
Volume of Solids of Revolution

1. The washer method
   - Cut initial region perpendicular to axis
2. The cylindrical shell method
   - Cut initial region parallel to axis

11.1 What Is a Sequence

Definition

A sequence is a function w/ domain $\mathbb{N}$



graph of a sequence

Convention

- function - "function w/ domain an interval"
  - x is the variable
  - $f(x)$ - value of the function f at x
  - Ex. $f(x) = \frac{3}{x+1}$
- sequence - "function w/ domain $\mathbb{N}$"
  - n is the variable
  - a_n - value of the sequence a at n
  - Ex. $a_n = \frac{3}{n+1} = \left\{ \frac{3}{1}, \frac{3}{2}, \frac{3}{3}, \ldots \right\}$

Ways to Describe a Sequence

1. With an equation

$$a_n = \frac{2^n n!}{n+1}$$

2. With the first few values

$$\{b_n\} = \{1, 2, 4, 8, 16\}$$

- Be careful here
- b_n might be 2^n

  or $2^n + n(n-1)(n-2)(n-3)(n-4)$

3. With words

$$p_n = n^{th} \text{ prime} = \{2, 3, 5, 7, 11, \ldots\}$$

4. With a recurrence relation

Ex. we define the Fibonacci sequence $\{F_n\}_0^\infty$ by

$$F_0 = 1$$

$$F_1 = 1$$

$$\forall n \geq 2, \quad F_n = F_{n-1} + F_{n-2}$$

- $F_2 = F_1 + F_0 = 1 + 1 = 2$
- $F_3 = F_2 + F_1 = 2 + 1 = 3$

Notation

- The "complete" notation for a sequence is

$$\{a_n\}_{n=0}^\infty$$

- We can abuse notation and write

$$\{a_n\}_{n\geq 0}, \ \{a_n\}_n^{\infty}, \ \{a_n\}_n, \ \{a_n\}_0^{\infty}, \ \{a_n\}, \ (a_n)$$

- Ex. $\left\{\frac{n}{k}\right\}_{k=2}^{\infty} = \left\{\frac{n}{2}, \frac{n}{3}, \frac{n}{4}, \ldots\right\}$

<u>General Definition</u>

A sequence is a function w/ domain

$$\{n \in \mathbb{Z} \mid n \geq n_0\} = \{n_0, n_0+1, n_0+2, \ldots\}$$

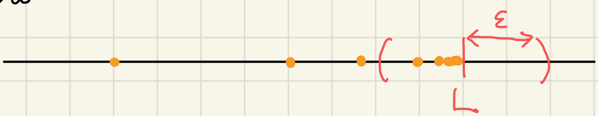
for some fixed $n_0 \in \mathbb{Z}$

- Ex. $\left\{\frac{1}{\ln n}\right\}_{n=2}^{\infty}$

11.2 The Limit of a Sequence

Ex. $\left\{ \frac{n}{n+1} \right\}_{n=0}^{\infty} = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \dots \right\}$

- $\lim_{n \to \infty} a_n = L$ means...



a "tail" of the sequence must stay inside

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t.

$\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow L - \varepsilon < a_n < L + \varepsilon$

Definition

The sequence $\{a_n\}_{n=0}^{\infty}$ converges to the number $L \in \mathbb{R}$ means

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t.

$\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow L - \varepsilon < a_n < L + \varepsilon$

- Equivalently, every open interval centered at L contains a tail of the sequence
  - Tail - all terms of the sequence after the

first few ones

- We write it $\lim_{n \to \infty} a_n = L$ or $a_n \to L$
    - By convention, we imply that $n \in \mathbb{N}$
- We say that the limit of $\{a_n\}_{n=0}^{\infty}$ is L
- $L - \varepsilon < a_n < L + \varepsilon$ is equivalent to $|L - a_n| < \varepsilon$

<u>Definitions</u>

A sequence is

- convergent when it has a limit
- divergent when it doesn't

$$
\begin{cases}
\text{convergent: } \{1/n\} = \{1, 1/2, 1/3, \ldots\} \\
\text{divergent} \begin{cases}
\text{to } \infty: \{n^2\} = \{1, 4, 9, \ldots\} \\
\text{to } -\infty: \{-n\} = \{-1, -2, -3, \ldots\} \\
\text{"oscillating": } \{(-1)^n\} = \{1, -1, 1, -1, \ldots\}
\end{cases}
\end{cases}
$$

11.3 Properties of Limits of Sequences

Limits of Sequences vs. Functions: What Works the Same

- Limit laws does
- The Squeeze Theorem does
- L'hopital's Rule doesn't
  - We cannot take derivatives of sequences
    - e.g. $\frac{d}{dn} n! = ???$

Sequences from Functions

- Let $c \in \mathbb{Z}$. Let f be a function defined on $[c, \infty)$.

  Define the sequence $\{a_n\}_{n=c}^{\infty}$ by $a_n = f(n)$
  - If $\lim_{x \to \infty} f(x) = L$, then $\lim_{n \to \infty} a_n = L$
  - If $\lim_{x \to \infty} f(x) = \infty$, then $\lim_{n \to \infty} a_n = \infty$
  - If $\lim_{x \to \infty} f(x)$ DNE, then $\lim_{n \to \infty} a_n$

    may or may not exist

Limit of a Composition

Theorem. Let $\{a_n\}$ be a sequence. Let f be a function.

Let $L \in \mathbb{R}$.

If $\begin{cases} a_n \to L \\ f \text{ is continuous at } L \end{cases}$

Then $f(a_n) \to f(L)$

Ex. Compute $\lim\limits_{n\to\infty} e^{1/n!}$

- $1/n! \to 0$
- $f(x) = e^x$ is continuous
- Therefore $e^{1/n!} \to e^0 = 1$

11.4 Monotonic and Bounded Functions

Definitions: Monotonic Sequences

A sequence $\{a_n\}_{n=0}^{\infty}$ is

- increasing when $\forall n \in \mathbb{N}$, $a_n < a_{n+1}$
  - Or equivalently, when $\forall n, m \in \mathbb{N}$,

    $n < m \implies a_n < a_m$
  - Ex. $\{n^2\}_{n=0}^{\infty} = \{0, 1, 4, 9, \dots\}$
- decreasing when $\forall n \in \mathbb{N}$, $a_n > a_{n+1}$
  - Ex. $\left\{\frac{1}{n+1}\right\}_{n=0}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$
- non-decreasing when $\forall n \in \mathbb{N}$, $a_n \leq a_{n+1}$
- non-increasing when $\forall n \in \mathbb{N}$, $a_n \geq a_{n+1}$
- monotonic when it is any of the above

- Ex. Is $\{n^3 e^{-n}\}_{n=0}^{\infty}$ monotonic?
  - This sequence comes from a function w/ domain $\mathbb{R}$
  - Let $f(x) = x^3 e^{-x}$. $f'(x) = x^2(3-x)e^{-x}$
  - f is decreasing on $[3, \infty)$

- The sequence $\{n^3 e^{-n}\}_{n=3}^{\infty}$ is decreasing
- The sequence $\{n^3 e^{-n}\}_{n=0}^{\infty}$ is eventually decreasing

<u>Definition</u>

The sequence $\{a_n\}_{n=0}^{\infty}$ is eventually decreasing when

$\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geq n_0 \Rightarrow a_n > a_{n+1}$

<u>Definition: Bounded Sequences</u>

A sequence $\{a_n\}_{n=0}^{\infty}$ is

- bounded below when $\exists A \in \mathbb{R}$ s.t.

  $\forall n \in \mathbb{N}$, $A \leq a_n$
- bounded above when $\exists B \in \mathbb{R}$ s.t.

  $\forall n \in \mathbb{N}$, $a_n < B$
- bounded when it is both bounded below and

  bounded above

<u>Theorem 1</u>

If a sequence is convergent,

Then it is bounded

<u>Theorem 2 (The Monotone Convergence Theorem for Sequences)</u>

If a sequence is

- (eventually) monotonic, and
- bounded

Then it is convergent

<u>Theorem 2A</u>

If a sequence is

- (eventually) non-decreasing, and
- bounded above

Then it is convergent

<u>Theorem 2B</u>

If a sequence is

- (eventually) non-increasing, and
- bounded below

Then it is convergent

<u>Theorem 3</u>

If a sequence is

- (eventually) increasing, and
- not bounded above

Then it is divergent to ∞

$$\text{A sequence may be } \begin{cases} \text{convergent} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{``oscillating''} \end{cases} \end{cases}$$

$$\text{An increasing sequence may be } \begin{cases} \text{convergent} \\ \text{divergent to } \infty \end{cases}$$

11.5 Every Convergent Sequence is Bounded

Theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.

If $\{a_n\}_{n=0}^{\infty}$ is convergent

Then $\{a_n\}_{n=0}^{\infty}$ is bounded

Definitions

- $\{a_n\}_{n=0}^{\infty}$ is bounded means there exists $A, B \in \mathbb{R}$ s.t.

  $\forall n \in \mathbb{N}, \ A \leq a_n \leq B$

- $\{a_n\}_{n=0}^{\infty}$ is convergent means there exists $L \in \mathbb{R}$ s.t.

  $\forall \varepsilon > 0, \ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, \ n \geq n_0 \Rightarrow |a_n - L| < \varepsilon$

  - All terms of the sequence, except finitely many, are close to L

Structure of the Proof

WTS $\exists A, B \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, \ A \leq a_n \leq B$

① Assume the sequence $\{a_n\}_{n=0}^{\infty}$ is convergent

② Take $A = ???$, $B = ???$

③ Let $n \in \mathbb{N}$ be arbitrary

④ I have proven that $A \le a_n \le B$

Rough Work

- We need $\forall n \in \mathbb{N}$, $A \le a_n \le B$

- From the definition of $\lim\limits_{n\to\infty} a_n = L$:

  $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$,

  $n \ge n_0 \implies L - \varepsilon < a_n < L + \varepsilon$

- Need $A < L - \varepsilon$ (choose one value of ε first)

- Also need $A \le a_0$, $A \le a_1, \dots, A \le a_{n_0 - 1}$

- Take $A = \min\{L - \varepsilon, a_0, a_1, \dots, a_{n_0-1}\}$

- Take $B = \max\{L + \varepsilon, a_0, a_1, \dots, a_{n_0-1}\}$

Proof

- Assume the sequence $\{a_n\}_{n=0}^{\infty}$ is convergent.

  Let L be the limit.

- I choose $\varepsilon = 1$ in the definition of $L = \lim\limits_{n\to\infty} a_n$

- We know $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$,

  $n \ge n_0 \implies L - 1 < a_n < L + 1$ (1)

- I take $\begin{cases} A = \min\{L-1, a_0, a_1, \dots, a_{n_0-1}\} \\ B = \max\{L+1, a_0, a_1, \dots, a_{n_0-1}\} \end{cases}$

- I will show that $\forall n \in \mathbb{N}$, $A \leq a_n \leq B$
- Let $n \in \mathbb{N}$
  - If $n \geq n_0$, then from (1),

    $A \leq L-1 < a_n < L+1 \leq B$
  - If $n < n_0$, then by definition of A and B,

    $A \leq a_n \leq B$

$\square$

Note: orange words outline the structure

11.6 The Monotone Convergence Theorem for Sequences

- The general theorem:

  If a sequence is (eventually) monotonic and bounded

  Then it is convergent

- Particular case we'll prove:

  If a sequence is increasing and bounded above

  Then it is convergent

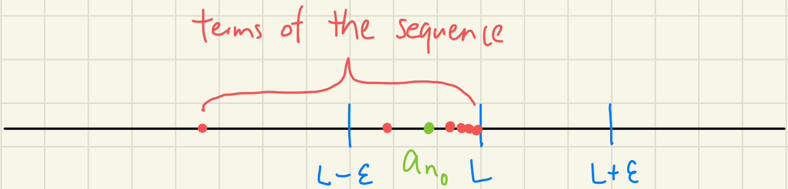
Structure of the Proof

- Assume $\{a_n\}_{n=0}^{\infty}$ is increasing and bounded above
- WTS $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$,

  $n \geq n_0 \Rightarrow L - \varepsilon < a_n < L + \varepsilon$
- I take $L = ???$
- Fix an arbitrary $\varepsilon > 0$
- I take $n_0 = ???$
- Fix an arbitrary $n \in \mathbb{N}$
- Assume $n \geq n_0$. I'll show $L - \varepsilon < a_n < L + \varepsilon$

Rough Work

- Increasing, bounded above sequence



- $L =$ sup of the sequence
- $L - \varepsilon$ is not an upper bound to the sequence
- $\exists n_0 \in \mathbb{N}$ s.t. $L - \varepsilon < a_{n_0} \leq L$
- $\forall n \geq n_0$, $a_{n_0} \leq a_n \leq L$

Proof

- Let $\{a_n\}_{n=0}^{\infty}$ be an increasing, bounded above sequence
- Consider the set $A = \{a_n \mid n \in \mathbb{N}\}$.

  It is nonempty and bounded above, so

  it has a supremum
- I take $L = \sup A$. I will prove that $L = \lim\limits_{n\to\infty} a_n$
- Fix an arbitrary $\varepsilon > 0$
- By definition of supremum, $\exists n_0 \in \mathbb{N}$ s.t. $L - \varepsilon < a_{n_0}$.

  I take that value of n_0

- Fix $n \in \mathbb{N}$. Assume $n \geq n_0$. WTS $L - \varepsilon < a_n < L + \varepsilon$
    - We know $L - \varepsilon < a_{n_0}$
    - Because the sequence is increasing, $a_{n_0} \leq a_n$
    - By definition of supremum, $a_n \leq L$
- Thus $L - \varepsilon < a_{n_0} \leq a_n \leq L < L + \varepsilon$

$\square$

Note: orange words outline the structure

11.7 The Big Theorem

- Ex. $\lim\limits_{n\to\infty} \dfrac{n^3+2n+1}{5n^3+3n^2} = \dfrac{1}{5}$

Justification: $L = \lim\limits_{n\to\infty} \dfrac{\cancel{n^3}\left[1+2n/n^3+1/n^3\right]}{\cancel{n^3}\left[5+3n^2/n^3\right]}$

$$= \lim_{n\to\infty} \frac{1+\overbrace{2/n^2}^{0}+\underbrace{1/n^3}_{0}}{5+\underbrace{3/n}_{0}} = \frac{1+0}{5+0} = \frac{1}{5}$$

- Note: $\lim\limits_{n\to\infty} \dfrac{n}{n^3} = \lim\limits_{n\to\infty} \dfrac{1}{n^2} = 0$

- Definition. Let $\{a_n\}$, $\{b_n\}$ be positive integers

$a_n \ll b_n$ means $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$

  - "a_n is much smaller than b_n"
  - "b_n is much larger than a_n"

- Ex. $n \ll n^3$ because $\lim\limits_{n\to\infty} \dfrac{n}{n^3} = 0$

- Theorem (The Big Theorem).

$$\ln n \ll \underset{(a>0)}{n^a} \ll \underset{(c>1)}{c^n} \ll n! \ll n^n$$

- Ex. $\lim_{n\to\infty} \frac{e^n + 2n^{100}}{\ln n + 5e^n} = \frac{1}{5}$

Justification: $L = \lim_{n\to\infty} \frac{e^n\left[1 + \underbrace{n^{100}/e^n}_{0}\right]}{e^n\left[\underbrace{\ln n/e^n}_{0} + 5\right]} = \frac{1+0}{0+5} = \frac{1}{5}$

11.8 Proof of the Big Theorem

- Theorem (The Big Theorem).

$$\ln n \ll n^a \ll c^n \ll n! \ll n^n$$

for every $a > 0$, $c > 1$

- We need to prove 4 claims

1. $\lim_{n\to\infty} \frac{\ln n}{n^a} = 0$
3. $\lim_{n\to\infty} \frac{c^n}{n!} = 0$
2. $\lim_{n\to\infty} \frac{n^a}{c^n} = 0$
4. $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

Use L'Hôpital's Rule

Harder as $n!$ only defined for $\mathbb{N}$

- Proof (of 3). Fix $c > 1$. Call $p_n = \frac{c^n}{n!}$.
  - $p_{n+1} = \frac{c}{n+1} p_n$
  - $\{p_n\}$ is eventually decreasing: $\forall n \geq c$, $p_{n+1} < p_n$
  - $\{p_n\}$ is bounded below by 0
  - By MCT $\{p_n\}$ is convergent. Call $L = \lim_{n\to\infty} p_n$

$$\lim_{n\to\infty} p_{n+1} = \lim_{n\to\infty} \frac{c}{n+1} \cdot \lim_{n\to\infty} p_n$$

$$L = 0 \cdot L$$

- We can only do this because we know all those limits exist

Thus $L = 0$. $\square$

- Proof (of 4)

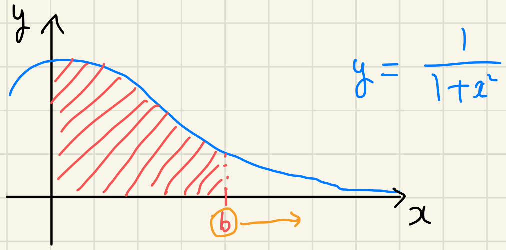
$$\frac{n!}{n^n} = \frac{1}{n}\underbrace{\left[\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n}\right]}_{\leq 1} \leq \frac{1}{n}$$

Thus, we have for all n: $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$

We know that $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$

Hence, by Squeeze Theorem, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. $\square$

12.1 Improper Integrals: Definition and Example 1



- $$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \left[\arctan x \,\Big|_{x=0}^{x=b} \right]$$

$$= \lim_{b\to\infty} \left[\arctan b - \underbrace{\arctan 0}_{0} \right] = \frac{\pi}{2}$$

Improper Integral "Type 1" (Unbounded Domain)

Definition. Let $a \in \mathbb{R}$.

Let f be a continuous function on $[a, \infty)$.

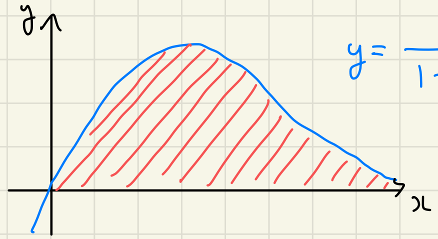
We define the integral of f from 0 to ∞ as

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \left[\int_a^b f(x)\,dx \right]$$

assuming this limit exists.

- The integral is convergent when the limit exists
- The integral is divergent when the limit DNE

12.2 Improper Integrals: Example 2



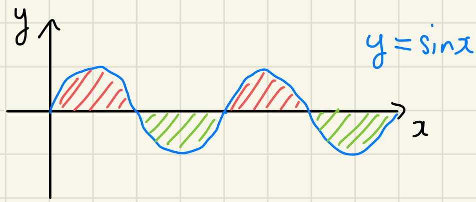
- $$\int_0^\infty \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_0^b \frac{2x}{x^2+1}\,dx$$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln(x^2+1)\Big|_{x=0}^{x=b}\right]$$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln(b^2+1) - \frac{1}{2}\underbrace{\ln 1}_{0}\right] = \infty$$

(Div.)

12.3 Improper Integrals : Example 3



$$\int_0^\infty \sin x \, dx = \lim_{b \to \infty} \int_0^b \sin x \, dx$$

$$= \lim_{b \to \infty} \left[-\cos x \Big|_{x=0}^{x=b} \right]$$

$$= \lim_{b \to \infty} \left[-\cos b + \underbrace{\cos 0}_{1} \right]$$

DNE
(divergent)

12.4 Improper Integrals: Example 4 ("p-Functions")

For which $p \in \mathbb{R}$ is $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

$$I_p = \int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$$

If $p \neq 1$:

- $$I_p = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1} \Bigg|_{x=1}^{x=b} \right]$$

$$= \lim_{b\to\infty} \left[\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1} \right] = \begin{cases} \infty \ \text{(Div)} & \text{if } -p+1 > 0 \ (p<1) \\ \frac{1}{p-1} \ \text{(Conv)} & \text{if } -p+1 < 0 \ (p>1) \end{cases}$$

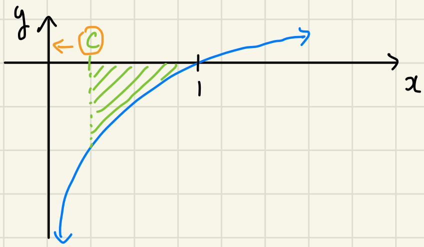
If $p = 1$:

- $$I_p = \lim_{b\to\infty} \left[\ln x \Big|_{x=1}^{x=b} \right] = \cdots = \infty \quad \text{(Div)}$$

Summary

- $\int_1^\infty \frac{dx}{x^p}$ is convergent $\iff p > 1$
- $\int_1^\infty \frac{dx}{x^p} = \infty \iff p \leq 1$

12.5 Improper Integrals: Example 5



- $\displaystyle\int_0^1 \ln x\,dx = \lim_{c\to 0^+} \int_c^1 \ln x\,dx$

$\displaystyle = \lim_{c\to 0^+} \left[(x\ln x - x)\Big|_{x=c}^{x=1} \right]$

$\displaystyle = \lim_{c\to 0^+} \Big[(1\cdot \underbrace{\ln 1}_{0} - 1) - (\underbrace{c\ln c}_{0} - \underbrace{c}_{0}) \Big] = -1$

- $\displaystyle\lim_{c\to 0^+} \underbrace{[c\cdot \ln c]}_{0\cdot(-\infty)} = \lim_{c\to 0^+} \frac{\ln c}{1/c} \qquad \left(\frac{-\infty}{\infty}\right)$

$\displaystyle (L'H) = \lim_{c\to 0^+} \frac{1/c}{-1/c^2} = \lim_{c\to 0^+} [-c] = \boxed{0}$

Improper Integral "Type 2" (Unbounded Function)

Definition. Let $a < b$.

Let f be a continuous function on $(a, b]$.

We define the integral of f from a to b as

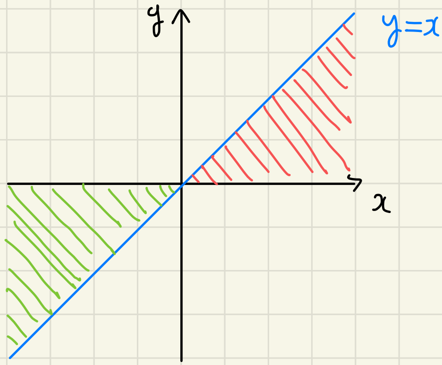
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

assuming this limit exists.

- The integral is convergent when the limit exists
- The integral is divergent when the limit DNE

12.6 Doubly Improper Integrals

Ex. $I = \int_{-\infty}^{\infty} x\,dx$



- Approach 1: $\lim_{R \to \infty} \int_{-R}^{R} x\,dx$

$$= \lim_{R \to \infty} \left[\frac{R^2}{2} - \frac{(-R^2)}{2} \right] = 0$$

- Approach 2: $\lim_{R \to \infty} \int_{1-R}^{1+R} x\,dx = \cdots = \infty$

- Both are incorrect (and they don't agree)

- Never cancel out ∞ and $-\infty$

- Approach 3: $I = \int_{-\infty}^{\infty} x\,dx$

Subst $\begin{cases} x = u+1 \\ dx = du \end{cases}$

$$= \int_{-\infty}^{\infty} (u+1)\,du = \underbrace{\int_{-\infty}^{\infty} u\,du}_{I} + \underbrace{\int_{-\infty}^{\infty} 1\,du}_{\infty}$$

$\Rightarrow I = I + \infty$ (can be easily contradicted)

- Rigorous approach:

$$I = \int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$$

$$= \underbrace{\lim_{R\to\infty} \int_{-R}^{0} x\,dx}_{-\infty} + \underbrace{\lim_{R\to\infty} \int_{0}^{R} x\,dx}_{\infty}$$

- For I to be convergent, we need both limits to exist separately
- In this case, I is divergent

General Strategy

- To calculate an integral A that is multiply improper:
  1. Break the domain into pieces so that the integral over each piece is improper only at one endpoint
  2. If each piece is convergent separately, then A is convergent
  3. If at least 1 piece is divergent, then A is divergent (not a number)
- The outcome won't depend on how we break the original domain

More Examples

$$\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x^2)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x^2)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x^2)}$$

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

12.7 The Basic Comparison Test for Improper Integrals

Improper Integrals of Positive Functions

- Let f be continuous on $[a, \infty)$, Let

  $$A = \int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

  Then A may be $\begin{cases} \text{convergent (to a number)} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{"oscillating"} \end{cases} \end{cases}$

- Assume $\forall x \geq a,\ f(x) > 0$

  Now A may only be $\begin{cases} \text{convergent (to a number)} \\ \text{divergent to } \infty \end{cases}$

  Proof. Call $F(b) = \int_a^b f(x)\,dx$.

  Then $A = \int_a^\infty f(x)\,dx = \lim_{b\to\infty} F(b)$

  By FTC, $F'(b) = f(b) > 0$, so F increasing.

  - If F is bounded above, then $\lim_{b\to\infty} F(b)$ exists (MCT)
  - If F is unbounded above, then $\lim_{b\to\infty} F(b) = \infty$

Notation (Only for improper integral of positive functions)

- $\int_a^\infty f(x)\,dx = \infty$ means divergent
- $\int_a^\infty f(x)\,dx < \infty$ means convergent

The Basic Comparison Test

Let $a \in \mathbb{R}$. Let f, g be continuous functions on $[a, \infty)$.

- Assume, for every $x \geq a$, $0 \leq f(x) \leq g(x)$
- Then:
    - If $\int_0^\infty f(x)\,dx = \infty$, then $\int_0^\infty g(x)\,dx = \infty$
    - If $\int_0^\infty g(x)\,dx < \infty$, then $\int_0^\infty f(x)\,dx < \infty$

12.8 The Basic Comparison Test: Examples

$$\int_1^\infty \frac{dx}{x^p} < \infty \iff p > 1$$

$$\int_1^\infty \frac{dx}{x^p} = \infty \iff p \leq 1$$

Ex. 1 $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

- For $x \geq 1$, $0 \leq \dfrac{\sin^2 x}{x^2} \leq \dfrac{1}{x^2}$

$$\int_1^\infty \frac{1}{x^2}\,dx < \infty$$

By BCT, $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx < \infty$

Ex. 2 $\displaystyle\int_1^\infty \frac{\sin^2 x}{x}\,dx$

- For $x \geq 1$, $0 \leq \dfrac{\sin^2 x}{x} \leq \dfrac{1}{x}$

$$\int_1^\infty \frac{1}{x}\,dx = \infty$$

BCT doesn't help

Ex. 3 $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

- Big Thm: $\ln x \ll x^{1/2}$ $\qquad \displaystyle\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = 0$

- For large x, $\dfrac{\ln x}{x^{1/2}} < 1$, $\ln x < x^{1/2}$

- $0 \leq \frac{\ln x}{x^2} < \frac{x^{1/2}}{x^2} < \frac{1}{x^{3/2}}$

  $\int_1^\infty \frac{1}{x^{3/2}}\,dx < \infty$

By BCT

$\int_1^\infty \frac{\ln x}{x^2}\,dx < \infty$

- Try $x^{1/2}$ instead of x

12.9 The Limit-Comparison Test for Improper Integrals

Limit-Comparison Test

Let $a \in \mathbb{R}$.

Let f and g be positive, continuous functions on $[a, \infty)$

If the limit $L = \lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists (is a number), and $L > 0$

Then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ are

both convergent or both divergent.

Ex. 1 $\int_1^{\infty} \frac{x^2+3x}{\sqrt{x^5+1}} dx$

- Call $f(x) = \frac{x^2+3x}{\sqrt{x^5+1}} \underset{x\to\infty}{\sim} \frac{x^2}{x^{5/2}} = \frac{1}{x^{1/2}} = g(x)$

- $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{x^2+3x}{\sqrt{x^5+1}}}{\frac{1}{x^{1/2}}} = \cdots = 1$

- $\int_1^\infty g(x)dx = \int \frac{1}{x^{1/2}} dx = \infty \to$ By LCT, $\int_1^\infty f(x)dx = \infty$

Ex. 2 $\int_1^{\infty} \sin\frac{1}{x^2} dx$

- As $x \to \infty$, $\frac{1}{x^2} \to 0$

- $\lim_{t \to 0} \frac{\sin t}{t} = 1$

- Call $\begin{cases} f(x) = \sin \frac{1}{x^2} \\ g(x) = \frac{1}{x^2} \end{cases}$

- $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{\sin 1/x^2}{1/x^2} = \cdots = 1$

- $\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^2} dx$ conv. $\to$ By LCT,

  $\int_1^\infty f(x)dx$ conv.

12.10 The Limit Comparison Test - Proof

Theorem (Limit-Comparison Test)

Let $a \in \mathbb{R}$.

Let f and g be positive, continuous functions on $[a, \infty)$

If the limit $L = \lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists (is a number), and $L > 0$

Then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ are

both convergent or both divergent.

Rough Work

- $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ means that for large x, $\frac{f(x)}{g(x)} \approx L$
- $\forall \varepsilon > 0, \exists M \in \mathbb{R}$ s.t. $x \geq M \Rightarrow \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$
  - $L - \varepsilon < \frac{f(x)}{g(x)} < L + \varepsilon$
    - $f(x) < (L + \varepsilon) g(x)$
    - $(L - \varepsilon) g(x) < f(x)$
  - Use BCT in both directions
  - Pick $\varepsilon = \frac{L}{2}$ so that $L - \varepsilon > 0$

Proof

- Since f and g are positive,

  each of $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ is convergent or ∞

- I take $\varepsilon = \frac{L}{2} > 0$ in the definition of $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$:

  $\exists M \in \mathbb{R}$ s.t. $x \geq M \Rightarrow \left| \frac{f(x)}{g(x)} - L \right| < \frac{L}{2}$

- Then, for all $x \geq M$, $\frac{L}{2} < \frac{f(x)}{g(x)} < \frac{3L}{2}$

- Notice that $\int_a^\infty \cdots = \int_a^M \cdots + \int_M^\infty \cdots$

  Thus, $\int_a^\infty \cdots$ is convergent iff $\int_M^\infty \cdots$ is convergent

① Assume $\int_M^\infty g(x)\,dx$ is convergent.

For all $x > M$, $0 < f(x) < \frac{3L}{2} g(x)$

By BCT, $\int_M^\infty f(x)\,dx$ is also convergent.

② Assume $\int_M^\infty f(x)\,dx$ is convergent

For all $x > M$, $0 < \frac{L}{2} g(x) < f(x)$

By BCT, $\int_M^\infty g(x)\,dx$ is convergent.

13.1 Infinite Sums: a Cautionary Tale

- Ex. $S = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

$$xS = x + x^2 + x^3 + \cdots$$

$$S - xS = 1$$

$$S = \frac{1}{1-x} \quad (\text{assuming } x \neq 1)$$

When $x = 2$: $\quad S = \frac{1}{1-2} = -1$

$$S = 1 + 2 + 4 + 8 + \cdots$$

Contradiction

- Ex. $T = \sum_{n=0}^{\infty} (-1)^n$

$$T = (1-1) + (1-1) + (1-1) + \cdots$$

$$= 0 + 0 + 0 + \cdots$$

$$= 0$$

$$T = 1 + (-1+1) + (-1+1) + \cdots$$

$$= 1 + 0 + 0 + \cdots$$

$$= 1$$

$T = 0 = 1$ Contradiction

Infinite Sums: The Right Way

1. What does adding up infinitely many numbers means?
   - Define what an infinite sum – a "series" – is
2. When is a series equal to a number?
   - When is a series convergent?
3. Which properties of finite sums carry over to infinite sums?

13.2 The Definition of Series

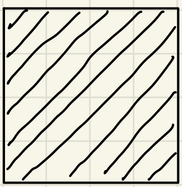
- A series is an infinite sum:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

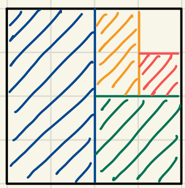
- A sequence is an infinite list.

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots$$

- Ex. $S = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$



1



$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1\,?$

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_k = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} = \sum_{n=1}^{k} \frac{1}{2^n} = 1 - \frac{1}{2^k}$$

$$S = \lim_{k \to \infty} S_k = 1$$

Definition

To sum the series $\sum_{n=1}^{\infty} a_n$:

Construct a sequence of partial sums $\{S_k\}_{n=1}^{\infty}$

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_k = a_1 + \cdots + a_k = \sum_{n=1}^{k} a_n$$

Then $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k$

- The series is convergent when this limit exists, and divergent otherwise

Improper Integrals vs. Series

- $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$
- $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n$

13.3 A Telescopic Series

- Ex. $S = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$

$$S_1 = \frac{1}{1^2+1} = \frac{1}{2}$$

$$S_2 = S_1 + \frac{1}{2^2+2} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = S_2 + \frac{1}{3^2+3} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$$

Conjecture: $\forall k \geq 1$, $S_k = \frac{k}{k+1}$

Pf: By induction

- Alternative approach

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+n} = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= \left[1 - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \cdots + \left[\frac{1}{k} - \frac{1}{k-1}\right]$$

$$= 1 - \frac{1}{k+1} = \frac{k}{k+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

13.4 Examples of Divergent Series From the Definition

- Ex. $S = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots = \infty$?

$$S = \lim_{k \to \infty} S_k$$

$$S_k = \sum_{n=1}^{k} 1 = \underbrace{1 + 1 + \cdots + 1}_{k \text{ times}} = k$$

$$S = \lim_{k \to \infty} k = \infty$$

$\sum_{n=1}^{\infty} 1$ is divergent

- Ex. $S = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots$

$$S = \lim_{k \to \infty} S_k$$

$S_0 = 1$

$S_1 = 1 - 1 = 0$

$S_2 = 1 - 1 + 1 = 1$

$\vdots$

$$S_k = \sum_{n=0}^{k} (-1)^n$$

$$S_k = \begin{cases} 0 & \text{if } k \text{ odd} \\ 1 & \text{if } k \text{ even} \end{cases}$$

$\lim_{k \to \infty} S_k$ DNE. $\sum_{n=0}^{\infty} (-1)^n$ is divergent

13.5 Geometric Series

Ex. For which $x \in \mathbb{R}$ is the series $\sum_{n=0}^{\infty} x^n$ convergent?

What is the value?

Let $x \in \mathbb{R}$. $S = \sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} S_k$, where $S_k = \sum_{n=0}^{k} x^n$

$$S_k = 1 + \boxed{x + x^2 + \cdots + x^k}$$

$$x S_k = \boxed{x + x^2 + x^3 + \cdots} + x^{k+1}$$

$$S_k - x S_k = 1 - x^{k+1} \qquad S_k = \frac{1 - x^{k+1}}{1-x} \quad (\text{if } x \neq 1)$$

$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1 - x^{k+1}}{1-x} = \frac{1 - \lim_{k\to\infty} x^{k+1}}{1-x}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{if } -1 < x < 1$$

$\sum_{n=0}^{\infty} x^n$ Divergent otherwise

13.6 Series Are Linear

$$\sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$$

$$\sum_{n=0}^{\infty} (ca_n) = c \sum_{n=0}^{\infty} a_n$$

A Surprising False Property

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots = \ln 2$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2} \ln 2$$

- We just reordered the elements

Theorem 1

If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent

Then $\sum_{n=0}^{\infty} (a_n + b_n)$ is also convergent, and

$$\sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$$

Theorem 2

Let $c \in \mathbb{R}$.

If $\sum_{n=0}^{\infty} a_n$ is convergent

Then $\sum_{n=0}^{\infty} (ca_n)$ is also convergent, and

$$\sum_{n=0}^{\infty} (ca_n) = c \sum_{n=0}^{\infty} a_n$$

Proof of Theorem 1

- $\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} S_k$, where $S_k = \sum_{n=0}^{k} a_n$
- $\sum_{n=0}^{\infty} b_n = \lim_{k\to\infty} T_k$, where $T_k = \sum_{n=0}^{k} b_n$
- $\sum_{n=0}^{\infty} (a_n + b_n) = \lim_{k\to\infty} R_k$, where $R_k = \sum_{n=0}^{k} (a_n + b_n)$
- By properties of finite sums, $R_k = S_k + T_k$
- By hypothesis, the first two limits exist
- Therefore, by limit laws.

$$\lim_{k\to\infty} R_k = \lim_{k\to\infty} S_k = \lim_{k\to\infty} T_k \qquad \square$$

13.7 The Tail of a Series

Theorem

$\sum_{n=0}^{\infty} a_n$ is convergent $\iff$ $\sum_{n=1}^{\infty} a_n$ is convergent

Moreover, in that case,

$$\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n$$

Notation: " $\sum_{n}^{\infty} a_n$ is convergent/divergent "

Typical Theorem

If for all $n \in \mathbb{N}$, (something about a_n)

Then the series $\sum_{n}^{\infty} a_n$ is convergent.

Generalized Theorem

If for all "large" $n \in \mathbb{N}$, (something about a_n)

$\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geq n_0 \Rightarrow$

Then the series $\sum_{n}^{\infty} a_n$ is convergent.

13.8 A Necessary Condition for Convergence of Series

Sequences vs. Series

- $\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} S_k$, where $S_k = \sum_{n=0}^{k} a_n$
- $\sum_{n=0}^{\infty} a_n$ is convergent $\iff$ $\{S_n\}_{n=0}^{\infty}$ is convergent

Theorem

If the series $\sum_{n=0}^{\infty} a_n$ is convergent,

Then $\lim_{n\to\infty} a_n = 0$

- If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=0}^{\infty} a_n$ may be convergent or divergent.
- If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=0}^{\infty} a_n$ is divergent

Examples

- $\sum_{n=0}^{\infty} \frac{\arctan n}{1+e^{-n}}$

  - $\lim_{n\to\infty} \frac{\arctan n}{1+e^{-n}} = \frac{\pi}{2} \neq 0$

    so $\sum_{n=0}^{\infty} \frac{\arctan n}{1+e^{-n}}$ is divergent

- $\sum_{n=0}^{\infty} \frac{1}{n}$ and $\sum_{n=0}^{\infty} \frac{1}{n^2}$

  - $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2} = 0$

    so we cannot draw any conclusion yet

Proof

- Assume $\sum_{n=0}^{\infty} a_n$ is convergent. WTS $\lim_{n\to\infty} a_n = 0$

- This means the following limit exists.

  $$S = \lim_{k\to\infty} S_k, \quad \text{where } S_k = \sum_{n=0}^{k} a_n$$

- Notice for every $n \geq 1$, $a_n = S_n - S_{n-1}$

- Then we can use limit laws:

  $$\lim_{n\to\infty} a_n = \left[\lim_{n\to\infty} S_n\right] - \left[\lim_{n\to\infty} S_{n-1}\right] = S - S = 0$$

$\square$

13.9 Positive Series

Definition

- We call series $\sum_{n=0}^{\infty} a_n$ positive when $\forall n \in \mathbb{N}$, $a_n > 0$
- We call series $\sum_{n=0}^{\infty} a_n$ negative when $\forall n \in \mathbb{N}$, $a_n < 0$
- We call series $\sum_{n=0}^{\infty} a_n$ non-negative when $\forall n \in \mathbb{N}$, $a_n \geq 0$
- etc.

Positive Series

A positive series may be $\begin{cases} \text{convergent} \\ \text{divergent to } \infty \end{cases}$

Proof

- In general, $\sum_{n=0}^{\infty} a_n = \lim_{k \to \infty} S_k$, where $S_k = \sum_{n=0}^{k} a_n$
- Assume the series $\sum_{n=0}^{\infty} a_n$ is positive.

  Then the sequence $\{S_n\}_{n=0}^{\infty}$ is increasing: $S_{n+1} - S_n = a_{n+1}$
- Use Monotone Convergence Theorem:

- An increasing, bounded sequence is convergent
- An increasing, unbounded sequence is divergent to ∞

Notation (Only for positive series)

- $\sum_{n}^{\infty} a_n = \infty$ means divergent
- $\sum_{n}^{\infty} a_n < \infty$ means convergent
- Same applies to $\begin{cases} \text{non-negative series} \\ \text{eventually non-negative series} \end{cases}$

How Does This Help?

- To prove a positive series is convergent, we only have to prove it does not diverge to ∞
- Useful theorems:
  1. Integral test
  2. Basic comparison test
  3. Limit-comparison test

13.10 The Integral Test

Question

Let f be a continuous function on $[1, \infty)$.

What is the relation between $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,dx$?

$$\sum_{n=1}^{\infty} f(n) = \lim_{k \to \infty} \sum_{n=1}^{k} f(n)$$

$$\int_1^{\infty} f(n)\,dn = \lim_{b \to \infty} \int_1^{b} f(x)\,dx$$

- We assume
  - f is positive, thus $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,dx$ are convergent to ∞
  - f is decreasing

$\int_1^5 f(x)\,dx$

$\sum_{n=1}^{4} f(n)$

$\sum_{n=2}^{5} f(n)$

$$\sum_{n=2}^{5} f(n) \le \int_1^5 f(x)\,dx \le \sum_{n=1}^{4} f(n)$$

$$\sum_{n=2}^{N} f(n) \le \int_1^N f(x)\,dx \le \sum_{n=1}^{N-1} f(n)$$

$$\sum_{n=2}^{\infty} f(n) \leq \int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} f(n)$$

Theorem (Integral Test)

Let $a \in \mathbb{R}$

Let f be a continuous, positive, decreasing on $[a, \infty)$

Then $\int_a^{\infty} f(x)\,dx$ is convergent $\Longleftrightarrow$ $\sum_n^{\infty} f(n)$ is convergent

- Notation: $\int_a^{\infty} f(x)\,dx \sim \sum_n^{\infty} f(n)$

13.11 The Integral Test: Examples

Theorem (Integral Test)

Let $a \in \mathbb{R}$

Let f be a continuous, positive, decreasing on $[a, \infty)$

Then $\int_a^\infty f(x)\,dx$ is convergent $\Longleftrightarrow$ $\sum_n^\infty f(n)$ is convergent

Ex. 1: "p-series"

For which values of $p > 0$ is the series $\sum_{n=1}^\infty \frac{1}{n^p}$ convergent?

- Let $f(x) = \frac{1}{x^p}$.

  For $x \geq 1$, f is continuous, positive, decreasing.

- By integral test, $\sum_{n=1}^\infty \frac{1}{n^p} \sim \int_1^\infty \frac{1}{x^p}\,dx$

- We know that $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent iff $p > 1$

- Thus, $\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent iff $p > 1$

Ex. 2

Is $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ convergent?

- Let $f(x) = \frac{1}{x \ln x}$.

  It's positive, decreasing, continuous for $x \geq 2$.

- By Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n \ln n} \sim \int_2^{\infty} \frac{1}{x \ln x} dx$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{b \to \infty} \int_2^{b} \frac{1}{x \ln x} dx = \lim_{b \to \infty} \left[\ln(\ln x) \Big|_2^b \right]$$

$$= \lim_{b \to \infty} \left[\ln(\ln b) - \ln(\ln 2) \right] = \infty$$

- Therefore, $\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty$

13.12 Comparison Tests for Series

- Comparison tests for series and improper integrals are the same
    - A positive series may only be convergent or divergent to ∞
    - To prove a positive series is convergent, we only need to prove it is not ∞

<u>Theorem (BCT for Series)</u>

Let $\sum_{n}^{\infty} a_n$ and $\sum_{n}^{\infty} b_n$ be two series

- Assume, for every $n \in \mathbb{N}$, $0 \le a_n \le b_n$
- Then
    1. If $\sum_{n}^{\infty} a_n = \infty$, then $\sum_{n}^{\infty} b_n = \infty$
    2. If $\sum_{n}^{\infty} b_n < \infty$, then $\sum_{n}^{\infty} a_n < \infty$

<u>Theorem (LCT for Series)</u>

Let $\sum_{n}^{\infty} a_n$ and $\sum_{n}^{\infty} b_n$ be two positive series

- If the limit $L = \frac{a_n}{b_n}$ exists and $L > 0$
- Then $\sum_{n}^{\infty} a_n$ and $\sum_{n}^{\infty} b_n$ are both convergent or both divergent

13.13 Alternating Series

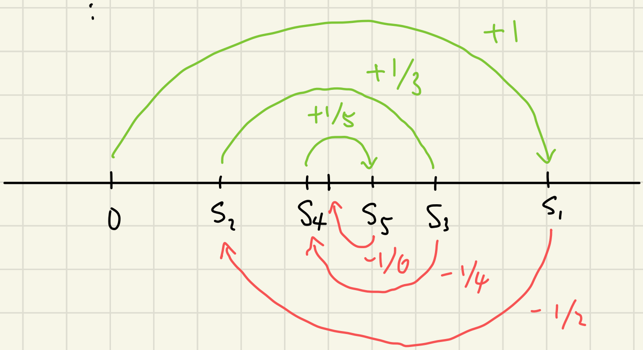
Definition

A series $\sum_{n}^{\infty} a_n$ is alternating when $\forall n,\ a_n a_{n+1} < 0$

- The terms "alternate" between positive and negative

- Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$
  - $S_1 = 1$
  - $S_2 = 1 - \frac{1}{2}$
  - $S_3 = 1 - \frac{1}{2} + \frac{1}{3}$
  - $S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

  $\vdots$



$S_2 < S_4 < S_6 < \cdots < S_5 < S_3 < S_1$

- $\{S_{2n}\}_n$ is increasing and bounded above (by S_1)
- $\{S_{2n+1}\}_n$ is decreasing and bounded below (by S_2)

- By MCT, both are convergent
- Call $A = \lim_{n\to\infty} S_{2n}$, $B = \lim_{n\to\infty} S_{2n+1}$

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} \frac{1}{2n+1}$$

$$B = A + 0$$

<u>Lemma</u>

Let $\{c_n\}_n^\infty$ be a sequence.

- If the sequences of even and odd term

  $\{c_{2n}\}_n^\infty$ and $\{c_{2n+1}\}_n^\infty$ are convergent to the same limit
- Then the full sequence $\{c_n\}_n^\infty$ is also convergent

  to the same limit

<u>Theorem (Alternating Series Test)</u>

Consider a series of the form $\sum_n^\infty (-1)^n b_n$ or $\sum_n^\infty (-1)^{n+1} b_n$

If: 1) $\forall n,\ b_n > 0$

2) the sequence $\{b_n\}_n^\infty$ is decreasing

3) $\lim_{n\to\infty} b_n = 0$

Then the series is convergent.

- Ex. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent

13.14 Estimating the Value of an Alternating Series

Theorem (AST - Part 2)

Consider a series of the form $\sum_n (-1)^n b_n$ or $\sum_n (-1)^{n+1} b_n$

- If (it satisfies the same 3 hypotheses as before)
- Then $|S - S_k| < b_{k+1}$

  where S_k is the k^{th} partial sum in the series

Example

Estimate the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ with an error smaller than 0.001

- By AST, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is convergent
- Actual value: $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \lim_{k\to\infty} S_k$
- Estimate: $S_k = \sum_{n=1}^{k} \frac{(-1)^n}{n^4}$ for some large k
- Error of estimation: $|S - S_k| < \frac{1}{(k+1)^4}$

Need to choose k so that $\frac{1}{(k+1)^4} < 0.001$

$k = 5$ works

- Estimate: $S_5 = -1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} \approx -0.94753$
- Actual value: $S \approx -0.94703$

13.15 Absolute Convergence vs. Conditional Convergence

- Ex. Is $\sum_{n}^{\infty} \frac{\sin n}{n^2}$ convergent?
    - We can't use comparison tests (the series is not positive)
    - The series is also not alternating
    - Consider $\sum_{n}^{\infty} \frac{|\sin n|}{n^2}$
        - We can use comparison tests for this one

Theorem (Absolute Convergence Test)

Let $\sum_{n}^{\infty} a_n$ be a series.

If $\sum_{n}^{\infty} |a_n|$ is convergent, then $\sum_{n}^{\infty} a_n$ is convergent.

Application

- We want to know whether $\sum_{n}^{\infty} a_n$ is convergent
- First, determine if $\sum_{n}^{\infty} |a_n|$ is convergent
    - If yes, then $\sum_{n}^{\infty} a_n$ is convergent
    - If no, then try something else

- Ex. Is $\sum_{n}^{\infty} \frac{\sin n}{n^2}$ convergent?

  - First, look at $\sum_{n}^{\infty} \frac{|\sin n|}{n^2}$

    Since it is a positive series, we can use

    comparison tests

  - $0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$.

    We know $\sum_{n}^{\infty} \frac{1}{n^2}$ is convergent

    By BCT, $\sum_{n}^{\infty} \frac{|\sin n|}{n^2}$ is also convergent.

  - By Absolute Convergence Test,

    $\sum_{n}^{\infty} \frac{\sin n}{n^2}$ is also convergent.

- Ex. Let $p_n = \begin{cases} 1/n & \text{if } n \text{ is prime} \\ -1/n & \text{otherwise} \end{cases}$. Is $\sum_{n}^{\infty} p_n$ convergent?

  - $\sum_{n}^{\infty} |p_n| = \sum_{n}^{\infty} \frac{1}{n} = \infty$

  - The Absolute Convergence Test does not apply.

    We do not know yet whether $\sum_{n}^{\infty} p_n$ is convergent.

Definition

A convergent series $\sum_{n}^{\infty} a_n$ is

- Absolutely convergent when $\sum_{n}^{\infty} |a_n|$ is also convergent
- Conditionally convergent when $\sum_{n}^{\infty} |a_n| = \infty$

	$\sum_{n}^{\infty} \|a_n\| < \infty$	$\sum_{n}^{\infty} \|a_n\| = \infty$
$\sum_{n}^{\infty} a_n$ convergent	Absolutely convergent	Conditionally convergent
$\sum_{n}^{\infty} a_n$ divergent	Impossible	Divergent

- Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent
    - $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent (AST)
    - $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (p-series w/ $p = 1$)

- Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is convergent (AST)

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (p-series w/ $p=2$)

<u>Remark</u>

- Conditionally convergent series are not commutative

13.16 Proof of the Absolute Convergence Test

Theorem (Absolute Convergence Test)

Let $\sum_{n}^{\infty} a_n$ be a series.

If $\sum_{n}^{\infty} |a_n|$ is convergent, then $\sum_{n}^{\infty} a_n$ is convergent.

Notation

- P.T - positive terms
- N.T - negative terms
- Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$
  - $\sum_{n}^{\infty} (P.T) = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$
  - $\sum_{n}^{\infty} (N.T) = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots$

Key Observation

When is $\sum_{n}^{\infty} a_n = \sum_{n}^{\infty} (P.T) + \sum_{n}^{\infty} (N.T)$?

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} \left[\sum_{n=1}^{k} (P.T) + \sum_{n=1}^{k} (N.T) \right]$$

Requires both limits to exist $= \lim_{k\to\infty} \sum_{n=1}^{k} (P.T) + \lim_{k\to\infty} \sum_{n=1}^{k} (N.T)$

$$= \sum_{k=1}^{\infty} (P.T) + \sum_{k=1}^{\infty} (N.T)$$

- If $\sum_{n}^{\infty} (P.T)$ and $\sum_{n}^{\infty} (N.T)$ are both convergent
- Then $\sum_{n}^{\infty} a_n = \sum_{n}^{\infty} (P.T) + \sum_{n}^{\infty} (N.T)$
- Worst case: $\sum_{n}^{\infty} (P.T) = \infty$ and $\sum_{n} (N.T) = -\infty$
  - Then we cannot conclude anything

<u>Proof of Abs. Conv. Test</u>

- Assume $\sum_{n}^{\infty} |a_n| < \infty$
- $\sum_{n}^{\infty} |P.T| \leq \sum_{n}^{\infty} |a_n|$ and $\sum_{n}^{\infty} |N.T| \leq \sum_{n}^{\infty} |a_n|$

By BCT, $\sum_{n}^{\infty} |P.T| < \infty$ and $\sum_{n}^{\infty} |N.T| < \infty$

- Therefore $\sum_{n}^{\infty} (P.T)$ and $\sum_{n}^{\infty} (N.T)$ are convergent
- By key observation $\sum_{n}^{\infty} a_n = \sum_{n}^{\infty} (P.T) + \sum_{n}^{\infty} (N.T)$

Thus $\sum_{n}^{\infty} a_n$ is convergent $\square$

13.17 Infinite Sums are Not Commutative

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots = \ln 2$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2} \ln 2$$

Theorem

- If a series is absolutely convergent, then it can be reordered. This won't change the value of the sum
- If a series is conditionally convergent, then
  - Reordering it may change its value
  - It can be reordered to make it convergent to any number we want
  - It can be reordered to make it divergent in any way we want

- Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

1. $\lim_{n\to\infty} \frac{1}{n} = 0$

2. $-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \cdots = -\infty$

   - b/c $\frac{-1}{2} \sum_{m=1}^{\infty} \frac{1}{m} = -\infty$

3. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \infty$

   - by comparison w/ negative terms

Every conditionally convergent series satisfies these three properties

- Ex. Reorder $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ so the sum is 7

  1. Add positive terms until a partial sum goes above 7

     - Possible b/c the positive terms add to ∞

  2. Add negative terms until a partial sum goes below 7

     - Possible b/c the negative terms add to $-\infty$

  3. Go back to Step 1

  - Since $\lim_{n\to\infty} \frac{1}{n} = 0$, each time the partial sums go above or below 7, they stay closer to 7.

    Thus, the limit of the sequence of partial sums is 7.

13.18 Ratio Test: The Theorem

Theorem (Ratio Test)

Let $\sum_{n}^{\infty} a_n$ be a series. Assume $\forall n,\ a_n \neq 0$.

Assume the limit $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists or is ∞.

- If $L < 1$, then $\sum_{n}^{\infty} a_n$ is absolutely convergent
- If $L = 1$, then we draw no conclusion
- If $L > 1$, then $\sum_{n}^{\infty} a_n$ is divergent

Idea of the Proof

- If $L > 1$, then $a_n \not\to 0$, so $\sum_{n}^{\infty} a_n$ is divergent
- Assume $0 \leq L < 1$
  - For "large n", $\left| \frac{a_{n+1}}{a_n} \right| \approx L$,

    or $|a_{n+1}| \approx |a_n| \cdot L$
  - For "large n", this series behaves like geometric series

$$\sum_{n}^{\infty} |a_n| \sim \sum_{n}^{\infty} L^n \text{, which is convergent}$$

- So $\sum_{n}^{\infty} a_n$ is absolutely convergent

13.19 Ratio Test: Examples

Ex. $\sum_{n}^{\infty} \frac{(-1)^n e^{2n}}{n!\,(n+3)}$

- Call $a_n = \frac{(-1)^n e^{2n}}{n!\,(n+3)}$

- $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{e^{2(n+1)}}{(n+1)!\,(n+4)}}{\frac{e^{2n}}{n!\,(n+3)}}$$

$$= \lim_{n\to\infty} \left[\frac{e^{2n+2}}{e^{2n}} \; \frac{n!}{(n+1)!} \; \frac{n+3}{n+4} \right]$$

$$= \lim_{n\to\infty} \frac{e^2(n+3)}{(n+1)(n+4)} = 0$$

- $0 < 1$, so by the Ratio Test, the original series converges absolutely.

Ex. $\sum_{n}^{\infty} \frac{n+1}{n^2+2}$

- Call $a_n = \frac{n+1}{n^2+2}$

- $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)+1}{(n+1)^2+2}}{\frac{n+1}{n^2+2}}$$

$$= \lim_{n\to\infty} \frac{(n+2)(n^2+2)}{(n^2+2n+3)(n+1)}$$

$$= 1$$

- Since the limit is 1, the ratio test is inconclusive

- Use LCT instead, compare w/ $\sum_{n}^{\infty} \frac{1}{n}$

14.1 Power Series: an Example

Ex. I want to define a function w/ this equation:

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n3^n}.$$ What is its domain?

- Domain $g = \{x \in \mathbb{R} : \text{the series } g(x) \text{ is convergent}\}$

- Ratio test: call $a_n = \dfrac{x^n}{n3^n}$

- $$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\dfrac{|x|^{n+1}}{(n+1)3^{n+1}}}{\dfrac{|x|^n}{n3^n}}$$

$$= \lim_{n\to\infty} \frac{|x|^{n+1} 3^n \, n}{|x|^n \cdot 3^{n+1} (n+1)} = \lim_{n\to\infty} \frac{|x| \, n}{3(n+1)} \xrightarrow{} 1$$

$$= \frac{|x|}{3}$$

- If $|x| < 3$, then $L = \dfrac{|x|}{3} < 1$

  By RT, $g(x)$ is Abs. Conv.

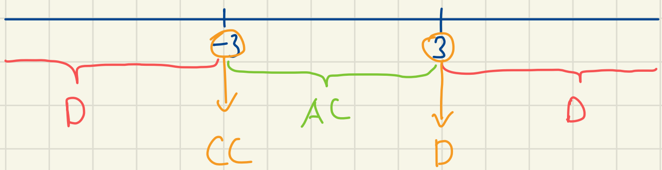
- If $|x| > 3$, then $L = \dfrac{|x|}{3} > 1$

By RT, $g(x)$ is Div.

- $g(3) = \sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (p-series w/ $p=1$)

- $g(-3) = \sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Conv (by AST)

  - Conditionally Convergent



- Dom $g = [-3, 3)$ = interval of convergence

  3 = radius of convergence

14.2 Power Series: the Main Theorem

Definition

Let $a \in \mathbb{R}$. A power series centered at a is a function f defined by an equation like

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where $c_0, c_1, c_2, \ldots \in \mathbb{R}$

- Domain $f = \{x \in \mathbb{R} : \text{the series } f(x) \text{ is convergent}\}$
- Note: $a \in$ Domain f
- Ultimate goal: write common functions as power series

Theorem

Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at $a \in \mathbb{R}$

1. The domain of f is an interval centered at a:

$(a-R, a+R)$ $\qquad$ $(a-R, a+R]$ $\qquad$ $\mathbb{R}$

$[a-R, a+R]$ $\qquad$ $[a-R, a+R)$ $\qquad$ $\{a\}$

   - We call this domain the interval of convergence

(IC) of f

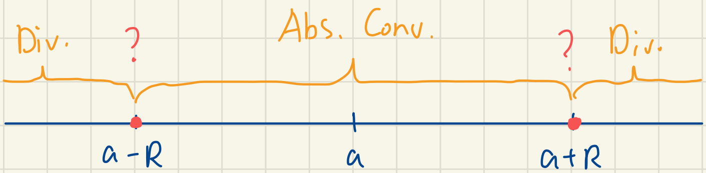
- We call its radius the radius of convergence

$0 \leq R \leq \infty$

2. In the interior of the IC, the series is Abs. Conv.

In the exterior of the IC, the series is Div.

At the endpoints (if any), anything may happen



3. In the interior of the IC, power series can be "treated like polynomials". They can be added, subtracted, composed, etc.

In particular, they can be differentiated or integrated "term by term", and this does not change the radius of convergence.

- $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

- $f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$

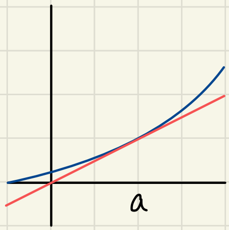
- $$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}$$

$$= c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \cdots$$

14.3 Taylor Polynomials (1): The Definition with the Limit

Goal

- Approximate functions w/ polynomials
    - f : function
    - $a \in$ Domain f
    - P : polynomial
- I want $P(x) \approx f(x)$ when x is close to a
- Ex. the tangent line



good approximation near a

- R : "remainder" or "error" $R(x) = f(x) - P(x)$

I want $R(x)$ to be "small"

- For a tangent line, $\lim_{x \to a} R(x) = 0$ "fast"
    - We need a scale for how "fast" it is

- 

$y = x$ slow
$y = x^2$ decent
$y = x^6$ fast

} when approaching 0

Definition

Let $a \in \mathbb{R}$. Let f and g be continuous functions at 0.

Let $n \in \mathbb{N}$.

We say that g is an approximation for f near a of order n

when $$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0$$

- This means that $f(x) = g(x) + R(x)$, and, as $x \to a$, $R(x) \to 0$ faster than $(x-a)^n \to 0$

First Definition of Taylor Polynomial

Let $a \in \mathbb{R}$. Let f be a continuous function defined at and near a. Let $n \in \mathbb{N}$.

The n^{th} Taylor polynomial for f at a is a polynomial P_n

- that is an approximation for f near a of order n

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$$

- with degree at most n

- Ex. $y = P_1(x)$ is the equation of the tangent line

14.4 Taylor Polynomials (2): The Definition with the Derivatives

Definitions

- A function f is called:
  - C^0 when f is continuous
  - C^1 when f' exists and is continuous
  - C^2 when f' and f'' exist and are continuous
  - $\vdots$
  - C^n when $f', f'', \dots, f^{(n)}$ exist and are continuous
  - C^∞ when all derivatives exist and are continuous

Intro

- Assume f and g are C^∞. We want to transform

  the condition $\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0$ into a condition

  about their derivatives

- Call $L = \lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n}$
  - If $f(a) - g(a) \neq 0$, then $L = \frac{\text{not } 0}{0} = \pm\infty$

- Assume $f(a) = g(a)$. We get $\frac{0}{0}$. Use L'Hôpital's.

- $$L \overset{(*)}{=} \lim_{x \to a} \frac{f'(x) - g'(x)}{n(x-a)^{n-1}}$$

  - If $f'(a) - g'(a) \neq 0$, then $L = \frac{\text{not } 0}{0} = \pm\infty$
  - Assume $f'(a) = g'(a)$. We get $\frac{0}{0}$. Use L'Hôpital's.

- $$L \overset{(*)}{=} \lim_{x \to a} \frac{f''(x) - g''(x)}{n(n-1)(x-a)^{n-2}} \quad \cdots$$

- After using L'Hôpital n times, we get

$$L \overset{(*)}{=} \lim_{x \to a} \frac{f^{(n)}(x) - g^{(n)}(x)}{n!} = \frac{f^{(n)}(a) - g^{(n)}(a)}{n!}$$

- $$L = 0 \iff \begin{cases} f(a) = g(a) \\ f'(a) = g'(a) \\ \vdots \\ f^{(n-1)}(a) = g^{(n-1)}(a) \\ f^{(n)}(a) = g^{(n)}(a) \end{cases}$$

- We have used f and g were C^n

Theorem

Let $a \in \mathbb{R}$. Let $n \in \mathbb{N}$.

Let f and g be C^n functions at a.

The following are equivalent:

1. $\lim_{x \to a} \frac{f(x) - f(a)}{(x-a)^n} = 0$
   - "g is a good approximation for f near a"

2. $f(a) = g(a)$, $f'(a) = g'(a)$, ..., $f^{(n)}(a) = g^{(n)}(a)$
   - "g and f have the same first few derivatives at a"

Second Definition of Taylor Polynomial

Let $a \in \mathbb{R}$. Let $n \in \mathbb{N}$. Let f be a C^n function at a.

The n^{th} Taylor Polynomial for f at a is

- a polynomial P_n, such that
- $P_n(a) = f(a)$, $P_n'(a) = f'(a)$, ..., $P_n^{(n)}(a) = f^{(n)}(a)$
- with degree at most n
- Ex. $y = P_1(x)$ is the equation of the tangent line

14.5 Taylor Polynomials (3): The Formula

Case: $a=0$

- $P_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
  - $P_n(0) = f(0)$
  - $P_n'(0) = f'(0)$
  - $P_n''(0) = f''(0)$

    $\vdots$
  - $P_n^{(n)}(0) = f^{(n)}(0)$
- $P_n^{(k)}(0) = k! \cdot c_k = f^{(k)}(0)$
  - $c_k = \dfrac{f^{(k)}(0)}{k!}$
  - $P_n(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$

General a

- $P_n(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n$
  - "change of basis" $x \to x-a$
- $P_n(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Third Definition of Taylor Polynomial

Let $a \in \mathbb{R}$. Let $n \in \mathbb{N}$. Let f be a C^n function at a.

The n^{th} Taylor Polynomial for f at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Degree $P_n \leq n$
- The Taylor Polynomials of a function are unique

Definition

Let $a \in \mathbb{R}$. Let f be a C^∞ function at a.

The Taylor series for f at a is the power series

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Equivalently, $\forall k \in \mathbb{N}$, $S^{(k)}(a) = f^{(k)}(a)$
- Ideal case: $f(x) = S(x)$
  - We call such functions analytic
- Maclaurin series means Taylor series at 0

14.6 The Four Main Maclaurin Series

Ex. 1 Maclaurin series for $f(x) = e^x$

- For all $k \in \mathbb{N}$, $f^{(k)}(x) = e^x$, $f^{(k)}(0) = 1$

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- $f(x) = e^x$ is analytic (proof later), so

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Ex. 1b Taylor series for $f(x) = e^x$ at c

- Write $f(x) = e^x$ in terms of powers of $(x - c)$

Substitution: $u = x - c$

- $e^x = e^{c+u} = e^c e^u = e^c \sum_{n=0}^{\infty} \frac{u^n}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!} (x - c)^n$$

Ex. 2 Maclaurin series for $g(x) = \sin x$

- $g(x) = \sin x$ $\qquad g(0) = 0$

  $g'(x) = \cos x$ $\qquad g'(0) = 1$

  $g''(x) = -\sin x$ $\qquad g''(0) = 0$

  $g'''(x) = -\cos x$ $\qquad g'''(0) = -1$

  $g^{(4)}(x) = \sin x$ $\qquad \vdots$

- $$S(x) = \sum_{n=0}^{\infty} g^{(n)}(0) \frac{x^n}{n!}$$

$$= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

- sin is analytic (proof later), so $\sin x = S(x)$

The Four Main Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all } x$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1$$

14.7 Analytic Functions and the Remainder Theorems

Definition

Let f be a C^∞ function defined on an open interval I.

- Let $a \in I$. Let $S_a(x)$ be the Taylor series of f at a. f is analytic at a when $\exists$ an open interval J_a centered at a s.t. $\forall x \in J_a$, $f(x) = S_a(x)$
- f is analytic when $\forall a \in I$, f is analytic at a

Results about Analytic Functions

- Polynomials are analytic
- Sums, products, quotients (not including divide by 0), and composition of analytic functions are analytic
- Derivatives and antiderivatives of analytic functions are analytic
- In the interior of the interval of convergence, a power series can be manipulated like a polynomial

- The Taylor series of a function at a point is unique
- Goal: prove that $f(x) = e^x$ and $g(x) = \sin x$ are analytic

Prove a Function is Analytic

Let f be a function defined on an open interval I.

Let $a \in I$.

- If f is C^n, we can write the n^{th} Taylor polynomial P_n at a:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k, \quad f(x) = P_n(x) + \underbrace{R_n(x)}_{\text{remainder/error}}$$

- If f is C^∞, we can write the Taylor series S at a:

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k, \quad \lim_{n\to\infty} P_n(x) = S(x)$$

- f is analytic when $f(x) = S(x)$
  - Equivalent to $\lim_{n\to\infty} R_n(x) = 0$

Two Different Limits

1. We know, from the definition of Taylor polynomial:

for fixed n, $\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0$

2. We want, for a function to be analytic:

for fixed x, $\lim_{n \to \infty} R_n(x) = 0$

Remainder Theorems

- Estimates the remainder of a Taylor polynomial
- Typical remainder theorem:
  - If (hypotheses)
  - Then (some formula for $R_n(x)$)
- At least 3 versions
  1. Lagrange's form
  2. Cauchy's form
  3. Integral form
- All proven w/ MVT or Rolle's Theorem

Lagrange's Remainder Theorem

- Let I be an open interval
- Let $n \in \mathbb{N}$

- Let f be a C^{n+1} function on I

- Let $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ be its n^{th} Taylor polynomial

- Let $R_n(x) = f(x) - P_n(x)$ be its remainder

Then $\exists \xi$ between a and x s.t.

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Usage

1. To prove a function is analytic

   - $\lim_{n \to \infty} R_n(x) = 0$

2. To estimate

   - Approximate $f(x)$ by $P_n(x)$ and bound the error

14.8 A Proof that the Exponential Function is Analytic

Goal

Prove that $f(x) = e^x$ is analytic at 0

- $f(x) = e^x$, $a = 0$
- $e^x = P_n(x) + R_n(x)$
- $P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ is the n^{th} Taylor polynomial
- $R_n(x)$ is the n^{th} remainder
- Need to prove $\forall n \in \mathbb{R}$, $\lim_{n\to\infty} R_n(x) = 0$
- Then, as a consequence, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots$

Proof

- Fix $x \in \mathbb{R}$
- Use Lagrange's Remainder Theorem for $f(x) = e^x$ and $a = 0$. $(\forall n \in \mathbb{N},)$ $\exists \xi$ between 0 and x s.t.

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} = \frac{e^{\xi}}{(n+1)!} x^{n+1}$$

- Then $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \left[e^{\xi} \frac{x^{n+1}}{(n+1)!} \right]$

- Cannot factor e^{ξ} out because ξ depends on n and x

- Case 1: $x > 0$. Then $0 < \xi < x$, so

$$0 \leq R_n(x) = e^{\xi} \frac{x^{n+1}}{(n+1)!} \leq e^{x} \frac{x^{n+1}}{(n+1)!}$$

  - From the Big Theorem, $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$
  - From the Squeeze Theorem, $\lim_{n\to\infty} R_n(x) = 0$

- Case 2: $x < 0$. Then $x < \xi < 0$, so

$$0 \leq |R_n(x)| = \left| e^{\xi} \frac{x^{n+1}}{(n+1)!} \right| \leq \left| e^{x} \frac{x^{n+1}}{(n+1)!} \right|$$

  - From the Big Theorem, $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$
  - From the Squeeze Theorem, $\lim_{n\to\infty} R_n(x) = 0$

$\square$

We have proven $e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

14.9 How to Write Functions as Power Series Quickly

The Slow Method

1. Start w/ a C^∞ function f
2. Obtain a Taylor Series at a.

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

3. Use the remainder theorems to prove $\lim_{n\to\infty} R_n(x) = 0$
4. Then $f(x) = S(x)$

The Fast Method

- In the interior of the radius of convergence,

  a power series can be manipulated like a polynomial
- Use the four Maclaurin Series
- Ex. 1 Write $f(x) = e^{-x}$ as power series at 0
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x
  - $e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$ for all x
- Ex. 2 Write $f(x) = x^3 \sin x^2$ as power series at 0

- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for all x

- $\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$

- $f(x) = x^3 \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+5}$ for all x

- Ex. 3 Write $f(x) = \frac{1}{x}$ as power series at 3

  - Change of variable $u = x - 3$

  - $\frac{1}{x} = \frac{1}{u+3} = \frac{1}{3} \cdot \frac{1}{1+u/3} = \frac{1}{3} \cdot \frac{1}{1-(-u/3)}$

    $= \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{u}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} u^n$

    $= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-3)^n$

  - Valid when $\left|-\frac{u}{3}\right| < 1$, $|x-3| < 3$

14.10 Logarithm as Power Series

- Let $g(x) = \ln x$. Domain $g = (0, \infty)$
  - Option 1: Taylor series centered at $a > 0$ of $g(x) = \ln x$
  - Option 2: Taylor series centered at $a = 0$ of $f(x) = \ln(1+x)$
- Write $f(x) = \ln(1+x)$ as power series at 0
  - $f'(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)}$

    $= \displaystyle\sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$ for $|x| < 1$
  - $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$ for $|x| < 1$
  - Evaluate at $x = 0$: $f(0) = 0 + C \Rightarrow C = 0$
- $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

  $= \displaystyle\sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m}$ for $|x| < 1$

  $= x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$

14.11 Taylor Series Application: Estimation

Ex. Estimate $\sqrt{e}$ w/ error < 0.001

- Let $f(x) = e^x$. Want $f(1/2)$.

- $f(1/2) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{2^k} = \underbrace{P_n(1/2)}_{\text{estimation}} + \underbrace{R_n(1/2)}_{\text{error}}$

  - $P_n(1/2) = \sum_{k=0}^{n} \frac{1}{k!} \frac{1}{2^k}$

  - Want $|R_n(1/2)| < 0.001$

- Use Lagrange's Remainder Theorem

$$\exists \xi \in \left(0, \tfrac{1}{2}\right) \text{ s.t. } R_n\left(\tfrac{1}{2}\right) = \frac{\overbrace{f^{(n+1)}(\xi)}^{=e^{\xi}}}{(n+1)!} \left[\tfrac{1}{2} - 0\right]^{n+1}$$

  - $0 < R_n\left(\frac{1}{2}\right) < \frac{e^{1/2}}{(n+1)!} \frac{1}{2^{n+1}} < \frac{2}{2^{n+1}(n+1)!} < 0.001$

  - Need $2^n (n+1)! > 1000$

  - $n = 4$ works: $2^4 \cdot 5! = 16 \cdot 120 > 1000$

- Estimation for $\sqrt{e}$

  - $P_4\left(\frac{1}{2}\right) = \sum_{k=0}^{4} \frac{1}{k! \, 2^k} = \cdots = \frac{211}{128} \approx \boxed{1.648}43\ldots$

$\sqrt{e} \approx \boxed{1.648}72\ldots$

14.12 Taylor Series Application: Integrals

Ex. Compute $I = \int_0^3 e^{-x^2}\,dx$

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x

- $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$

- $I = \int_0^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^3 x^{2n}\,dx$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}\Bigg|_{x=0}^{x=3} = \sum_{n=0}^{\infty} \frac{(-1)^n\, 3^{2n+1}}{n!\,(2n+1)}$$

Ex. $\int \frac{1}{1-x^8}\,dx$

- Directly finding an antiderivative takes very long and would result in a long (and thus unhelpful) expression

- $\int \frac{1}{1-x^8}\,dx = \int \sum_{n=0}^{\infty} x^{8n}\,dx = \sum_{n=0}^{\infty} \frac{x^{8n+1}}{8n+1} + C$

  - Valid for $|x| < 1$

14.13 Taylor Series Application: Limits

Ex. $\displaystyle\lim_{x\to 0} \frac{2x^3 + x^4 + 11x^6}{5x^3 + x^5 - 7x^6}$

$$= \lim_{x\to 0} \frac{\cancel{x^3}[2 + x + 11x^3]}{\cancel{x^3}[5 + x^2 - 7x^3]} = \frac{2}{5}$$

Ex. $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^3}$

$$= \lim_{x\to 0} \frac{\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right] - x}{x^3}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots}{x^3}$$

$$= \lim_{x\to 0} \left[-\frac{1}{6} + \frac{1}{120}x^2 + \cdots\right]$$

$$= -\frac{1}{6}$$

- Strategy: look for the smallest nonzero term

Ex. $\displaystyle\lim_{x\to 0} \frac{3x^2 - e^{x^2} + \cos(2x)}{x\sin x - \ln(1+x^2)}$

- Numerator

$\oplus$ $3x^2$

$\ominus$ $e^{x^2} = 1 + (x^2) + \frac{1}{2!}(x^2)^2 + \frac{1}{3!}(x^2)^3 + \cdots$

$\oplus$ $\cos(2x) = 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6 + \cdots$

- $3x^2 - e^{x^2} + \cos(2x) = [-1+1] + \left[3 - 1 - \frac{4}{2}\right]x^2$

$+ \left[-\frac{1}{2} + \frac{2^4}{4!}\right]x^4 + \cdots = \frac{1}{6}x^4 + (\text{h.o.t.})$

- "higher-order terms"

- Denominator

$\oplus$ $x\sin x = x\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right]$

$\ominus$ $\ln(1+x^2) = (x^2) - \frac{1}{2}(x^2)^2 + \frac{1}{3}(x^2)^3 - \cdots$

- $x\sin x - \ln(1+x^2)$

$= [1-1]x^2 + \left[-\frac{1}{6} + \frac{1}{2}\right]x^4 + \cdots$

$= \frac{1}{3}x^4 + (\text{h.o.t.})$

- $\lim_{x\to 0} \frac{3x^2 - e^{x^2} + \cos(2x)}{x\sin x - \ln(1+x^2)} = \lim_{x\to 0} \frac{1/6\, x^4 + (\text{h.o.t.})}{1/3\, x^4 + (\text{h.o.t.})}$

$= \frac{1/6}{1/3} = \frac{1}{2}$

14.14 Taylor Series Application: Computing Sums

Ex. Compute $A = \sum_{n=1}^{\infty} \frac{n}{2^n}$

- Want $\sum_{n=1}^{\infty} n x^n$ where $n = \frac{1}{2}$

  - $\sum_{n=1}^{\infty} n x^{n-1} = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$

  - $\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$

- Evaluate at $x = \frac{1}{2}$: $\sum_{n=1}^{\infty} n \frac{1}{2^n} = \frac{1/2}{(1-1/2)^2} = 2$

Ex. Compute $B = \sum_{n=0}^{\infty} \frac{2^n}{(n+2)n!}$

- Want $\sum_{n=0}^{\infty} \frac{x^n}{(n+2)n!}$ when $x = 2$

- Know $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

  - $\int x e^x \, dx = \int \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \, dx = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!} + C$

  - $(x-1)e^x = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!} + C$

- Evaluate at $x = 0$. $-1 = 0 + C \Rightarrow C = -1$

- $$\sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!} = (x-1)e^x + 1$$

- $$\sum_{n=0}^{\infty} \frac{x^{n}}{(n+2)n!} = \frac{(x-1)e^x + 1}{x^2}$$

- Evaluate at $x = 2$: $$\sum_{n=0}^{\infty} \frac{2^{n}}{(n+2)n!} = \frac{e^2 + 1}{4}$$

14.15 Taylor Series Applications: Physics

Physics

- Use physics principles to derive equations that regulate the behaviour of a system
- The equations are too complicated, so we approximate them w/ Taylor polynomials

Kinetic Energy of a Particle

- Classical physics:

$$T = \frac{1}{2} m_0 v^2$$

$\frac{v}{c}$ "small"

m_0 = mass "at rest"
v = velocity
c = speed of light
m = mass

- Relativity:

$$T = mc^2 - m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-(v/c)^2}} - m_0 c^2$$

Proof of the First Equation Approximates the Second

- $$T = m_0 c^2 \left[\frac{1}{\sqrt{1-(v/c)^2}} - 1 \right]$$

- $$f(x) = (1-x)^{-1/2} \approx f(0) + f'(0)x + f''(0)x^2$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

- $$\frac{1}{\sqrt{1-x^2}} \approx 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4$$

- $$T = m_0 c^2 \left[\frac{1}{\sqrt{1-(v/c)^2}} - 1 \right]$$

$$\approx m_0 c^2 \left[\frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 \right]$$

$$= \underbrace{\frac{1}{2} m_0 v^2 + \frac{3 m_0 v^4}{8 c^2}}_{\text{first relativistic correction}}$$

- The classical expression of kinetic energy is the first non-zero term of the Taylor series expansion on the relativistic expression

- We can always make the approximations better