



*MAT*237

*Multivariable Calculus With Proofs*

# 1.1 Curves

## Motion

- A parametric curve  $\gamma: I \rightarrow \mathbb{R}^n$  for some interval  $I \subseteq \mathbb{R}$

describes the motion of an object moving in  $\mathbb{R}^n$

- Position at time  $t$  is  $\gamma(t)$

- Component functions:  $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$

for  $n$  single-variable functions  $\gamma_i: I \rightarrow \mathbb{R}$

- $\gamma'(t) = (\gamma'_1(t), \dots, \gamma'_n(t))$

- Speed at time  $t$  is  $\|\gamma'(t)\|$

$$= \sqrt{|\gamma'_1(t)|^2 + \dots + |\gamma'_n(t)|^2}$$

- Direction of motion: unit tangent vector,

denoted  $T = T(t)$  defined by

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

- $\gamma'(t)$  is a scalar multiple of  $T(t)$

- Acceleration is  $\gamma''$  so

$$\gamma''(t) = (\gamma''_1(t), \dots, \gamma''_n(t))$$

## Frenet Frame in 3 Dimensions

- Define the (principal) unit normal to be

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

- T and N are orthogonal

- $T''(t)$  is a scalar multiple of  $N(t)$

- Define the binormal unit vector B to be the unique unit vector s.t.  $\{T, N, B\}$  form a positively-oriented ordered orthogonal basis in  $\mathbb{R}^3$

- $\{T, N, B\}$  forms the Frenet frame

(or Frenet-Serret frame or TNB frame)

describing the motion of an object in 3D

- $B = T \times N$

- Cross-product only works for vectors in  $\mathbb{R}^3$ :

let  $a = (a_1, a_2, a_3)$ ,  $b = (b_1, b_2, b_3)$ :

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

## Geometry of Curves

- Trace of a parametric curve  $\gamma: I \rightarrow \mathbb{R}^n$  is the image of  $\gamma$ , i.e.,  $\gamma(I)$ 
  - "Path traced out by  $\gamma$ "
  - Trace of a parametric curve is a set  $C \subseteq \mathbb{R}^n$ .  
this set is a Curve
- Let  $C \subseteq \mathbb{R}^n$  be a set.  $C$  is a curve in  $\mathbb{R}^n$  if  $C$  is the trace of a continuous<sup>2</sup> parametric curve  
 $\gamma: I \rightarrow \mathbb{R}^n$

## 1.2 Real-valued Functions

### Scalar Fields and Densities

- Real-valued functions:  $\mathbb{R}^n \rightarrow \mathbb{R}$ 
  - Also called scalar fields, scalar functions, potentials
- Real-valued functions that are non-negative: densities
  - Shorthand:  $f \geq 0$
  - Density:  $\frac{\text{quantity}}{\text{unit of measurement}}$

### Graphs, Level Sets, and Slices

- Let  $A \subseteq \mathbb{R}^n$ . The graph of a function  $f: A \rightarrow \mathbb{R}$  is the set in  $\mathbb{R}^{n+1}$  given by  $\{(x, f(x)): x \in A\}$
- Let  $A \subseteq \mathbb{R}^n$  and  $f: A \rightarrow \mathbb{R}$ . Fix  $k \in \mathbb{R}$ .  
The level set of  $f$  at  $k$  is  $\{x \in \mathbb{R}^n: f(x) = k\}$ 
  - Also called the  $k$ -level set
  - Contour: level set in  $\mathbb{R}^2$
  - For graphs of 2-variable functions, we can create a contour plot by plotting the level sets

for a few different values

- Heat maps: colour gradient that corresponds to the values of the function
  - Continuous version of a contour plot
- Slicing: fixing a variable
  - Let  $A \subseteq \mathbb{R}^2$  and  $f: A \rightarrow \mathbb{R}$ 
    - For fixed  $a \in \mathbb{R}$ , the  $x$ -slice at  $a$  of the graph  $f$  is  $\{(y, z) : z = f(a, y)\}$
    - For fixed  $b \in \mathbb{R}$ , the  $y$ -slice at  $b$  of the graph  $f$  is  $\{(x, z) : z = f(x, b)\}$
    - Slices for 3-variable functions can be defined similarly

# 1.3 Vector Fields

- An ( $n$ -dimensional) vector field is a function  $F$  w/ domain and codomain lying in  $\mathbb{R}^n$ 
  - "F is a vector field in  $\mathbb{R}^n$ "
- To plot a 2D vector field, draw vectors for points on the grid
- Notation
  - $F(x,y,z) = (x^2, yx, -z)$  most common
  - $F(x,y,z) = \langle x^2, yx, -z \rangle$
  - $F = [x^2, yx, -z]$
  - $F = x^2\hat{i} + yx\hat{j} - z\hat{k}$

## 1.4 Coordinate Transformations

- Transformation: map between 2 subsets lying in the same dimension
- A coordinate transformation  $f: A \rightarrow B$  refers to a continuous bijective transformation
  - Domain A and map f form a coordinate system for the codomain B

### Polar Coordinates

- Define the polar coordinate transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $T(r, \theta) = (r\cos\theta, r\sin\theta)$ 
  - $r$ : radius
  - $\theta$ : polar angle
  - Not bijective
- The polar coordinate transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $T(r, \theta) = (r\cos\theta, r\sin\theta)$  maps the subset  $(0, \infty) \times (-\pi, \pi)$  bijectively to the subset  $\mathbb{R}^2 \setminus \{(x, 0) . x \leq 0\}$

## Cylindrical Coordinates

- Define the cylindrical coordinate transformation

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ by } T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$$

- $r$ : polar radius
  - $\theta$ : polar angle
  - $z$ : usual rectangular coordinate
- The cylindrical coordinate transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

defined above maps the subset  $(0, \infty) \times (-\pi, \pi) \times \mathbb{R}$

bijectively to the subset  $\mathbb{R}^3 \setminus \{(x, 0, z) : x \leq 0, z \in \mathbb{R}\}$

## Spherical Coordinates

- Define the spherical coordinate transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by

$$T(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$

- $\rho$ : spherical radius
    - $\theta$ : polar angle
    - $\phi$ : azimuthal angle
  - The spherical coordinate transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$
- defined above maps the subset  $(0, \infty) \times (-\pi, \pi) \times (0, \pi)$
- bijectively to the subset  $\mathbb{R}^3 \setminus \{(x, 0, z) : x \leq 0, z \in \mathbb{R}\}$

## 1.5 Surfaces

### Parametric Surfaces

- Described by maps  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  where  $n < m$
- Let  $m, n \in \mathbb{N}^+$  with  $n < m$ . Let  $S \subseteq \mathbb{R}^m$ ,  $A \subseteq \mathbb{R}^n$ , and  $g: A \rightarrow \mathbb{R}^m$  be a cts map.  
If  $S = \{g(x) : x \in A\} = \text{img}(g)$ , then the pair  $(S, g)$  is a parametric surface
  - $S$  is parameterized by  $g$

### Explicit Surfaces

- Let  $m, n \in \mathbb{N}^+$ ,  $A \subseteq \mathbb{R}^n$ . The graph of a function  $f: A \rightarrow \mathbb{R}^m$  is the set  $S = \{(x, f(x)) : x \in A\} \subseteq \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ 
  - Variables can be reordered
- A set  $S \subseteq \mathbb{R}^n$  is an explicit surface if  $S$  is a graph of a cts function

## Implicit Surfaces

- "Implicit": does not explicitly express one variable in terms of the others
- A set  $S \subseteq \mathbb{R}^n$  is an **implicit surface** if there exists a constant  $c \in \mathbb{R}^m$ , a set  $A \subseteq \mathbb{R}^n$ , and a cts function  $f: A \rightarrow \mathbb{R}^m$  s.t.  
$$S = f^{-1}(\{c\}) = \{x \in \mathbb{R}^n : f(x) = c\}$$
  - $f^{-1}$  is **not** the inverse function, it is the preimage of a set under a function
  - Explicit surfaces are also implicit surfaces
- Dim of implicit surface  
= # of variables - # of equations

## 1.6 Projections

- Projections: maps of the form  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  where  $n > m$
- For  $i \in \{1, \dots, n\}$ , the map  $\pi_i: \mathbb{R}^n \rightarrow \mathbb{R}$  given by  
 $\pi_i(x_1, \dots, x_n) = x_i$  is the  $i^{\text{th}}$  coordinate map
- For  $i \in \{1, \dots, n\}$ , the map  $\Pi_i: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  given by  
 $\Pi_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  is the  $i^{\text{th}}$  coordinate plane projection

## 2.1 Sets

### Balls and Spheres

- Distance in  $\mathbb{R}^n$ :  $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$
- Let  $r \geq 0$ ,  $a \in \mathbb{R}^n$ :
  - The **open ball** of radius  $r$  centered at  $a$  is the set  $B_r(a) = \{x \in \mathbb{R}^n : \|x-a\| < r\}$
  - The **closed ball** of radius  $r$  centered at  $a$  is the set  $\{x \in \mathbb{R}^n : \|x-a\| \leq r\}$
  - The **sphere** of radius  $r$  centered at  $a$  is the set  $\{x \in \mathbb{R}^n : \|x-a\| = r\}$
- An open ball or closed ball is **punctured** if it excludes the centre, i.e.  $B_r(a) \setminus \{a\}$
- Balls are solid, spheres are hollow
- The  $(n-1)$ -dimensional unit sphere in  $\mathbb{R}^n$  is the sphere of radius 1 centered at the origin and is denoted  $S^{n-1}$ 
  - $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$

## Rectangles

- A (closed) rectangle in  $\mathbb{R}^n$  is a set  $R$  of the form

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

$$= \{(x_1, \dots, x_n) : x_i \in [a_i, b_i] \text{ for } 1 \leq i \leq n\}$$

where  $a_1, b_1, \dots, a_n, b_n \in \mathbb{R}$  and  $a_i < b_i$  for all  $1 \leq i \leq n$

- The set  $(a_1, b_1) \times \cdots \times (a_n, b_n)$  is an open rectangle

- An ( $n$ -dimensional) hypercube is a set in  $\mathbb{R}^n$

$$\text{of the form } [a, b]^n = [a, b] \times \cdots \times [a, b]$$

- The ( $n$ -dimensional) unit hypercube is the set  $[0, 1]^n$

## 2.2 Interior, Boundary, and Closure

### Interior

- Let  $A \subseteq \mathbb{R}^n$  be a set. A point  $p \in \mathbb{R}^n$  is an interior point of  $A$  if there exists  $\varepsilon > 0$  s.t.  
 $B_\varepsilon(p) \subseteq A$
- Let  $A \subseteq \mathbb{R}^n$  be a set. The interior of  $A$ , denoted  $A^\circ$  or  $\text{int}(A)$ , is the set of interior points of  $A$
- Let  $A, B \subseteq \mathbb{R}^n$ , then
  - $A^\circ \subseteq A$
  - $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$
  - $(A \cap B)^\circ = A^\circ \cap B^\circ$
  - $(A \times B)^\circ = A^\circ \times B^\circ$

### Boundary

- Let  $A \subseteq \mathbb{R}^n$ . A point  $p \in \mathbb{R}^n$  is a boundary point of  $A$  if for every  $\varepsilon > 0$ , the sets  $B_\varepsilon(p) \cap A$  and  $B_\varepsilon(p) \cap A^c$  are both non-empty

- Let  $A \subseteq \mathbb{R}^n$ . The (topological) boundary of  $A$ , denoted  $\partial A$ , is the set of boundary points of  $A$
- For any set  $A \subseteq \mathbb{R}^n$ ,  $A^\circ$  and  $\partial A$  are disjoint

### Closure

- Let  $A \subseteq \mathbb{R}$ . A point  $p \in \mathbb{R}^n$  is a limit point if for every  $\epsilon > 0$ ,  $B_\epsilon(p) \setminus \{p\}$  contains points in  $A$ 
  - Denoted  $A^*$
- Let  $A \subseteq \mathbb{R}$ . The closure of  $A$ , denoted  $\bar{A}$  or  $\text{cl}(A)$ , is the set  $A$  along with its limit points.
  - i.e.  $\bar{A} = A \cup A^*$
- Let  $A, B \subseteq \mathbb{R}^n$ . Then
  - $A \subseteq \bar{A}$
  - $\text{cl}(A \cup B) = \bar{A} \cup \bar{B}$
  - $\text{cl}(A \cap B) \subseteq \bar{A} \cap \bar{B}$
  - $\text{cl}(A \times B) = \bar{A} \times \bar{B}$
- Let  $A \subseteq \mathbb{R}^n$ . Then  $\bar{A} = A^\circ \cup \partial A$  and  $\partial A = \bar{A} \setminus A^\circ$

## 2.3 Sequences

- A sequence in  $\mathbb{R}^n$  is a function w/  
 $\{k \in \mathbb{Z} : k \geq k_0\} \rightarrow \mathbb{R}^n$  for some fixed  $k_0 \in \mathbb{Z}$

### Convergence of Sequences

- Let  $\{x(k)\}_k$  be a sequence in  $\mathbb{R}^n$ . Then  
 $\{x(k)\}_k$  converges to  $p \in \mathbb{R}^n$  if  $\forall \varepsilon > 0$ ,  $\exists K \in \mathbb{N}$  s.t.  
 $\forall k \in \mathbb{N}, k \geq K \Rightarrow \|x(k) - p\| < \varepsilon$ 
  - $\lim_{k \rightarrow \infty} x(k) = p$  or  $x(k) \rightarrow p$
- The sequence  $\{x(k)\}_k$  converges if there exists  $p \in \mathbb{R}^n$  s.t.  $\lim_{k \rightarrow \infty} x(k) = p$ 
  - Otherwise,  $\{x(k)\}_k$  diverges
- A sequence  $\{x(k)\}_k$  in  $\mathbb{R}^n$  converges to  $x$  if every open ball centered at  $x$  contains all but finitely many points of the sequence  $\{x(k)\}_k$
- Let  $\{x(k)\}_k$  be a sequence in  $\mathbb{R}^n$  w/  
 $x(k) = (x_1(k), \dots, x_n(k))$ .  $\{x(k)\}_k$  converges iff  
 $\{x_i(k)\}_k$  converges for all  $i = 1, 2, \dots, n$

## Limit Points, Boundary Points, and Interior Points

- Let  $A \subseteq \mathbb{R}^n$  be a set. A point  $p \in \mathbb{R}^n$  is a limit point of  $A$  iff there exists a sequence of points in  $A \setminus \{p\}$  which converges to  $p$
- Let  $A \subseteq \mathbb{R}^n$  be a set. Let  $p \in \mathbb{R}^n$  be a point.
  - The point  $p$  is an interior point of  $A$  iff for every sequence  $\{x(k)\}_k$  of points converging to  $p$ , there exists  $K \in \mathbb{N}^+$  s.t  $\{x(k)\}_{k=K}^{\infty} \subseteq A$
  - The point  $p$  is a boundary point of  $A$  iff there exists a sequence of points in  $A$  converging to  $p$  and there exists a sequence of points in  $A^c$  converging to  $p$

## 2.4 Open Sets and Closed Sets

- If the set  $A$  is the domain of a function, we want a sequence of approximations lying inside  $A$  to converge to a point within  $A$ 
  - ① If a sequence in  $\mathbb{R}^n$  converges to a point  $a \in A$ , then the tail of the sequence belongs to  $A$
  - ② If a sequence in  $A$  converges to a point  $a \in \mathbb{R}^n$ , then  $a$  must belong to  $A$

### Open Sets

- A set  $A \subseteq \mathbb{R}^n$  is open if every point in  $A$  is an interior point of  $A$ 
  - Such set satisfies ①
- The interior of a set  $A \subseteq \mathbb{R}^n$  is open
- Let  $A \subseteq \mathbb{R}^n$ . Then all of the following are equivalent:
  - $A$  is open
  - $A = A^\circ$
  - $A \cap \partial A = \emptyset$

## Closed Sets

- A set  $A \subseteq \mathbb{R}$  is closed if every limit point of  $A$  belongs to  $A$ 
  - Such set satisfies ②
  - Any convergent sequence in a closed set  $A$  must converge to a point in  $A$
- The closure of a set  $A$  is closed
- Let  $A \subseteq \mathbb{R}^n$ . All of the following are equivalent:
  - $A$  is closed
  - $A = \bar{A}$
  - $\partial A \subseteq A$

## Set Operations

- A set  $A \subseteq \mathbb{R}^n$  is open iff  $A^c$  is closed
- Clopen: both closed and open
  - E.g.  $\emptyset, \mathbb{R}^n$
- All the following are true for sets in  $\mathbb{R}^n$ :
  - A finite intersection of open sets is open
  - Any union of open sets is open

- A finite union of closed sets is closed
- Any intersection of closed sets is closed
- A finite Cartesian product of open/closed sets  
is open/closed, respectively
- Infinite intersection of open sets may not be open

E.g.  $\bigcap_{\varepsilon > 0} (-\varepsilon, \varepsilon) = \{0\}$

- Infinite union of closed sets may not be closed

E.g.  $\bigcup_{0 < \varepsilon < 1} [-\varepsilon, \varepsilon] = (0, 1)$

## 2.5 Compact Sets

- Suppose a set  $A \subseteq \mathbb{R}$  is the domain of a real-valued function  $f$ . Construct a sequence of points  $\{x(k)\}_{k=1}^{\infty}$  in  $A$ , where each value  $f(x(k))$  is attempting to approximate the max value of  $f$

- As  $k \rightarrow \infty$ ,  $\{x(k)\}_{k=1}^{\infty}$  converges to  $p \in A$ , where  $f(p)$  is the max value of  $f$

(?) How to ensure  $\{x(k)\}_{k=1}^{\infty}$  converges? (to  $p \in \mathbb{R}^n$ )

(?) How to ensure the limiting point  $p$  belongs to  $A$ ?

- (?) can be addressed by assuming  $A$  is closed

### Definition of Compactness

- Let  $x: \mathbb{N}^+ \rightarrow \mathbb{R}^n$  be a sequence, let  $m: \mathbb{N}^+ \rightarrow \mathbb{N}^+$  be a strictly increasing function. The sequence  $\{x(m(k))\}_{k=1}^{\infty}$  is a subsequence of the sequence  $\{x(k)\}_{k=1}^{\infty}$ 
  - Domain of  $x$  must = codomain of  $m$
- A set  $A \subseteq \mathbb{R}$  is compact if every sequence of  $A$  has a subsequence which converges to a point in  $A$

- A set  $A \subseteq \mathbb{R}^n$  is bounded if  $\exists R > 0$  s.t.

$$A \subseteq \{x \in \mathbb{R}^n : \|x\| < R\}$$

- Unbounded: not bounded

- Bolzano-Weierstrass Theorem: A set in  $\mathbb{R}^n$  is compact iff it is both closed and bounded

### Set Operations and Subsets

- All of the following are true for sets in  $\mathbb{R}^n$ 
  - A finite union of compact sets is compact
  - Any intersection of compact sets is compact
  - A finite Cartesian product of compact sets is compact
- Let  $A$  be a compact set in  $\mathbb{R}^n$ . If  $B \subseteq A$  and  $B$  is closed then  $B$  is compact

## 2.6 Limits

### Formal Definitions

- Let  $f: A \rightarrow \mathbb{R}^m$  be a function w/  $A \subseteq \mathbb{R}^n$ .

Let  $a \in \mathbb{R}^n$  be a limit point of  $A$  and let  $b \in \mathbb{R}^m$ .

Define  $b$  to be the limit of  $f$  at  $a$  provided

$\forall \epsilon > 0, \exists \delta > 0$  s.t.  $\forall x \in A,$

$$0 < \|x - a\| < \delta \Rightarrow \|f(x) - b\| < \epsilon$$

- $\lim_{x \rightarrow a} f(x) = b$

- $f(x) \rightarrow b$  as  $x \rightarrow a$

- Limit only defined at limit points

$$\left\{ \begin{array}{l} \text{Defined} \\ \text{Not Defined} \end{array} \right\} \left\{ \begin{array}{l} \text{Exists} \\ \text{DNE} \end{array} \right\}$$

- Let  $A \subseteq \mathbb{R}^n$  be a set and let  $f: A \rightarrow \mathbb{R}^m$  be a function.

Let  $a \in \mathbb{R}^n$  be a limit point of  $A$  and let  $b \in \mathbb{R}^m$ .

Then  $\lim_{x \rightarrow a} f(x) = b$  iff for every sequence of points

$\{x(k)\}_k$  in  $A \setminus \{a\}$  w/  $x(k) \rightarrow a$ , the sequence of points  $\{f(x(k))\}_k$  in  $\mathbb{R}^n$  converges to  $b$ ; i.e.  $f(x(k)) \rightarrow b$

### Computing Limits

- Let  $f: A \rightarrow \mathbb{R}^m$  where  $A \subseteq \mathbb{R}^n$ . Let  $a$  be a limit point of  $A$  and let  $b = (b_1, \dots, b_m) \in \mathbb{R}^m$ .

Let  $f_1, \dots, f_m$  be the coordinate functions of  $f$

so  $f = (f_1, \dots, f_m)$ . Then  $\lim_{x \rightarrow a} f(x) = b$  iff

for all  $i = \{1, \dots, m\}$ ,  $\lim_{x \rightarrow a} f_i(x) = b_i$

- Let  $A \subseteq \mathbb{R}^n$  be a set and let  $a$  be a limit point of  $A$ .

Let  $f, g$  be  $\mathbb{R}^m$ -valued functions defined on  $A$ .

Let  $\phi$  be a real-valued function defined on  $A$ .

Let  $\lambda \in \mathbb{R}$ ,  $b \in \mathbb{R}^m$  be constants. Then:

- (Constants)  $\lim_{x \rightarrow a} b = b$  and  $\lim_{x \rightarrow a} x = a$
- (Linearity) If  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist then  
 $\lim_{x \rightarrow a} (f(x) + \lambda g(x))$  exists and  
 $\lim_{x \rightarrow a} (f(x) + \lambda g(x)) = \lim_{x \rightarrow a} f(x) + \lambda \lim_{x \rightarrow a} g(x)$
- (Dot product) If  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist then  
 $\lim_{x \rightarrow a} (f(x) \cdot g(x))$  exists and

$$\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \left( \lim_{x \rightarrow a} f(x) \right) \cdot \left( \lim_{x \rightarrow a} g(x) \right)$$

- (Scalar product) If  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} \phi(x)$  exist then

$$\lim_{x \rightarrow a} (\phi(x) f(x)) \text{ exists and}$$

$$\lim_{x \rightarrow a} (\phi(x) f(x)) = \left( \lim_{x \rightarrow a} \phi(x) \right) \left( \lim_{x \rightarrow a} f(x) \right)$$

- Squeeze Theorem: Let  $A \subseteq \mathbb{R}$  be a set and

let  $a$  be a limit point of  $A$ .

Let  $f, g, h$  be real-valued functions w/ domain  $A$ .

Assume there exists  $\delta > 0$  s.t.

$$\forall x \in A, 0 < \|x - a\| < \delta \Rightarrow f(x) \leq g(x) \leq h(x).$$

If  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = b$  for some  $b \in \mathbb{R}$  then  $\lim_{x \rightarrow a} g(x) = b$ .

### Limits with Infinity

- Let  $A \subseteq \mathbb{R}^n$  be unbounded. Let  $f: A \rightarrow \mathbb{R}^m$  and let  $b \in \mathbb{R}^m$ .

Define  $b$  to be the limit of  $f(x)$  as  $\|x\| \rightarrow \infty$  provided

$$\forall \epsilon > 0, \exists M > 0 \text{ s.t. } \forall x \in A, \|x\| > M \Rightarrow \|f(x) - b\| < \epsilon.$$

- $\lim_{\|x\| \rightarrow \infty} f(x) = b$

- $f(x) \rightarrow b$  as  $\|x\| \rightarrow \infty$

- Let  $A \subseteq \mathbb{R}^n$  be a set. Let  $a$  be a limit point of  $A$ .

Let  $f: A \rightarrow \mathbb{R}$  be a real-valued function

The limit of  $f$  at  $a$  diverges to  $+\infty$  provided

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A, 0 < |x-a| < \delta \Rightarrow f(x) > M$$

- $\lim_{x \rightarrow a} f(x) = +\infty$
- $f(x) \rightarrow +\infty$  as  $x \rightarrow a$
- $f(x) \rightarrow -\infty$  as  $x \rightarrow a$  can be defined similarly

## 2.7 Continuity

### Formal Definition

- Let  $f: A \rightarrow \mathbb{R}^m$  be a function w/ domain  $A \subseteq \mathbb{R}^n$ .

Let  $a \in A$  be a point. The function  $f$  is **continuous** at  $a$  provided  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.

$$\forall x \in A, \|x - a\| < \delta \Rightarrow \|f(x) - f(a)\| < \varepsilon$$

- If  $a$  is an isolated point of  $A$ , then

$f$  is cts at  $a$

- If  $a$  is a limit point of  $A$ , then

$f$  is cts at  $a$  iff  $\lim_{x \rightarrow a} f(x) = f(a)$

- $f$  is cts at  $a$  is equivalent to

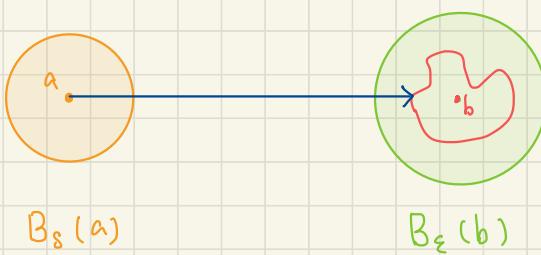
- $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $\forall x \in A,$

$$x \in B_\delta(a) \Rightarrow f(x) \in B_\varepsilon(f(a))$$

- $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $\forall x \in A \cap B_\delta(a),$

$$f(x) \in B_\varepsilon(f(a))$$

- $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  $f(A \cap B_\delta(a)) \subseteq B_\varepsilon(f(a))$



- Let  $f: A \rightarrow \mathbb{R}^m$  be a function w/ domain  $A \subseteq \mathbb{R}^n$ .  
 Let  $a \in A$  be a point. Then  $f$  is cts at  $a$  iff  
 for every sequence  $\{x(k)\}_k$  in  $A$  converging to  $a$ ,  
 the sequence  $\{f(x(k))\}_k$  in  $\mathbb{R}^m$  converges to  $f(a)$ .
- Let  $f: A \rightarrow \mathbb{R}^m$  be a function w/ domain  $A \subseteq \mathbb{R}^n$ .  
 For a subset  $S \subseteq A$ , the function  $f$  is continuous  
 on  $S$  if  $f$  is cts at  $a$  for every  $a \in S$ .
  - $f$  is continuous if  $f$  is cts on its domain  $A$

### Basic Properties

- The map  $f = (f_1, \dots, f_m): A \rightarrow \mathbb{R}^m$  is cts at  $a \in A$  iff  
 for each  $i \in \{1, \dots, m\}$ , the component function  $f_i$  is  
 cts at  $a$ .
- Every linear transformation  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  is cts
- Let  $A \subseteq \mathbb{R}^n$  and let  $a \in A$ . Let  $f, g: A \rightarrow \mathbb{R}^m$ .

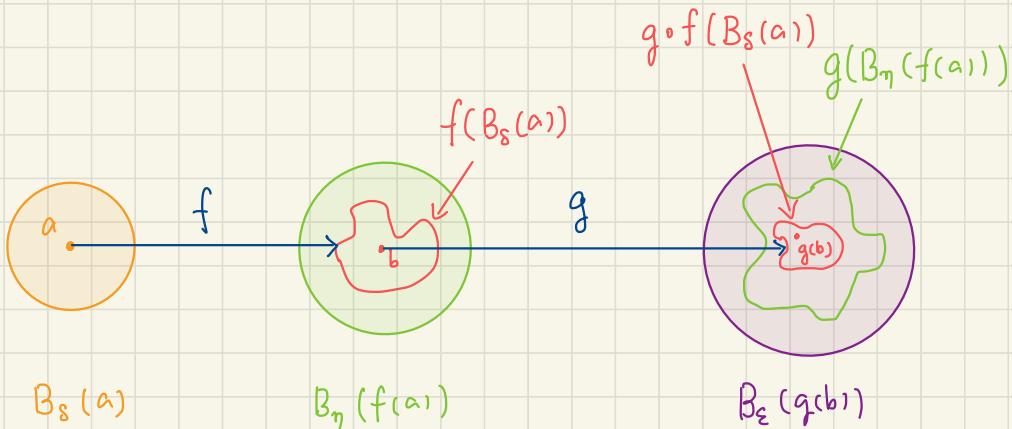
Let  $\phi: A \rightarrow \mathbb{R}$ . Let  $a \in A$ . Then

- If  $f, g$  are cts at  $a$ , then  $f + \lambda g$  is cts at  $a$
- If  $f, g$  are cts at  $a$ , then  $f \cdot g$  is cts at  $a$
- If  $f, \phi$  are cts at  $a$ , then  $\phi f$  is cts at  $a$
- Let  $f: A \rightarrow B$  where  $A \subseteq \mathbb{R}^n$  and  $B \subseteq \mathbb{R}^m$ .

Let  $g: B \rightarrow \mathbb{R}^k$ .

- Let  $a \in A$ . If  $f$  is cts at  $a$  and  $g$  is cts at  $f(a)$  then  $g \circ f$  is cts at  $a$
- Let  $a$  be a limit point of  $A$  and let  $b \in B$ .  
If  $\lim_{x \rightarrow a} f(x) = b$  and  $g$  is cts at  $b$  then

$$\lim_{x \rightarrow a} g \circ f(x) = g(b)$$



- A monomial in the  $n$  variables  $x_1, \dots, x_n$  is a function of the form  $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  for some  $\alpha_1, \dots, \alpha_n \in \mathbb{N}$ .
- A polynomial in the  $n$  variables  $x_1, \dots, x_n$  is a linear combination of monomials in  $n$  variables w/ real coefficients.
- All polynomials in  $n$  variables are cts on  $\mathbb{R}^n$

### Topological Properties

- Cts functions preserve topological properties of sets (under image and preimage)
- Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a function. The following are equivalent:
  - $f$  is cts on  $\mathbb{R}^n$
  - The preimage  $f^{-1}(U)$  is open for every open set  $U \subseteq \mathbb{R}^m$
  - The preimage  $f^{-1}(V)$  is closed for every closed set  $V \subseteq \mathbb{R}^m$
- If  $A$  is a compact subset of  $\mathbb{R}^n$  and  $f$  is an  $\mathbb{R}^m$ -valued function that is cts on  $A$  then  $f(A)$  is a cpt subset of  $\mathbb{R}^m$

## 2.8 Path-Connected Sets

- A set  $S \subseteq \mathbb{R}^n$  is path-connected if for every pair of points  $p, q \in S$  there exists a cts function  $\gamma: [a, b] \rightarrow \mathbb{R}^n$  s.t.  $\gamma(a) = p$  and  $\gamma(b) = q$  and  $\text{img}(\gamma) \subseteq S$ 
  - $S$  is  $C^k$  path-connected if  $\gamma$  is required to be continuously  $k$ -times differentiable
- A set  $S \subseteq \mathbb{R}^n$  is convex if the line segment between any two points  $p, q \in S$  lies inside  $S$
- Let  $S \subseteq \mathbb{R}^n$  be a path-connected set. Let  $f: S \rightarrow \mathbb{R}^m$ . If  $f$  is cts on  $S$  then  $f(S)$  is path-connected.
- Intermediate Value Theorem: Let  $f: [a, b] \rightarrow \mathbb{R}$ . If  $f$  is cts on  $[a, b]$  then  $f([a, b])$  is path-connected

## 2.9 Global Extrema

### Definition of Global Extrema

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}$ 
  - A point  $p \in A$  is a (global) maximum point of  $f$  on  $A$  if  $f(p) \geq f(x)$  for all  $x \in A$
  - $f(p)$  is the (global) maximum value of  $f$  on  $A$
- If a maximum point of  $f$  on  $A$  exists, then  $f$  attains a (global) maximum on  $A$
- Minimum point, minimum value, and attaining minimum can be defined similarly
- Extremum: minimum or maximum

### Extreme Value Theorem

- Extreme value theorem: If  $A \subseteq \mathbb{R}^n$  is a compact set and  $f: A \rightarrow \mathbb{R}$  is cts then  $f$  attains maximum and minimum values at points of  $A$

- Let  $A \subseteq \mathbb{R}^n$  be closed and unbounded. Let  $f: A \rightarrow \mathbb{R}$  be cts. If  $f(x) \rightarrow -\infty$  as  $\|x\| \rightarrow \infty$  in  $A$ , then  $f$  attains a maximum on  $A$

- Chain rule:** Let  $A, B \subseteq \mathbb{R}$ . Let  $\varphi: A \rightarrow B$  and let  $f: B \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$  s.t.  $\varphi(a) \in B^\circ$ . If  $\varphi$  is differentiable at  $a$  and  $f$  is differentiable at  $\varphi(a)$ , then  $(f \circ \varphi)'(a) = f'(\varphi(a)) \varphi'(a)$ .

# 3.1 Derivatives of One Variable

## Definition

- Let  $A \subseteq \mathbb{R}$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .

The derivative of  $f$  at  $a$  is defined to be

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ provided the limit exists.}$$

- $f$  is differentiable at  $a$

$$\bullet \quad f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

## Basic Properties

- Let  $A \subseteq \mathbb{R}$  and let  $f = (f_1, \dots, f_m): A \rightarrow \mathbb{R}^m$ .

Let  $a \in A^\circ$ .  $f$  is differentiable at  $a$  iff

for every  $i \in \{1, \dots, m\}$ ,  $f_i$  is differentiable at  $a$ .

$$\bullet \quad f'(a) = (f'_1(a), \dots, f'_m(a)) = \begin{bmatrix} f'_1(a) \\ \vdots \\ f'_m(a) \end{bmatrix}$$

- Let  $A \subseteq \mathbb{R}$  and let  $f, g: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .

Let  $\lambda \in \mathbb{R}$  and let  $\varphi: A \rightarrow \mathbb{R}$ . Then

- (Linearity) If  $f, g$  are differentiable at  $a$ , then

$f + \lambda g$  is differentiable at  $a$  and

$$(f + \lambda g)'(a) = f'(a) + \lambda g'(a).$$

- (Scalar product) If  $f, \varphi$  are differentiable at  $a$ ,  
then  $\varphi f$  is differentiable at  $a$  and

$$(\varphi f)'(a) = \varphi'(a)f(a) + \varphi(a)f'(a)$$

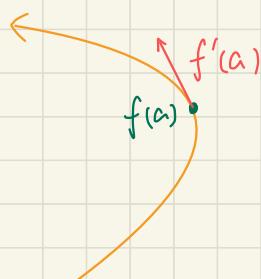
- (Dot product) If  $f, g$  are differentiable at  $a$ ,  
then  $f \cdot g$  is differentiable at  $a$  and

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

## Four Viewpoints

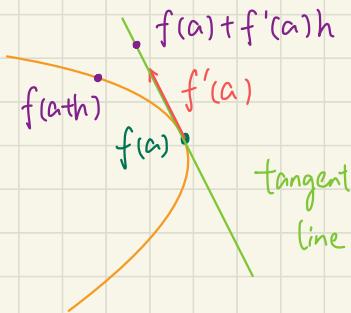
Let  $A \subseteq \mathbb{R}$  and let  $f: A \rightarrow \mathbb{R}^m$  be a parametric curve  
differentiable at  $a \in A$ .

- Physical viewpoint
  - If  $f$  is the position of a particle, then  $f'(a)$  is the instantaneous velocity of the particle at time  $a$  and position  $f(a)$



- Geometric viewpoint

- The tangent line at  $f(a)$  on  $f$  is (presumably) given by the set  $\{f(a) + hf'(a) : h \in \mathbb{R}\}$
- $f'(a)$  is the direction vector



- Analytic viewpoint

- The linear approximation of  $f$  at  $a$  is the function  $l: \mathbb{R} \rightarrow \mathbb{R}^m$  defined by

$$l(x) = f(a) + f'(a)(x-a)$$

- $f(x) \approx l(x)$  for  $x$  near  $a$

- Algebraic viewpoint

- Let  $A \subseteq \mathbb{R}$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .

$f$  is differentiable at  $a$  iff there exists a linear map

$$L: \mathbb{R} \rightarrow \mathbb{R}^m \text{ s.t. } \lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - L(h)}{h} = 0,$$

in which case  $L(h) = f'(a)h$

- Let  $A \subseteq \mathbb{R}$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .
  - If  $f$  is differentiable at  $a$ , then the linear map  $df_a: \mathbb{R} \rightarrow \mathbb{R}^m$  defined by  $df_a(h) = f'(a)h$  is the differential of  $f$  at  $a$ .
- For  $h$  near 0,  $f(a+h) \approx f(a) + f'(a)h = f(a) + df_a(h)$
- Chain rule in terms of differentials:
$$d(g \circ f)_a = dg_{f(a)} \circ df_a$$
  - Differential of the composition is composition of differentials

## 3.2 Partial Derivatives

### Definition

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .  
Fix  $1 \leq j \leq n$ . Let  $\{e_1, \dots, e_n\}$  be the standard basis for  $\mathbb{R}^n$ . The  $j$ th partial derivative of  $f$  at  $a$  is
$$\partial_j f(a) := \lim_{h \rightarrow 0} \frac{f(a + he_j) - f(a)}{h}$$
provided that the limit exists.
  - The  $j$ th partial derivative of  $f$  is the function
$$\partial_j f: U \rightarrow \mathbb{R}^m$$
, where  $U$  is the set of points  $a \in A$  s.t.  $\partial_j f(a)$  exists.
- Equivalent notations for  $\partial_j f$ :
  - $\frac{\partial f}{\partial x_j}$     $D_{x_j} f$     $f_{x_j}$     $D_j f$     $\partial_{x_j} f$

### Computations

- Let  $A \subseteq \mathbb{R}^n$  and let  $f = (f_1, \dots, f_m): A \rightarrow \mathbb{R}^m$ .  
Let  $a \in A^\circ$ .  $\partial_j f(a)$  exists iff for every  $i \in \{1, \dots, m\}$ ,  $\partial_j f_i(a)$  exists. If so,  $\partial_j f(a) = (\partial_j f_1(a), \dots, \partial_j f_m(a))$ .
- Let  $A \subseteq \mathbb{R}^n$  and let  $f, g: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .

Fix  $1 \leq j \leq n$ . Let  $\lambda \in \mathbb{R}$  and let  $\varphi: A \rightarrow \mathbb{R}$ .

- (Linearity) If  $\partial_j f(a)$  and  $\partial_j g(a)$  both exist,  
then  $\partial_j(f + \lambda g)(a)$  exists and  
$$\partial_j(f + \lambda g)(a) = \partial_j f(a) + \lambda \partial_j g(a)$$
- (Scalar product) If  $\partial_j f(a)$  and  $\partial_j \varphi(a)$  both exist,  
then  $\partial_j(\varphi f)(a)$  exists and  
$$\partial_j(\varphi f)(a) = f(a) \partial_j \varphi(a) + \varphi(a) \partial_j f(a)$$
- (Dot product) If  $\partial_j f(a)$  and  $\partial_j g(a)$  both exist,  
then  $\partial_j(f \cdot g)(a)$  exists and  
$$\partial_j(f \cdot g)(a) = \partial_j f(a) \cdot g(a) + f(a) \cdot \partial_j g(a)$$

### 3.3 Directional Derivatives

#### Definition

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .  
Fix  $v \in \mathbb{R}^n$ . The directional derivative of  $f$  in the direction  $v$  at  $a$  is given by  $D_v f(a) = \lim_{h \rightarrow 0} \frac{f(a+hv) - f(a)}{h}$ 
  - The directional derivative of  $f$  in the direction  $v$  is the function  $D_v f: U \rightarrow \mathbb{R}^m$ , where  $U$  is the set of points  $a \in A$  s.t.  $D_v f(a)$  exists
  - $D_{e_j} f = \partial_j f$

#### Computations

- Let  $A \subseteq \mathbb{R}^n$  and let  $f = (f_1, \dots, f_m): A \rightarrow \mathbb{R}^m$ .  
Let  $a \in A^\circ$ . Fix  $v \in \mathbb{R}^n$ . Then  $D_v f(a)$  exists iff  $D_v f_i(a)$  exists for every  $i \in \{1, \dots, m\}$ . If so,  
$$D_v f(a) = (D_v f_1(a), \dots, D_v f_m(a))$$
- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .  
If  $f$  is differentiable at  $a$ , then

for all  $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ ,  $D_v f(a) = \sum_{j=1}^n v_j \partial_j f(a)$

## Geometry of Directional Derivatives

- The directional derivative of  $f$  at  $a$  in the direction  $v$  outputs a vector  $D_v f(a)$  that is tangent to the parametric curve  $\gamma(t) = f(a + tv)$  at  $t = 0$

- $$\gamma'(0) = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t} = \lim_{t \rightarrow 0} \frac{f(a + tv) - f(a)}{t} = D_v f(a)$$

## 3.4 Gradient

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}$ . Let  $a \in A^\circ$ .

The gradient of  $f$  at  $a$  is denoted  $\nabla f(a)$  and given by

$$\nabla f(a) = (\partial_1 f(a), \dots, \partial_n f(a))$$

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}$ . Let  $a \in A^\circ$ .

If  $f$  is differentiable at  $a$ , then

for all  $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ ,  $D_v f(a)$  exists and

$$D_v f(a) = \nabla f(a) \cdot v = (\nabla f(a))^T v$$

### Gradient Vector Field

- Let  $U \subseteq \mathbb{R}^n$  be open and let  $f: U \rightarrow \mathbb{R}$ . Assume all partial derivatives of  $f$  exists on  $U$ . The gradient of  $f$

(or gradient vector field of  $f$ ) is the function

$$\nabla f: U \rightarrow \mathbb{R}^n \text{ given by } \nabla f(a) = (\partial_1 f(a), \dots, \partial_n f(a))$$

for all  $a \in U$ .

- $\nabla f(a)$  points in the direction of steepest ascent,  
orthogonal to tangent line

## Direction of Steepest Ascent

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}$ . Let  $a \in A^\circ$ .
  - Assume  $f$  is differentiable at  $a$  and  $\nabla f(a) \neq 0$ , then:
    - The max of  $D_u f(a)$  over all unit vectors  $u$  occurs when  $u = +\frac{\nabla f(a)}{\|\nabla f(a)\|}$  and the max value is  $\|\nabla f(a)\|$
    - The min of  $D_u f(a)$  over all unit vectors  $u$  occurs when  $u = -\frac{\nabla f(a)}{\|\nabla f(a)\|}$  and the min value is  $-\|\nabla f(a)\|$
  - $\nabla f(a)$  points in the direction of steepest ascent at  $a$
  - $\|\nabla f(a)\|$  is the rate of change of  $f$  in this direction

## Orthogonality to Level Sets

- Every tangent vector  $v$  at a point  $p$  of an implicit surface  $S$  is orthogonal to  $\nabla f(p)$ 
  - $\nabla f(p)$  is orthogonal to the tangent plane of  $S$  at  $p$

## 3.5 Differentials and the Jacobian

### Definitions

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .  
 $f$  is differentiable at  $a$  if there exists a linear map  
 $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$  s.t.  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - L(h)}{\|h\|} = 0$ 
  - $L$  is the differential of  $f$  at  $a$ , denoted  $df_a$ 
    - Aka total derivative of  $f$  at  $a$
  - The error  $f(a+h) - f(a) - L(h)$  tends to 0 faster than  $\|h\|$  tends to the 0 scalar

### Properties

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .  
Then  $f$  is differentiable at  $a$  iff each of its component functions  $f^1, f^2, \dots, f^m$  is. If so,  
 $df_a = (df_a^1, \dots, df_a^m)$
- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .  
If  $f$  is differentiable at  $a$ , then  $f$  is cts at  $a$

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .

If  $f$  is differentiable at  $a$ , then  $\forall v \in \mathbb{R}^n$ ,  $D_v f(a)$  exists and  $d_{fa}(v) = D_v f(a)$

### Matrix of the Differential

- Let  $A \subseteq \mathbb{R}^n$  and let  $f = (f_1, \dots, f_m): A \rightarrow \mathbb{R}^m$ .

Let  $a \in A^\circ$ . Assume all the partials of  $f$  exist at  $a$ .

The **Jacobian** of  $f$  at  $a$  is the  $m \times n$  matrix  $Df(a)$

given by  $Df(a) = [\partial_j f_i(a)]_{i,j}$

$$= \begin{bmatrix} 1 & & 1 \\ \partial_1 f_1(a) & \cdots & \partial_n f_1(a) \\ & \ddots & \\ 1 & & 1 \end{bmatrix} = \begin{bmatrix} \partial_1 f_1(a) & \cdots & \partial_n f_1(a) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(a) & \cdots & \partial_n f_m(a) \end{bmatrix}$$

- Also **Jacobian matrix**
- Equivalent notations:  $f'(a)$ ,  $Jf(a)$ ,  $J_f(a)$ ,  $J_{\mathbf{f}}(a)$

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .

If  $f$  is differentiable at  $a$ , then the matrix of the differential of  $f$  at  $a$  is the  $m \times n$  Jacobian matrix

of  $f$  at  $a$ , i.e.  $\forall v \in \mathbb{R}^n$ ,  $d_{fa}(v) = Df(a)v$

$$d_{fa}(v) = Df(a)v = D_v f(a) = \sum_{j=1}^n v_j \partial_j f(a)$$

## 3.6 Differentiability

### Continuously Differentiable Functions

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .
  - f is continuously differentiable at a (or  $C'$  at a or of class  $C'$  at a) if  $\partial_1 f, \dots, \partial_n f$  are defined on an open set containing a and are all cts at a
- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $U \subseteq \mathbb{R}^n$ .
  - f is continuously differentiable on U if f is  $C'$  at every point  $a \in U$
  - f is continuously differentiable if f is  $C'$  on its domain
- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .
  - f is  $C'$  at a iff each component function  $f_i$  is  $C'$  at a for all  $i \in \{1, \dots, m\}$
- Let  $A \subseteq \mathbb{R}^n$ . Let  $f, g: A \rightarrow \mathbb{R}^m$ . Let  $\phi, \psi: A \rightarrow \mathbb{R}$ . Fix  $\lambda \in \mathbb{R}$  and let  $a \in A^\circ$ .
  - If  $f, g$  are  $C'$  at a then  $f + \lambda g$  is  $C'$  at a

- If  $f, g$  are  $C^1$  at  $a$  then  $f \cdot g$  is  $C^1$  at  $a$
- If  $f, \phi$  are  $C^1$  at  $a$  then  $\phi f$  is  $C^1$  at  $a$
- If  $\phi, \psi$  are  $C^1$  at  $a$  and  $\psi(a) \neq 0$   
then  $\phi/\psi$  is  $C^1$  at  $a$

### Differentiability Criterion

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}^m$ . Let  $a \in A^\circ$ .  
If  $f$  is  $C^1$  at  $a$  then  $f$  is differentiable at  $a$ 
  - Converse is not true

## 3.7 Chain Rule

- Let  $A \subseteq \mathbb{R}^n$  and let  $a \in A^\circ$ . Fix  $\lambda \in \mathbb{R}$ .  
If  $f: A \rightarrow \mathbb{R}^m$  and  $g: A \rightarrow \mathbb{R}^n$  are differentiable at  $a$ ,  
then  $f + \lambda g$  is differentiable at  $a$  and  
 $d(f + \lambda g)_a = df_a + \lambda dg_a$
- **Chain rule:** Let  $U \subseteq \mathbb{R}^n$  and  $V \subseteq \mathbb{R}^m$  be open.  
If  $f: U \rightarrow V$  and  $g: V \rightarrow \mathbb{R}^k$  are differentiable at  
 $a \in U$  and  $f(a) \in V$  respectively, then  $g \circ f$  is  
differentiable at  $a$ .
  - $dh_a = dg_{f(a)} \circ df_a$
  - $Dh(a) = Dg(f(a)) Df(a)$
  - "The differential of a composition is the  
composition of differentials"
- Let  $U \subseteq \mathbb{R}^n$ ,  $V \subseteq \mathbb{R}^m$ . If  $f: U \rightarrow V$  is  $C^1$  and  
 $g: V \rightarrow \mathbb{R}^k$  is  $C^1$ , then  $g \circ f: U \rightarrow \mathbb{R}^k$  is  $C^1$

## Leibniz Notation and Chain Rule Trees

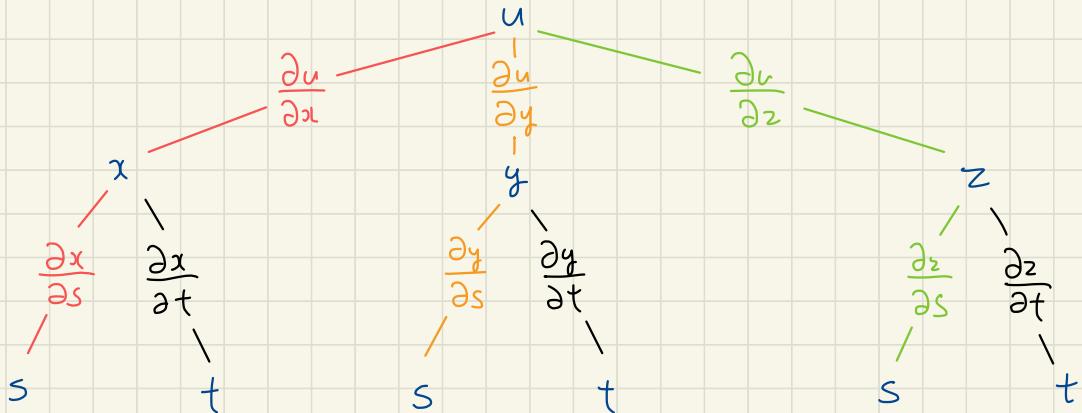
o  $\partial_u g$  is same as  $\frac{\partial g}{\partial x}$

o Define  $g: \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ . Define  $u = g(x, y, z)$

and  $(x, y, z) = f(s, t)$ . Then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t}$$



## 3.8 Local Extrema and Critical Points

### Local Extreme Value Theorem

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}$ . Let  $a \in A$ .
  - $a$  is a local maximum point of  $f$  on  $A$  if  $\exists \delta > 0$  s.t.  $f(a) \geq f(x)$  for all  $x \in A \cap B_\delta(a)$ 
    - $f(a)$  is a local maximum value of  $f$  on  $A$
    - $f$  attains a local maximum on  $A$
  - Def of local minimum point, local minimum value, attaining a local minimum are similar
- Local extremum: local max or local min
- Local extreme value theorem: Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}$ .  
Let  $a \in A^\circ$ . If  $a$  is a local extremum of  $f$  and  $f$  is differentiable at  $a$ , then  $\nabla f(a) = 0$ 
  - Gives no information on  $\partial A$

### Critical Points

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}$ . A point  $a \in A$  is a critical point of  $f$  at  $a$  if  $a \in A^\circ$  and either

$$\nabla f(a) = 0 \text{ or } \nabla f(a) \text{ DNE}$$

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}$ . If  $a \in A$  is a local extremum of  $f$ , then either  $a \in \partial A$  or  $a$  is a critical point of  $f$

## 3.9 Optimization

1. Determine whether global extrema must exist
  - Use EVT if possible
2. Identify the crit points on the interior of the domain
  - Gradient equals 0
3. Check the boundary for possible extrema
  - Parametrize boundary if possible
4. Plug the candidate points back into the function

## 3.10 Tangent Space

### Tangent Vectors, Spaces, and Planes

- Let  $S \subseteq \mathbb{R}^n$  and let  $p \in S$ . A vector  $v \in \mathbb{R}^n$  is a tangent vector of  $S$  at  $p$  if there exists an open interval  $I \subseteq \mathbb{R}$  containing  $0$  and a differentiable parametric curve  $\gamma: I \rightarrow \mathbb{R}^n$  w/  $\gamma(I) \subseteq S$ ,  $\gamma(0) = p$ , and  $\gamma'(0) = v$ 
  - "There exists a particle moving along  $S$  through  $p$  w/ velocity  $v$ "
- Let  $S \subseteq \mathbb{R}^n$  and let  $p \in S$ . The tangent space of  $S$  at  $p$ , denoted  $T_p S$ , is the set of tangent vectors to  $S$  at  $p$ 
  - $T_p S = \{v \in \mathbb{R}^n : v \text{ is a tangent vector of } S \text{ at } p\}$
  - " $T_p S$  is the set of all possible velocities for a particle moving along  $S$  through  $p$ "
- Let  $S \subseteq \mathbb{R}^n$  and  $p \in S$ . The tangent plane of  $S$  at  $p$ , denoted  $p + T_p S$ , is the tangent space translated to  $p$

$$\circ \quad p + T_p S = \{p + v : v \in T_p S\}$$

## Tangent Space of a Graph

- Let  $S \subseteq \mathbb{R}^n$  be the graph of a function  $F: U \rightarrow \mathbb{R}^{n-k}$  where  $U \subseteq \mathbb{R}^k$  is open. Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a parametric curve where  $I \subseteq \mathbb{R}$  is open. Then  $\gamma$  is differentiable and  $\gamma(I) \subseteq S$  iff  $\gamma(t) = (g(t), F(g(t)))$  for some differentiable function  $g: I \rightarrow U$ .
- Let  $S \subseteq \mathbb{R}^n$  be the graph of a differentiable function  $F: U \rightarrow \mathbb{R}^{n-k}$  where  $U \subseteq \mathbb{R}^k$  is open. For any  $a \in U$ , the tangent space of  $S$  at  $p = (a, f(a))$  is  $T_p S = \{(w, dF_a(w)) : w \in \mathbb{R}^k\}$ , and  $T_p S$  is a  $k$ -dimensional subspace of  $\mathbb{R}^n$ .

## 3.11 Regular Surfaces

### Definition of a Regular Surface

- Let  $S \subseteq \mathbb{R}^n$  and let  $p \in S$ .  $S$  is a  $k$ -dimensional regular surface at  $p$  if  $\exists \varepsilon > 0$  s.t.  $B_\varepsilon(p) \cap S$  is a graph of a  $C^1$  function  $f: U \rightarrow \mathbb{R}^{n-k}$  where  $U \subseteq \mathbb{R}^k$  is open
  - Choice of  $f$  may not be unique
  - A set is not a regular surface at a point when the tangent space fails to be a subspace w/ the expected dimension
- A set  $S \subseteq \mathbb{R}$  is a  $k$ -dimensional regular surface if  $S$  is a  $k$ -dimensional regular surface at every point in  $S$ 
  - Regular curve: 1-dimensional regular surface
- Let  $S \subseteq \mathbb{R}^n$  be the graph of a  $C^1$  function  $F: U \rightarrow \mathbb{R}^{n-k}$  w/  $U \subseteq \mathbb{R}^k$  open.  $S$  is a  $k$ -dimensional regular surface

### Tangent Space of a Regular Surface

- Let  $S \subseteq \mathbb{R}^n$  and let  $p \in S$ . If  $S$  is a  $k$ -dimensional regular surface at  $p$ , then  $T_p S$  is a  $k$ -dimensional subspace of  $\mathbb{R}^n$

## 4.1 Diffeomorphisms

### Global Diffeomorphisms

- Let  $U, V \subseteq \mathbb{R}^n$ . The (global) inverse of  $F: U \rightarrow V$  is a map  $G: V \rightarrow U$  satisfying  $G \circ F(x) = x$  for all  $x \in U$  and  $F \circ G(y) = y$  for all  $y \in V$ .
  - Equivalently,  $\forall x \in U, \forall y \in V,$   
 $y = F(x) \Leftrightarrow x = G(y)$
  - The inverse of  $F$  is unique and is denoted  $F^{-1}$
- Let  $U, V$  be open subsets of  $\mathbb{R}^n$ . A function  $F: U \rightarrow V$  is a (global) diffeomorphism if  $F$  is bijective,  $F$  is  $C^1$ , and its inverse function  $F^{-1}: V \rightarrow U$  is  $C^1$ .

### Properties of Diffeomorphisms

- Let  $U, V$  be open subsets of  $\mathbb{R}$ . Assume  $F: U \rightarrow V$  is bijective.  $F$  is a diffeomorphism iff  $F'$  is a diffeomorphism
- Let  $U, V, W$  be open subsets of  $\mathbb{R}$ . If  $F: U \rightarrow V$  and  $G: V \rightarrow W$  are diffeomorphisms, then  $G \circ F: U \rightarrow W$

is a diffeomorphism

- Let  $U, V$  be open subsets of  $\mathbb{R}^n$ . Let  $F: U \rightarrow V$  be a diffeomorphism. For every  $S \subseteq U$ :

- $S$  is open iff  $F(S)$  is open
- $S$  is closed iff  $F(S)$  is closed
- $S$  is cpt iff  $F(S)$  is cpt
- $S$  is path-connected iff  $F(S)$  is path-connected

### Local Diffeomorphisms

- Let  $A, B$  be open subsets of  $\mathbb{R}^n$ . Fix  $a \in A$ .  
A function  $F: A \rightarrow B$  is a local diffeomorphism at  $a$  if there exists an open subset  $U \subseteq A$  containing  $a$  s.t.  $F(U)$  is open and the restriction  $F|_U: U \rightarrow F(U)$  is a diffeomorphism.
  - The inverse function  $G = F|_U^{-1}: F(U) \rightarrow U$  is the local inverse of  $F$  at  $a$
- Let  $A, B$  be open subsets of  $\mathbb{R}^n$ . If  $F: A \rightarrow B$  is a global diffeomorphism, then  $F$  is a local diffeomorphism at every  $a \in A$ .

## 4.2 Inverse Function Theorem

### Derivatives of Diffeomorphisms

- Let  $U, V$  be open subsets of  $\mathbb{R}^n$ . Assume  $F: U \rightarrow V$  is a diffeomorphism. For every  $x \in U$ , the Jacobian  $DF(x)$  is an invertible  $n \times n$  matrix and the Jacobian of the inverse function  $G = F^{-1}: V \rightarrow U$  satisfies  $DG(y) = [DF(x)]^{-1}$  for every  $x \in U$  and  $y = F(x)$ .
- Let  $A, B$  be open subsets of  $\mathbb{R}^n$ . Fix  $a \in A$ . Let  $F: A \rightarrow B$  be a  $C^1$  function. If  $F$  is a local diffeomorphism at  $a$ , then the Jacobian  $DF(a)$  is an invertible matrix
  - "If a non-linear map is invertible, then its linear approximation must be invertible"

### Inverse Function Theorem

- Inverse function theorem:** Let  $A, B$  be open subsets of  $\mathbb{R}^n$ . Fix  $a \in A$ . Let  $F: A \rightarrow B$  be  $C^1$ . If the Jacobian  $DF(a)$  is an invertible  $n \times n$  matrix, then  $F$  is a local diffeomorphism at  $a$ .

## 4.3 Nonlinear Systems

### Single Nonlinear Equation

- Let  $U \subseteq \mathbb{R}^n \times \mathbb{R}$  be an open set. Let  $f: U \rightarrow \mathbb{R}$  be  $C^1$ .

Let  $(a, b) \in U$  so  $a = (a_1, \dots, a_n) \in \mathbb{R}^n$  and  $b \in \mathbb{R}$ .

Assume  $f(a, b) = 0$ . The equation  $f(x_1, \dots, x_n, y) = 0$

locally defines  $y$  as a  $C^1$  function of  $x = (x_1, \dots, x_n)$

near  $(a, b)$  if there exists an open set  $V \subseteq \mathbb{R}^n$

containing  $a$ , an open set  $W \subseteq \mathbb{R}$  containing  $b$ ,

and a  $C^1$  function  $\phi: V \rightarrow W$  s.t.  $V \times W \subseteq U$  and

$\forall (x, y) \in V \times W, f(x, y) = 0 \Leftrightarrow y = \phi(x)$

$$\bullet \quad \{(x, y) \in V \times W : f(x, y) = 0\} = \{(x, \phi(x)) : x \in V\}$$

### Many Nonlinear Equations

- Let  $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$  be an open set. Let  $F: U \rightarrow \mathbb{R}^k$  be  $C^1$ .

Let  $(a, b) \in U$  so  $a = (a_1, \dots, a_n) \in \mathbb{R}^n$  and

$b \in (b_1, \dots, b_k) \in \mathbb{R}^k$ . Assume  $F(a, b) = 0$ . The equation

$F(x_1, \dots, x_n, y_1, \dots, y_k) = 0$  locally defines  $y$  as a

$C^1$  function of  $x$  near  $(a, b)$  if there exists an open set

$V \subseteq \mathbb{R}^n$  containing  $a$ , an open set  $W \subseteq \mathbb{R}^k$  containing  $b$ ,

and a  $C^1$  function  $\phi: V \rightarrow W$  s.t.  $V \times W \subseteq U$  and

for all  $(x, y) \in V \times W$ ,  $F(x, y) = 0 \Leftrightarrow y = \phi(x)$

- Let  $A$  be a  $k \times n$  matrix and let  $B$  be a  $k \times k$  matrix. If  $B$  is invertible then the system of  $k$  linear equations w/  $n+k$  variables

$$[A \mid B] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ globally defines}$$

$y = (y_1, \dots, y_k) \in \mathbb{R}^k$  as a  $C^1$  function of

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$

## 4.4 Implicit Function Theorem

### Implicit Differentiation for One Variable

- Let  $U \subseteq \mathbb{R}^n \times \mathbb{R}$  be an open set. Let  $f: U \rightarrow \mathbb{R}$  be  $C^1$ .

Let  $(a, b) \in U$  so  $a = (a_1, \dots, a_n) \in \mathbb{R}^n$  and  $b \in \mathbb{R}$ .

Assume  $f(a, b) = 0$  and  $f$  is not constant.

If the equation  $f(x_1, \dots, x_n, y) = 0$  locally defines  $y$

as a  $C^1$  function  $\phi: V \rightarrow W$  of  $x = (x_1, \dots, x_n)$  near  $(a, b)$

then for every point  $(v, w) = (v_1, \dots, v_n, w) \in V \times W$  and  
every  $j \in \{1, 2, \dots, n\}$ ,

$$\frac{\partial f}{\partial x_j}(v, w) + \frac{\partial f}{\partial y}(v, w) \frac{\partial \phi}{\partial x_j}(v) = 0$$

### Implicit Function Theorem for One Variable

- Implicit Function Theorem:** Let  $U \subseteq \mathbb{R}^n \times \mathbb{R}$  be open and let  $f: U \rightarrow \mathbb{R}$  be  $C^1$ . Let  $(a, b) \in U$  so  $a \in \mathbb{R}^n$ ,  $b \in \mathbb{R}$ .  
If  $f(a, b) = 0$  and  $\frac{\partial f}{\partial y}(a, b) \neq 0$ , then the equation  $f(x, y) = 0$  defines  $y$  locally as a  $C^1$  function  $\phi$  of  $x$  near  $(a, b)$

- "Assume a solution exists to a nonlinear equation.  
If one can solve an approximate linear equation,  
then one can locally solve the nonlinear equation."

### Implicit Function Theorem for Many Variables

- Let  $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$  be open. Let  $F: U \rightarrow \mathbb{R}^k$  be  $C^1$ .  
Let  $(a, b) \in U$  so  $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ ,  $b = (b_1, \dots, b_k) \in \mathbb{R}^k$ .  
Assume  $F(a, b) = 0$ . If the equation  $F(x_1, \dots, x_n, y_1, \dots, y_k) = 0$   
locally defines  $y$  as a  $C^1$  function  $\phi: V \rightarrow W$  of  $x$   
near  $(a, b)$ , then for  $(v, w) \in V \times W$ , the Jacobian  
 $D\phi(w)$  is a  $k \times k$  matrix satisfying

$$\frac{\partial F}{\partial x}(v, w) + \frac{\partial F}{\partial y}(v, w) D\phi(v) = 0$$

- $\frac{\partial F}{\partial x} = \frac{\partial(F_1, \dots, F_k)}{\partial(x_1, \dots, x_n)} := \left( \frac{\partial F_i}{\partial x_j} \right)_{i,j}$  is a  $k \times n$  matrix

and  $\frac{\partial F}{\partial y} = \frac{\partial(F_1, \dots, F_k)}{\partial(y_1, \dots, y_k)} := \left( \frac{\partial F_i}{\partial y_j} \right)_{i,j}$  is a  $k \times k$  matrix

- $\frac{\partial F}{\partial x}$  and  $\frac{\partial F}{\partial y}$  are submatrices of the  $k \times (n+k)$   
Jacobian  $DF = \left[ \begin{array}{c|c} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{array} \right]$

• Implicit Function Theorem: Let  $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$  be open.

Let  $F: U \rightarrow \mathbb{R}^k$  be  $C^1$ . Let  $(a, b) \in U$  so  $a \in \mathbb{R}^n, b \in \mathbb{R}^k$ .

If  $F(a, b) = 0$  and the  $k \times k$  matrix

$$\frac{\partial F}{\partial y}(a, b) = \frac{\partial(F_1, \dots, F_k)}{\partial(y_1, \dots, y_k)}(a, b) := \left( \frac{\partial F_i}{\partial y_j}(a, b) \right)_{i,j}$$

is invertible, then the equation  $F(x, y) = 0$  locally defines

$y = (y_1, \dots, y_k)$  as a  $\mathbb{R}^k$ -valued  $C^1$  function  $\phi$  of  $x = (x_1, \dots, x_n)$

near  $(a, b)$

## 4.5 Implicit Surfaces

### Level Sets and Gradients

- Let  $U \subseteq \mathbb{R}^n$  be open. Let  $f: U \rightarrow \mathbb{R}$  be  $C^1$ . Assume  $S = \{x \in U : f(x) = 0\} = f^{-1}(\{0\})$  is nonempty.  
Fix  $p \in S$ . If  $\nabla f(p) \neq 0$ , then  $S$  is a  $(n-1)$ -dimensional regular surface at  $p$ .
  - A vector  $v \in \mathbb{R}^n$  is a tangent vector of  $S$  at  $p$  iff  $\nabla f(p) \cdot v = 0$ , i.e.  $T_p S = \{v \in \mathbb{R}^n : \nabla f(p) \cdot v = 0\}$
  - $p + T_p S = \{x \in \mathbb{R}^n : \nabla f(p) \cdot (x - p) = 0\}$
  - "A level set in  $\mathbb{R}^n$  is a regular  $(n-1)$ -dimensional surface wherever the gradient does not vanish, and the gradient is orthogonal to the  $(n-1)$ -dimensional tangent plane."
  - Converse may not hold

### Implicit Surfaces and Kernels

- Fix  $k, n \in \mathbb{N}^+$  w/  $k < n$ . Let  $U \subseteq \mathbb{R}^n$  be open and let  $F: U \rightarrow \mathbb{R}^k$  be  $C^1$ . Assume that  $S = F^{-1}(\{0\})$

is nonempty. Fix  $p \in S$ . If  $dF_p$  has full rank,

then  $S$  is a  $(n-k)$ -dimensional regular surface at  $p$ .

- The tangent space to  $S$  at  $p$  is a

$(n-k)$ -dimensional subspace of  $\mathbb{R}^n$  given by

$$T_p S = \ker dF_p = \{v \in \mathbb{R}^n : DF(p)v = 0\}$$

## 4.6 Lagrange Multipliers

### Extrema on Subsets

- Let  $A \subseteq \mathbb{R}^n$  and let  $f: A \rightarrow \mathbb{R}$ . Let  $S \subseteq A$ .

$f$  has a global maximum on  $S$  (at  $a$ ) if

$$\forall x \in S, f(x) \leq f(a).$$

$f$  has a local maximum on  $S$  (at  $a$ ) if

$$\exists \varepsilon > 0 \text{ s.t. } \forall x \in S \cap B_\varepsilon(a), f(x) \leq f(a).$$

- Global/local minima on  $S$  are defined similarly

- $f$  has a global/local max/min on  $S$  at  $a$  iff

$$f|_S: S \rightarrow \mathbb{R} \text{ has a global/local max/min at } a$$

- Let  $A \subseteq \mathbb{R}^n$  and  $f: A \rightarrow \mathbb{R}$ . Let  $S \subseteq A$ .

Assume  $a \in S^\circ$ .  $f$  has a local max on  $S$  at  $a$  iff

$f$  has a local max at  $a$ .

- The same equivalence holds for local minima

- A  $k$ -dimensional regular surface  $S$  in  $\mathbb{R}^n$  has empty interior

## Lagrange Multipliers with One Constraint

- Lagrange multipliers with one constraint: Let  $U \subseteq \mathbb{R}^n$  be open.

Let  $f: U \rightarrow \mathbb{R}$  and  $g: U \rightarrow \mathbb{R}$  be  $C^1$ . Fix  $c \in \mathbb{R}$ .

Let  $S = \{x \in U : g(x) = c\}$  and assume

$\nabla g(p) \neq 0$  for any  $p \in S$ . If  $f$  has a local extremum on  $S$  at  $a$ , then there exists  $\lambda \in \mathbb{R}$  s.t.  $\nabla f(a) = \lambda \nabla g(a)$ .

- $\lambda$  is the Lagrange multiplier
- Solutions  $a \in U$ ,  $\lambda \in \mathbb{R}$  to the Lagrange system

$$\nabla f(a) = \lambda \nabla g(a); \quad g(a) = c$$

are candidates for local extrema of  $f$  on  $S$

## Lagrange Multipliers with Many Constraints

- Lagrange multipliers with many constraints: Let  $U \subseteq \mathbb{R}^n$  be open.

Sps  $f: U \rightarrow \mathbb{R}$  is differentiable,  $g_1, \dots, g_k: U \rightarrow \mathbb{R}$  is  $C^1$ ,

and  $c_1, \dots, c_k \in \mathbb{R}$

Let  $S = \{x \in U : g_1(x) = c_1, \dots, g_k(x) = c_k\}$ .

Assume for every  $p \in S$  that  $\nabla g_1(p), \dots, \nabla g_k(p)$  are linearly independent. If  $a$  is a local extremum of  $f$  on  $S$

then there exists  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$  s.t.  $\nabla f(a) = \sum_{i=1}^k \lambda_i \nabla g_i(a)$ .

## 4.7 Optimization with Constraints

- The Lagrange multiplier  $\lambda$  is the rate of change of the optimum value of  $f$  as  $c$  increases, where  $g(x) = c$  is the constraint
- For  $c \in \mathbb{R}$ , let  $x(c) \in \mathbb{R}^n$  be the optimum point and  $\lambda(c) \in \mathbb{R}$  be the Lagrange multiplier corresponding to the optimum value of  $f(x)$  subject to the constraint  $g(x) = c$ . If  $\lambda(c)$  and  $x(c)$  are locally defined as  $C^1$  functions of  $c$ , then  $\lambda(c) = \frac{d}{dc} f(x(c))$

# 5.1 Mean Value Theorem

- Mean Value Theorem: Let  $U \subseteq \mathbb{R}^n$  be open and let  $a, b \in U$ . Let  $f: U \rightarrow \mathbb{R}$  be diff. If  $U$  contains the line segment  $L$  from  $a$  to  $b$  then there exists  $c \in L$  s.t.  $f(b) - f(a) = \nabla f(c) \cdot (b-a)$ 
  - Does not hold for vector-valued functions
    - Different  $c$ s for different components
  - Let  $U \subseteq \mathbb{R}^n$  be open and  $C'$  path-connected.  
Let  $F: U \rightarrow \mathbb{R}^m$  be diff.  $DF(x)$  is the  $m \times n$  zero matrix for all  $x \in U$  iff  $F$  is a constant map
  - Let  $U \subseteq \mathbb{R}^n$  be open and  $C'$  path-connected.  
Let  $F: U \rightarrow \mathbb{R}^m$  and  $G: U \rightarrow \mathbb{R}^m$  be diff.  
If  $DF(x) = DG(x)$  for all  $x \in U$ , then  $\exists C \in \mathbb{R}^m$  s.t.  
 $F(x) = G(x) + C$  for all  $x \in U$

## 5.2 Second Order Derivatives

### Definition

- Let  $U \subseteq \mathbb{R}^n$  be open. Let  $f: U \rightarrow \mathbb{R}^m$  be  $C^1$ .

Fix  $i, j \in \{1, \dots, n\}$  and  $a \in U$ .

The second order partial derivative  $\partial_i \partial_j f$  at  $a$  is

defined by  $\partial_i \partial_j f(a) := \partial_i (\partial_j f)(a)$

- If  $i \neq j$  then the partial is mixed

- Order matters

- If  $i = j$  then the partial is pure

- Can write  $\partial_i^2$

- Notations:  $f_{xy} = (f_x)_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$

- $\partial_i \partial_j f(a, b) = \lim_{h \rightarrow 0} \frac{\partial_i f(a+h, b) - \partial_i f(a, b)}{h}$

### Clairaut's Theorem

- Let  $U \subseteq \mathbb{R}^n$  be an open set. A function  $f: U \rightarrow \mathbb{R}^m$  is

twice continuously differentiable (or  $C^2$ ) provided

for all  $i, j \in \{1, \dots, n\}$ ,  $\partial_i \partial_j f$  exist and are cts everywhere in  $U$

- Every polynomial is  $C^2$
- Every linear map is  $C^2$
- Clairaut's Theorem: Let  $U \subseteq \mathbb{R}^n$  be open and let  $f: U \rightarrow \mathbb{R}^m$ . If  $f$  is  $C^2$ , then  $\forall i, j = \{1, \dots, n\}, \partial_i \partial_j f = \partial_j \partial_i f$ 
  - "Mixed partials commute"
- Let  $U \subseteq \mathbb{R}^n$  be open. If  $f: U \rightarrow \mathbb{R}^m$  is  $C^2$ , then for  $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ ,  $D_h^2 f(p) := D_h(D_h f)(p)$ 

$$= \sum_{i=1}^n h_i^2 (\partial_i^2 f)(p) + \sum_{i=1}^n \sum_{j=i+1}^n 2h_i h_j (\partial_i \partial_j f)(p)$$

### Hessian Matrix

- Let  $f$  be a real-valued function which is  $C^2$  at  $a \in \mathbb{R}^n$ . The Hessian of  $f$  at  $a$  is the  $n \times n$  matrix  $Hf(a)$  defined by  $Hf(a) = [\partial_i \partial_j f(a)]_{i,j}$ 
  - The Hessian is a symmetric matrix by Clairaut's theorem

## 5.3 Higher Order Derivatives

### Generalized Clairaut's Theorem

- Let  $k \in \mathbb{N}^+$ . Let  $U \subseteq \mathbb{R}^n$  be an open set.

A function  $f: U \rightarrow \mathbb{R}^m$  is  $k$ -times continuously differentiable (or  $C^k$ ) provided all of its  $k^{\text{th}}$  order partials exist and are cts everywhere in  $U$

- For all  $i_1, \dots, i_k \in \{1, \dots, n\}$ , the  $k^{\text{th}}$  partial derivative  $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f$  exists and is cts everywhere in  $U$
- A function  $f$  is smooth (or  $C^\infty$ ) if it is  $C^k$  for every  $k \in \mathbb{N}^+$
- Every polynomial is  $C^\infty$

- Generalized Clairaut's Theorem: Let  $U \subseteq \mathbb{R}^n$  be open.

If  $f: U \rightarrow \mathbb{R}^m$  is  $C^k$ , then the mixed partial derivatives of  $f$  up to order  $k$  commute

- For any integers  $1 \leq i_1, i_2, \dots, i_k \leq n$  and any reordering  $j_1, \dots, j_k$  of  $i_1, \dots, i_k$ ,

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f = \partial_{j_1} \partial_{j_2} \cdots \partial_{j_k} f$$

## Chain Rule with Higher Derivatives

- Let  $u = u(x, y)$ ,  $x = x(s, t)$ ,  $y = y(s, t)$ .

Assume all variables are  $C^2$

- $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s}$

- $\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s} \frac{\partial u}{\partial s} = \frac{\partial}{\partial s} \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} \right) + \frac{\partial}{\partial s} \left( \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \right)$

$$= \underbrace{\frac{\partial^2 u}{\partial s \partial x} \frac{\partial x}{\partial s}}_{\text{product rule}} + \underbrace{\frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial s^2}}_{\text{product rule}} + \underbrace{\frac{\partial^2 u}{\partial s \partial y} \frac{\partial y}{\partial s}}_{\text{product rule}} + \underbrace{\frac{\partial u}{\partial y} \frac{\partial^2 y}{\partial s^2}}_{\text{product rule}}$$

product rule

$$= \frac{\partial}{\partial x} \left( \underbrace{\frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s}}_{\text{from first derivative}} \right) \frac{\partial x}{\partial s} + \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial s^2}$$

$$+ \frac{\partial}{\partial y} \left( \underbrace{\frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s}}_{\text{from first derivative}} \right) \frac{\partial y}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial^2 y}{\partial s^2}$$

from first derivative

$$= \left( \frac{\partial^2 u}{\partial^2 x} \frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial s} \right) \frac{\partial x}{\partial s} + \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial s^2}$$

$$+ \left( \frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial s} \right) \frac{\partial y}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial^2 y}{\partial s^2}$$

## Multi-Index Notation and Polynomials

- For any integers  $1 \leq i_1, \dots, i_k \leq n$  and a  $C^k$  function  $f$ ,

we can reorder the partials  $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f$

where  $\alpha_i \geq 0$  denotes the number of times  $\partial_e$  appears on the left side

- $\alpha_1 + \dots + \alpha_n = k$
- An element  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  is a **multi-index**.  
The **degree** of  $\alpha$  is the non-negative integer  $|\alpha| = \alpha_1 + \dots + \alpha_n$ .  
The **factorial**  $\alpha!$  is the positive integer  $\alpha_1! \alpha_2! \dots \alpha_n!$
- Let  $U \subseteq \mathbb{R}^n$ . Let  $f: U \rightarrow \mathbb{R}^m$  be  $C^k$ . For any multi-index  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  w/ degree  $|\alpha| \leq k$ , define the  $\alpha$ -partial derivative by  $\partial^\alpha f = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f$
- Let  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  be a multi-index and let  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . The **monomial**  $x^\alpha$  in the variables  $x_1, \dots, x_n$  is defined by  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ 
  - $x^\alpha$  has degree equal to  $|\alpha|$  in the variables  $x_1, \dots, x_n$
- A **polynomial** is a finite linear combination of monomials.  
The **degree** of a polynomial is the maximum of the degree of its monomials (w/ nonzero coefficients)
- Let  $\alpha, \beta \in \mathbb{N}^n$  be any multi-indices. Define  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  by  $f(x) = x^\beta$ . Then

1. If  $\alpha = \beta$  then  $\partial^\alpha f(x) = \alpha!$  for  $x \in \mathbb{R}^n$
  2. If  $|\alpha| > |\beta|$  then  $\partial^\alpha f(x) = 0$  for  $x \in \mathbb{R}^n$
  3. If  $\alpha \neq \beta$  then  $\partial^\alpha f(0) = 0$
- Let  $P: \mathbb{R}^n \rightarrow \mathbb{R}$  be a polynomial in  $n$  variables of degree  $\leq k$ ,
- so  $P(x) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq k}} C_\alpha x^\alpha$  for some constants  $C_\alpha \in \mathbb{R}$
- w/  $\alpha \in \mathbb{N}^n$  and  $|\alpha| \leq k$ . Then  $C_\alpha = \frac{\partial^\alpha P(0)}{\alpha!}$  for every such  $\alpha$ .
- For a single-variable polynomial  $P: \mathbb{R} \rightarrow \mathbb{R}$  of degree  $k \in \mathbb{N}^+$ ,  $P(x) = \sum_{n=0}^k \frac{P^{(n)}(0)}{n!} x^n$

## 5.4 Taylor Polynomials

### Explicit Formula

- Let  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  be a multi-index and

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

$$\circ \quad \alpha! = \alpha_1! \cdots \alpha_n!$$

$$\circ \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

- Let  $f: B_\varepsilon(a) \rightarrow \mathbb{R}$  be  $C^N$ . The  $N^{\text{th}}$  Taylor polynomial

of  $f$  at  $a$  is  $P_N(x) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq N}} \frac{\partial^\alpha f(a)}{\alpha!} (x-a)^\alpha$

$$\circ \quad P_N(x) = \sum_{k=0}^N \sum_{\substack{\alpha \in \mathbb{N}^n \\ \alpha_1 + \cdots + \alpha_n = k}} \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot \frac{\partial^k f(a)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \cdot$$

$$(x_1 - a_1)^{\alpha_1} \cdots (x_n - a_n)^{\alpha_n}$$

- Let  $f: B_\varepsilon(a) \rightarrow \mathbb{R}$  be  $C^2$ . For  $x \in \mathbb{R}^n$ ,

$$\circ \quad P_0(x) = f(a)$$

$$\circ \quad P_1(x) = f(a) + \nabla f(a) \cdot (x-a)$$

$$\circ \quad P_2(x) = f(a) + \nabla f(a) \cdot (x-a) + \frac{1}{2} (x-a)^T H f(a) (x-a)$$

## Matching Derivatives

- Let  $f: B_\epsilon(a) \rightarrow \mathbb{R}$  be  $C^N$ . Then a polynomial  $P$  is the  $N^{\text{th}}$  Taylor polynomial of  $f$  at  $a$  iff  
 $P$  is a polynomial of degree  $\leq N$  s.t.  
for all multi-indices  $\alpha \in \mathbb{N}^n$  w/  $|\alpha| \leq N$ ,  
 $\partial^\alpha f(a) = \partial^\alpha P(a)$

## Higher Order Approximations

- Let  $\alpha \in \mathbb{N}^n$ . If  $|\alpha| \geq N+1$  then  $\lim_{x \rightarrow 0} \frac{x^\alpha}{\|x\|^N} = 0$
- Let  $f, g: B_\epsilon(a) \rightarrow \mathbb{R}$ .  $g$  is an  $N^{\text{th}}$  order approximation of  $f$  at  $a$  if  $\lim_{x \rightarrow a} \frac{f(x) - g(x)}{\|x-a\|^N} = 0$
- Taylor's Theorem: Let  $f: B_\epsilon(a) \rightarrow \mathbb{R}$  be  $C^{N+1}$ .  
A polynomial  $P$  is the  $N^{\text{th}}$  order Taylor polynomial of  $f$  at  $a$  iff  $P$  is the unique degree  $\leq N$  polynomial which is an  $N^{\text{th}}$  order approximation of  $f$  at  $a$

## 5.5 Classification of Critical Points

- Let  $f: B_\varepsilon(a) \rightarrow \mathbb{R}$  be  $C^2$ . The quadratic form of  $f$  at  $a$  is the function  $q: \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\forall v \in \mathbb{R}^n, q(v) = v^T Hf(a)v$$

- If  $a$  is a crit point of  $f$  at  $q$  is the quadratic form of  $f$  at  $a$ , then

$$P_2(x) = f(a) + \frac{1}{2}q(x-a)$$

- $q$   $\begin{cases} \text{always pos} \\ \text{always neg on } \mathbb{R}^n \setminus \{0\}, \text{ then} \\ \text{pos and neg} \end{cases}$

$$P_2 \text{ will have } \begin{cases} \text{a global maximum} \\ \text{a global minimum at } a \\ \text{no global extrema} \end{cases}$$

Quadratic Forms

integral

- Let  $A$  be a real  $n \times n$  symmetric matrix. The quadratic form associated to  $A$  is the function  $q: \mathbb{R}^n \rightarrow \mathbb{R}$  given by

$$q(v) = v^T A v \text{ for } v \in \mathbb{R}^n.$$

- Let  $A$  be a  $n \times n$  real symmetric matrix. Let  $q: \mathbb{R}^n \rightarrow \mathbb{R}$  be its associated quadratic form. If  $v \in \mathbb{R}^n$  is an eigenvector of  $A$  w/ eigenvalue  $\lambda \in \mathbb{R}$ , then
 
$$q(v) = \lambda \|v\|^2$$
  - $q(v) = v^T A v = v^T (\lambda v) = \lambda (v^T v) = \lambda \|v\|^2$
- Spectral Theorem:** Every  $n \times n$  real symmetric matrix has an orthogonal basis of eigenvectors w/ real eigenvalues
- Let  $A$  be an  $n \times n$  real symmetric matrix. Let  $q: \mathbb{R}^n \rightarrow \mathbb{R}$  be its quadratic form defined by  $q(v) = v^T A v$  for  $v \in \mathbb{R}^n$ .  
 The max and min values of  $q$  on the unit sphere  $S^{n-1}$  are respectively equal to the max and min eigenvalues of  $A$

### Second Derivative Test

- Second Derivative Test:** Let  $f: B_\epsilon(a) \rightarrow \mathbb{R}$  be  $C^3$ .  
 If  $a$  is a crit point of  $f$ , then  $f$  has
  - A local min if all eigenvalues of  $Hf(a)$  are positive
  - A local max if all eigenvalues of  $Hf(a)$  are negative
  - A saddle point if  $Hf(a)$  has a pos and a neg eigenvalue
  - Inconclusive if  $Hf(a)$  has a zero eigenvalue and the

other eigenvalues are all pos or all neg

- Cannot say anything about global extrema
- Let  $f: B_\epsilon(p) \rightarrow \mathbb{R}^2$  where  $f$  is  $C^3$  and  $p \in \mathbb{R}^2$ .

If  $p$  is a crit point of  $f$ , then  $f$  has

① A local min if  $f_{xx}(p)f_{yy}(p) - (f_{xy}(p))^2 > 0$  and

$$f_{xx}(p) > 0$$

② A local max if  $f_{xx}(p)f_{yy}(p) - (f_{xy}(p))^2 > 0$  and

$$f_{xx}(p) < 0$$

③ A saddle point if  $f_{xx}(p)f_{yy}(p) - (f_{xy}(p))^2 < 0$

$$\begin{vmatrix} f_{xx}(p) & f_{xy}(p) \\ f_{xy}(p) & f_{yy}(p) \end{vmatrix} = f_{xx}(p)f_{yy}(p) - (f_{xy}(p))^2 = \lambda_1\lambda_2$$

## 5.6 Proof of Taylor's Theorem

### Another Explicit Formula

- Let  $f$  be a real-valued  $C^k$  function. The  $k^{\text{th}}$  iterated directional derivative of  $f$  is a map defined by

$$D_h^k f = \underbrace{D_h(D_h(\cdots(D_h f)))}_{k \text{ times}}$$

- Let  $f: B_\varepsilon(a) \rightarrow \mathbb{R}$  be  $C^k$  where  $a \in \mathbb{R}^n$ . For all  $h \in \mathbb{R}^n$ ,

$$\frac{D_h^k f(a)}{k!} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha$$

- Let  $f: B_\varepsilon(a) \rightarrow \mathbb{R}$  be  $C^N$  where  $a \in \mathbb{R}^n$ . If  $P_N$  is the  $N^{\text{th}}$  Taylor polynomial of  $f$  at  $a$ , then for  $h \in \mathbb{R}^n$ ,

$$P_N(a+h) = \sum_{k=0}^N \frac{D_h^k f(a)}{k!}$$

- Follows from the previous equation

### Lagrange's Remainder Theorem

- If  $P_N$  is the  $N^{\text{th}}$  Taylor polynomial of  $f$  at  $a$ , define the  $N^{\text{th}}$  remainder of  $f$  at  $a$  by

$$R_N(x) = f(x) - P_N(x)$$

- Lagrange's Remainder Theorem: Let  $N \in \mathbb{N}$ .

If  $f$  is  $C^{N+1}$  on an open set  $D$  containing the line segment  $L$  from  $a \in \mathbb{R}^n$  to  $a+h \in \mathbb{R}^n$ , then there exists a point  $\xi \in L$  s.t. the

$$N^{\text{th}} \text{ remainder of } f \text{ satisfies } R_N(a+h) = \frac{D_h^{N+1} f(\xi)}{(N+1)!}$$

### Completing the Proof

- Let  $Q$  be the polynomial in  $n$  variables of degree  $\leq N$ .

Then  $Q$  is the zero polynomial iff  $\lim_{x \rightarrow 0} \frac{Q(x)}{\|x\|^n} = 0$

- If  $Q$  is nonzero, since its degree is  $\leq N$ ,  
the limit DNE

# 6.1 Partitions

## Constructing Partitions

- A rectangle in  $\mathbb{R}$  is a closed interval  $[a,b]$  where  $a, b \in \mathbb{R}$  and  $a < b$ 
  - The length of  $[a,b]$  is defined to be
$$\text{length}([a,b]) = b - a$$
  - A partition  $P$  of  $[a,b]$  is a finite set s.t.
$$\{a,b\} \subseteq P \subseteq [a,b]$$
    - Often written as  $P = \{x_0, x_1, \dots, x_k\}$ , where it is assumed that
$$a = x_0 < x_1 < \dots < x_{k-1} < x_k = b$$
    - For  $1 \leq i \leq k$ , the interval  $[x_{i-1}, x_i]$  is referred to as a subinterval of  $[a,b]$
  - Trivial partition:  $P = \{a,b\}$
  - Regular partition: each subinterval has the same length



- If  $P = \{x_0, \dots, x_k\}$  is a partition of  $[a, b]$ , then

$$\text{length}([a, b]) = \sum_{i=1}^k \text{length}([x_{i-1}, x_i])$$

- A rectangle in  $\mathbb{R}^2$  is a set  $R = [a, b] \times [c, d]$

where  $a, b, c, d \in \mathbb{R}$  and  $a < b$  and  $c < d$

- The area of a rectangle is  $\text{area}(R) = (b-a)(d-c)$
- A partition  $P$  of the rectangle  $R$  is a collection of subrectangles

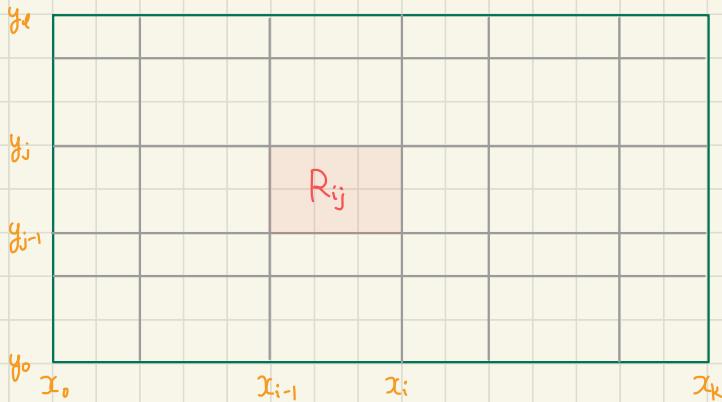
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad \forall 1 \leq i \leq k, 1 \leq j \leq l$$

where  $\{x_0, x_1, \dots, x_k\}$  and  $\{y_0, y_1, \dots, y_l\}$  are partitions of  $[a, b]$  and  $[c, d]$  respectively

$$\begin{aligned} P &= \{R_{ij}\}_{i,j} = \{R_{ij} : 1 \leq i \leq k, 1 \leq j \leq l\} \\ &= \{[x_0, x_1], \dots, [x_{k-1}, x_k]\} \times \{[y_0, y_1], \dots, [y_{l-1}, y_l]\} \\ &= \{[x_{i-1}, x_i] : 1 \leq i \leq k\} \times \{[y_{j-1}, y_j] : 1 \leq j \leq l\} \\ &\approx \{[x_{i-1}, x_i] \times [y_{j-1}, y_j] : 1 \leq i \leq k, 1 \leq j \leq l\} \end{aligned}$$

- If  $P = \{R_{ij} : 1 \leq i \leq k, 1 \leq j \leq l\}$  is a partition of a rectangle  $R = [a, b] \times [c, d] \subset \mathbb{R}^2$ , then

$$\sum_{i=1}^k \sum_{j=1}^l \text{area}(R_{ij}) = \text{area}(R)$$



- A rectangle in  $\mathbb{R}^n$  is a set  $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$

where  $a_1, b_1, \dots, a_n, b_n \in \mathbb{R}$  and  $a_i < b_i$  for all  $1 \leq i \leq n$

- The volume of a rectangle is

$$\text{vol}(R) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$$

- A partition  $P$  of the rectangle  $R$  is a

collection of subrectangles

$$R_{i_1, \dots, i_n} = [x_{1,i_1}, x_{1,i_1+1}] \times \cdots \times [x_{n,i_n}, x_{n,i_n+1}],$$

$\forall 1 \leq i_1 \leq k_1, \dots, 1 \leq i_n \leq k_n$  where for  $1 \leq j \leq n$ ,

$\{x_{j,0}, x_{j,1}, \dots, x_{j,k_j}\}$  is a partition of the

interval  $[a_j, b_j]$

- Can write  $P = \{R_i\}_i$  or  $P = \{R_i\}_{i \in I}$

where  $I$  is a finite set of multi-indices

$$(i_1, \dots, i_n)$$

- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . If  $P = \{R_i\}_{i \in I}$  is a partition of  $R$ , then  $\sum_{i \in I} \text{vol}(R_i) = \text{vol}(R)$
- A partition  $P$  of a rectangle  $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$  is *regular* if  $P$  is constructed from a regular partition of the interval  $[a_j, b_j]$  for every  $1 \leq j \leq n$

### Refining Partitions

- Let  $P = \{R_i\}_i$  and  $P' = \{R'_j\}_j$  be two partitions of a rectangle  $R$ .  $P'$  is a refinement of  $P$  if for every subrectangle  $R'_j$  of  $P'$ , there is a subrectangle  $R_i$  of  $P$  s.t.  $R'_j \subseteq R_i$
- Let  $P, P', P''$  be partitions of a rectangle  $R \subseteq \mathbb{R}^n$ . If  $P''$  is a refinement of  $P'$  and  $P'$  is a refinement of  $P$ , then  $P''$  is a refinement of  $P$
- Let  $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ . Let  $P'$  be a partition of  $R$  constructed from the partitions  $P'_i$  of  $[a_i, b_i]$

- for  $1 \leq i \leq n$ . Let  $P''$  be the partition of  $R$  constructed from the partitions  $P_i''$  of  $[a_i, b_i]$  for  $1 \leq i \leq n$ . The common refinement of  $P'$  and  $P''$  is the partition  $P$  of  $R$  constructed from the partitions  $P_i' \cup P_i''$  of  $[a_i, b_i]$  for  $1 \leq i \leq n$ .
- Let  $P'$  and  $P''$  be partitions of the rectangle  $R \subset \mathbb{R}^n$ . If  $P$  is the common refinement of  $P'$  and  $P''$ , then  $P$  is a refinement of both  $P'$  and  $P''$ .
  - Let  $P$  be a partition of a rectangle  $R \subset \mathbb{R}^n$ . The norm of  $P$ , denoted  $\|P\|$ , is the max diameter of all of its subrectangles.
    - The diameter of a rectangle is the max distance between any two points in the rectangle
  - Let  $R$  be a rectangle in  $\mathbb{R}^n$ . For every  $\delta > 0$ , there exists a partition  $P$  of  $R$  w/ norm  $\|P\| < \delta$

## 6.2 Upper Sums and Lower Sums

### Definition of Upper and Lower Sums

- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f: R \rightarrow \mathbb{R}$  be bdd.  
Let  $P = \{R_{ij}: 1 \leq i \leq k, 1 \leq j \leq l\}$  be a partition of  $R$ .  
The  $P$ -lower sum and  $P$ -upper sum of  $f$  are respectively defined by

$$L_p(f) = \sum_{i=1}^k \sum_{j=1}^l m_{ij} \text{area}(R_{ij}) \quad U_p(f) = \sum_{i=1}^k \sum_{j=1}^l M_{ij} \text{area}(R_{ij})$$

where  $\forall 1 \leq i \leq k, 1 \leq j \leq l$ ,

$$m_{ij} = \inf_{x \in R_{ij}} f(x) \quad \text{and} \quad M_{ij} = \sup_{x \in R_{ij}} f(x)$$

- Always defined for a bdd function
- Upper sum: overestimate, lower sum: underestimate
- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f: R \rightarrow \mathbb{R}$  be bdd.  
Let  $P = \{R_i\}_{i \in I}$  be a partition of  $R$  where  $I$  is a finite set of multi-indices. The  $P$ -lower sum of  $f$  and  $P$ -upper sum of  $f$  are respectively

$$L_p(f) = \sum_{i \in I} m_i \text{vol}(R_i) \text{ and } U_p(f) = \sum_{i \in I} M_i \text{vol}(R_i)$$

where  $\forall i \in I, m_i = \inf_{x \in R_i} f(x)$  and  $M_i = \sup_{x \in R_i} f(x)$

## Properties of Upper and Lower Sums



- Let  $P$  be a partition of a rectangle  $R \subseteq \mathbb{R}^n$ .  
Let  $f: R \rightarrow \mathbb{R}$  be bdd. Then  $L_p(f) \leq U_p(f)$
- Let  $P$  and  $P'$  be partitions of a given rectangle  $R \subseteq \mathbb{R}^n$ .  
Let  $f: R \rightarrow \mathbb{R}$  be bdd. If  $P'$  is a refinement of  $P$ ,  
then  $L_p(f) \leq L_{p'}(f)$  and  $U_{p'}(f) \leq U_p(f)$
- If  $P'$  and  $P''$  are two partitions of a rectangle  $R \subseteq \mathbb{R}^n$ ,  
then  $L_{p'}(f) \leq U_{p''}(f)$ 
  - $L_{p'}(f) \leq L_p(f) \leq U_p(f) \leq U_{p''}(f)$  for some common refinement  $P$  of  $P'$  and  $P''$
- Let  $P = \{R_i\}_i$  be a partition of a rectangle  $R \subseteq \mathbb{R}^n$ .  
Let  $f: R \rightarrow \mathbb{R}$  and  $g: R \rightarrow \mathbb{R}$  be bdd. Then:
  - ①  $U_p(f+g) \leq U_p(f) + U_p(g)$

$$\textcircled{2} \quad U_p(\lambda f) = \lambda U_p(f) \text{ for any } \lambda > 0$$

$$\textcircled{3} \quad U_p(-f) = -L_p(f)$$

$$\textcircled{4} \quad \text{If } f \leq g \text{ on } R, \text{ then } U_p(f) \leq U_p(g)$$

- Lower sum lemmas are similar

### Definition of Riemann Sums

- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f: R \rightarrow \mathbb{R}$  be bdd.

Let  $P = \{R_i\}_{i \in I}$  be a partition of  $R$  where  $I$  is a finite set of multi-indices. For each  $i \in I$ ,

choose a sample point  $x_i^* \in R_i$ . Then

$S_p^*(f) = \sum_{i \in I} f(x_i^*) \text{vol}(R_i)$  is the Riemann sum

for  $f$  w/  $P$  and these sample points

- Tagged partitions: partitions and the associated sample points

◦ Each sample point is a "tag" for the corresponding subrectangle

- We don't know whether the Riemann sum is an overestimate or underestimate

Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f: R \rightarrow \mathbb{R}$  and  $g: R \rightarrow \mathbb{R}$

be bdd. Let  $P = \{R_i\}_{i \in I}$  be a partition of  $\mathbb{R}$  where  $I$  is a finite set of multi-indices. For each  $i \in I$ , choose a sample point  $x_i^* \in R_i$ . Then

$$\textcircled{1} \quad S_p^*(f + \lambda g) = S_p^*(f) + \lambda S_p^*(g) \text{ for any } \lambda \in \mathbb{R}$$

$$\textcircled{2} \quad \text{If } f \leq g \text{ on } \mathbb{R}, \text{ then } S_p^*(f) \leq S_p^*(g)$$

## 6.3 Integration Over a Rectangle

### Definition of the Integral

- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f: R \rightarrow \mathbb{R}$  be bdd.

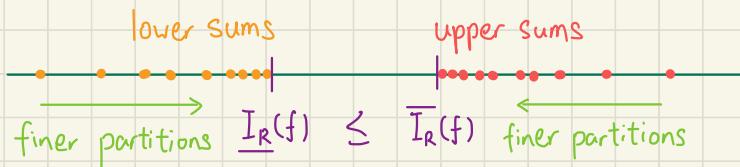
The lower integral of  $f$  on  $R$  and the

upper integral of  $f$  on  $R$  are defined by

$$\underline{I}_R(f) = \sup_P L_p(f) \quad \text{and} \quad \overline{I}_R(f) = \inf_P U_p(f),$$

where the supremum and infimum are over all partitions  $P$

of the rectangle  $R$

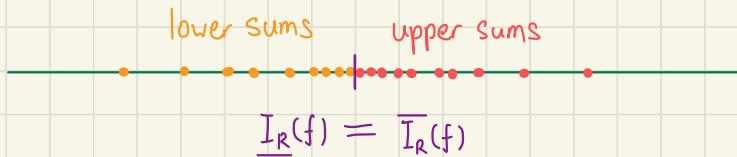


- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . If  $f: R \rightarrow \mathbb{R}$  is bdd, then both  $\underline{I}_R(f)$  and  $\overline{I}_R(f)$  exist, and  $\underline{I}_R(f) \leq \overline{I}_R(f)$
- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f: R \rightarrow \mathbb{R}$  be bdd  
If  $\underline{I}_R(f) = \overline{I}_R(f)$ , then  $f$  is integrable on  $R$  and  
the integral of  $f$  on  $R$  is defined by

$$\int_R f dV := \underline{I}_R(f) = \overline{I}_R(f).$$

If  $\underline{I}_R(f) < \overline{I}_R(f)$ , then  $f$  is non-integrable

- $dV$  stands for "volume element"
- Integrand is  $f$ , not  $f(x)$
- This integral is the Darboux integral



- $\varepsilon$ -characterization of integrability: Let  $R$  be a rectangle in  $\mathbb{R}^n$ .

Let  $f: R \rightarrow \mathbb{R}$  be bdd.  $f$  is integrable on  $R$  iff

$$\forall \varepsilon > 0, \exists \text{ a partition } P \text{ of } R \text{ s.t. } U_p(f) - L_p(f) < \varepsilon$$

- Other notations:  $\int \cdots \int_R f dV$  or  $\int_R f$ 
  - For rectangles in  $\mathbb{R}^3$ :  $\iiint_R f dV$
  - For rectangles in  $\mathbb{R}^2$ :  $\iint_R f dA$  or  $\int \int_R f dA$ 
    - $dA$  stands for "area element"
  - Do **not** write  $dx dy$  instead of  $dA$  or  $dx, \dots dx_n$  instead of  $dV$

### Properties of Integrals Over Rectangles

- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . The identity function  $I$  is integrable on  $R$  and  $\int_R I dV = \text{vol}(R)$
- Linearity: Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f$  and  $g$  be

bdd functions on  $\mathbb{R}$ . Let  $f$  and  $g$  are

integrable on  $\mathbb{R}$ , then  $f + \lambda g$  is integrable on  $\mathbb{R}$  and

$$\int_{\mathbb{R}} (f + \lambda g) dV = \int_{\mathbb{R}} f dV + \lambda \int_{\mathbb{R}} g dV$$

- **Monotonicity:** Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f$  and  $g$  be bdd functions on  $R$ . If  $f$  and  $g$  are integrable on  $R$  and  $f \leq g$  on  $R$ , then  $\int_R f dV \leq \int_R g dV$
- **Triangle inequality:** Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f$  be a bdd function on  $R$ . If  $f$  is integrable on  $R$ ,

then  $|f|$  is integrable on  $R$  and  $\left| \int_R f dV \right| \leq \int_R |f| dV$

- **Cauchy-Schwarz inequality:** Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f$  and  $g$  be bdd functions on  $R$ . If  $f$  and  $g$  are integrable on  $R$ , then  $fg$  is integrable on  $R$  and
- $$\int_R f g dV \leq \left( \int_R f^2 dV \right)^{1/2} \left( \int_R g^2 dV \right)^{1/2}$$
- **Additivity:** Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f$  be a bdd function on  $R$ . Sps  $R = R' + R''$  is a union of two subrectangles  $R'$  and  $R''$  w/ disjoint interiors.  $f$  is integrable on  $R$  iff  $f$  is integrable on both  $R'$  and  $R''$ , in which case

$$\int_R f dV = \int_{R'} f dV + \int_{R''} f dV$$

- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f: R \rightarrow \mathbb{R}$  be bdd.

Let  $P_1, P_2, \dots$  be a sequence of partitions of  $R$  s.t.

$\|P_N\| \rightarrow 0$  as  $N \rightarrow \infty$ . For each partition  $P = \{R_i\}_{i \in I}$

in the sequence, pick a sample point  $x_i^* \in R_i$

for every  $i \in I$ . If  $f$  is integrable on  $R$ , then

$$\int_R f dV = \lim_{N \rightarrow \infty} S_{P_N}^*(f)$$

# 6.4 Uniform Continuity and Integration

## Uniform Continuity

- $\forall x \in A$ ,  $f$  cts at  $x \Leftrightarrow \forall x \in A, \forall \varepsilon > 0, \exists \delta > 0$  s.t.  
 $\forall y \in A, \|x-y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$ 
  - $\delta$  may depend on  $x$
- Let  $A \subseteq \mathbb{R}^n$  be a set. A function  $f: A \rightarrow \mathbb{R}^m$  is uniformly continuous if  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.  
 $\forall x, y \in A, \|x-y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$ 
  - For a subset  $S \subseteq A$ ,  $f$  is uniformly continuous on  $S$  if  $f|_S: S \rightarrow \mathbb{R}^m$  is uniformly continuous
- Uniform continuity is a global property of a function
  - Does not depend on any particular point in the domain
- If  $f: A \rightarrow \mathbb{R}^m$  is uniformly cts, then  $f$  is cts
- Let  $A \subseteq \mathbb{R}^n$ . Let  $f: A \rightarrow \mathbb{R}^m$ . If  $A$  is cpt and  $f$  is cts, then  $f$  is uniformly cts

## Continuous Functions are Integrable

- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f: R \rightarrow \mathbb{R}$ .

If  $f$  is cts on  $\mathbb{R}$ , then  $f$  is integrable on  $\mathbb{R}$

- Let  $a, b \in \mathbb{R}$  w/  $a < b$ . Let  $f: [a, b] \rightarrow \mathbb{R}$ .  
If  $f$  is cts on  $[a, b]$ , then  $f$  is  
integrable on  $[a, b]$

# 6.5 Sets With Zero Jordan Measure

## Definition of Zero Volume

- A set  $S \subseteq \mathbb{R}$  has zero Jordan measure (or zero volume) if for every  $\epsilon > 0$ , there exists finitely many non-trivial intervals  $I_1, \dots, I_N \subseteq \mathbb{R}$  s.t.

$$S \subseteq \bigcup_{i=1}^N I_i \text{ and } \sum_{i=1}^N \text{length}(I_i) < \epsilon$$

- A set  $S \subseteq \mathbb{R}^n$  has zero Jordan measure (or zero volume) if for every  $\epsilon > 0$ , there exists finitely many rectangles  $R_1, \dots, R_N$  in  $\mathbb{R}^n$  s.t.  
$$S \subseteq \bigcup_{i=1}^N R_i \text{ and } \sum_{i=1}^N \text{vol}(R_i) < \epsilon$$
- Any unbounded set of  $\mathbb{R}^n$  does not have zero Jordan measure
- Any set of  $\mathbb{R}^n$  with non-empty interior does not have zero Jordan measure

## Properties of Zero Volume Sets

- For sets in  $\mathbb{R}^n$ :
  - Any subset of a zero volume set has zero volume
  - Any finite union of zero volume sets has zero volume

- The closure of a zero volume set has zero volume
- Let  $k < n$ . Let  $R$  be a rectangle in  $\mathbb{R}^k$ .  
If  $f$  is a  $\mathbb{R}^n$ -valued function that is  $C^1$  on an open set containing  $R$ , then  $f(R) = \{f(x) : x \in R\}$  has zero Jordan measure in  $\mathbb{R}^n$

# 6.6 Jordan Measurable Sets and Volume

## Intro

- Let  $S \subseteq \mathbb{R}^n$ . The indicator function of  $S$ , denoted  $\chi_S$ , is defined by  $\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$

## Jordan Measurable Sets

- The set  $S \subseteq \mathbb{R}^n$  is Jordan measurable if  $S$  is bdd and  $\partial S$  has zero Jordan measure
- Let  $S, T \subseteq \mathbb{R}^n$  be sets. Then:
  - If  $S$  has zero Jordan measure, then  $S$  is Jordan measurable
  - If  $S$  is Jordan measurable, then  $\bar{S}, S^\circ, \partial S$  are Jordan measurable
  - If  $S, T$  are Jordan measurable, then  $S \cup T$  and  $S \cap T$  are Jordan measurable
- Let  $S \subseteq \mathbb{R}^n$ . If  $S$  is Jordan measurable, then  $\chi_S$  is integrable on any rectangle in  $\mathbb{R}^n$  containing  $S$

## Definition of Volume

- Let  $S$  be a Jordan measurable set.

Define the **Jordan measure** of  $S$  (or **volume** of  $S$ )

to be  $\text{vol}(S) := \int_R \chi_S dV$  for a rectangle  $R$  containing  $S$

- For  $\mathbb{R}^2$ , this can be referred to as the **area** of  $S$  and is denoted as  $\text{area}(S)$

- Let  $S \subseteq \mathbb{R}^n$  be a Jordan measurable set.

If  $R$  and  $R'$  are rectangles each containing  $S$ , then

$\chi_S$  is integrable on  $R$  and on  $R'$ , and

$$\int_R \chi_S dV = \int_{R'} \chi_S dV$$

- Therefore the Jordan measure is well-defined
- We can write  $\text{vol}(S) = \int \chi_S dV = \int_S 1 dV$   
and drop the rectangle notation

- Let  $S, T \subseteq \mathbb{R}^n$  be Jordan measurable sets. Then

$$\textcircled{1} \quad \text{If } S \subseteq T \text{ then } \text{vol}(S) \leq \text{vol}(T)$$

$$\textcircled{2} \quad \text{vol}(S \cup T) = \text{vol}(S) + \text{vol}(T) - \text{vol}(S \cap T)$$

- Let  $S \subseteq \mathbb{R}^n$ . If  $S$  has zero Jordan measure,

$$\text{then } \text{vol}(S) = \int \chi_S dV = 0$$

## Integrability of Indicator Functions

- Let  $P$  be a partition of a rectangle  $R \subseteq \mathbb{R}^n$ .  
Let  $S$  be a rectangle lying inside  $R$ . Then  $P$  subdivides  $S$  if  $S$  can be written as a union of rectangles belonging to  $P$
- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $S_1, \dots, S_m$  be a finite collection of rectangles lying inside  $R$ . Then:
  - ① There exists a partition of  $R$  subdividing every  $S_1, \dots, S_m$
  - ② Let  $P$  be a partition of  $R$ . If  $P$  subdivides every  $S_1, \dots, S_m$  and  $P'$  is a refinement of  $P$ , then  $P'$  also subdivides every  $S_1, \dots, S_m$

# 6.7 Integration Over a Non-Rectangle

## Definition of the Integral

- $\chi_S f : \mathbb{R}^n \rightarrow \mathbb{R}$  is defined by

$$\chi_S f(x) = \begin{cases} f(x) & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

- Even if  $f$  is not defined outside of  $S$ ,  
this function is defined everywhere in  $\mathbb{R}^n$
- Let  $S \subseteq \mathbb{R}^n$  be bdd. Let  $f: S \rightarrow \mathbb{R}$  be bdd.  
 $f$  is integrable on  $S$  if the function  $\chi_S f: \mathbb{R}^n \rightarrow \mathbb{R}$  is  
integrable on a rectangle containing  $S$ .

If so, the integral of  $f$  over  $S$  is  $\int_S f dV := \int_R \chi_S f dV$   
for a rectangle  $R$  containing  $S$

## Criteria For Integrability

- Let  $S \subseteq \mathbb{R}$  be bdd. Let  $f: S \rightarrow \mathbb{R}$  be bdd.  
If  $S$  is Jordan measurable and the set of discontinuities  
of  $f$  on  $S$  has zero Jordan measure, then  
 $f$  is integrable on  $S$

- Let  $S \subseteq \mathbb{R}^n$  be bdd. Let  $f: S \rightarrow \mathbb{R}$  be bdd.
  - If  $S$  has zero volume, then  $f$  is integrable on  $S$  and  $\int_S f dV = 0$ 
    - Holds for any bdd function  $f$
  - If  $f = 0$  on  $S$  except on a set of zero volume, then  $f$  is integrable on  $S$  and  $\int_S f dV = 0$ 
    - Holds for any bdd set  $S$

### Properties of Integrals Over the Set

- Let  $S \subseteq \mathbb{R}^n$  be Jordan measurable. Fix  $\lambda \in \mathbb{R}$ .  
 The constant function  $\lambda$  is integrable on  $S$  and  

$$\int_S \lambda dV = \lambda \text{vol}(S)$$
- Linearity:** Let  $S \subseteq \mathbb{R}^n$  be bdd. Let  $f, g: S \rightarrow \mathbb{R}$  be bdd.  
 Let  $\lambda \in \mathbb{R}$ . If  $f$  and  $g$  are integrable on  $S$ , then  
 $f + \lambda g$  is integrable on  $S$  and  

$$\int_S (f + \lambda g) dV = \int_S f dV + \lambda \int_S g dV$$
- Monotonicity:** Let  $S \subseteq \mathbb{R}^n$  be bdd. Let  $f, g: S \rightarrow \mathbb{R}$  be bdd.  
 If  $f$  and  $g$  are integrable on  $S$  and  $f \leq g$  on  $S$ , then  

$$\int_S f dV \leq \int_S g dV$$

- Triangle inequality: Let  $S \subseteq \mathbb{R}^n$  be bdd.

Let  $f: S \rightarrow \mathbb{R}$  be bdd. If  $f$  is integrable on  $S$ ,

then  $|f|$  is integrable on  $S$  and  $\left| \int_S f dV \right| \leq \int_S |f| dV$

- Cauchy-Schwarz inequality: Let  $S \subseteq \mathbb{R}^n$  be bdd.

Let  $f, g: S \rightarrow \mathbb{R}$  be bdd. If  $f$  and  $g$  are integrable on  $S$ ,

then  $fg$  is integrable on  $S$  and

$$\int_S fg dV \leq \left( \int_S f^2 dV \right)^{1/2} \left( \int_S g^2 dV \right)^{1/2}$$

- Additivity: Let  $S \subseteq \mathbb{R}^n$  be bdd. Let  $f: S \rightarrow \mathbb{R}$  be bdd.

Spz  $S = S' \cup S''$  s.t.  $S' \cap S''$  has zero Jordan measure.

If  $f$  is integrable on both  $S'$  and  $S''$ , then

$f$  is integrable on  $S$  and  $\int_S f dV = \int_{S'} f dV + \int_{S''} f dV$

- Let  $S \subseteq \mathbb{R}^n$  be Jordan measurable. Let  $f, g: S \rightarrow \mathbb{R}$  be bdd.

If  $f = g$  on  $S$  except on a set of zero volume,

then  $f$  is integrable on  $S$  iff  $g$  is integrable on  $S$ .

If so,  $\int_S f dV = \int_S g dV$

- An integral does not change value if it is modified on a set of zero volume

## Integration on a Rectangle With Few Discontinuities

- Let  $R \subseteq \mathbb{R}^n$  be a rectangle. Let  $f: R \rightarrow \mathbb{R}$  be bdd.  
If the set of discontinuities of  $f$  inside  $R$  has zero Jordan measure, then  $f$  is integrable on  $R$

## 6.8 Volumes, Averages, and Mass

### Volume Under a Graph

- Let  $S \subseteq \mathbb{R}^n$  be a cpt Jordan measurable set,  
Let  $f: S \rightarrow [0, \infty)$  be cts. The  $(n+1)$ -dimensional set  
 $T = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in S, 0 \leq y \leq f(x)\}$  is a  
cpt Jordan measurable set and satisfies  $\text{vol}(T) = \int_S f(x) dV$

### Averages and Totals

- Let  $S \subseteq \mathbb{R}^n$  be a Jordan measurable set with nonzero volume. Let  $f$  be integrable on  $S$ .  
The average value of  $f$  on  $S$  is  $\frac{1}{\text{vol}(S)} \int_S f dV$
- Total value of  $f$  over  $S$   
 $= (\text{average value of } f \text{ on } S) \times (\text{volume of } S)$
- Integral mean value theorem: Let  $S \subseteq \mathbb{R}^n$  be Jordan measurable. Let  $f: S \rightarrow \mathbb{R}$  be cts. If  $S$  is cpt and path-connected, then  $\exists p \in S$  s.t.  
 $\int_S f dV = f(p) \text{vol}(S)$
- Fix  $p \in \mathbb{R}^n$ . Let  $f$  be a real-valued function.

If  $f$  is cts on an open set containing  $p$ , then

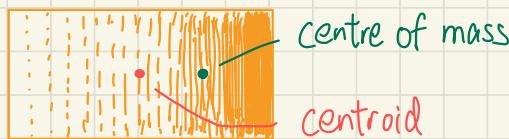
$$f(p) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\text{vol}(B_\epsilon(p))} \int_{B_\epsilon(p)} f dV$$

## Mass

- Let  $\delta: S \rightarrow [0, \infty)$  be the density function for an object  $S \subseteq \mathbb{R}^n$ . Assume  $S$  is bdd and  $\delta$  is integrable on  $S$ . Its mass  $m = \text{mass}(S)$  and average density  $\rho$  are respectively

$$m = \int_S \delta dV \quad \rho = \frac{1}{\text{vol}(S)} \int_S \delta dV$$

- Let  $\delta: S \rightarrow [0, \infty)$  be the density function for an object  $S \subseteq \mathbb{R}^n$ . Assume  $S$  is bdd and  $\delta$  is integrable on  $S$ . Its centre of mass  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^n$  is  $\bar{x} = \frac{1}{m} \int_S x \delta(x) dV$  where  $m$  is the mass of  $S$ . Its centroid is the centre of mass assuming density is uniform



- If  $f: S \rightarrow \mathbb{R}^k$  w/ components  $f_1, \dots, f_k$ , then

$$\int_S f dV := \left( \int_S f_1 dV, \dots, \int_S f_k dV \right) \in \mathbb{R}^k$$

# 6.9 Probability

## Sample Space and Events

- A sample space  $\Omega$  is a non-empty set that contains all possible outcomes
- The event space  $\Sigma$  is a collection of subsets of  $\Omega$ 
  - Every element  $A \in \Sigma$  is an event
- Let  $\Omega \subseteq \mathbb{R}^n$  be Jordan measurable. Define the event space  $\Sigma$  to be

$\Sigma = \{A \subseteq \Omega : A \text{ is Jordan measurable}\}$ , then

①  $\Omega \in \Sigma$

- Some outcome must happen

② If  $A \in \Sigma$ , then  $\Omega \setminus A \in \Sigma$

- An event either occurs or does not occur

③ If  $A_1, \dots, A_N \in \Sigma$ , then  $A_1 \cup \dots \cup A_N \in \Sigma$

- Any finite union of events is an event

## Probability Spaces and Densities

- A probability function  $P: \Sigma \rightarrow [0,1]$  is a function

assigning each event in the event space a probability between 0 and 1

- If  $A \in \Sigma$ ,  $P(A)$  is the probability that  $A$  occurs
- A probability space is the triple  $(\Omega, \Sigma, P)$
- Let  $\Omega \subseteq \mathbb{R}^n$  be Jordan measurable. Let  $\Sigma$  be all Jordan measurable subsets of  $\Omega$ .

Let  $\phi: \Omega \rightarrow [0, \infty)$  be cts on  $\Sigma$  except for a set of zero Jordan measure. Assume  $\int_{\Omega} \phi dV = 1$ .

For every  $A \in \Sigma$ , define  $P(A) = \int_A \phi dV$ . Then:

①  $P(\Omega) = 1$

②  $P(A)$  exists and  $0 \leq P(A) \leq 1$  for every  $A \in \Sigma$

③ If  $A_1, \dots, A_n \in \Sigma$  are pairwise disjoint,

then  $P(A_1 \cup \dots \cup A_n) = \sum_{i=1}^n P(A_i)$

•  $(\Omega, \Sigma, P)$  is a cts probability space in  $\mathbb{R}^n$

•  $\phi$  is the probability density function

of  $P: \Sigma \rightarrow [0, 1]$

- Let  $(\Omega, \Sigma, P)$  be a cts probability space in  $\mathbb{R}^n$ .

$\mathbb{P}$  is uniform if its PDF  $\phi: \Omega \rightarrow [0, \infty)$  is constant.

That is,  $\phi(x) = \frac{1}{\text{vol}(\Omega)}$  for all  $x \in \Omega$

### Limitations of the Darboux Integral and the Jordan Measure

- $\Omega$  is Jordan measurable and hence bdd
- $\Omega \subseteq \mathbb{R}^n$  for a finite  $n \in \mathbb{N}^+$
- The event space  $\Sigma$  of Jordan measurable subsets excludes many reasonable sets
- $\mathbb{P}$  satisfies finite additivity, but not countable additivity

# 7.1 Fubini's Theorem in 2D

## Intro

- Let  $f: [a,b] \rightarrow \mathbb{R}$  be bdd. If  $f$  is integrable on  $[a,b]$ ,  
then  $\int_{[a,b]} f dV = \int_a^b f(x) dx$

## Integral of Slices

- Let  $f: [a,b] \times [c,d] \rightarrow \mathbb{R}$ .
  - A  $x$ -slice of  $f$  is the function  $f^x: [c,d] \rightarrow \mathbb{R}$  of the form  $f^x(y) = f(x,y)$  for some fixed  $x \in [a,b]$
  - A  $y$ -slice of  $f$  is the function  $f^y: [a,b] \rightarrow \mathbb{R}$  of the form  $f^y(x) = f(x,y)$  for some fixed  $y \in [c,d]$
- Let  $f: [a,b] \times [c,d] \rightarrow \mathbb{R}$ 
  - If  $f$  is bdd, then every slice of  $f$  is bdd
  - If  $f$  is cts, then every slice of  $f$  is cts
- If  $f: [a,b] \times [c,d] \rightarrow \mathbb{R}$  is cts, then both  $F$  and  $G$  defined by  $F(x) = \int_c^d f(x,y) dy$  and  $G(y) = \int_a^b f(x,y) dx$  is cts

## Iterated Double Integrals

- Let  $f: [a,b] \times [c,d] \rightarrow \mathbb{R}$  be bdd. The quantities  $\int_a^b \left( \int_c^d f(x,y) dy \right) dx$  and  $\int_c^d \left( \int_a^b f(x,y) dx \right) dy$  are **iterated double integrals**
- There is no obvious relationship between  $\int_a^b \left( \int_c^d f(x,y) dy \right) dx$ ,  $\int_c^d \left( \int_a^b f(x,y) dx \right) dy$ , and  $\iint_{[a,b] \times [c,d]} f dA$

## Fubini's Theorem in Two Dimensions

- Fubini's Theorem:** Let  $R = [a,b] \times [c,d]$  and let  $f: R \rightarrow \mathbb{R}$  be bdd. For  $x \in [a,b]$ , define  $f^x: [c,d] \rightarrow \mathbb{R}$  by  $f^x(y) = f(x,y)$ . Assume
  - $f^x$  is integrable on  $[c,d]$  for every  $x \in [a,b]$
  - $f$  is integrable on  $[a,b] \times [c,d]$

Then

- $\int_c^d f(x,y) dy$  exists for every  $x \in [a,b]$
- $\int_a^b \left( \int_c^d f(x,y) dy \right) dx$  exists and equals  $\iint_R f dA$
- Fubini's Corollary:** If  $f: [a,b] \times [c,d] \rightarrow \mathbb{R}$  is cts, then  $\int_a^b \left( \int_c^d f(x,y) dy \right) dx$  and  $\int_c^d \left( \int_a^b f(x,y) dx \right) dy$  both exist and are equal to  $\iint_{[a,b] \times [c,d]} f dA$

## 7.2 Fubini's Theorem

### Integrals of Slices in 3D

- Let  $R = [a,b] \times [c,d] \times [e,f] \subseteq \mathbb{R}^3$ . Let  $\varphi: R \rightarrow \mathbb{R}$ .  
A  $(x,y)$ -slice of  $\varphi$  is the function  $\varphi^{x,y}: [e,f] \rightarrow \mathbb{R}$  of the form  $\varphi^{x,y}(z) = \varphi(x,y,z)$  for some fixed  $x \in [a,b], y \in [c,d]$ 
  - $(y,z)$ -slice and  $(x,z)$ -slice are defined similarly
- Let  $R = [a,b] \times [c,d] \times [e,f] \subseteq \mathbb{R}^3$ . Let  $\varphi: R \rightarrow \mathbb{R}$ .  
A  $x$ -slice of  $\varphi$  is the function  $\varphi^x: [c,d] \times [e,f] \rightarrow \mathbb{R}$  of the form  $\varphi^x(y,z) = \varphi(x,y,z)$  for some fixed  $x \in [a,b]$ 
  - $y$ -slice and  $z$ -slice are defined similarly

### Iterated Triple Integrals and Fubini's Theorem in 3D

- Let  $\varphi: [a,b] \times [c,d] \times [e,f] \rightarrow \mathbb{R}$  be bdd.  
The quantity  $\int_a^b \int_c^d \int_e^f \varphi(x,y,z) dz dy dx$  is an iterated triple integral
- Fubini's Theorem: Let  $R = [a,b] \times [c,d] \times [e,f]$ .

Let  $\varphi: R \rightarrow \mathbb{R}$  be bdd. If:

- ① For every  $x \in [a, b]$ ,  $y \in [c, d]$ , the  $(x, y)$ -slice  $\varphi^{(x, y)}$  is integrable on  $[e, f]$
- ② For every  $x \in [a, b]$ , the  $x$ -slice  $\varphi^x$  is integrable on  $[c, d] \times [e, f]$
- ③  $\varphi$  is integrable on  $R = [a, b] \times [c, d] \times [e, f]$

Then the iterated triple integral  $\int_a^b \int_c^d \int_e^f \varphi(x, y, z) dz dy dx$  exists and is equal to  $\iiint_R \varphi dV$

- Let  $R = [a, b] \times [c, d] \times [e, f] \subseteq \mathbb{R}^3$ .

If  $\varphi: R \rightarrow \mathbb{R}$  is cts, then every iterated triple integral of  $\varphi$  on  $R$  exists and equals  $\iiint_R \varphi dV$

### Iterated Integrals and Fubini's Theorem in Any Dimension

- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f: R \rightarrow \mathbb{R}$ . A function  $g$  is a slice of  $f$  on  $R$  if  $g$  is defined by fixing one or more coordinates of  $f$  in  $R$
- Let  $R$  be a rectangle in  $\mathbb{R}^n$ . Let  $f: R \rightarrow \mathbb{R}$ .
  - ① If  $f$  is bdd, then every slice of  $f$  is bdd
  - ② If  $f$  is cts, then every slice of  $f$  is cts

- Let  $R = [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$ . Let  $\varphi: R \rightarrow \mathbb{R}$  be bdd.

The quantity  $\int_{a_1}^{b_1} \int_{a_2}^{b_2} \dots \int_{a_n}^{b_n} \varphi(x_1, \dots, x_n) dx_n \dots dx_2 dx_1$  is an **iterated n-fold integral**

- Fubini's Theorem:** Let  $R = [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$ .

Let  $f: R \rightarrow \mathbb{R}$  be bdd. If  $f$  is integrable on  $R$  and every slice of  $f$  on  $R$  is integrable on its domain, then every iterated integral of  $f$  on  $R$  exists and equal to the integral of  $f$  on  $R$ , i.e.

$\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} f(x_1, \dots, x_n) dx_n \dots dx_1$  exists and all  $n!$  orderings of this iterated integral exist and equal to  $\int_R f dV$

- Let  $R \subseteq \mathbb{R}^n$  be a rectangle. If  $f: R \rightarrow \mathbb{R}$  is cts, then every iterated integral of  $f$  on  $R$  exists and equal to the integral of  $f$  on  $R$

- Fubini's Theorem:** Let  $R \subseteq \mathbb{R}^n$  be a rectangle.

Let  $\varphi: R \times [a, b] \rightarrow \mathbb{R}$  be bdd. For every  $t \in [a, b]$ , define the slice  $\varphi^t: R \rightarrow \mathbb{R}$  by  $\varphi^t(x) = \varphi(x, t)$ .

If  $\varphi$  is integrable on  $R \times [a, b]$ , and for every  $t \in [a, b]$  the slice  $\varphi^t$  is integrable on  $R$ , then the function

$t \mapsto \int_R \varphi^t dV$  is integrable on  $[a, b]$  and

$$\int_{R \times [a, b]} \varphi dV = \int_a^b \left( \int_R \varphi^t dV \right) dt$$

## 7.3 Double Integrals

- A set  $S \subseteq \mathbb{R}^2$  is  $x$ -simple if there exists

cts  $f: [a,b] \rightarrow \mathbb{R}$  and  $g: [a,b] \rightarrow \mathbb{R}$  s.t.

$$S = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\}$$

$$\iint_S \varphi dA = \int_a^b \int_{f(x)}^{g(x)} \varphi(x,y) dy dx$$

- A set  $T \subseteq \mathbb{R}^2$  is  $y$ -simple if there exists

cts  $p: [c,d] \rightarrow \mathbb{R}$  and  $q: [c,d] \rightarrow \mathbb{R}$  s.t.

$$T = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d, p(y) \leq x \leq q(y)\}$$

$$\iint_T \varphi dA = \int_c^d \int_{p(y)}^{q(y)} \varphi(x,y) dx dy$$

- Can often break non-simple regions into a union of simple pieces
- Strategies for computing double integrals:

- Sketch the region and describe in several ways
- Directly calculate with FTC
- Swap the order of integration w/ Fubini's theorem
- Apply symmetries of the integrand or region
- Break up region into smaller pieces
- Interpret geometrically as a volume of classic object

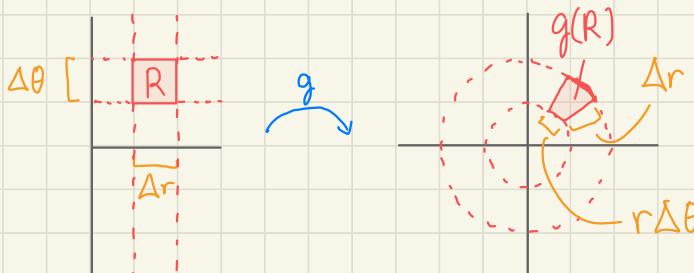
# 7.4 Double Integrals in Polar Coordinates

## Regions in Polar Coordinates

- The set  $A = \{(r, \theta) \in \mathbb{R}^2 : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$  is **not** the same as the set  $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$
- Instead,  $B = g(A)$ , where  $g(r, \theta) = (r \cos \theta, r \sin \theta)$  is the polar coordinate transformation
- We could informally say " $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$ " is the unit disk  $x^2 + y^2 \leq 1$  in polar form"
- If  $r < 0$ , then the angle becomes  $\theta + \pi$

## Derivation of Integrals in Polar Coordinates

- For a fixed  $r > 0$  and  $\theta \in \mathbb{R}$ , the small rectangle  $R = [r, r + \Delta r] \times [\theta, \theta + \Delta\theta]$  in the  $(r, \theta)$ -plane transforms under  $g$  to a piece of washer  $g(R)$  in the  $(x, y)$ -plane



• Let  $\Omega \subseteq \mathbb{R}^2$  be a Jordan measurable set s.t.

the restricted polar coordinate transformation  $g|_{\Omega}: \Omega \rightarrow g(\Omega)$

given by  $g(r, \theta) = (r\cos\theta, r\sin\theta)$  is a bijection.

If  $f: g(\Omega) \rightarrow \mathbb{R}$  is integrable on  $g(\Omega)$ , then

$F: \Omega \rightarrow \mathbb{R}$  given by  $F(r, \theta) = (f \circ g)(r, \theta) \cdot r$  is

integrable on  $\Omega$  and  $\iint_{g(\Omega)} f dA = \iint_{\Omega} F dA$

## 7.5 Triple Integrals

- Let  $S \subseteq \mathbb{R}^3$  be the region of integration

### Projection Method

- Project  $S$  into a plane (e.g.  $(x,y)$ -plane) and call the resulting set  $T \subseteq \mathbb{R}^2$
- For any  $(x,y) \in T$ ,  $(x,y,z) \in \mathbb{R}^3$  lies in  $S$  iff  $(x,y,z)$  lies between  $z = \gamma_1(x,y)$  and  $z = \gamma_2(x,y)$ .

$$\text{Then } \text{vol}(S) = \iint_T \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} dz dA$$

- Integrate over  $T$

### Slicing Method

- Find a slice of  $S$  (e.g.  $z$ -slice),  $S_z$ , for  $a \leq z \leq b$   
Then  $\text{vol}(S) = \int_a^b \text{area}(S_z) dz$
- Calculate  $\text{area}(S_z)$

### Using Geometry

- Observe characteristics of the volume, break down into easy-to-calculate pieces if necessary
- Sketch pictures

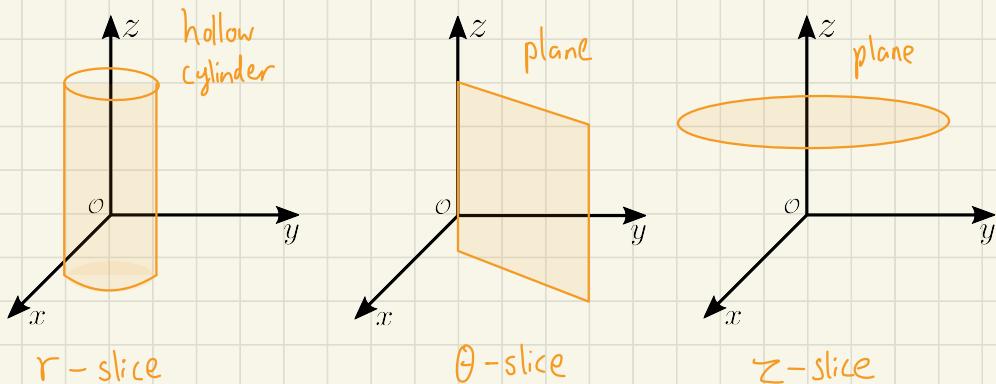
# 7.6 Triple Integrals in Cylindrical Coordinates

## Intro

- Define  $g: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  as  $g(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$ 
  - Domain:  $(r, \theta, z)$ -space; codomain:  $(x, y, z)$ -space
- Want to integrate over regions in  $(x, y, z)$ -space by integrating in  $(r, \theta, z)$ -space

## Regions in Cylindrical Coordinates

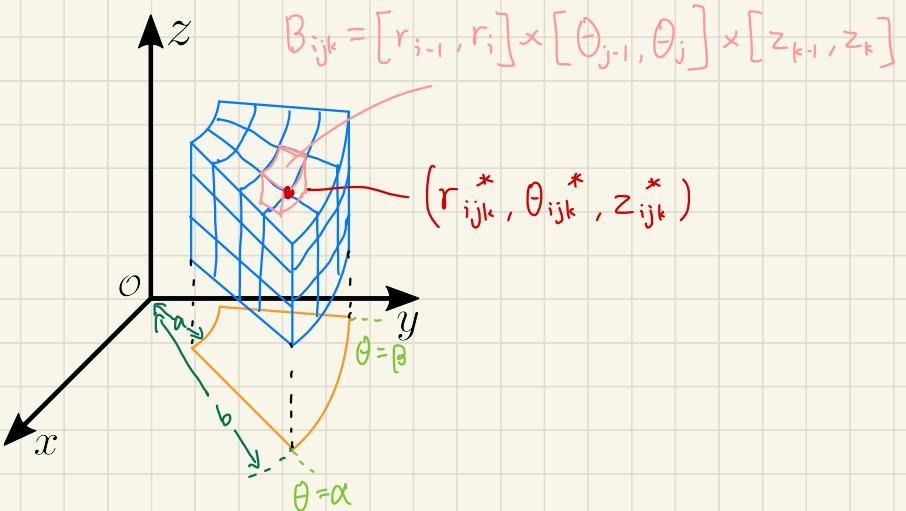
- Slices



## Derivation of Integrals in Cylindrical Coordinates

- Let  $R = [r, r+\Delta r] \times [\theta, \theta+\Delta\theta] \times [z, z+\Delta z]$
- $\text{vol}(g(R)) \approx r \Delta r \Delta\theta \Delta z = r \text{vol}(R)$
- For a rectangle  $\Omega = \{(r, \theta, z) : a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\}$ ,

the transformed region  $D = g(\Omega)$  is



- $\iiint_{g(\Omega)} f dV = \int_c^d \int_{\alpha}^{\beta} \int_a^b (f \circ g)(r, \theta, z) |r| dr d\theta dz$

- Let  $\Omega \subseteq \mathbb{R}^3$  be Jordan measurable and the restricted cylindrical coordinate transformation  $g|_{\Omega}: \Omega \rightarrow g(\Omega)$  is a bijection. If a function  $f: g(\Omega) \rightarrow \mathbb{R}$  is integrable on  $g(\Omega)$ , then the function  $F: \Omega \rightarrow \mathbb{R}$  given by

$F(r, \theta, z) = (f \circ g)(r, \theta, z) \cdot |r|$  is integrable on  $\Omega$  and

$$\iiint_{g(\Omega)} f dV = \iiint_{\Omega} F dV$$

- $dx dy dz = dV = r dr d\theta dz$

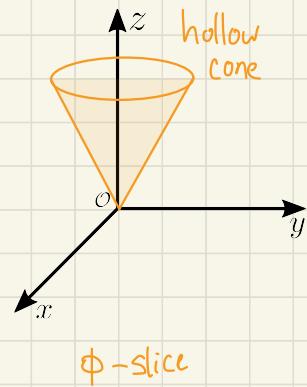
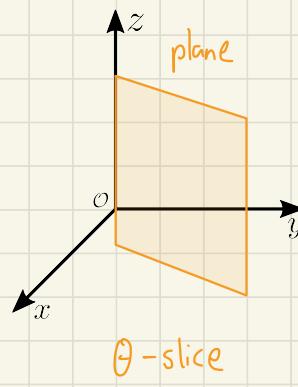
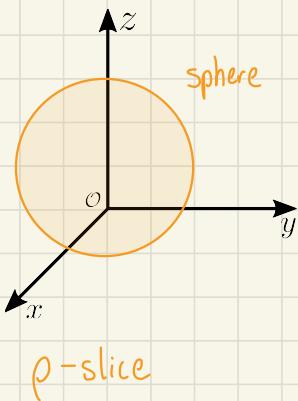
# 7.7 Triple Integrals in Spherical Coordinates

## Intro

- Define  $g: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  as  $g(\rho, \theta, \phi) = (\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi)$ 
  - Domain:  $(\rho, \theta, \phi)$ -space ; codomain:  $(x, y, z)$ -space
- Want to integrate over regions in  $(x, y, z)$ -space by integrating in  $(\rho, \theta, \phi)$ -space

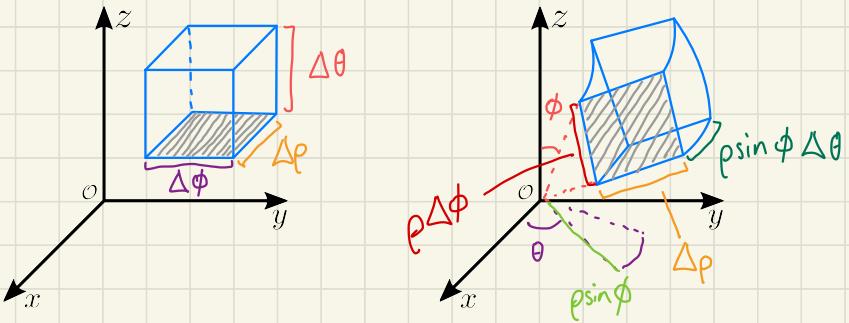
## Regions in Spherical Coordinates

- Slices



## Derivation of Integrals in Spherical Coordinates

- Let  $R = [\rho, \rho + \Delta\rho] \times [\theta, \theta + \Delta\theta] \times [\phi, \phi + \Delta\phi]$



- $\text{vol}(g(\Omega)) \approx \rho^2 \sin \phi \Delta \rho \Delta \theta \Delta \phi = \rho^2 \sin \phi \text{vol}(\Omega)$
- For a region  $\Omega = [a, b] \times [\alpha, \beta] \times [\lambda, \mu]$ ,  

$$\iiint_{g(\Omega)} f dV = \int_\lambda^\mu \int_\alpha^\beta \int_a^b (f \circ g)(\rho, \theta, \phi) |\rho^2 \sin \theta| d\rho d\theta d\phi$$
- Let  $\Omega \subseteq \mathbb{R}^3$  be Jordan measurable and the restricted spherical coordinate transformation  $g|_{\Omega}: \Omega \rightarrow g(\Omega)$  is a bijection. If a function  $f: g(\Omega) \rightarrow \mathbb{R}$  is integrable on  $g(\Omega)$ , then the function  $F: \Omega \rightarrow \mathbb{R}$  given by  

$$F(\rho, \theta, \phi) = (f \circ g)(\rho, \theta, \phi) \cdot |\rho^2 \sin \phi|$$
 is integrable on  $\Omega$  and  $\iiint_{g(\Omega)} f dV = \iiint_{\Omega} F dV$
- $dxdydz = dV = \rho^2 \sin \phi d\rho d\theta d\phi$

## 7.8 Change of Variables

### Derivation of Change of Variables

- Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be linear. For every rectangle  $R \subseteq \mathbb{R}^n$ ,
$$\text{vol}(T(R)) = |\det(T)| \text{vol}(R)$$
  - The "stretch factor" of a linear map is the absolute value of its determinant
- We can linearly approximate a transformed rectangle  $g(R)$  where  $g$  is a nonlinear map
  - $\forall x \in \mathbb{R}^n, g(x) = g(a) + Dg_a(x-a)$
  - $Dg_a(R)$  is a translated parallelogram
  - $$\begin{aligned}\text{vol}(g(R)) &\approx \text{vol}(Dg_a(R)) = |\det(Dg_a)| \text{vol}(R) \\ &= |\det Dg_a| \text{vol}(R)\end{aligned}$$
  - The "stretch factor" of a nonlinear map is the absolute value of its linear approximation determinant

### Statement and Consequences

- Change of Variables Theorem:** Let  $U, V \subseteq \mathbb{R}^n$ . Let  $g: U \rightarrow V$  be a diffeomorphism. Let  $\Omega \subseteq U$

be cpt and Jordan measurable. The function  $f$  is integrable on  $g(\Omega)$  iff  $(f \circ g)|\det Dg|$  is integrable on  $\Omega$ .

If so,  $\int_{g(\Omega)} f dV = \int_{\Omega} (f \circ g) |\det Dg| dV$ .

- If Fubini's theorem is satisfied for both integrals, then

$$\int \cdots \int_{g(\Omega)} f(x) dx_1 \cdots dx_n$$

$$= \int \cdots \int_{\Omega} (f \circ g)(u) |\det Dg(u)| du_1 \cdots du_n$$

- Integration in  $\mathbb{R}^n$  does not depend on the choice of coordinate system

- Let  $U, V \subseteq \mathbb{R}$  be open sets w/  $[a, b] \subseteq U$ .

Let  $g: U \rightarrow V$  be C' and increasing. If  $f$  is integrable on  $[g(a), g(b)]$ , then  $f \circ g$  is integrable on  $[a, b]$  and

$$\int_{g(a)}^{g(b)} f(x) dx = \int_a^b f(g(u)) g'(u) du.$$

- Let  $U, V \subseteq \mathbb{R}^n$  be open. Let  $g: U \rightarrow V$  be a diffeomorphism.

For any cpt Jordan measurable set  $\Omega \subseteq U$ ,

$$\text{vol}(g(\Omega)) = \int_{\Omega} |\det Dg| dV$$

- Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be invertible and linear. For any cpt

Jordan measurable set  $\Omega \subseteq \mathbb{R}^n$ ,  $\text{vol}(T(\Omega)) = |\det(T)| \text{vol}(\Omega)$ .

- Let  $U, V \subseteq \mathbb{R}^n$  be open. If  $g: U \rightarrow V$  is bijective, and  $\det Dg(x) \neq 0$  for all  $x \in U$ , then  $g$  is a diffeomorphism.

## 7.9 Improper Integrals

### Local Integrability

- Let  $\Omega \subseteq \mathbb{R}^n$ . A real-valued function  $f$  is **locally integrable** on  $\Omega$  if  $f$  is integrable on every cpt Jordan measurable subset of  $\Omega$ .
  - Neither  $\Omega$  nor  $f$  need to be bdd.
- Let  $\Omega \subseteq \mathbb{R}^n$ . If a real-valued function  $f$  is cts on  $\Omega$ , then  $f$  is locally integrable on  $\Omega$ .
- Let  $\Omega \subseteq \mathbb{R}^n$  be Jordan measurable. Let  $f: \Omega \rightarrow \mathbb{R}$  be bdd. If  $f$  is integrable on  $\Omega$ , then  $f$  is locally integrable on  $\Omega$ .

### Exhaustions

- Let  $\Omega \subseteq \mathbb{R}^n$ . A sequence of cpt Jordan measurable sets  $\{\Omega_k\}_{k=1}^{\infty}$  is an **exhaustion** of  $\Omega$  if  $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$  and  $\forall k \geq 1, \Omega_k \subseteq \Omega_{k+1}^o$ .
- Let  $\Omega \subseteq \mathbb{R}^n$ . If there exists an exhaustion of  $\Omega$  by cpt Jordan measurable sets  $\{\Omega_k\}_{k=1}^{\infty}$ , then  $\Omega$  is open

## Improper Integrals and Monotone Convergence

- Let  $\Omega \subseteq \mathbb{R}^n$ . Let  $\{\Omega_k\}_{k=1}^{\infty}$  be an exhaustion of  $\Omega$  by cpt Jordan measurable sets. Let  $f: \Omega \rightarrow \mathbb{R}$  be locally integrable. The **improper integral** of  $f$  over  $\Omega$  is given by  $\int_{\Omega} f dV = \lim_{k \rightarrow \infty} \int_{\Omega_k} f dV$  provided that the limit does not depend on the choice of exhaustion. If so:
  - The improper integral **converges** when the limit exists
  - The improper integral **diverges** when the limit DNE
  - The improper integral **diverges to  $\infty$  ( $-\infty$ )** when the limit is  $\infty$  ( $-\infty$ )If the limit depends on the choice of exhaustion, then the improper integral **diverges**
- Monotone Convergence Theorem:** Let  $\Omega \subseteq \mathbb{R}^n$  have an exhaustion by cpt Jordan measurable sets. Let  $f$  be a real-valued locally integrable function on  $\Omega$ . If  $f > 0$  on  $\Omega$  then the improper integral  $\int_{\Omega} f dV$  either converges or diverges to  $\infty$
- Let  $\Omega \subseteq \mathbb{R}^n$  have an exhaustion by cpt Jordan

measurable sets. If  $f$  is integrable on  $\Omega$ , then the

improper integral of  $f$  on  $\Omega$  converges and its value is equal to the integral of  $f$  on  $\Omega$ .

- Let  $\Omega \subseteq \mathbb{R}^n$  have an exhaustion by cpt Jordan measurable sets. Let  $f: \Omega \rightarrow \mathbb{R}$  and  $g: \Omega \rightarrow \mathbb{R}$  be locally integrable. Fix  $\lambda \in \mathbb{R}$ . If the improper integrals  $\int_{\Omega} f dV$  and  $\int_{\Omega} g dV$  both converge, then the improper integral  $\int_{\Omega} (f + \lambda g) dV$  converges and
$$\int_{\Omega} (f + \lambda g) dV = \int_{\Omega} f dV + \lambda \int_{\Omega} g dV$$

### A Family of Improper Integrals

- Fix  $p \in \mathbb{R}$ . For the given improper integrals in  $\mathbb{R}^n$ , one has

$$\int_{\|x\| > 1} \frac{1}{\|x\|^p} dV \begin{cases} \text{converges if } p > n \\ \text{diverges to } \infty \text{ if } p \leq n \end{cases}$$

$$\int_{0 < \|x\| < 1} \frac{1}{\|x\|^p} dV \begin{cases} \text{diverges to } \infty \text{ if } p \geq n \\ \text{converges if } p < n \end{cases}$$

### Basic Comparison Test

- Basic comparison test:** Let  $\Omega \subseteq \mathbb{R}^n$  be open. Let  $f$  and  $g$  be real-valued locally integrable functions on  $\Omega$ .

① If  $0 \leq f \leq g$  on  $\Omega$  and  $\int_{\Omega} gdV$  converges

then  $\int_{\Omega} fdV$  converges

② If  $0 \leq f \leq g$  on  $\Omega$  and  $\int_{\Omega} gdV$  diverges

then  $\int_{\Omega} fdV$  diverges

### Absolute Convergence

- Let  $\Omega \subseteq \mathbb{R}^n$ . If  $f$  is locally integrable on  $\Omega$ ,  
then  $|f|$  is locally integrable on  $\Omega$
- Let  $\Omega \subseteq \mathbb{R}^n$  have an exhaustion by cpt Jordan  
measurable sets. Let  $f$  be a real-valued function  
locally integrable on  $\Omega$ . The improper integral  
 $\int_{\Omega} fdV$  absolutely converges if the improper integral  
 $\int_{\Omega} |f|dV$  converges
- Let  $\Omega \subseteq \mathbb{R}^n$  have an exhaustion by cpt Jordan  
measurable sets. Let  $f$  be a real-valued function  
locally integrable on  $\Omega$ . If the improper integral  
 $\int_{\Omega} |f|dV$  converges, then the improper integral  
 $\int_{\Omega} fdV$  converges

## Proof of the Monotone Convergence Theorem

- Heine-Borel Theorem: Let  $A \subseteq \mathbb{R}^n$  be cpt.

Let  $\{V_j\}_{j=1}^{\infty}$  be a sequence of open sets s.t.

$V_j \subseteq V_{j+1}$  for  $j \geq 1$ . If  $A \subseteq \bigcup_{j=1}^{\infty} V_j$ , then

there exists  $k \in \mathbb{N}^+$  s.t.  $A \subseteq V_k$

# 8.1 Parametrized Curves

## Simple Regular Parametrizations

- A map  $\gamma: [a,b] \rightarrow \mathbb{R}^n$  is a **parametrization** of a set  $C \subseteq \mathbb{R}^n$  if  $C = \gamma([a,b])$  and  $\gamma$  is cts on  $[a,b]$ 
  - If we momentarily stop ( $\gamma'(t) = 0$ ), then we can rigidly change direction
- A map  $\gamma: [a,b] \rightarrow \mathbb{R}^n$  is a **regular parametrization** of a set  $C \subseteq \mathbb{R}^n$  when
  - ①  $C = \gamma([a,b])$  and  $\gamma$  is cts on  $[a,b]$
  - ②  $\gamma$  is  $C^1$  on  $(a,b)$  and  $\gamma'(t) \neq 0$  for any  $t \in (a,b)$ 
    - If  $\gamma$  is not injective on its domain, then we could visit the same point multiple times
- A map  $\gamma: [a,b] \rightarrow \mathbb{R}^n$  is a **simple regular parametrization** of a set  $C \subseteq \mathbb{R}^n$  when
  - ①  $C = \gamma([a,b])$  and  $\gamma$  is cts on  $[a,b]$
  - ②  $\gamma$  is  $C^1$  on  $(a,b)$  and  $\gamma'(t) \neq 0$  for any  $t \in (a,b)$
  - ③  $\gamma$  is injective except possibly with  $\gamma(a) = \gamma(b)$

- If additionally  $\gamma(a) = \gamma(b)$ , then  $\gamma$  is closed
- If  $\gamma$  satisfies only ① and ③,  
then  $\gamma$  is a simple parametrization
- If a map  $\gamma: [a,b] \rightarrow \mathbb{R}^n$  is a simple regular parametrization  
of a set  $C \subseteq \mathbb{R}^n$ , then  $C$  is a 1-D regular surface  
at  $\gamma(c)$  for every  $c \in (a,b)$ 
  - Does not include endpoints  $\{a,b\}$

### Curves and Piecewise Curves

- A set  $C \subseteq \mathbb{R}^n$  is a (parametrized simple regular) curve  
if there exists a simple regular parametrization of  $C$ 
  - The Curve is also closed if the parametrization  
is closed
- Every parametrized simple regular curve is a  
1-D regular surface
- A set  $C \subseteq \mathbb{R}^n$  is a piecewise (parametrized simple regular)  
curve if  $C$  can be written as a finite union of  
parametrized simple regular curves  $C_1, \dots, C_k$  s.t.  
 $C_i \cap C_j$  is finite for any pair of distinct curves  $C_i, C_j$

## Reparametrizations and Orientation

- Let  $\gamma_1: [a,b] \rightarrow \mathbb{R}^n$  and  $\gamma_2: [c,d] \rightarrow \mathbb{R}^n$  be simple regular parametrizations of a set  $C \subseteq \mathbb{R}^n$   
 $\gamma_1$  is a **reparametrization** of  $\gamma_2$ , if there exists a cts invertible  $\varphi: [a,b] \rightarrow [c,d]$  s.t.  $\varphi$  is  $C'$  on  $(a,b)$  w/  $\varphi'$  never zero, and  $\gamma_1 = \gamma_2 \circ \varphi$ 
  - If  $\varphi' > 0$  on  $(a,b)$ , then  $\gamma_1$  has the same orientation as  $\gamma_2$
  - If  $\varphi' < 0$  on  $(a,b)$ , then  $\gamma_1$  has the opposite orientation as  $\gamma_2$
- Let  $\gamma_1: [a,b] \rightarrow \mathbb{R}^n$ ,  $\gamma_2: [c,d] \rightarrow \mathbb{R}^n$ ,  $\gamma_3: [e,f] \rightarrow \mathbb{R}^n$  be simple regular parametrizations of a set  $C \subseteq \mathbb{R}^n$  Then
  - Reflexive:**  $\gamma_1$  is a reparametrization of itself
  - Symmetry:** If  $\gamma_1$  is a reparametrization of  $\gamma_2$ , then  $\gamma_2$  is a reparametrization of  $\gamma_1$
  - Transitive:** If  $\gamma_1$  is a reparametrization of  $\gamma_2$  and  $\gamma_2$  is a reparametrization of  $\gamma_3$ , then  $\gamma_1$  is a reparametrization of  $\gamma_3$

## 8.2 Arc Length

### Definition and Invariance

- The arc length (or length) of a piecewise curve  $C \subseteq \mathbb{R}^n$  is  $\ell(C) = \int_a^b \|\gamma'(t)\| dt$  where  $\gamma: [a,b] \rightarrow \mathbb{R}^n$  is a parametrization of  $C$ 
  - "The distance a particle travels is the integral of its speed"
- Invariance of arc length theorem: Let  $C \subseteq \mathbb{R}^n$  be a curve. Let  $\gamma_1: [a,b] \rightarrow \mathbb{R}^n$  and  $\gamma_2: [c,d] \rightarrow \mathbb{R}^n$  be parametrizations of  $C$ . If  $\gamma_1$  is a reparametrization of  $\gamma_2$ , then  $\int_a^b \|\gamma_1'(t)\| dt = \int_c^d \|\gamma_2'(t)\| dt$

### Arc Length Parametrization

- Let  $\gamma: [a,b] \rightarrow \mathbb{R}^n$  be a parametrization of a piecewise curve  $C \subseteq \mathbb{R}^n$ . The arc length parameter of  $\gamma$  is the function  $s: [a,b] \rightarrow [0, \infty)$  given by
$$s(t) = \int_a^t \|\gamma'(u)\| du, \quad a \leq t \leq b$$
  - Depends on the parametrization  $\gamma$

- "Length of  $\gamma$  on the interval  $[a, t]$  for  $a \leq t \leq b$ "
- "Distance travelled by the particle from time  $a$  to time  $t$ "
- Let  $\gamma: [a, b] \rightarrow \mathbb{R}^n$  be a parametrization of a piecewise curve in  $\mathbb{R}^n$ . The map  $\gamma$  is parametrized by arc length if  $\|\gamma'(t)\| = 1$  for  $a \leq t \leq b$
- In this case we write  $\gamma(s)$  instead of  $\gamma(t)$

### Derivation of Arc Length

- If  $\gamma: [a, b] \rightarrow \mathbb{R}^n$  parametrizes a piecewise curve  $C \subseteq \mathbb{R}^n$ , then  $\int_a^b \|\gamma'(t)\| dt = \sup_P \left\{ \sum_{i=1}^n \|\gamma(t_i) - \gamma(t_{i-1})\| \right\}$  where the supremum is over all partitions  $P = \{t_0, t_1, \dots, t_n\}$  of  $[a, b]$

### Line Integrals of Scalar Functions

- Let  $C \subseteq \mathbb{R}^n$  be a piecewise curve parametrized by  $\gamma: [a, b] \rightarrow \mathbb{R}^n$ . Let  $f: C \rightarrow \mathbb{R}$  be bdd.

The line integral of  $f$  over  $C$  is

$$\int_C f ds := \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt.$$

If this integral exists, then  $f$  is integrable on the curve  $C$ .

- $ds$ : Arc length element
  - "The infinitesimal length of the curve"
  - "Net area of the surface traced out by  $f$  along  $C$ "
- Invariance of line integrals theorem: Let  $C$  be a piecewise curve in  $\mathbb{R}^n$ . Let  $f: C \rightarrow \mathbb{R}$  be bdd. Let  $\gamma_1: [a, b] \rightarrow \mathbb{R}^n$  and  $\gamma_2: [c, d] \rightarrow \mathbb{R}^n$  be parametrizations of  $C$ .  $(f \circ \gamma_1) \|\gamma_1'\|$  is integrable on  $[a, b]$  iff  $(f \circ \gamma_2) \|\gamma_2'\|$  is integrable on  $[c, d]$ . If so,
 
$$\int_a^b f(\gamma_1(t)) \|\gamma_1'(t)\| dt = \int_c^d f(\gamma_2(t)) \|\gamma_2'(t)\| dt$$
  - We can integrate on a curve in  $\mathbb{R}^n$  independent of the choice of parametrization

## 8.3 Line Integrals

### Oriented Curves

- Let  $\gamma_1: [a,b] \rightarrow \mathbb{R}^n$  and  $\gamma_2: [c,d] \rightarrow \mathbb{R}^n$  be parametrizations of a curve  $C$ .
  - If  $\gamma_1$  is a reparam of  $\gamma_2$  w/ the same orientation, then  $\forall s \in (a,b), \forall t \in (c,d)$ ,
$$\gamma_1(s) = \gamma_2(t) \Rightarrow \frac{\gamma_1'(s)}{\|\gamma_1'(s)\|} = \frac{\gamma_2'(t)}{\|\gamma_2'(t)\|}$$
  - If  $\gamma_1$  is a reparam of  $\gamma_2$  w/ the opposite orientation, then  $\forall s \in (a,b), \forall t \in (c,d)$ ,
$$\gamma_1(s) = \gamma_2(t) \Rightarrow \frac{\gamma_1'(s)}{\|\gamma_1'(s)\|} = -\frac{\gamma_2'(t)}{\|\gamma_2'(t)\|}$$
  - "The unit tangent vector of two parametrizations w/ the same orientation remains the same at every point along the curve"- An oriented curve  $C$  is a set of parametrizations that are reparams of each other w/ the same orientation
  - Formally a set of maps, informally a set traced out by a parametrization w/ a direction

- Let  $C$  be an oriented curve parametrized by  $\gamma: [a, b] \rightarrow \mathbb{R}^n$ .  
 The oppositely oriented curve  $-C$  is the set of parametrizations that are reparams of  $\gamma$  w/ the opposite orientation



- Let  $C_1$  and  $C_2$  be oriented curves in  $\mathbb{R}^n$ .  
 The concatenation of  $C_1$  and  $C_2$ , denoted  $C = C_1 + C_2$ , is the set of cts maps  $\gamma: [a, b] \rightarrow \mathbb{R}^n$  s.t.  $\exists c \in (a, b)$  where  $\gamma|_{[a, c]}$  and  $\gamma|_{[c, b]}$  are parametrizations of  $C_1$  and  $C_2$  respectively



- Can concatenate any number of curves
- $C_1 - C_2$  is equivalent to  $C_1 + (-C_2)$
- A piecewise closed curve in  $\mathbb{R}^n$  is the concatenation of finitely many oriented curves in  $\mathbb{R}^n$

## Line Integrals of Vector Fields

- Let  $C$  be an oriented curve parametrized by  $\gamma: [a, b] \rightarrow \mathbb{R}^n$  w/ unit tangent vector  $T$ . Let  $F: C \rightarrow \mathbb{R}^n$ .

The line integral of  $F$  over  $C$  is given by

$$\int_C F \cdot T ds := \int_a^b F(\gamma(t)) \cdot T(t) \| \gamma'(t) \| dt$$

provided this integral exists

- Equivalently, this is the work done by  $F$

along the curve  $C$

- $\int_C F \cdot T ds = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt$

- $\int_C F \cdot T ds = \int_C F \cdot d\gamma \text{ or } \int_C F \cdot dr$

- We can treat  $d\gamma$  as  $\gamma'(t)dt$   
 $= (\gamma'_1(t)dt, \dots, \gamma'_n(t)dt)$

- $\int_C F \cdot d\gamma = \int_C F_1 dx_1 + F_2 dx_2 + \dots + F_n dx_n$

- Think of  $d\gamma$  as  $(dx_1, \dots, dx_n)$  where

$$dx_j = \gamma'_j(t)dt$$

- Let  $C, C_1, C_2$  be oriented curves in  $\mathbb{R}^n$ . Let  $F, G$  be oriented curves in  $\mathbb{R}^n$  defined on  $C, C_1, C_2$ . Then

$$\textcircled{1} \quad \int_{-C} F \cdot T ds = - \int_C F \cdot T ds$$

$$\textcircled{2} \quad \int_C (F + \lambda G) \cdot T ds = \int_C F \cdot T ds + \lambda \int_C G \cdot T ds$$

for  $\lambda \in \mathbb{R}$

$$\textcircled{3} \quad \int_{C_1 + C_2} F \cdot T ds = \int_{C_1} F \cdot T ds + \int_{C_2} F \cdot T ds$$

# 8.4 Fundamental Theorem of Line Integrals

## Statement and Proof

- **Fundamental theorem of line integrals:** Let  $C$  be an oriented piecewise curve in  $\mathbb{R}^n$  parametrized by  $\gamma: [a, b] \rightarrow \mathbb{R}^n$ . Let  $f$  be a real-valued function that is  $C'$  on an open set containing  $C$ . Then

$$\int_C \nabla f \cdot d\gamma = f(\gamma(b)) - f(\gamma(a))$$

- The line integral does **not** depend on the path
- The line integral of a gradient vector field over a **closed** curve evaluates to  $0$

## Conservative Vector Fields and Potentials

- A vector field  $F$  is **conservative** on an open set  $U \subseteq \mathbb{R}^n$  if there exists a function  $f: U \rightarrow \mathbb{R}$  s.t.  
 $F = \nabla f$  on  $U$ 
  - $f$  is the **potential function** or **scalar potential** of  $F$
  - $F$  is a **gradient vector field**

- If  $\mathbf{F}$  is a conservative vector field,  
then  $\int_C \mathbf{F} \cdot d\gamma$  only depends on the endpoints of  $C$
- To determine whether  $\mathbf{F}$  is conservative, assume that it has potential  $f$ , then solve the system of PDE  

$$\frac{\partial f}{\partial x} = F_1(x, y), \quad \frac{\partial f}{\partial y} = F_2(x, y)$$

### Irrational Vector Fields

- A  $C^1$  vector field  $\mathbf{F} = (F_1, \dots, F_n)$  is irrational on an open set  $U \subseteq \mathbb{R}^n$  if  $\forall 1 \leq i < j \leq n, \partial_i F_j = \partial_j F_i$  on  $U$
- Let  $\mathbf{F}$  be a vector field in  $\mathbb{R}^n$  that is  $C^1$  on an open set  $U$ . If  $\mathbf{F}$  is conservative on  $U$ , then  $\mathbf{F}$  is irrational on  $U$

# 8.5 Conservative Vector Fields

## Physical Viewpoints

- If the work done by the vector field does not depend on the path taken by the particle, then the field is **path-independent**

## Path Independence

- A vector field is conservative iff it satisfies path independence
- Let  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  that is cts on an open path-connected set  $U \subseteq \mathbb{R}^n$ . The following are equivalent:
  - ①  $\exists f: U \rightarrow \mathbb{R}$  s.t.  $f$  is  $C^1$  and  $F = \nabla f$
  - ②  $\int_{C_1} F \cdot d\gamma = \int_{C_2} F \cdot d\gamma$  for oriented piecewise curves  $C_1, C_2 \subseteq U$  with the same endpoints
  - ③  $\int_C F \cdot d\gamma = 0$  for any closed piecewise curve  $C \subseteq U$
  - Gradient, path-independent, and conservative vector fields are equivalent

## Irrational Vector Fields and Sets Without Holes

- Poincaré's lemma: Let  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  that is  $C^1$  on the

open set  $U \subseteq \mathbb{R}^n$ . If  $U$  is convex and  $F$  is irrotational on  $U$ , then  $F$  is conservative on  $U$

- Jordan curve theorem: Let  $C \subseteq \mathbb{R}^2$  be a simple closed curve.

Then  $C$  divides  $\mathbb{R}^2$  into two regions:

- ① An open bdd region  $\Omega$  (the inside)
- ② An unbdd region  $\mathbb{R}^2 \setminus \Omega$  (the outside)

Moreover,  $\Omega$  is Jordan measurable and  $\partial\Omega = C$

- A set  $D \subseteq \mathbb{R}^2$  is a simply connected domain if  $D$  is open, path-connected, and, for every simple closed curve lying in  $D$ , its inside is a subset of  $D$
- A set  $D \subseteq \mathbb{R}^3$  is a simply connected domain if  $D$  is open, path-connected, and any simple closed curve can shrink continuously to a point while staying entirely in  $D$
- Let  $F$  be a vector field in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  that is  $C^1$  on the open path-connected set  $D$ . If  $D$  is simply connected and  $F$  is irrotational on  $D$ , then  $F$  is conservative on  $D$ 
  - "Irrotational vector fields are conservative if the vector field domain has no loop holes"

## 8.6 Circulation and Flux in 2D

### Circulation and Curl in 2D

- Let  $F: C \rightarrow \mathbb{R}^2$  where  $C \subseteq \mathbb{R}^2$  is an oriented curve.  
Assume  $C$  is simple and closed. The **circulation** of  $F$  along  $C$  is  $\int_C F \cdot T ds$ 
  - The work done by  $F$  along a **closed** curve  $C$  is identical to the circulation of  $F$  along  $C$
- Let  $F = (F_1, F_2)$  be a  $C^1$  vector field in  $\mathbb{R}^2$ .  
The **curl** of  $F$  is the cts real-valued function  
 $\text{curl}(F) = \partial_1 F_2 - \partial_2 F_1$ ,
  - $F$  is irrotational iff  $\text{curl}(F) = 0$ , such vector fields are **curl-free**
- Let  $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . Fix  $p \in \mathbb{R}^2$ . If  $F$  is  $C^1$  on a neighbourhood of  $p$ , then
$$(\text{curl } F)(p) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\text{area}(B_\varepsilon(p))} \oint_{\partial B_\varepsilon(p)} (F \cdot T) ds$$
where  $\partial B_\varepsilon(p)$  is the circle oriented counterclockwise
  - "Curl is infinitesimal circulation"

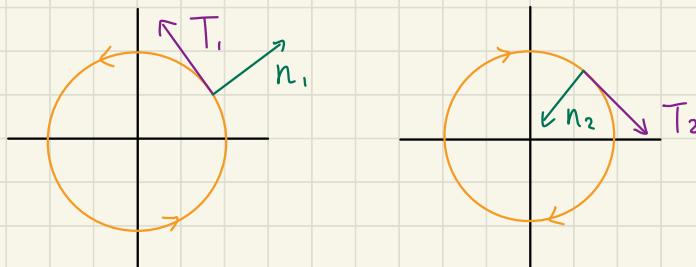
## Unit Normal and Positive Orientation

- Let  $C$  be an oriented closed curve in  $\mathbb{R}^2$  parametrized by

$\gamma: [a, b] \rightarrow \mathbb{R}^n$  w/ unit tangent vector  $T$ .

The **unit normal** of  $C$  is the cts function

$n: [a, b] \rightarrow \mathbb{R}^2$  s.t. for every  $t \in (a, b)$ ,  $n(t)$  is a unit vector orthogonal to  $T(t)$  and  $\{n(t), T(t)\}$  is a positively oriented ordered basis for  $\mathbb{R}^2$



- Let  $C$  be an oriented closed curve in  $\mathbb{R}^2$

①  $C$  is **positively oriented** if the unit normal  $n$  of  $C$

points **outward**, or the unit tangent  $T$

traverses  $C$  **counterclockwise**

②  $C$  is **negatively oriented** if the unit normal  $n$  of  $C$

points **inward**, or the unit tangent  $T$

traverses  $C$  **clockwise**

## Flux and Divergence in 2D

- Let  $F: C \rightarrow \mathbb{R}^2$  where  $C \subseteq \mathbb{R}^2$  is an oriented curve.  
Assume  $C$  is simple and closed. The flux of  $F$  across  $C$  is  $\int_C F \cdot \mathbf{n} ds$ 
  - Outward flux corresponds to a curve  $C$  oriented counterclockwise
  - Inward flux corresponds to a curve  $C$  oriented clockwise
  - Measures how much  $F$  aligns w/ the unit normal  $\mathbf{n}$  of a curve  $C$
- Let  $F = (F_1, F_2)$  be a  $C^1$  vector field in  $\mathbb{R}^2$ .  
The divergence of  $F$  is the cts real-valued function  
 $\text{div}(F) = \partial_1 F_1 + \partial_2 F_2$ 
  - Equivalent notation:  $\nabla \cdot F$
  - If  $G$  is a  $C^1$  vector field in  $\mathbb{R}^n$ ,  
then  $\text{div}(G) = \partial_1 G_1 + \dots + \partial_n G_n$
- A point  $p \in \mathbb{R}^2$  is a source if  $\text{div}(F)(p) > 0$  and a sink if  $\text{div}(F)(p) < 0$

- A vector field is sourceless if  $\operatorname{div}(F) = 0$
- Let  $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . Fix  $p \in \mathbb{R}^2$ . If  $F$  is  $C^1$  on a neighbourhood of  $p$ , then

$$(\operatorname{div} F)(p) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\operatorname{area}(B_\varepsilon(p))} \oint_{\partial B_\varepsilon(p)} (F \cdot n) ds$$

where  $\partial B_\varepsilon(p)$  is a positively oriented circle

- "Divergence is infinitesimal flux"

# 8.7 Green's Theorem and Curl

## Regular Regions and Orienting the Boundary

- A cpt Jordan measurable set  $R \subseteq \mathbb{R}^2$  is a regular region if  $\overline{R^\circ} = R$ 
  - "A regular region is a cpt set whose boundary always touches the interior"
- Let  $R \subseteq \mathbb{R}^2$  be a regular region whose boundary  $\partial R$  is a piecewise curve. The boundary  $\partial R$  is positively oriented (resp. negatively oriented) if the unit normal along the curve points outward away from  $R$  (resp. inward towards  $R$ ).  
That is, the region always stays to the left (resp. right) as one traverses the boundary

## Statement and Proof

- Green's theorem - curl form: Let  $F$  be a vector field in  $\mathbb{R}^n$  that is  $C^1$  on a regular region  $R \subseteq \mathbb{R}^2$ . If  $\partial R$  is a positively oriented piecewise curve, then

$$\oint_{\partial R} (F \cdot T) ds = \iint_R \text{curl}(F) dA$$

- If  $C$  is negatively oriented, then apply Green's theorem with  $-C$
- "The total infinitesimal circulation over  $R$  is the circulation along its boundary  $\partial R$ "

# 8.8 Green's Theorem and Divergence

## Regular Regions and Orienting the Boundary

- Right hand rule: thumb points out of the page,  
index finger:  $n$ , middle finger:  $T$
- The boundary of a regular region is positively oriented  
if its unit normal points outward as we traverse the boundary

## Statement and Proof

- Green's theorem - divergence form: Let  $F$  be a vector field in  $\mathbb{R}^2$  that is  $C^1$  on a regular region  $R \subseteq \mathbb{R}^2$ .  
If  $\partial R$  is a positively oriented piecewise curve, then
$$\oint_{\partial R} (F \cdot n) ds = \iint_R \operatorname{div}(F) dA$$
  - "The total infinitesimal flux over  $R$  is the flux across  $\partial R$ "

# 9.1 Parametrized Surfaces

## Simple Regular Parametrizations

- Let  $S \subseteq \mathbb{R}^3$ . Let  $U \subseteq \mathbb{R}^2$  be cpt. A map  $G: U \rightarrow \mathbb{R}^3$  is a (2-variable) parametrization of  $S$  if  $\text{img}(G) = S$ , and  $G$  is cts
- Let  $U \subseteq \mathbb{R}^2$  be Jordan measurable and cpt. A map  $G: U \rightarrow \mathbb{R}^3$  is regular if  $G$  is  $C^1$  and  $\{\partial_1 G, \partial_2 G\}$  is linearly independent at every point in  $U$  except for a set of zero Jordan measure in  $\mathbb{R}^2$ .
- Let  $U \subseteq \mathbb{R}^2$  be Jordan measurable and cpt. A map  $G: U \rightarrow \mathbb{R}^3$  is simple if  $G$  is injective on  $U$  except possibly along the boundary, i.e.  
 $\forall x, y \in U, G(x) = G(y) \Rightarrow x \in y \text{ or } x, y \in \partial U$
- Let  $U \subseteq \mathbb{R}^2$ . If a map  $G: U \rightarrow \mathbb{R}^3$  is a simple regular parametrization of a set  $S \subseteq \mathbb{R}^3$ , then  $S$  is a 2D regular surface at  $G(c)$  for every  $c \in U^\circ$

## Surfaces and Piecewise Surfaces

- A set  $S \subseteq \mathbb{R}^3$  is a (parametrized simple regular) surface in  $\mathbb{R}^3$  if there exists a simple regular two-variable parametrization of  $S$
- A set  $S \subseteq \mathbb{R}^3$  is a piecewise (parametrized simple regular) surface if  $S$  can be constructed by glueing together finitely many parametrized simple regular surfaces along their boundaries

## Reparametrizations and Orientation

- Let  $G: U \rightarrow \mathbb{R}^3$  and  $H: V \rightarrow \mathbb{R}^3$  be simple regular parametrizations of a set  $S \subseteq \mathbb{R}^3$ . Define  $G$  to be a reparametrization of  $H$  if there exists a cts invertible  $\varphi: U \rightarrow V$  s.t.  $\varphi$  is  $C^1$  on  $U^\circ$  w/  
 $\det D\varphi$  never zero, and  $G = H \circ \varphi$ 
  - ① If  $\det D\varphi > 0$  on  $U^\circ$ , then  $G$  has the same orientation as  $H$
  - ② If  $\det D\varphi < 0$  on  $U^\circ$ , then  $G$  has the opposite orientation as  $H$

- Let  $G_1 : U_1 \rightarrow \mathbb{R}^n$ ,  $G_2 : U_2 \rightarrow \mathbb{R}^n$ ,  $G_3 : U_3 \rightarrow \mathbb{R}^n$  be simple regular 2-variable parametrizations of a set  $S \subseteq \mathbb{R}^3$ . Then

- ① **Reflexive:**  $G_1$  is a reparam of itself
- ② **Symmetry:** If  $G_1$  is a reparam of  $G_2$ , then  $G_2$  is a reparam of  $G_1$ .
- ③ **Transitive:** If  $G_1$  is a reparam of  $G_2$  and  $G_2$  is a reparam of  $G_3$ , then  $G_1$  is a reparam of  $G_3$

## 9.2 Surface Area

### Surface Area

- Let  $S \subseteq \mathbb{R}^3$  be a surface parametrized by  $G: U \rightarrow \mathbb{R}^3$ .

The **Surface area** of  $S$  is defined as

$$A(S) = \iint_U \| \partial_1 G \times \partial_2 G \| dA \text{ provided the integral exists}$$

- The cross product can be computed by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{vmatrix}$$

- $\| \mathbf{a} \times \mathbf{b} \|$  represents the area of the parallelogram defined by  $\mathbf{a}$  and  $\mathbf{b}$

- Properties of the cross product: let  $a, b, c \in \mathbb{R}^3$  and  $\lambda \in \mathbb{R}$

$$\textcircled{1} \quad \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$\textcircled{2} \quad \mathbf{a} \times \lambda \mathbf{b} = \lambda \mathbf{a} \times \mathbf{b} = \lambda (\mathbf{a} \times \mathbf{b})$$

$$\textcircled{3} \quad (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

$$\textcircled{4} \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

- Invariance of surface area theorem: Let  $G: U \rightarrow S$  and

$H: V \rightarrow S$  be parametrizations of the surface  $S \subseteq \mathbb{R}^3$ .

Assume  $G$  is a reparam of  $H$ . The function  $\|\partial_1 G \times \partial_2 G\|$  is integrable on  $U$  iff  $\|\partial_1 H \times \partial_2 H\|$  is integrable on  $V$ .

If so,  $\iint_U \|\partial_1 G \times \partial_2 G\| dA = \iint_V \|\partial_1 H \times \partial_2 H\| dA$

- The surface element is  $dS = \|\partial_1 G \times \partial_2 G\| dA$  where  $G: U \rightarrow \mathbb{R}^3$  is a parametrization of a surface  $S \subseteq \mathbb{R}^3$ 
  - Represents the area of a small piece of  $S$
  - $A(S) = \iint_S 1 dS$
  - The surface  $S$  and the  $S$  in  $dS$  have no relation

### Surface Integrals of Scalar Functions

- Let  $S \subseteq \mathbb{R}^3$  be a surface parametrized by  $G: U \rightarrow \mathbb{R}^3$ .

Let  $f: S \rightarrow \mathbb{R}$  be bdd. The (scalar) surface integral of  $f$  over  $S$  is given by

$$\iint_S f dS := \iint_U (f \circ G) \|\partial_1 G \times \partial_2 G\| dA$$

If this integral exists, then  $f$  is integrable on the surface  $S$

- Invariance of scalar surface integrals theorem: Let  $S$  be a surface in  $\mathbb{R}^3$ . Let  $G: U \rightarrow \mathbb{R}^3$  and  $H: V \rightarrow \mathbb{R}^3$  be

parametrizations of  $S$ . Let  $f: S \rightarrow \mathbb{R}$  be bdd.

The function  $(f \circ G) \| \partial_1 G \times \partial_2 G \|$  is integrable on  $U$   
iff  $(f \circ H) \| \partial_1 H \times \partial_2 H \|$  is integrable on  $V$ . If so

$$\iint_U (f \circ G) \| \partial_1 G \times \partial_2 G \| dA = \iint_V (f \circ H) \| \partial_1 H \times \partial_2 H \| dA$$

## 9.3 Orientation and Boundary of Surfaces

### Unit Normal

- Let  $G: U \rightarrow \mathbb{R}^3$  be a parametrization of a surface in  $\mathbb{R}^3$ .

The unit normal (of the parametrization  $G$ ) is given by

$\frac{\partial_1 G \times \partial_2 G}{\|\partial_1 G \times \partial_2 G\|}$ , which is a  $C^1$  function defined on  $U \subseteq \mathbb{R}^2$

except for a set of zero Jordan measure

- Let  $G: U \rightarrow \mathbb{R}^3$  and  $H: V \rightarrow \mathbb{R}^3$  be parametrizations of a surface  $S \subseteq \mathbb{R}^3$ . Assume  $G$  is a reparam of  $H$  with  $\varphi: U \rightarrow V$  satisfying  $G = H \circ \varphi$ .

- If  $G$  is a reparam of  $H$  with the same orientation,

then 
$$\frac{(\partial_1 G \times \partial_2 G)(u, v)}{\|(\partial_1 G \times \partial_2 G)(u, v)\|} = \frac{(\partial_1 H \times \partial_2 H)(s, t)}{\|(\partial_1 H \times \partial_2 H)(s, t)\|}$$

- If  $G$  is a reparam of  $H$  with the opposite orientation,

then 
$$\frac{(\partial_1 G \times \partial_2 G)(u, v)}{\|(\partial_1 G \times \partial_2 G)(u, v)\|} = - \frac{(\partial_1 H \times \partial_2 H)(s, t)}{\|(\partial_1 H \times \partial_2 H)(s, t)\|}$$

for  $(u, v) \in U$  and  $(s, t) = \varphi(u, v) \in V$  except for a zero Jordan measure subset of  $U$

## Oriented Surfaces

- An oriented surface  $S$  is a set of (two-variable regular simple) parametrizations that are reparams of each other with the same orientation
  - May abuse notations by having  $S$  to also denote the set of points in  $\mathbb{R}^3$
- Let  $S$  be an oriented surface in  $\mathbb{R}^3$ . A unit normal  $n: S \rightarrow S^2$  is a cts function from  $S \subseteq \mathbb{R}^3$  to the set of unit vectors  $S^2$  in  $S^3$  which, for any parametrization  $G: U \rightarrow \mathbb{R}^3$  of the oriented surface  $S$ , is defined by
$$n(G(u,v)) = \frac{(\partial_1 G \times \partial_2 G)(u,v)}{\|(\partial_1 G \times \partial_2 G)(u,v)\|} \text{ for all } (u,v) \in U$$
aside from a zero Jordan measure subset of  $U$ 
  - We could write  $n(u,v)$  or  $n$
- There are non-orientable surfaces, e.g. Möbius strip

## Relative Boundary of a Surface

- Let  $S$  be a surface in  $\mathbb{R}^3$ ,
  - ① A point  $p \in S$  is a (relative) boundary point of  $S$  if there exists an open set  $V \subseteq \mathbb{R}^3$  containing  $p$ ,

an open set  $U \subseteq \mathbb{R}^2$ , a cts invertible map

$$\psi: U \cap \{(x,y) \in \mathbb{R}^2 : y \geq 0\} \rightarrow V \cap S \text{ s.t.}$$

$\psi^{-1}$  is cts and  $\psi^{-1}(p)$  lies on the  $x$ -axis

- ② The (relative) boundary of  $S$ , denoted  $\partial S$ ,  
is the set of its relative boundary points

## 9.4 Surface Integrals

### Surface Integrals and Vector Fields

- Let  $S$  be an oriented surface in  $\mathbb{R}^3$  parametrized by  $G: U \rightarrow \mathbb{R}^3$  w/ unit normal  $n$ . Let  $F$  be a vector field on  $S$ . The surface integral of  $F$  over  $S$  is given by  $\iint_S F \cdot n dS := \iint_U (F \circ G) \cdot (\partial_1 G \times \partial_2 G) dA$  provided it exists
  - Equivalently, this is the flux of  $F$  across the surface  $S$  (in the  $n$  direction)

### Basic Properties

- Invariance of flux theorem: Let  $S$  be an oriented surface in  $\mathbb{R}^3$  w/ unit normal  $n$ . Let  $F$  be a vector field defined on  $S$ . Let  $G: U \rightarrow \mathbb{R}^3$  and  $H: V \rightarrow \mathbb{R}^3$  be parametrizations of  $S$  with the same orientation. The function  $(F \circ G) \cdot (\partial_1 G \times \partial_2 G)$  is integrable on  $U$  iff  $(F \circ H) \cdot (\partial_1 H \times \partial_2 H)$  is integrable on  $V$ . If so,

$$\iint_U (F \circ G) \cdot (\partial_1 G \times \partial_2 G) dA = \iint_V (F \circ H) \cdot (\partial_1 H \times \partial_2 H) dA$$

- Let  $S$  be an oriented surface in  $\mathbb{R}^3$ . Its Oppositely oriented surface  $-S$  is the reparam of  $S$  w/ the opposite orientation
- Let  $S, T$  be oriented surfaces in  $\mathbb{R}^3$ . The concatenation of  $S$  and  $T$  is the piecewise oriented surface  $S+T$  which may be formed by gluing together all or some of their relative boundaries
  - The unit normals do not need to appear "consistent"
- Let  $S, T$  be oriented surfaces in  $\mathbb{R}^3$ . Let  $F, G$  be cts vector fields in  $\mathbb{R}^3$  defined on  $S$  and  $T$ . Then
  - $\iint_{-S} F \cdot n dS = - \iint_S F \cdot n dS$
  - For  $\lambda \in \mathbb{R}$ ,  $\begin{aligned} \iint_S (F + \lambda G) \cdot n dS \\ = \iint_S F \cdot n dS + \lambda \iint_S G \cdot n dS \end{aligned}$
  - If  $S+T$  is an oriented surface in  $\mathbb{R}^3$ ,  
then  $\iint_{S+T} F \cdot n dS = \iint_S F \cdot n dS + \iint_T F \cdot n dS$

# 9.5 Flux and Divergence in 3D

## Flux Over Closed Surfaces

- A piecewise surface  $S$  in  $\mathbb{R}^3$  is **closed** if its relative boundary  $\partial S$  is empty
  - A surface integral over a closed surface is denoted  $\iint_S$

## Divergence

- Let  $F = (F_1, F_2, F_3)$  be a  $C^1$  vector field in  $\mathbb{R}^3$ .  
The **divergence** of  $F$  is the cts real-valued function  
 $\text{div}(F) = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$ 
  - $\text{div}(F) = \nabla \cdot F = (\partial_1, \partial_2, \partial_3) \cdot (F_1, F_2, F_3)$
- Let  $F$  be a vector field in  $\mathbb{R}^3$ . Fix  $p \in \mathbb{R}^3$  in its domain.  
If  $F$  is  $C^1$  on an open set containing  $p$ , then  
 $(\text{div } F)(p) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\text{vol}(B_\varepsilon(p))} \iint_{\partial B_\varepsilon(p)} (F \cdot n) dS$   
where  $\partial B_\varepsilon(p)$  has outward unit normal
  - "Divergence is infinitesimal flux (or flux density)"
- Let  $F$  be a  $C^1$  vector field in  $\mathbb{R}^3$ . A point  $p \in \mathbb{R}^3$

is a source if  $(\operatorname{div} F)(p) > 0$  and a sink if  $(\operatorname{div} F)(p) < 0$ .

- $F$  is sourceless if  $\operatorname{div}(F) = 0$  everywhere
- Let  $F$  and  $G$  be  $C^1$  vector fields in  $\mathbb{R}^3$  w/  
domain  $U \subseteq \mathbb{R}^3$ . Fix a  $C^1$  real-valued function  $f$  on  $U$   
and fix  $\lambda \in \mathbb{R}$ . All of the following hold on  $U$ :
  - ①  $\operatorname{div}(F + \lambda G) = \operatorname{div}(F) + \lambda \operatorname{div}(G)$
  - ②  $\operatorname{div}(fF) = (\nabla f) \cdot F + f \operatorname{div}(F)$
  - ③  $\operatorname{div}(\nabla f) = \partial_1^2 f + \partial_2^2 f + \partial_3^2 f$  provided  $f$  is  $C^2$
- The Laplacian  $\Delta f := \partial_1^2 f + \partial_2^2 f + \partial_3^2 f = \nabla \cdot \nabla f$

# 9.6 Divergence Theorem

## Regular Regions and Orienting the Boundary

- Let  $R \subseteq \mathbb{R}^3$  be a regular region whose boundary  $\partial R$  is a closed piecewise surface.  $\partial R$  is **positively oriented** (resp. **negatively oriented**) if the unit normal along the surface points outward (resp. inward) w.r.t.  $R$ 
  - $\partial^2 R = \underline{\partial}(\partial R) = \emptyset$   
**relative topological**
  - "Outward" is w.r.t a region

## Divergence Theorem

- Divergence theorem:** Let  $F$  be a vector field in  $\mathbb{R}^3$  that is  $C^1$  on a regular region  $R \subseteq \mathbb{R}^3$ . If  $\partial R$  is a closed piecewise surface and is positively oriented, then
$$\oint_{\partial R} F \cdot \mathbf{n} dS = \iiint_R \operatorname{div}(F) dV$$
  - "The total infinitesimal flux over  $R$  is the flux across  $\partial R$ "
  - "The total flow inside is the net flow across the edge"

# 9.7 Circulation and Curl in 3D

## Circulation

- Let  $F: C \rightarrow \mathbb{R}^3$  where  $C \subseteq \mathbb{R}^3$  is an oriented curve.  
Assume  $C$  is simple and closed. The circulation of  $F$  along  $C$  is  $\int_C F \cdot T ds$

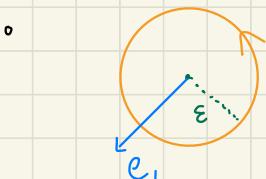
## Definition of Curl

- Let  $F$  be a  $C^1$  vector field on  $\mathbb{R}^3$ . The curl of  $F$  is the cts  $\mathbb{R}^3$ -valued function given by
$$\text{curl}(F) = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1)$$
  - $\text{curl}(F) = \nabla \times F = (\partial_1, \partial_2, \partial_3) \times (F_1, F_2, F_3)$
- A  $C^1$  vector field  $F$  in  $\mathbb{R}^3$  iff  $\text{curl}(F) = 0$  everywhere on its domain
- Irrational vector fields in  $\mathbb{R}^3$  are also called curl-free vector fields

## Geometry of Curl

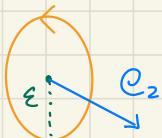
- When viewing the vector field from above,  
 $(\text{curl } F) \cdot e_3 = \partial_1 F_2 - \partial_2 F_1$ , looks like 2D curl

- The quantity  $(\operatorname{curl} F) \cdot e_3$  is the infinitesimal circulation of shrinking positively oriented  $\varepsilon$ -circles lying on the 2D plane orthogonal to  $e_3$



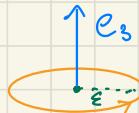
$$(\operatorname{curl} F) \cdot e_1$$

$$= \partial_2 F_3 - \partial_3 F_2$$



$$(\operatorname{curl} F) \cdot e_2$$

$$= \partial_3 F_1 - \partial_1 F_3$$



$$(\operatorname{curl} F) \cdot e_3$$

$$= \partial_1 F_2 - \partial_2 F_1$$

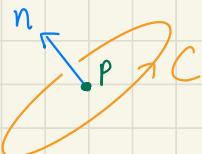
- Given a unit vector  $n$  and a point  $p \in \mathbb{R}^3$ ,

$(\operatorname{curl} F)(p) \cdot n$  measures the infinitesimal circulation

around the axis defined by  $n$  according to the

right hand rule

- Thumb:  $n$ , other fingers: tangential direction



- $|(\operatorname{curl} F)(p) \cdot n|$  is the speed at which the fluid swirls around  $n$

- Sign of  $(\operatorname{curl} F)(p) \cdot n$  is +ve if the fluid swirls counterclockwise and -ve if clockwise

## Properties of Curl

- Let  $F$  be a  $C^1$  vector field in  $\mathbb{R}^3$ . Then

- The maximum of  $\text{curl } F(p) \cdot n$  over all

unit vectors  $n \in \mathbb{R}^3$  occurs when  $n = + \frac{\nabla \times F(p)}{\|\nabla \times F(p)\|}$

and the max value is  $\|\nabla \times F(p)\|$

- The minimum of  $\text{curl } F(p) \cdot n$  over all

unit vectors  $n \in \mathbb{R}^3$  occurs when  $n = - \frac{\nabla \times F(p)}{\|\nabla \times F(p)\|}$

and the min value is  $-\|\nabla \times F(p)\|$

- The vector  $\nabla \times F(p)$  points in the direction of fastest counterclockwise spin of  $F$  at  $p$ , and its norm  $\|\nabla \times F(p)\|$  is the speed of the spin of  $F$  in this direction

- Let  $F$  and  $G$  be  $C^1$  vector fields in  $\mathbb{R}^3$  w/ domain  $U \subseteq \mathbb{R}^3$ . Fix a  $C^2$  real-valued function on  $U$  and fix  $\lambda \in \mathbb{R}$ . All of the following hold on  $U$ :

$$① \quad \nabla \times (F + \lambda G) = \nabla \times F + \lambda \nabla \cdot G$$

$$② \quad \nabla \times (fG) = f \nabla \times G + (\nabla f) \times G$$

$$③ \quad \nabla(F \times G) = (G \cdot \nabla)F + (\nabla \cdot G)F + (F \cdot \nabla)G + (\nabla \cdot F)G$$

where  $(G \cdot \nabla)F = \sum_{j=1}^3 G_j \partial_j F$  and

$$(F \cdot \nabla)G = \sum_{j=1}^3 F_j \partial_j G$$

- If  $F$  is a  $C^2$  vector field in  $\mathbb{R}^3$  and

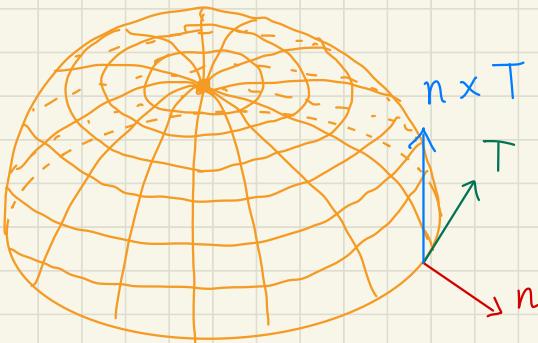
$f$  is a  $C^2$  real-valued function, then

$$\text{curl}(\nabla f) = (0, 0, 0) \text{ and } \text{div}(\text{curl}(F)) = 0$$

## 9.8 Stokes' Theorem

### Stokes' Orientation

- Given an oriented surface  $S$ , its relative boundary  $\partial S$  has the *Stokes orientation* if  $S$  is always on the left as one traverses the boundary  $\partial S$  as one's head pointing in the unit normal direction
- If  $n$  is the unit normal of the oriented surface  $S$ , and  $T$  is the unit tangent of the oriented boundary  $\partial S$ , then  $\partial S$  has the *Stokes orientation* provided  $n \times T$  points towards  $S$



### Statement

- Stokes' theorem:* Let  $S \subseteq \mathbb{R}^3$  be a surface

oriented w/ normal  $\mathbf{n}$  and whose boundary  $\partial S$  is a closed piecewise curve. Let  $\mathbf{F}$  be a vector field in  $\mathbb{R}^3$  that is  $C^1$  on an open set containing  $S$ . If  $S$  is endowed w/ the Stokes orientation, then

$$\oint_{\partial S} (\mathbf{F} \cdot \mathbf{T}) ds = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dS$$

- $\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dS$  is the total circulation of  $\mathbf{F}$  over  $S$
- "The total infinitesimal circulation over  $S$  is the circulation along its boundary  $\partial S$ "

# 9.9 Div, Grad, and Curl

## Gradient and Curl

- Gradient vector fields are curl-free, i.e.

$$\operatorname{curl}(\operatorname{grad}(f)) = 0$$

- Let  $F$  be a vector field in  $\mathbb{R}^3$  that is  $C^1$  on an open set  $U \subseteq \mathbb{R}^3$ . Assume  $U$  is convex.

If  $F$  is curl-free on  $U$ , then  $F$  is a gradient vector field, i.e. there exists a scalar-valued  $C^2$  function  $f$  on  $U$  s.t.  $F = \operatorname{grad}(f)$  on  $U$

## Curl and Divergence

- Curl vector fields are divergence-free, i.e.

$$\operatorname{div}(\operatorname{curl}(G)) = 0$$

- Let  $F$  be a vector field in  $\mathbb{R}^3$  that is  $C^1$  on an open set  $U \subseteq \mathbb{R}^3$ . Assume  $U$  is convex.

If  $F$  is divergence-free on  $U$ , then  $F$  is a curl vector field, i.e. there exists a vector-valued  $C^2$  function  $G$  on  $U$  s.t.  $F = \operatorname{curl}(G)$  on  $U$

## A Unified View

- Let  $U \subseteq \mathbb{R}^3$ . Let  $C^\infty(U)$  be the set of real-valued functions  $f: U \rightarrow \mathbb{R}$  w/ infinitely many partial derivatives. Call the space of  $C^\infty$  scalar fields  $V = C^\infty(U)$  and the space of  $C^\infty$  vector fields  $V^3$ 
  - grad:  $V \rightarrow V^3$  is a linear map
  - curl:  $V^3 \rightarrow V^3$  is a linear map
  - div:  $V^3 \rightarrow V$  is a linear map

$$V \xrightarrow{\text{grad}} V^3 \xrightarrow{\text{curl}} V^3 \xrightarrow{\text{div}} V$$