MAT237

Multivariable Calculus With Proofs

1.1 Curves

Motion

- A parametric curve $\gamma : I \to \mathbb{R}^n$ for some interval $I \subseteq \mathbb{R}$ describes the motion of an object moving in $\mathbb{R}^n$
  - Position at time t is $\gamma(t)$
  - Component functions : $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$ for n single-variable functions $\gamma_i : I \to \mathbb{R}$
  - $\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t))$
  - Speed at time t is $\|r'(t)\|$ $= \sqrt{|r_1'(t)|^2 + \dots + |r_n'(t)|^2}$
  - Direction of motion : unit tangent vector, denoted $T = T(t)$ defined by

    $$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

    - $\gamma'(t)$ is a scalar multiple of $T(t)$
  - Acceleration is γ'' so

    $$\gamma''(t) = (\gamma_1''(t), \dots, \gamma_n''(t))$$

Frenet Frame in 3 Dimensions

- Define the (principal) unit normal to be

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

  - T and N are orthogonal
  - $\gamma''(t)$ is a scalar multiple of $N(t)$

- Define the binormal unit vector B to be the unique unit vector s.t. $\{T, N, B\}$ form a positively-oriented ordered orthogonal basis in $\mathbb{R}^3$
  - $\{T, N, B\}$ forms the Frenet frame (or Frenet - Serret frame or TNB frame) describing the motion of an object in 3D
  - $B = T \times N$

- Cross-product only works for vectors in $\mathbb{R}^3$:

let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$:

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Geometry of Curves

- Trace of a parametric curve $\gamma : I \to \mathbb{R}^n$ is the image of γ, i.e. $\gamma(I)$
  - "Path traced out by γ"
  - Trace of a parametric curve is a set $C \subseteq \mathbb{R}^n$. this set is a curve
- Let $C \subseteq \mathbb{R}^n$ be a set. C is a curve in $\mathbb{R}^n$ if C is the trace of a continuous² parametric curve $\gamma : I \to \mathbb{R}^n$

1.2 Real-valued Functions

Scalar Fields and Densities

- Real-valued functions: $\mathbb{R}^n \to \mathbb{R}$
  - Also called scalar fields, scalar functions, potentials
- Real-valued functions that are non-negative: densities
  - Shorthand: $f \geq 0$
  - Density: $\dfrac{\text{quantity}}{\text{unit of measurement}}$

Graphs, Level Sets, and Slices

- Let $A \subseteq \mathbb{R}^n$. The graph of a function $f : A \to \mathbb{R}$ is the set in $\mathbb{R}^{n+1}$ given by $\{(x, f(x)) : x \in A\}$
- Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$. Fix $k \in \mathbb{R}$.
  The level set of f at k is $\{x \in \mathbb{R}^n : f(x) = k\}$
  - Also called the k-level set
  - Contour: level set in $\mathbb{R}^2$
  - For graphs of 2-variable functions, we can create a contour plot by plotting the level sets

for a few different values

- Heat maps: colour gradient that corresponds to the values of the function
    - Continuous version of a contour plot
- Slicing: fixing a variable
- Let $A \subseteq \mathbb{R}^2$ and $f: A \to \mathbb{R}$
    - For fixed $a \in \mathbb{R}$, the x-slice at a of the graph f is $\{(y,z) : z = f(a,y)\}$
    - For fixed $b \in \mathbb{R}$, the y-slice at b of the graph f is $\{(x,z) : z = f(x,b)\}$
    - Slices for 3-variable functions can be defined similarly

1.3 Vector Fields

- An (n-dimensional) vector field is a function F w/ domain and codomain lying in $\mathbb{R}^n$
  - "F is a vector field in $\mathbb{R}^n$"
- To plot a 2D vector field, draw vectors for points on the grid
- Notation
  - $F(x,y,z) = (x^2, yx, -z)$ most common
  - $F(x,y,z) = \langle x^2, yx, -z \rangle$
  - $F = [x^2, yx, -z]$
  - $F = x^2\hat{i} + yx\hat{j} - z\hat{k}$

1.4 Coordinate Transformations

- Transformation: map between 2 subsets lying in the same dimension
- A coordinate transformation $f: A \to B$ refers to a continuous bijective transformation
  - Domain A and map f form a coordinate system for the codomain B

Polar Coordinates

- Define the polar coordinate transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(r, \theta) = (r\cos\theta, r\sin\theta)$
  - r : radius
  - θ : polar angle
  - Not bijective
- The polar coordinate transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(r, \theta) = (r\cos\theta, r\sin\theta)$ maps the subset $(0, \infty) \times (-\pi, \pi)$ bijectively to the subset $\mathbb{R}^2 \setminus \{(x, 0) . x \le 0\}$

Cylindrical Coordinates

- Define the cylindrical coordinate transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$
    - r: polar radius
    - θ: polar angle
    - z: usual rectangular coordinate
- The cylindrical coordinate transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined above maps the subset $(0, \infty) \times (-\pi, \pi) \times \mathbb{R}$ bijectively to the subset $\mathbb{R}^3 \setminus \{(x, 0, z) : x \leq 0, z \in \mathbb{R}\}$

Spherical Coordinates

- Define the spherical coordinate transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$
    - ρ: spherical radius
    - θ: polar angle
    - ϕ: azimuthal angle
- The spherical coordinate transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined above maps the subset $(0, \infty) \times (-\pi, \pi) \times (0, \pi)$ bijectively to the subset $\mathbb{R}^3 \setminus \{(x, 0, z) : x \leq 0, z \in \mathbb{R}\}$

1.5 Surfaces

Parametric Surfaces

- Described by maps $\mathbb{R}^n \to \mathbb{R}^m$ where $n < m$
- Let $m, n \in \mathbb{N}^+$ with $n < m$. Let $S \subseteq \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$, and $g: A \to \mathbb{R}^m$ be a cts map.

  If $S = \{g(x) : x \in A\} = \operatorname{img}(g)$, then the pair (S, g) is a parametric surface
  - S is parameterized by g

Explicit Surfaces

- Let $m, n \in \mathbb{N}^+$, $A \subseteq \mathbb{R}^n$. The graph of a function $f: A \to \mathbb{R}^m$ is the set

  $S = \{(x, f(x)) : x \in A\} \subseteq \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{m+n}$
  - Variables can be reordered
- A set $S \subseteq \mathbb{R}^n$ is an explicit surface if S is a graph of a cts function

Implicit Surfaces

- "Implicit" : does not explicitly express one variable in terms of the others
- A set $S \subseteq \mathbb{R}^n$ is an implicit surface if there exists a constant $c \in \mathbb{R}^m$, a set $A \subseteq \mathbb{R}^n$, and a cts function $f : A \to \mathbb{R}^m$ s.t.
  $S = f^{-1}(\{c\}) = \{x \in \mathbb{R}^n : f(x) = c\}$
  - f^{-1} is not the inverse function, it is the preimage of a set under a function
  - Explicit surfaces are also implicit surfaces
- Dim of implicit surface
  $=$ # of varichles $-$ # of equations

1.6 Projections

- Projections: maps of the form $\mathbb{R}^n \to \mathbb{R}^m$ where $n > m$
- For $i \in \{1, \dots, n\}$, the map $\pi_i : \mathbb{R}^n \to \mathbb{R}$ given by $\pi_i(x_1, \dots, x_n) = x_i$ is the i^{th} coordinate map
- For $i \in \{1, \dots, n\}$, the map $\Pi_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ given by $\Pi_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ is the i^{th} coordinate plane projection

2.1 Sets

Balls and Spheres

- Distance in $\mathbb{R}^n$: $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$
- Let $r \geq 0$, $a \in \mathbb{R}^n$:
  - The open ball of radius r centered at a is the set $B_r(a) = \{x \in \mathbb{R}^n : \|x-a\| < r\}$
  - The closed ball of radius r centered at a is the set $\{x \in \mathbb{R}^n : \|x-a\| \leq r\}$
  - The sphere of radius r centered at a is the set $\{x \in \mathbb{R}^n : \|x-a\| = r\}$
- An open ball or closed ball is punctured if it excludes the centre, i.e. $B_r(a) \setminus \{a\}$
- Balls are solid, spheres are hollow
- The $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ is the sphere of radius 1 centered at the origin and is denoted S^{n-1}
  - $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$

Rectangles

- A (closed) rectangle in $\mathbb{R}^n$ is a set R of the form

  $$R = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

  $$= \{(x_1, \dots, x_n) : x_i \in [a_i, b_i] \text{ for } 1 \leq i \leq n\}$$

  where $a_1, b_1, \dots, a_n, b_n \in \mathbb{R}$ and $a_i < b_i$ for all $1 \leq i \leq n$

  - The set $(a_1, b_1) \times \cdots \times (a_n, b_n)$ is an open rectangle

- An (n-dimensional) hypercube is a set in $\mathbb{R}^n$ of the form $[a,b]^n = [a,b] \times \cdots \times [a,b]$

  - The (n-dimensional) unit hypercube is the set $[0,1]^n$

2.2 Interior, Boundary, and Closure

Interior

- Let $A \subseteq \mathbb{R}^n$ be a set. A point $p \in \mathbb{R}^n$ is an interior point of A if there exists $\varepsilon > 0$ s.t. $B_\varepsilon(p) \subseteq A$
- Let $A \subseteq \mathbb{R}^n$ be a set. The interior of A, denoted A° or $\mathrm{int}(A)$, is the set of interior points of A
- Let $A, B \subseteq \mathbb{R}^n$, then
  - $A^\circ \subseteq A$
  - $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$
  - $(A \cap B)^\circ = A^\circ \cap B^\circ$
  - $(A \times B)^\circ = A^\circ \times B^\circ$

Boundary

- Let $A \subseteq \mathbb{R}^n$. A point $p \in \mathbb{R}^n$ is a boundary point of A if for every $\varepsilon > 0$, the sets $B_\varepsilon(p) \cap A$ and $B_\varepsilon(p) \cap A^c$ are both non-empty

- Let $A \subseteq \mathbb{R}^n$. The (topological) boundary of A, denoted ∂A, is the set of boundary points of A
- For any set $A \subseteq \mathbb{R}^n$, A° and ∂A are disjoint

<u>Closure</u>

- Let $A \subseteq \mathbb{R}$. A point $p \in \mathbb{R}^n$ is a limit point if for every $\varepsilon > 0$, $B_\varepsilon(p) \setminus \{p\}$ contains points in A
  - Denoted A^*
- Let $A \subseteq \mathbb{R}$. The closure of A, denoted $\bar{A}$ or $cl(A)$, is the set A along with its limit points
  - i.e. $\bar{A} = A \cup A^*$
- Let $A, B \subseteq \mathbb{R}^n$. Then
  - $A \subseteq \bar{A}$
  - $cl(A \cup B) = \bar{A} \cup \bar{B}$
  - $cl(A \cap B) \subseteq \bar{A} \cap \bar{B}$
  - $cl(A \times B) = \bar{A} \times \bar{B}$
- Let $A \subseteq \mathbb{R}^n$. Then $\bar{A} = A^\circ \cup \partial A$ and $\partial A = \bar{A} \setminus A^\circ$

2.3 Sequences

- A sequence in $\mathbb{R}^n$ is a function w/ $\{k \in \mathbb{Z} : k \geq k_0\} \to \mathbb{R}^n$ for some fixed $k_0 \in \mathbb{Z}$

Convergence of Sequences

- Let $\{x(k)\}_k$ be a sequence in $\mathbb{R}^n$. Then $\{x(k)\}_k$ converges to $p \in \mathbb{R}^n$ if $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\forall k \in \mathbb{N}$, $k \geq K \Rightarrow \|x(k) - p\| < \varepsilon$
  - $\lim_{k \to \infty} x(k) = p$ or $x(k) \to p$
- The sequence $\{x(k)\}_k$ converges if there exists $p \in \mathbb{R}^n$ s.t. $\lim_{k \to \infty} x(k) = p$
  - Otherwise, $\{x(k)\}_k$ diverges
- A sequence $\{x(k)\}_k$ in $\mathbb{R}^n$ converges to x if every open ball centered at x contains all but finitely many points of the sequence $\{x(k)\}_k$
- Let $\{x(k)\}_k$ be a sequence in $\mathbb{R}^n$ w/ $x(k) = (x_1(k), \dots, x_n(k))$. $\{x(k)\}_k$ converges iff $\{x_i(k)\}_k$ converges for all $i = 1, 2, \dots, n$

Limit Points, Boundary Points, and Interior Points

- Let $A \subseteq \mathbb{R}^n$ be a set. A point $p \in \mathbb{R}^n$ is a limit point of A iff there exists a sequence of points in $A \setminus \{p\}$ which converges to p
- Let $A \subseteq \mathbb{R}^n$ be a set. Let $p \in \mathbb{R}^n$ be a point.
  - The point p is an interior point of A iff for every sequence $\{x(k)\}_k$ of points converging to p, there exists $K \in \mathbb{N}^+$ s.t $\{x(k)\}_{k=K}^{\infty} \subseteq A$
  - The point p is a boundary point of A iff there exists a sequence of points in A converging to p and there exists a sequence of points in A^c converging to p

2.4 Open Sets and Closed Sets

- If the set A is the domain of a function, we want a sequence of approximations lying inside A to converge to a point within A
  - ① If a sequence in $\mathbb{R}^n$ converges to a point $a \in A$, then the tail of the sequence belongs to A
  - ② If a sequence in A converges to a point $a \in \mathbb{R}^n$, then a must belong to A

Open Sets

- A set $A \subseteq \mathbb{R}^n$ is open if every point in A is an interior point of A
  - Such set satisfies ①
- The interior of a set $A \subseteq \mathbb{R}^n$ is open
- Let $A \subseteq \mathbb{R}^n$. Then all of the following are equivalent:
  - A is open
  - $A = A^\circ$
  - $A \cap \partial A = \varnothing$

Closed Sets

- A set $A \subseteq \mathbb{R}$ is closed if every limit point of A belongs to A
    - Such set satisfies ②
    - Any convergent sequence in a closed set A must converge to a point in A
- The closure of a set A is closed
- Let $A \subseteq \mathbb{R}^n$. All of the following are equivalent:
    - A is closed
    - $A = \bar{A}$
    - $\partial A \subseteq A$

Set Operations

- A set $A \subseteq \mathbb{R}^n$ is open iff A^c is closed
- Clopen: both closed and open
    - E.g. $\emptyset$, $\mathbb{R}^n$
- All the following are true for sets in $\mathbb{R}^n$:
    - A finite intersection of open sets is open
    - Any union of open sets is open

    - A finite union of closed sets is closed
    - Any intersection of closed sets is closed
    - A finite Cartesian product of open/closed sets is open/closed, respectively
- Infinite intersection of open sets may not be open

  E.g. $\bigcap_{\varepsilon>0} (-\varepsilon, \varepsilon) = \{0\}$
- Infinite union of closed sets may not be closed

  E.g. $\bigcup_{0<\varepsilon<1} [-\varepsilon, \varepsilon] = (0,1)$

2.5 Compact Sets

- Sps a set $A \subseteq \mathbb{R}$ is the domain of a real-valued function f. Construct a sequence of points $\{x(k)\}_{k=1}^{\infty}$ in A, where each value $f(x(k))$ is attempting to approximate the max value of f
  - As $k \to \infty$, $\{x(k)\}_{k=1}^{\infty}$ converges to $p \in A$, where $f(p)$ is the max value of f

?1 How to ensure $\{x(k)\}_{k=1}^{\infty}$ converges? (to $p \in \mathbb{R}^n$)

?2 How to ensure the limiting point p belongs to A?

  - ?2 can be addressed by assuming A is closed

Definition of Compactness

- Let $x : \mathbb{N}^+ \to \mathbb{R}^n$ be a sequence, let $m : \mathbb{N}^+ \to \mathbb{N}^+$ be a strictly increasing function. The sequence $\{x(m(k))\}_{k=1}^{\infty}$ is a subsequence of the sequence $\{x(k)\}_{k=1}^{\infty}$
  - Domain of x must = codomain of m
- A set $A \subseteq \mathbb{R}$ is compact if every sequence of A has a subsequence which converges to a point in A

- A set $A \subseteq \mathbb{R}^n$ is bounded if $\exists R > 0$ s.t.

  $A \subseteq \{x \in \mathbb{R}^n : \|x\| < R\}$

  - Unbounded: not bounded

- Bolzano-Weierstrass Theorem: A set in $\mathbb{R}^n$ is compact iff it is both closed and bounded

Set Operations and Subsets

- All of the following are true for sets in $\mathbb{R}^n$
  - A finite union of compact sets is compact
  - Any intersection of compact sets is compact
  - A finite Cartesian product of compact sets is compact

- Let A be a compact set in $\mathbb{R}^n$. If $B \subseteq A$ and B is closed then B is compact

2.6 Limits

Formal Definitions

- Let $f : A \to \mathbb{R}^m$ be a function w/ $A \subseteq \mathbb{R}^n$.

  Let $a \in \mathbb{R}^n$ be a limit point of A and let $b \in \mathbb{R}^m$.

  Define b to be the limit of f at a provided

  $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in A$,

  $0 < \|x - a\| < \delta \implies \|f(x) - b\| < \varepsilon$

  - $\lim_{x \to a} f(x) = b$
  - $f(x) \to b$ as $x \to a$

- Limit only defined at limit points

$$\begin{cases} \text{Defined} \begin{cases} \text{Exists} \\ \text{DNE} \end{cases} \\ \text{Not Defined} \end{cases}$$

- Let $A \subseteq \mathbb{R}^n$ be a set and let $f : A \to \mathbb{R}^m$ be a function.

  Let $a \in \mathbb{R}^n$ be a limit point of A and let $b \in \mathbb{R}^m$.

  Then $\lim_{x \to a} f(x) = b$ iff for every sequence of points

$\{x(k)\}_k$ in $A \setminus \{a\}$ w/ $x(k) \to a$, the sequence of points $\{f(x(k))\}_k$ in $\mathbb{R}^n$ converges to b; i.e. $f(x(k)) \to b$

Computing Limits

- Let $f: A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$. Let a be a limit point of A and let $b = (b_1, \dots, b_m) \in \mathbb{R}^m$. Let $f_1, \dots, f_m$ be the coordinate functions of f so $f = (f_1, \dots, f_m)$. Then $\lim_{x \to a} f(x) = b$ iff for all $i = \{1, \dots, m\}$, $\lim_{x \to a} f_i(x) = b_i$
- Let $A \subseteq \mathbb{R}^n$ be a set and let a be a limit point of A. Let f, g be $\mathbb{R}^m$-valued functions defined on A. Let ϕ be a real-valued function defined on A, Let $\lambda \in \mathbb{R}$, $b \in \mathbb{R}^m$ be constants. Then:
    - (Constants) $\lim_{x \to a} b = b$ and $\lim_{x \to a} x = a$
    - (Linearity) If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist then $\lim_{x \to a} (f(x) + \lambda g(x))$ exists and $\lim_{x \to a} (f(x) + \lambda g(x)) = \lim_{x \to a} f(x) + \lambda \lim_{x \to a} g(x)$
    - (Dot product) If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist then $\lim_{x \to a} (f(x) \cdot g(x))$ exists and

$$\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x) \right) \cdot \left(\lim_{x \to a} g(x) \right)$$

- (Scalar product) If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} \phi(x)$ exist then $\lim_{x \to a} \left(\phi(x) f(x) \right)$ exists and

$$\lim_{x \to a} (\phi(x) f(x)) = \left(\lim_{x \to a} \phi(x) \right) \left(\lim_{x \to a} f(x) \right)$$

- Squeeze Theorem: Let $A \subseteq \mathbb{R}$ be a set and let a be a limit point of A.

  Let f, g, h be real-valued functions w/ domain A.

  Assume there exists $\delta > 0$ s.t.

  $\forall x \in A, \; 0 < \|x - a\| < \delta \Rightarrow f(x) \leq g(x) \leq h(x)$.

  If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = b$ for some $b \in \mathbb{R}$ then $\lim_{x \to a} g(x) = b$.

<u>Limits With Infinity</u>

- Let $A \subseteq \mathbb{R}^n$ be unbounded. Let $f : A \to \mathbb{R}^m$ and let $b \in \mathbb{R}^m$.

  Define b to be the limit of $f(x)$ as $\|x\| \to \infty$ provided

  $\forall \varepsilon > 0, \; \exists M > 0$ s.t. $\forall x \in A, \; \|x\| > M \Rightarrow \|f(x) - b\| < \varepsilon$.

  - $\lim_{\|x\| \to \infty} f(x) = b$
  - $f(x) \to b$ as $\|x\| \to \infty$

- Let $A \subseteq \mathbb{R}^n$ be a set. Let a be a limit point of A.

  Let $f : A \to \mathbb{R}$ be a real-valued function

The limit of f at a diverges to $+\infty$ provided

$\forall M > 0,\ \exists \delta > 0$ s.t. $\forall x \in A,\ 0 < \|x-a\| < \delta \Rightarrow f(x) > M$

- $\lim_{x \to a} f(x) = +\infty$
- $f(x) \to +\infty$ as $x \to a$

- $f(x) \to -\infty$ as $x \to a$ can be defined similarly

2.7 Continuity

Formal Definition

- Let $f: A \to \mathbb{R}^m$ be a function w/ domain $A \subseteq \mathbb{R}^n$.
  Let $a \in A$ be a point. The function f is continuous at a provided $\forall \varepsilon > 0, \exists \delta > 0$ s.t.
  $\forall x \in A, \|x-a\| < \delta \Rightarrow \|f(x) - f(a)\| < \varepsilon$
  - If a is an isolated point of A, then f is cts at a
  - If a is a limit point of a, then f is cts at a iff $\lim_{x \to a} f(x) = f(a)$
- f is cts at a is equivalent to
  - $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$, $x \in B_\delta(a) \Rightarrow f(x) \in B_\varepsilon(f(a))$
  - $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A \cap B_\delta(a)$, $f(x) \in B_\varepsilon f(a)$
  - $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $f(A \cap B_\delta(a)) \subseteq B_\varepsilon(f(a))$

$B_\delta(a)$ $B_\varepsilon(b)$

- Let $f: A \to \mathbb{R}^m$ be a function w/ domain $A \subseteq \mathbb{R}^n$.
  Let $a \in A$ be a point. Then f is cts at a iff for every sequence $\{x(k)\}_k$ in A converging to a, the sequence $\{f(x(k))\}_k$ in $\mathbb{R}^m$ converges to $f(a)$
- Let $f: A \to \mathbb{R}^m$ be a function w/ domain $A \subseteq \mathbb{R}^n$.
  For a subset $S \subseteq A$, the function f is continuous on S if f is cts at a for every $a \in S$.
  - f is continuous if f is cts on its domain A

<u>Basic Properties</u>

- The map $f = (f_1, \ldots, f_m): A \to \mathbb{R}^m$ is cts at $a \in A$ iff for each $i \in \{1, \ldots, m\}$, the component function f_i is cts at a.
- Every linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ is cts
- Let $A \subseteq \mathbb{R}^n$ and let $a \in A$. Let $f, g: A \to \mathbb{R}^m$.

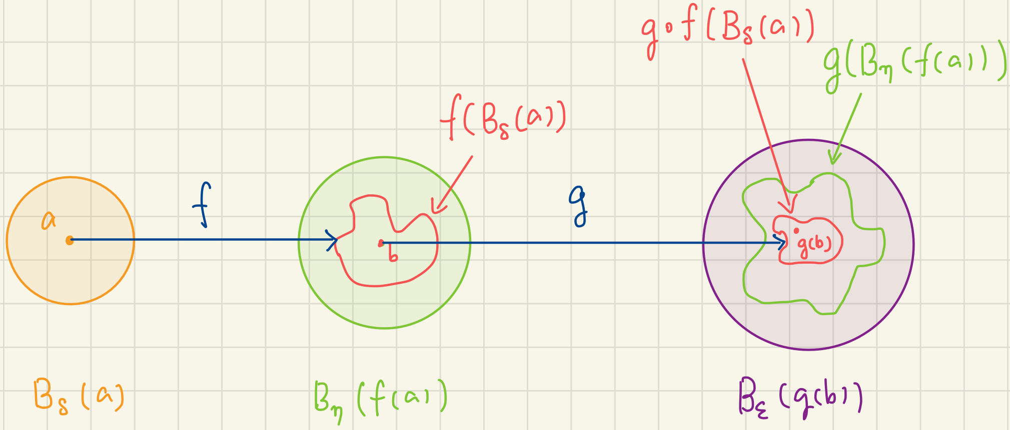
Let $\phi : A \to \mathbb{R}$. Let $\lambda \in \mathbb{R}$. Then

- If f, g are cts at a, then $f + \lambda g$ is cts at a
- If f, g are cts at a, then $f \cdot g$ is cts at a
- If f, ϕ are cts at a, then ϕf is cts at a

- Let $f : A \to B$ where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$.

  Let $g : B \to \mathbb{R}^k$.

  - Let $a \in A$. If f is cts at a and g is cts at $f(a)$ then $g \circ f$ is cts at a
  - Let a be a limit point of A and let $b \in B$. If $\lim_{x \to a} f(x) = b$ and g is cts at b then $\lim_{x \to a} g \circ f(x) = g(b)$



- A monomial in the n variables $x_1, \dots, x_n$ is a function of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{N}$.
- A polynomial in the n variables $x_1, \dots, x_n$ is a linear combination of monomials in n variables w/ real coefficients
- All polynomials in n variables are cts on $\mathbb{R}^n$

Topological Properties

- Cts functions preserve topological properties of sets (under image and preimage)
- Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a function. The following are equivalent:
    - f is cts on $\mathbb{R}^n$
    - The preimage $f^{-1}(U)$ is open for every open set $U \subseteq \mathbb{R}^m$
    - The preimage $f^{-1}(V)$ is closed for every closed set $V \subseteq \mathbb{R}^m$
- If A is a compact subset of $\mathbb{R}^n$ and f is an $\mathbb{R}^m$-valued function that is cts on A then $f(A)$ is a cpt subset of $\mathbb{R}^m$

2.8 Path-Connected Sets

- A set $S \subseteq \mathbb{R}^n$ is path-connected if for every pair of points $p, q \in S$ there exists a cts function $\gamma : [a,b] \to \mathbb{R}^n$ s.t. $\gamma(a) = p$ and $\gamma(b) = q$ and $\operatorname{img}(\gamma) \subseteq S$
    - S is C^k path-connected if γ is required to be continuously k-times differentiable
- A set $S \subseteq \mathbb{R}^n$ is convex if the line segment between any two points $p, q \in S$ lies inside S
- Let $S \subseteq \mathbb{R}^n$ be a path-connected set. Let $f : S \to \mathbb{R}^m$. If f is cts on S then $f(S)$ is path-connected.
- Intermediate Value Theorem: Let $f : [a,b] \to \mathbb{R}$. If f is cts on $[a,b]$ then $f([a,b])$ is path-connected

2.9 Global Extrema

Definition of Global Extrema

- Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}$
  - A point $p \in A$ is a (global) maximum point of f on A if $f(p) \geq f(x)$ for all $x \in A$
    - $f(p)$ is the (global) maximum value of f on A
  - If a maximum point of f on A exists, then f attains a (global) maximum on A
  - Minimum point, minimum value, and attaining minimum can be defined similarly
- Extremum: minimum or maximum

Extreme Value Theorem

- Extreme value theorem: If $A \subseteq \mathbb{R}^n$ is a compact set and $f : A \to \mathbb{R}$ is cts then f attains maximum and minimum values at points of A

- Let $A \subseteq \mathbb{R}^n$ be closed and unbounded. Let $f: A \to \mathbb{R}$ be cts. If $f(x) \to -\infty$ as $\|x\| \to \infty$ in A, then f attains a maximum on A

- Chain rule: Let $A, B \subseteq \mathbb{R}$. Let $\varphi: A \to B$ and let $f: B \to \mathbb{R}^m$. Let $a \in A^\circ$ s.t. $\varphi(a) \in B^\circ$. If φ is differentiable at a and f is differentiable at $\varphi(a)$, then $(f \circ \varphi)'(a) = f'(\varphi(a))\,\varphi'(a)$.

3.1 Derivatives of One Variable

Definition

- Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  The derivative of f at a is defined to be

  $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ provided the limit exists.

  - f is differentiable at a
  - $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Basic Properties

- Let $A \subseteq \mathbb{R}$ and let $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$.

  Let $a \in A^\circ$. f is differentiable at a iff

  for every $i \in \{1, \dots, m\}$, f_i is differentiable at a.

  - $f'(a) = (f_1'(a), \dots, f_m'(a)) = \begin{bmatrix} f_1'(a) \\ \vdots \\ f_m'(a) \end{bmatrix}$

- Let $A \subseteq \mathbb{R}$ and let $f, g : A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  Let $\lambda \in \mathbb{R}$ and let $\varphi : A \to \mathbb{R}$. Then

  - (Linearity) If f, g are differentiable at a, then

    $f + \lambda g$ is differentiable at a and

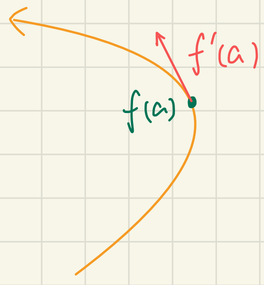
$$(f+\lambda g)'(a) = f'(a) + \lambda g'(a).$$

- (Scalar product) If f, φ are differentiable at a, then φf is differentiable at a and
$$(\varphi f)'(a) = \varphi'(a) f(a) + \varphi(a) f'(a)$$
- (Dot product) If f, g are differentiable at a, then $f \cdot g$ is differentiable at a and
$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

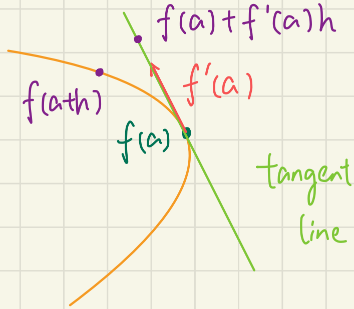
Four Viewpoints

Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}^m$ be a parametric curve differentiable at $a \in A$.

- Physical viewpoint
    - If f is the position of a particle, then $f'(a)$ is the instantaneous velocity of the particle at time a and position $f(a)$



- Geometric viewpoint
    - The tangent line at $f(a)$ on f is (presumably) given by the set $\{f(a)+hf'(a) : h \in \mathbb{R}\}$
        - $f'(a)$ is the direction vector



- Analytic viewpoint
    - The linear approximation of f at a is the function $\ell : \mathbb{R} \to \mathbb{R}^m$ defined by $\ell(x) = f(a) + f'(a)(x-a)$
    - $f(x) \approx \ell(x)$ for x near a
- Algebraic viewpoint
    - Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}^m$. Let $a \in A^\circ$. f is differentiable at a iff there exists a linear map $L : \mathbb{R} \to \mathbb{R}^m$ s.t. $\lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{h} = 0$, in which case $L(h) = f'(a)h$

- Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  If f is differentiable at a, then the linear map $df_a : \mathbb{R} \to \mathbb{R}^m$ defined by $df_a(h) = f'(a)h$ is the differential of f at a

- For h near 0, $f(a+h) \approx f(a) + f'(a)h = f(a) + df_a(h)$

- Chain rule in terms of differentials:

  $d(g \circ f)_a = dg_{f(a)} \circ df_a$

  - Differential of the composition is composition of differentials

3.2 Partial Derivatives

Definition

- Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  Fix $1 \leq j \leq n$. Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$. The jth partial derivative of f at a is

  $\partial_j f(a) := \lim_{h \to 0} \dfrac{f(a+he_j) - f(a)}{h}$ provided that the limit exists.

  - The jth partial derivative of f is the function $\partial_j f : U \to \mathbb{R}^m$, where U is the set of points $a \in A$ s.t. $\partial_j f(a)$ exists

- Equivalent notations for $\partial_j f$:
  - $\dfrac{\partial f}{\partial x_j}$ $\quad D_{e_j} f$ $\quad f_{x_j}$ $\quad D_j f$ $\quad \partial_{x_j} f$

Computations

- Let $A \subseteq \mathbb{R}^n$ and let $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$.

  Let $a \in A^\circ$. $\partial_j f(a)$ exists iff for every $i \in \{1, \dots, m\}$, $\partial_j f_i(a)$ exists. If so, $\partial_j f(a) = \big(\partial_j f_1(a), \dots, \partial_j f_m(a)\big)$.

- Let $A \subseteq \mathbb{R}^n$ and let $f, g : A \to \mathbb{R}^m$. Let $a \in A^\circ$.

Fix $1 \le j \le n$. Let $\lambda \in \mathbb{R}$ and let $\varphi : A \to \mathbb{R}$.

- (Linearity) If $\partial_j f(a)$ and $\partial_j g(a)$ both exist, then $\partial_j (f + \lambda g)(a)$ exists and
  $\partial_j (f + \lambda g)(a) = \partial_j f(a) + \lambda \partial_j g(a)$
- (Scalar product) If $\partial_j f(a)$ and $\partial_j \varphi(a)$ both exist, then $\partial_j (\varphi f)(a)$ exists and
  $\partial_j (\varphi f)(a) = f(a) \partial_j \varphi(a) + \varphi(a) \partial_j f(a)$
- (Dot product) If $\partial_j f(a)$ and $\partial_j g(a)$ both exist, then $\partial_j (f \cdot g)(a)$ exists and
  $\partial_j (f \cdot g)(a) = \partial_j f(a) \cdot g(a) + f(a) \cdot \partial_j g(a)$

3.3 Directional Derivatives

Definition

- Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  Fix $v \in \mathbb{R}^n$. The directional derivative of f in the direction v at a is given by $D_v f(a) = \lim_{h \to 0} \frac{f(a+hv) - f(a)}{h}$

  - The directional derivative of f in the direction v is the function $D_v f : U \to \mathbb{R}^m$, where U is the set of points $a \in A$ s.t. $D_v f(a)$ exists
  - $D_{e_j} f = \partial_j f$

Computations

- Let $A \subseteq \mathbb{R}^n$ and let $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$.

  Let $a \in A^\circ$. Fix $v \in \mathbb{R}^n$. Then $D_v f(a)$ exists iff $D_v f_i(a)$ exists for every $i \in \{1, \dots, m\}$. If so,

  $D_v f(a) = (D_v f_1(a), \dots, D_v f_m(a))$

- Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  If f is differentiable at a, then for all $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, $D_v f(a) = \sum_{j=1}^{n} v_j \partial_j f(a)$

Geometry of Directional Derivatives

- The directional derivative of f at a in the direction v outputs a vector $D_v f(a)$ that is tangent to the parametric curve $\gamma(t) = f(a+tv)$ at $t=0$
    - $\gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t} = \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t} = D_v f(a)$

3.4 Gradient

- Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}$. Let $a \in A^\circ$.

  The gradient of f at a is denoted $\nabla f(a)$ and given by

  $\nabla f(a) = (\partial_1 f(a), \dots, \partial_n f(a))$

- Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}$. Let $a \in A^\circ$.

  If f is differentiable at a, then

  for all $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, $D_v f(a)$ exists and

  $D_v f(a) = \nabla f(a) \cdot v = (\nabla f(a))^T v$

Gradient Vector Field

- Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$. Assume all partial derivatives of f exists on U. The gradient of f (or gradient vector field of f) is the function

  $\nabla f : U \to \mathbb{R}^n$ given by $\nabla f(a) = (\partial_1 f(a), \dots, \partial_n f(a))$

  for all $a \in U$.

  - $\nabla f(a)$ points in the direction of steepest ascent, orthogonal to tangent line

Direction of Steepest Ascent

- Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}$. Let $a \in A^\circ$.

  Assume f is differentiable at a and $\nabla f(a) \neq 0$, then:

  - The max of $D_u f(a)$ over all unit vectors u occurs when $u = +\frac{\nabla f(a)}{\|\nabla f(a)\|}$ and the max value is $\|\nabla f(a)\|$
  - The min of $D_u f(a)$ over all unit vectors u occurs when $u = -\frac{\nabla f(a)}{\|\nabla f(a)\|}$ and the min value is $-\|\nabla f(a)\|$

- $\nabla f(a)$ points in the direction of steepest ascent at a
- $\|\nabla f(a)\|$ is the rate of change of f in this direction

Orthogonality to Level Sets

- Every tangent vector v at a point p of an implicit surface S is orthogonal to $\nabla f(p)$
  - $\nabla f(p)$ is orthogonal to the tangent plane of S at p

3.5 Differentials and the Jacobian

Definitions

- Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  f is differentiable at a if there exists a linear map

  $L: \mathbb{R}^n \to \mathbb{R}^m$ s.t. $\lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{\|h\|} = 0$

  - L is the differential of f at a, denoted df_a
    - Aka total derivative of f at a
  - The error $f(a+h) - f(a) - L(h)$ tends to 0

    faster than $\|h\|$ tends to the 0 scalar

Properties

- Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  Then f is differentiable at a iff each of its

  component functions $f^1, f^2, \dots, f^m$ is. If so,

  $df_a = (df_a^1, \dots, df_a^m)$

- Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  If f is differentiable at a, then f is cts at a

- Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  If f is differentiable at a, then $\forall v \in \mathbb{R}^n$, $D_v f(a)$ exists

  and $df_a(v) = D_v f(a)$

Matrix of the Differential

- Let $A \subseteq \mathbb{R}^n$ and let $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$.

  Let $a \in A^\circ$. Assume all the partials of f exist at a.

  The Jacobian of f at a is the $m \times n$ matrix $Df(a)$

  given by $Df(a) = [\partial_j f_i(a)]_{i,j}$

  $$= \begin{bmatrix} | & & | \\ \partial_1 f(a) & \cdots & \partial_n f(a) \\ | & & | \end{bmatrix} = \begin{bmatrix} \partial_1 f_1(a) & \cdots & \partial_n f_1(a) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(a) & \cdots & \partial_n f_m(a) \end{bmatrix}$$

  - Aka Jacobian matrix
  - Equivalent notations: $f'(a)$, $Jf(a)$, $J_f(a)$, $Jac_f(a)$

- Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  If f is differentiable at a, then the matrix of the differential of f at a is the $m \times n$ Jacobian matrix of f at a, i.e. $\forall v \in \mathbb{R}^n$, $df_a(v) = Df(a)v$

  - $df_a(v) = Df(a)v = D_v f(a) = \sum_{j=1}^{n} v_j \partial_j f(a)$

3.6 Differentiability

Continuously Differentiable Functions

- Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  f is continuously differentiable at a (or C^1 at a or of class C^1 at a) if $\partial_1 f, \dots, \partial_n f$ are defined on an open set containing a and are all cts at a

- Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}^m$. Let $U \subseteq \mathbb{R}^n$.
  - f is continuously differentiable on U if f is C^1 at every point $a \in U$
  - f is continuously differentiable if f is C^1 on its domain

- Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  f is C^1 at a iff each component function f_i is C^1 at a for all $i \in \{1, \dots, m\}$

- Let $A \subseteq \mathbb{R}^n$. Let $f, g: A \to \mathbb{R}^m$. Let $\phi, \psi: A \to \mathbb{R}$. Fix $\lambda \in \mathbb{R}$ and let $a \in A^\circ$.
  - If f, g are C^1 at a then $f + \lambda g$ is C^1 at a

- If f, g are C^1 at a then $f \cdot g$ is C^1 at a
- If f, ϕ are C^1 at a then ϕf is C^1 at a
- If ϕ, ψ are C^1 at a and $\psi(a) \neq 0$

  then ϕ / ψ is C^1 at a

<u>Differentiability Criterion</u>

- Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. Let $a \in A^\circ$.

  If f is C^1 at a then f is differentiable at a

  - Converse is not true

3.7 Chain Rule

- Let $A \subseteq \mathbb{R}^n$ and let $a \in A^\circ$. Fix $\lambda \in \mathbb{R}$.

  If $f: A \to \mathbb{R}^m$ and $g: A \to \mathbb{R}^n$ are differentiable at a, then $f + \lambda g$ is differentiable at a and

  $d(f + \lambda g)_a = df_a + \lambda dg_a$

- Chain rule: Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open.

  If $f: U \to V$ and $g: V \to \mathbb{R}^k$ are differentiable at $a \in U$ and $f(a) \in V$ respectively, then $g \circ f$ is differentiable at a.

  - $dh_a = dg_{f(a)} \circ df_a$
  - $Dh(a) = Dg(f(a))\,Df(a)$
  - "The differential of a composition is the composition of differentials"

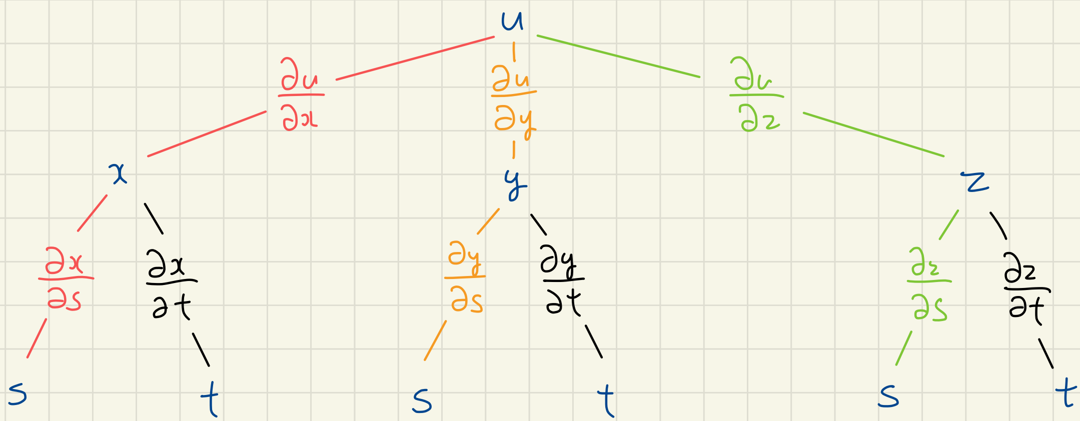
- Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$. If $f: U \to V$ is C^1 and $g: V \to \mathbb{R}^k$ is C^1, then $g \circ f: U \to \mathbb{R}^k$ is C^1

Leibniz Notation and Chain Rule Trees

- $\partial_1 g$ is same as $\dfrac{\partial g}{\partial x}$
- Define $g : \mathbb{R}^3 \to \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}^3$. Define $u = g(x, y, z)$ and $(x, y, z) = f(s, t)$. Then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$



3.8 Local Extrema and Critical Points

Local Extreme Value Theorem

- Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}$. Let $a \in A$.
  - a is a local maximum point of f on a if $\exists \delta > 0$ s.t. $f(a) \geq f(x)$ for all $x \in A \cap B_\delta(a)$
    - $f(a)$ is a local maximum value of f on A
    - f attains a local maximum on A
- Def of local minimum point, local minimum value, attaining a local minimum are similar
- Local extremum: local max or local min
- Local extreme value theorem: Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}$. Let $a \in A^\circ$. If a is a local extremum of f and f is differentiable at a, then $\nabla f(a) = 0$
  - Gives no information on ∂A

Critical Points

- Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}$. A point $a \in A$ is a critical point of f at a if $a \in A^\circ$ and either

$\nabla f(a) = 0$ or $\nabla f(a)$ DNE

- Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}$. If $a \in A$ is a local extremum of f, then either $a \in \partial A$ or a is a critical point of f

3.9 Optimization

1. Determine whether global extrema must exist
   - Use EVT if possible
2. Identify the crit points on the interior of the domain
   - Gradient equals 0
3. Check the boundary for possible extrema
   - Parametrize boundary if possible
4. Plug the candidate points back into the function

3.10 Tangent Space

Tangent Vectors, Spaces, and Planes

- Let $S \subseteq \mathbb{R}^n$ and let $p \in S$. A vector $v \in \mathbb{R}^n$ is a tangent vector of S at p if there exists an open interval $I \subseteq \mathbb{R}$ containing 0 and a differentiable parametric curve $\gamma : I \to \mathbb{R}^n$ w/ $\gamma(I) \subseteq S$, $\gamma(0) = p$, and $\gamma'(0) = v$
    - "There exists a particle moving along S through p w/ velocity v"
- Let $S \subseteq \mathbb{R}^n$ and let $p \in S$. The tangent space of S at p, denoted T_pS, is the set of tangent vectors to S at p
    - $T_pS = \{v \in \mathbb{R}^n : v \text{ is a tangent vector of } S \text{ at } p\}$
    - "T_pS is the set of all possible velocities for a particle moving along S through p"
- Let $S \subseteq \mathbb{R}^n$ and $p \in S$. The tangent plane of S at p, denoted $p + T_pS$, is the tangent space translated to p

- $p + T_pS = \{p+v : v \in T_pS\}$

Tangent Space of a Graph

- Let $S \subseteq \mathbb{R}^n$ be the graph of a function $F : U \to \mathbb{R}^{n-k}$ where $U \subseteq \mathbb{R}^k$ is open. Let $\gamma : I \to \mathbb{R}^n$ be a parametric curve where $I \subseteq \mathbb{R}$ is open. Then γ is differentiable and $\gamma(I) \subseteq S$ iff $\gamma(t) = (g(t), F(g(t)))$ for some differentiable function $g : I \to U$
- Let $S \subseteq \mathbb{R}^n$ be the graph of a differentiable function $F : U \to \mathbb{R}^{n-k}$ where $U \subseteq \mathbb{R}^k$ is open. For any $a \in U$, the tangent space of S at $p = (a, f(a))$ is $T_pS = \{(w, dF_a(w)) : w \in \mathbb{R}^k\}$, and T_pS is a k-dimensional subspace of $\mathbb{R}^n$

3.11 Regular Surfaces

Definition of a Regular Surface

- Let $S \subseteq \mathbb{R}^n$ and let $p \in S$. S is a k-dimensional regular surface at p if $\exists \varepsilon > 0$ s.t. $B_\varepsilon(p) \cap S$ is a graph of a C^1 function $f: U \to \mathbb{R}^{n-k}$ where $U \subseteq \mathbb{R}^k$ is open
  - Choice of f may not be unique
  - A set is not a regular surface at a point when the tangent space fails to be a subspace w/ the expected dimension
- A set $S \subseteq \mathbb{R}$ is a k-dimensional regular surface if S is a k-dimensional regular surface at every point in S
  - Regular curve: 1-dimensional regular surface
- Let $S \subseteq \mathbb{R}^n$ be the graph of a C^1 function $F: U \to \mathbb{R}^{n-k}$ w/ $U \subseteq \mathbb{R}^k$ open. S is a k-dimensional regular surface

Tangent Space of a Regular Surface

- Let $S \subseteq \mathbb{R}^n$ and let $p \in S$. If S is a k-dimensional regular surface at p, then T_pS is a k-dimensional subspace of $\mathbb{R}^n$

4.1 Diffeomorphisms

Global Diffeomorphisms

- Let $U, V \subseteq \mathbb{R}^n$. The (global) inverse of $F: U \to V$ is a map $G: V \to U$ satisfying $G \circ F(x) = x$ for all $x \in U$ and $F \circ G(y) = y$ for all $y \in V$.
  - Equivalently, $\forall x \in U$, $\forall y \in V$,
    $y = F(x) \iff x = G(y)$
  - The inverse of F is unique and is denoted F^{-1}
- Let U, V be open subsets of $\mathbb{R}^n$. A function $F: U \to V$ is a (global) diffeomorphism if F is bijective, F is C^1, and its inverse function $F^{-1}: V \to U$ is C^1

Properties of Diffeomorphisms

- Let U, V be open subsets of $\mathbb{R}$. Assume $F: U \to V$ is bijective. F is a diffeomorphism iff F^{-1} is a diffeomorphism
- Let U, V, W be open subsets of $\mathbb{R}$. If $F: U \to V$ and $G: V \to W$ are diffeomorphisms, then $G \circ F: U \to W$

is a diffeomorphism

- Let U, V be open subsets of $\mathbb{R}^n$. Let $F: U \to V$ be a diffeomorphism. For every $S \subseteq U$:
    1. S is open iff $F(S)$ is open
    2. S is closed iff $F(S)$ is closed
    3. S is cpt iff $F(S)$ is cpt
    4. S is path-connected iff $F(S)$ is path-connected

Local Diffeomorphisms

- Let A, B be open subsets of $\mathbb{R}^n$. Fix $a \in A$. A function $F: A \to B$ is a local diffeomorphism at a if there exists an open subset $U \subseteq A$ containing a s.t. $F(U)$ is open and the restriction $F|_U : U \to F(U)$ is a diffeomorphism.
    - The inverse function $G = F|_U^{-1} : F(U) \to U$ is the local inverse of F at a
- Let A, B be open subsets of $\mathbb{R}^n$. If $F: A \to B$ is a global diffeomorphism, then F is a local diffeomorphism at every $a \in A$.

4.2 Inverse Function Theorem

Derivatives of Diffeomorphisms

- Let U, V be open subsets of $\mathbb{R}^n$. Assume $F: U \to V$ is a diffeomorphism. For every $x \in U$, the Jacobian $DF(x)$ is an invertible $n \times n$ matrix and the Jacobian of the inverse function $G = F^{-1}: V \to U$ satisfies $DG(y) = [DF(x)]^{-1}$ for every $x \in U$ and $y = F(x)$
- Let A, B be open subsets of $\mathbb{R}^n$. Fix $a \in A$. Let $F: A \to B$ be a C^1 function. If F is a local diffeomorphism at a, then the Jacobian $DF(a)$ is an invertible matrix
  - "If a non-linear map is invertible, then its linear approximation must be invertible"

Inverse Function Theorem

- Inverse function theorem: Let A, B be open subsets of $\mathbb{R}^n$. Fix $a \in A$. Let $F: A \to B$ be C^1. If the Jacobian $DF(a)$ is an invertible $n \times n$ matrix, then F is a local diffeomorphism at a.

4.3 Nonlinear Systems

Single Nonlinear Equation

- Let $U \subseteq \mathbb{R}^n \times \mathbb{R}$ be an open set. Let $f: U \to \mathbb{R}$ be C^1.

  Let $(a,b) \in U$ so $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

  Assume $f(a,b) = 0$. The equation $f(x_1, \dots, x_n, y) = 0$

  locally defines y as a C^1 function of $x = (x_1, \dots, x_n)$

  near (a,b) if there exists an open set $V \subseteq \mathbb{R}^n$

  containing a, an open set $W \subseteq \mathbb{R}$ containing b,

  and a C^1 function $\phi: V \to W$ s.t. $V \times W \subseteq U$ and

  $\forall (x,y) \in V \times W$, $f(x,y) = 0 \iff y = \phi(x)$

  - $\{(x,y) \in V \times W : f(x,y) = 0\} = \{(x, \phi(x)) : x \in V\}$

Many Nonlinear Equations

- Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be an open set. Let $F: U \to \mathbb{R}^k$ be C^1.

  Let $(a,b) \in U$ so $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and

  $b \in (b_1, \dots, b_k) \in \mathbb{R}^k$. Assume $F(a,b) = 0$. The equation

  $F(x_1, \dots, x_n, y_1, \dots, y_k) = 0$ locally defines y as a

  C^1 function of x near (a,b) if there exists an open set

$V \subseteq \mathbb{R}^n$ containing a, an open set $W \subseteq \mathbb{R}^k$ containing b, and a C^1 function $\phi : V \to W$ s.t. $V \times W \subseteq U$ and for all $(x,y) \in V \times W$, $F(x,y) = 0 \iff y = \phi(x)$

- Let A be a $k \times n$ matrix and let B be a $k \times k$ matrix. If B is invertible then the system of k linear equations w/ $n+k$ variables

$$[\, A \mid B \,] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ globally defines

$y = (y_1, \dots, y_k) \in \mathbb{R}^k$ as a C^1 function of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

4.4 Implicit Function Theorem

Implicit Differentiation for One Variable

- Let $U \subseteq \mathbb{R}^n \times \mathbb{R}$ be an open set. Let $f: U \to \mathbb{R}$ be C^1.

  Let $(a,b) \in U$ so $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

  Assume $f(a,b) = 0$ and f is not constant.

  If the equation $f(x_1, \dots, x_n, y) = 0$ locally defines y

  as a C^1 function $\phi : V \to W$ of $x = (x_1, \dots, x_n)$ near (a,b)

  then for every point $(v,w) = (v_1, \dots, v_n, w) \in V \times W$ and

  every $j \in \{1, 2, \dots, n\}$,

  $$\frac{\partial f}{\partial x_j}(v,w) + \frac{\partial f}{\partial y}(v,w)\frac{\partial \phi}{\partial x_j}(v) = 0$$

Implicit Function Theorem for One Variable

- Implicit Function Theorem: Let $U \subseteq \mathbb{R}^n \times \mathbb{R}$ be open and

  let $f: U \to \mathbb{R}$ be C^1. Let $(a,b) \in U$ so $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.

  If $f(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) \neq 0$, then the equation

  $f(x,y) = 0$ defines y locally as a C^1 function ϕ of x

  near (a,b)

- "Assume a solution exists to a nonlinear equation. If one can solve an approximate linear equation, then one can locally solve the nonlinear equation."

<u>Implicit Function Theorem for Many Variables</u>

- Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be open. Let $F : U \to \mathbb{R}^k$ be C^1.

  Let $(a,b) \in U$ so $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, $b = (b_1, \dots, b_k) \in \mathbb{R}^k$.

  Assume $F(a,b) = 0$. If the equation $F(x_1, \dots, x_n, y_1, \dots, y_k) = 0$ locally defines y as a C^1 function $\phi : V \to W$ of x near (a,b), then for $(v,w) \in V \times W$, the Jacobian $D\phi(w)$ is a $k \times k$ matrix satisfying

  $$\frac{\partial F}{\partial x}(v,w) + \frac{\partial F}{\partial y}(v,w)\, D\phi(v) = 0$$

  - $\dfrac{\partial F}{\partial x} = \dfrac{\partial(F_1, \dots, F_k)}{\partial(x_1, \dots, x_n)} := \left(\dfrac{\partial F_i}{\partial x_j}\right)_{i,j}$ is a $k \times n$ matrix

    and $\dfrac{\partial F}{\partial y} = \dfrac{\partial(F_1, \dots, F_k)}{\partial(y_1, \dots, y_k)} := \left(\dfrac{\partial F_i}{\partial y_j}\right)_{i,j}$ is a $k \times k$ matrix

  - $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$ are submatrices of the $k \times (n+k)$ Jacobian $DF = \left[\dfrac{\partial F}{\partial x} \;\middle|\; \dfrac{\partial F}{\partial y}\right]$

- **Implicit Function Theorem:** Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be open.

Let $F: U \to \mathbb{R}^k$ be C^1. Let $(a,b) \in U$ so $a \in \mathbb{R}^n$, $b \in \mathbb{R}^k$.

If $F(a,b) = 0$ and the $k \times k$ matrix

$$\frac{\partial F}{\partial y}(a,b) = \frac{\partial(F_1, \dots, F_k)}{\partial(y_1, \dots, y_k)}(a,b) := \left(\frac{\partial F_i}{\partial y_j}(a,b) \right)_{i,j}$$

is invertible, then the equation $F(x,y) = 0$ locally defines

$y = (y_1, \dots, y_k)$ as a $\mathbb{R}^k$-valued C^1 function ϕ of $x = (x_1, \dots, x_n)$

near (a,b)

4.5 Implicit Surfaces

Level Sets and Gradients

- Let $U \subseteq \mathbb{R}^n$ be open. Let $f: U \to \mathbb{R}$ be C^1.

  Assume $S = \{x \in U : f(x) = 0\} = f^{-1}(\{0\})$ is nonempty.

  Fix $p \in S$. If $\nabla f(p) \neq 0$, then S is a $(n-1)$-dimensional regular surface at p.

  - A vector $v \in \mathbb{R}^n$ is a tangent vector of S at p iff $\nabla f(p) \cdot v = 0$, i.e. $T_pS = \{v \in \mathbb{R}^n : \nabla f(p) \cdot v = 0\}$
  - $p + T_pS = \{x \in \mathbb{R}^n : \nabla f(p) \cdot (x - p) = 0\}$
  - "A level set in $\mathbb{R}^n$ is a regular $(n-1)$-dimensional surface wherever the gradient does not vanish, and the gradient is orthogonal to the $(n-1)$-dimensional tangent plane."
  - Converse may not hold

Implicit Surfaces and Kernels

- Fix $k, n \in \mathbb{N}^+$ w/ $k < n$. Let $U \subseteq \mathbb{R}^n$ be open and let $F: U \to \mathbb{R}^k$ be C^1. Assume that $S = F^{-1}(\{0\})$

is nonempty. Fix $p \in S$. If dF_p has full rank, then S is a $(n-k)$-dimensional regular surface at p.

- The tangent space to S at p is a $(n-k)$-dimensional subspace of $\mathbb{R}^n$ given by

$$T_pS = \ker dF_p = \{v \in \mathbb{R}^n : DF(p)v = 0\}$$

4.6 Lagrange Multipliers

Extrema on Subsets

- Let $A \subseteq \mathbb{R}^n$ and let $f: A \to \mathbb{R}$. Let $S \subseteq A$.

  f has a global maximum on S (at a) if

  $\forall x \in S,\ f(x) \le f(a)$.

  f has a local maximum on S (at a) if

  $\exists \varepsilon > 0$ s.t. $\forall x \in S \cap B_\varepsilon(a),\ f(x) \le f(a)$.

  - Global/local minima on S are defined similarly
  - f has a global/local max/min on S at a iff

    $f|_S : S \to \mathbb{R}$ has a global/local max/min at a

- Let $A \subseteq \mathbb{R}^n$ and $f: A \to \mathbb{R}$. Let $S \subseteq A$.

  Assume $a \in S^\circ$. f has a local max on S at a iff

  f has a local max at a.

  - The same equivalence holds for local minima

- A k-dimensional regular surface S in $\mathbb{R}^n$ has empty interior

Lagrange Multipliers with One Constraint

- Lagrange multipliers with one constraint: Let $U \subseteq \mathbb{R}^n$ be open. Let $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ be C^1. Fix $c \in \mathbb{R}$. Let $S = \{x \in U : g(x) = c\}$ and assume $\nabla g(p) \neq 0$ for any $p \in S$. If f has a local extremum on S at a, then there exists $\lambda \in \mathbb{R}$ s.t. $\nabla f(a) = \lambda \nabla g(a)$.
  - λ is the Lagrange multiplier
  - Solutions $a \in U$, $\lambda \in \mathbb{R}$ to the Lagrange system

    $\nabla f(a) = \lambda \nabla g(a)$; $\quad g(a) = c$

    are candidates for local extrema of f on S

Lagrange Multipliers with Many Constraints

- Lagrange multipliers with many constraints: Let $U \subseteq \mathbb{R}^n$ be open. Sps $f : U \to \mathbb{R}$ is differentiable, $g_1, \dots, g_k : U \to \mathbb{R}$ is C^1, and $c_1, \dots, c_k \in \mathbb{R}$

  Let $S = \{x \in U : g_1(x) = c_1, \dots, g_k(x) = c_k\}$.

  Assume for every $p \in S$ that $\nabla g_1(p), \dots, \nabla g_k(p)$ are linearly independent. If a is a local extremum of f on S then there exists $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ s.t. $\nabla f(a) = \sum_{i=1}^{k} \lambda_i \nabla g_i(a)$.

4.7 Optimization with Constraints

- The Lagrange multiplier λ is the rate of change of the optimum value of f as c increases, where $g(x) = c$ is the constraint
- For $c \in \mathbb{R}$, let $x(c) \in \mathbb{R}^n$ be the optimum point and $\lambda(c) \in \mathbb{R}$ be the Lagrange multiplier corresponding to the optimum value of $f(x)$ subject to the constraint $g(x) = c$. If $\lambda(c)$ and $x(c)$ are locally defined as C^1 functions of c, then $\lambda(c) = \frac{d}{dc} f(x(c))$

5.1 Mean Value Theorem

- Mean Value Theorem: Let $U \subseteq \mathbb{R}^n$ be open and let $a, b \in U$. Let $f: U \to \mathbb{R}$ be diff. If U contains the line segment L from a to b then there exists $c \in L$ s.t. $f(b) - f(a) = \nabla f(c) \cdot (b-a)$
    - Does not hold for vector-valued functions
        - Different cs for different components
- Let $U \subseteq \mathbb{R}^n$ be open and C^1 path-connected. Let $F: U \to \mathbb{R}^m$ be diff. $DF(x)$ is the $m \times n$ zero matrix for all $x \in U$ iff F is a constant map
- Let $U \subseteq \mathbb{R}^n$ be open and C^1 path-connected. Let $F: U \to \mathbb{R}^m$ and $G: U \to \mathbb{R}^m$ be diff. If $DF(x) = DG(x)$ for all $x \in U$, then $\exists C \in \mathbb{R}^m$ s.t. $F(x) = G(x) + C$ for all $x \in U$

5.2 Second Order Derivatives

Definition

- Let $U \subseteq \mathbb{R}^n$ be open. Let $f: U \to \mathbb{R}^m$ be C^1.

  Fix $i, j \in \{1, \dots, n\}$ and $a \in U$.

  The second order partial derivative $\partial_i \partial_j f$ at a is

  defined by $\partial_i \partial_j f(a) := \partial_i (\partial_j f)(a)$

  - If $i \neq j$ then the partial is mixed
    - Order matters
  - If $i = j$ then the partial is pure
    - Can write ∂_i^2
  - Notations: $f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$
  - $\partial_1 \partial_2 f(a,b) = \lim_{h \to 0} \frac{\partial_2 f(a+h, b) - \partial_2 f(a,b)}{h}$

Clairaut's Theorem

- Let $U \subseteq \mathbb{R}^n$ be an open set. A function $f: U \to \mathbb{R}^m$ is

  twice continuously differentiable (or C^2) provided

  for all $i, j \in \{1, \dots, n\}$, $\partial_i \partial_j f$ exist and are cts everywhere in U

- Every polynomial is C^2
- Every linear map is C^2

- Clairaut's Theorem: Let $U \subseteq \mathbb{R}^n$ be open and let $f: U \to \mathbb{R}^m$. If f is C^2, then $\forall i,j = \{1,\dots,n\}$, $\partial_i \partial_j f = \partial_j \partial_i f$
  - "Mixed partials commute"
- Let $U \subseteq \mathbb{R}^n$ be open. If $f: U \to \mathbb{R}^m$ is C^2, then for $h = (h_1, \dots, h_n) \in \mathbb{R}^n$, $D_h^2 f(p) := D_h(D_h f)(p)$

$$= \sum_{i=1}^{n} h_i^2 (\partial_i^2 f)(p) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} 2 h_i h_j (\partial_i \partial_j f)(p)$$

Hessian Matrix

- Let f be a real-valued function which is C^2 at $a \in \mathbb{R}^n$. The Hessian of f at a is the $n \times n$ matrix $Hf(a)$ defined by $Hf(a) = [\partial_i \partial_j f(a)]_{i,j}$
  - The Hessian is a symmetric matrix by Clairaut's theorem

5.3 Higher Order Derivatives

Generalized Clairaut's Theorem

- Let $k \in \mathbb{N}^+$. Let $U \subseteq \mathbb{R}^n$ be an open set.

  A function $f: U \to \mathbb{R}^m$ is k-times continuously differentiable (or C^k) provided all of its k^{th} order partials exist and are cts everywhere in U

  - For all $i_1, \dots, i_k \in \{1, \dots, n\}$, the k^{th} partial derivative $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f$ exists and is cts everywhere in U
- A function f is smooth (or C^∞) if it is C^k for every $k \in \mathbb{N}^+$
  - Every polynomial is C^∞
- Generalized Clairaut's Theorem: Let $U \subseteq \mathbb{R}^n$ be open.

  If $f: U \to \mathbb{R}^m$ is C^k, then the mixed partial derivatives of f up to order k commute

  - For any integers $1 \le i_1, i_2, \dots, i_k \le n$ and any reordering $j_1, \dots, j_k$ of $i_1, \dots, i_k$,

    $$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f = \partial_{j_1} \partial_{j_2} \cdots \partial_{j_k} f$$

Chain Rule with Higher Derivatives

- Let $u = u(x,y)$, $x = x(s,t)$, $y = y(s,t)$.

  Assume all variables are C^2

  - $$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

  - $$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s}\frac{\partial u}{\partial s} = \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}\right) + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\right)$$

    $$= \underbrace{\frac{\partial^2 u}{\partial s \partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial s^2}}_{\text{product rule}} + \underbrace{\frac{\partial^2 u}{\partial s \partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial s^2}}$$

    $$= \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\right)}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial s^2}$$

    $$+ \frac{\partial}{\partial y}\underbrace{\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\right)}_{\text{from first derivative}}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial s^2}$$

    $$= \left(\frac{\partial^2 u}{\partial^2 x}\frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial s} + \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial s^2}$$

    $$+ \left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial s}\right)\frac{\partial y}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial s^2}$$

Multi-Index Notation and Polynomials

- For any integers $1 \le i_1, \ldots, i_k \le n$ and a C^k function f,

  we can reorder the partials $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k} f = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n} f$

where $\alpha_\ell \geq 0$ denotes the number of times ∂_ℓ appears on the left side

- $\alpha_1 + \cdots + \alpha_n = k$

- An element $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index.
  The degree of α is the non-negative integer $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
  The factorial $\alpha!$ is the positive integer $\alpha_1! \, \alpha_2! \cdots \alpha_n!$

- Let $U \subseteq \mathbb{R}^n$. Let $f: U \to \mathbb{R}^m$ be C^k. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ w/ degree $|\alpha| \leq k$, define the α-partial derivative by $\partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$

- Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be a multi-index and let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The monomial x^α in the variables $x_1, \ldots, x_n$ is defined by $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

  - x^α has degree equal to $|\alpha|$ in the variables $x_1, \ldots, x_n$

- A polynomial is a finite linear combination of monomials.
  The degree of a polynomial is the maximum of the degree of its monomials (w/ nonzero coefficients)

- Let $\alpha, \beta \in \mathbb{N}^n$ be any multi-indices. Define $f: \mathbb{R}^n \to \mathbb{R}$ by $f(x) = x^\beta$. Then

1. If $\alpha = \beta$ then $\partial^\alpha f(x) = \alpha!$ for $x \in \mathbb{R}^n$

2. If $|\alpha| > |\beta|$ then $\partial^\alpha f(x) = 0$ for $x \in \mathbb{R}^n$

3. If $\alpha \neq \beta$ then $\partial^\alpha f(0) = 0$

- Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial in n variables of degree $\leq k$, so $P(x) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq k}} C_\alpha x^\alpha$ for some constants $C_\alpha \in \mathbb{R}$ w/ $\alpha \in \mathbb{N}^n$ and $|\alpha| \leq k$. Then $C_\alpha = \dfrac{\partial^\alpha P(0)}{\alpha!}$ for every such α.

  - For a single-variable polynomial $P : \mathbb{R} \to \mathbb{R}$ of degree $k \in \mathbb{N}^+$, $P(x) = \sum_{n=0}^{k} \dfrac{P^{(n)}(0)}{n!} x^n$

5.4 Taylor Polynomials

Explicit Formula

- Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$
  - $\alpha! = \alpha_1! \cdots \alpha_n!$
  - $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
- Let $f : B_\varepsilon(a) : \mathbb{R}$ be C^N. The N^{th} Taylor polynomial of f at a is $P_N(x) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \le N}} \frac{\partial^\alpha f(a)}{\alpha!} (x-a)^\alpha$.
  - $$P_N(x) = \sum_{k=0}^{N} \sum_{\substack{\alpha \in \mathbb{N}^n \\ \alpha_1 + \cdots + \alpha_n = k}} \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot \frac{\partial^k f(a)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \cdot (x_1 - a_1)^{\alpha_1} \cdots (x_n - a_n)^{\alpha_n}$$
- Let $f : B_\varepsilon(a) \to \mathbb{R}$ be C^2. For $x \in \mathbb{R}^n$,
  - $P_0(x) = f(a)$
  - $P_1(x) = f(a) + \nabla f(a) \cdot (x-a)$
  - $P_2(x) = f(a) + \nabla f(a) \cdot (x-a) + \frac{1}{2}(x-a)^T Hf(a)(x-a)$

Matching Derivatives

- Let $f: B_\varepsilon(a) \to \mathbb{R}$ be C^N. Then a polynomial P is the N^{th} Taylor polynomial of f at a iff P is a polynomial of degree $\leq N$ s.t. for all multi-indices $\alpha \in \mathbb{N}^n$ w/ $|\alpha| \leq N$,

  $\partial^\alpha f(a) = \partial^\alpha P(a)$

Higher Order Approximations

- Let $\alpha \in \mathbb{N}^n$. If $|\alpha| \geq N+1$ then $\lim_{x \to 0} \frac{x^\alpha}{\|x\|^N} = 0$
- Let $f, g: B_\varepsilon(a) \to \mathbb{R}$. g is an N^{th} order approximation of f at a if $\lim_{x \to a} \frac{f(x) - g(x)}{\|x-a\|^N} = 0$

- Taylor's Theorem: Let $f: B_\varepsilon(a) \to \mathbb{R}$ be C^{N+1}. A polynomial P is the N^{th} order Taylor polynomial of f at a iff P is the unique degree $\leq N$ polynomial which is an N^{th} order approximation of f at a

5.5 Classification of Critical Points

- Let $f: B_\varepsilon(a) \to \mathbb{R}$ be C^2. The quadratic form of f at a is the function $q: \mathbb{R}^n \to \mathbb{R}$ defined by
  $\forall v \in \mathbb{R}^n, \quad q(v) = v^T Hf(a) v$
  - If a is a crit point of f at q is the quadratic form of f at a, then
    $P_2(x) = f(a) + \frac{1}{2} q(x-a)$
  - $$q \begin{cases} \text{always pos} \\ \text{always neg} \\ \text{pos and neg} \end{cases} \text{on } \mathbb{R}^2 \setminus \{0\}, \text{ then}$$
    $$P_2 \text{ will have } \begin{cases} \text{a global maximum} \\ \text{a global minimum} \\ \text{no global extrema} \end{cases} \text{at } a$$

<u>Quadratic Forms</u> integral

- Let A be a real $n \times n$ symmetric matrix. The quadratic form associated to A is the function $q: \mathbb{R}^n \to \mathbb{R}$ given by
  $q(v) = v^T A v$ for $v \in \mathbb{R}^n$.

- Let A be a $n \times n$ real symmetric matrix. Let $q: \mathbb{R}^n \to \mathbb{R}$ be its associated quadratic form. If $v \in \mathbb{R}^n$ is an eigenvector of A w/ eigenvalue $\lambda \in \mathbb{R}$, then $q(v) = \lambda \|v\|^2$
  - $q(v) = v^T A v = v^T(\lambda v) = \lambda(v^T v) = \lambda \|v\|^2$
- Spectral Theorem: Every $n \times n$ real symmetric matrix has an orthogonal basis of eigenvectors w/ real eigenvalues
- Let A be an $n \times n$ real symmetric matrix. Let $q: \mathbb{R}^n \to \mathbb{R}$ be its quadratic form defined by $q(v) = v^T A v$ for $v \in \mathbb{R}^n$. The max and min values of q on the unit sphere S^{n-1} are respectively equal to the max and min eigenvalues of A

Second Derivative Test

- Second Derivative Test: Let $f: B_\varepsilon(a) \to \mathbb{R}$ be C^3. If a is a crit point of f, then f has
  1. A local min if all eigenvalues of $Hf(a)$ are positive
  2. A local max if all eigenvalues of $Hf(a)$ are negative
  3. A saddle point if $Hf(a)$ has a pos and a neg eigenvalue
  - Inconclusive if $Hf(a)$ has a zero eigenvalue and the

other eigenvalues are all pos or all neg

- Cannot say anything about global extrema

- Let $f: B_\varepsilon(p) \to \mathbb{R}^2$ where f is C^3 and $p \in \mathbb{R}^2$.

If p is a crit point of f, then f has

1. A local min if $f_{xx}(p) f_{yy}(p) - (f_{xy}(p))^2 > 0$ and $f_{xx}(p) > 0$

2. A local max if $f_{xx}(p) f_{yy}(p) - (f_{xy}(p))^2 > 0$ and $f_{xx}(p) < 0$

3. A saddle point if $f_{xx}(p) f_{yy}(p) - (f_{xy}(p))^2 < 0$

- $\begin{vmatrix} f_{xx}(p) & f_{xy}(p) \\ f_{xy}(p) & f_{yy}(p) \end{vmatrix} = f_{xx}(p) f_{yy}(p) - (f_{xy}(p))^2 = \lambda_1 \lambda_2$

5.6 Proof of Taylor's Theorem

Another Explicit Formula

- Let f be a real-valued C^k function. The k^{th} iterated directional derivative of f is a map defined by

$$D_h^k f = \underbrace{D_h(D_h(\cdots(D_h f)))}_{k \text{ times}}$$

- Let $f: B_\varepsilon(a) \to \mathbb{R}$ be C^k where $a \in \mathbb{R}^n$. For all $h \in \mathbb{R}^n$,

$$\frac{D_h^k f(a)}{k!} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha$$

- Let $f: B_\varepsilon(a) \to \mathbb{R}$ be C^N where $a \in \mathbb{R}^n$. If P_N is the N^{th} Taylor polynomial of f at a, then for $h \in \mathbb{R}^n$,

$$P_N(a+h) = \sum_{k=0}^{N} \frac{D_h^k f(a)}{k!}$$

  - Follows from the previous equation

Lagrange's Remainder Theorem

- If P_N is the N^{th} Taylor polynomial of f at a, define the N^{th} remainder of f at a by

$$R_N(x) = f(x) - P_N(x)$$

- **Lagrange's Remainder Theorem:** Let $N \in \mathbb{N}$.

  If f is C^{N+1} on an open set D containing the line segment L from $a \in \mathbb{R}^n$ to $a+h \in \mathbb{R}^n$, then there exists a point $\xi \in L$ s.t. the

  N^{th} remainder of f satisfies $R_N(a+h) = \dfrac{D_h^{N+1} f(\xi)}{(N+1)!}$

Completing the Proof

- Let Q be the polynomial in n variables of degree $\leq N$.

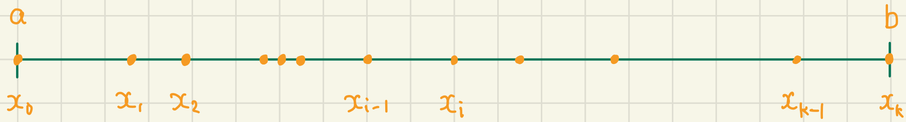
  Then Q is the zero polynomial iff $\lim_{x \to 0} \dfrac{Q(x)}{\|x\|^N} = 0$

  - If Q is nonzero, since its degree is $\leq N$, the limit DNE

6.1 Partitions

Constructing Partitions

- A rectangle in $\mathbb{R}$ is a closed interval $[a,b]$ where $a,b \in \mathbb{R}$ and $a < b$
  - The length of $[a,b]$ is defined to be $\text{length}([a,b]) = b-a$
  - A partition P of $[a,b]$ is a finite set s.t. $\{a,b\} \subseteq P \subseteq [a,b]$
    - Often written as $P = \{x_0, x_1, \ldots, x_k\}$, where it is assumed that $a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$
    - For $1 \leq i \leq k$, the interval $[x_{i-1}, x_i]$ is referred to as a subinterval of $[a,b]$
  - Trivial partition: $P = \{a,b\}$
  - Regular partition: each subinterval has the same length

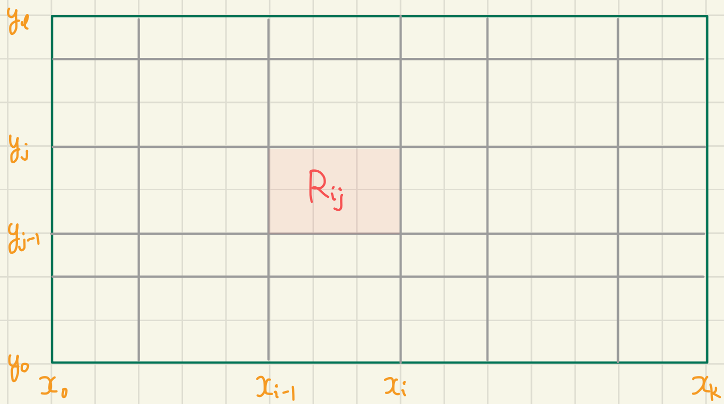


- If $P = \{x_0, \dots, x_k\}$ is a partition of $[a,b]$, then
$$\text{length}([a,b]) = \sum_{i=1}^{k} \text{length}([x_{i-1}, x_i])$$

- A rectangle in $\mathbb{R}^2$ is a set $R = [a,b] \times [c,d]$ where $a,b,c,d \in \mathbb{R}$ and $a < b$ and $c < d$
  - The area of a rectangle is $\text{area}(R) = (b-a)(d-c)$
  - A partition P of the rectangle R is a collection of subrectangles
  $$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad \forall\, 1 \le i \le k,\ 1 \le j \le \ell$$
  where $\{x_0, x_1, \dots, x_k\}$ and $\{y_0, y_1, \dots, y_\ell\}$ are partitions of $[a,b]$ and $[c,d]$ respectively
  - $$\begin{aligned} P = \{R_{ij}\}_{i,j} &= \{R_{ij} : 1 \le i \le k,\ 1 \le j \le \ell\} \\ &= \{[x_0,x_1], \dots, [x_{k-1}, x_k]\} \times \{[y_0,y_1], \dots, [y_{\ell-1}, y_\ell]\} \\ &= \{[x_{i-1}, x_i] : 1 \le i \le k\} \times \{[y_{j-1}, y_j] : 1 \le j \le \ell\} \\ &= \{[x_{i-1}, x_i] \times [y_{j-1}, y_j] : 1 \le i \le k,\ 1 \le j \le \ell\} \end{aligned}$$

- If $P = \{R_{ij} : 1 \le i \le k,\ 1 \le j \le \ell\}$ is a partition of a rectangle $R = [a,b] \times [c,d] \subset \mathbb{R}^2$, then

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell} \text{area}(R_{ij}) = \text{area}(R)$$



- A rectangle in $\mathbb{R}^n$ is a set $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ where $a_1, b_1, \ldots, a_n, b_n \in \mathbb{R}$ and $a_i < b_i$ for all $1 \le i \le n$
    - The volume of a rectangle is

      $$\text{vol}(R) = (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n)$$

    - A partition P of the rectangle R is a collection of subrectangles

      $$R_{i_1,\ldots,i_n} = [x_{1,i_1-1}, x_{1,i_1}] \times \cdots \times [x_{n,i_n-1}, x_{n,i_n}],$$

      $\forall\, 1 \le i_1 \le k_1, \ldots, 1 \le i_n \le k_n$ where for $1 \le j \le n$, $\{x_{j,0}, x_{j,1}, \ldots, x_{j,k_j}\}$ is a partition of the interval $[a_j, b_j]$

        - Can write $P = \{R_i\}_i$ or $P = \{R_i\}_{i \in I}$

where I is a finite set of multi-indices $(i_1, \dots, i_n)$

- Let R be a rectangle in $\mathbb{R}^n$. If $P = \{R_i\}_{i \in I}$ is a partition of R, then $\sum_{i \in I} \text{vol}(R_i) = \text{vol}(R)$

- A partition P of a rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ is regular if P is constructed from a regular partition of the interval $[a_j, b_j]$ for every $1 \le j \le n$

Refining Partitions

- Let $P = \{R_i\}_i$ and $P' = \{R_j'\}_j$ be two partitions of a rectangle R. P' is a refinement of P if for every subrectangle R_j' of P', there is a subrectangle R_i of P s.t. $R_j' \subseteq R_i$

- Let P, P', P'' be partitions of a rectangle $R \subset \mathbb{R}^n$. If P'' is a refinement of P' and P' is a refinement of P, then P'' is a refinement of P

- Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$. Let P' be a partition of R constructed from the partitions P_i' of $[a_i, b_i]$

for $1 \leq i \leq n$. Let P'' be the partition of R constructed from the partitions P_i'' of $[a_i, b_i]$ for $1 \leq i \leq n$. The common refinement of P' and P'' is the partition P of R constructed from the partitions $P_i' \cup P_i''$ of $[a_i, b_i]$ for $1 \leq i \leq n$

- Let P' and P'' be partitions of the rectangle $R \subset \mathbb{R}^n$. If P is the common refinement of P' and P'', then P is a refinement of both P' and P''
- Let P be a partition of a rectangle $R \subset \mathbb{R}^n$. The norm of P, denoted $\|P\|$, is the max diameter of all of its subrectangles
  - The diameter of a rectangle is the max distance between any two points in the rectangle
- Let R be a rectangle in $\mathbb{R}^n$. For every $\delta > 0$, there exists a partition P of R w/ norm $\|P\| < \delta$

6.2 Upper Sums and Lower Sums

Definition of Upper and Lower Sums

- Let R be a rectangle in $\mathbb{R}^2$. Let $f: R \to \mathbb{R}$ be bdd.

  Let $P = \{R_{ij} : 1 \leq i \leq k, 1 \leq j \leq l\}$ be a partition of R.

  The P-lower sum and P-upper sum of f are respectively defined by

  $$L_P(f) = \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} \operatorname{area}(R_{ij}) \qquad U_P(f) = \sum_{i=1}^{k} \sum_{j=1}^{l} M_{ij} \operatorname{area}(R_{ij})$$

  where $\forall 1 \leq i \leq k, 1 \leq j \leq l$,

  $$m_{ij} = \inf_{x \in R_{ij}} f(x) \quad \text{and} \quad M_{ij} = \sup_{x \in R_{ij}} f(x)$$

  - Always defined for a bdd function
  - Upper sum: overestimate, lower sum: underestimate

- Let R be a rectangle in $\mathbb{R}^n$. Let $f: R \to \mathbb{R}$ be bdd.

  Let $P = \{R_i\}_{i \in I}$ be a partition of R where I is a finite set of multi-indices. The P-lower sum of f and P-upper sum of f are respectively

$L_P(f) = \sum_{i \in I} m_i \operatorname{vol}(R_i)$ and $U_P(f) = \sum_{i \in I} M_i \operatorname{vol}(R_i)$

where $\forall i \in I$, $m_i = \inf_{x \in R_i} f(x)$ and $M_i = \sup_{x \in R_i} f(x)$

Properties of Upper and Lower Sums

- Let P be a partition of a rectangle $R \subseteq \mathbb{R}^n$.

  Let $f : R \to \mathbb{R}$ be bdd. Then $L_P(f) \le U_P(f)$

- Let P and P' be partitions of a given rectangle $R \subseteq \mathbb{R}^n$.

  Let $f : R \to \mathbb{R}$ be bdd. If P' is a refinement of P,

  then $L_P(f) \le L_{P'}(f)$ and $U_{P'}(f) \le U_P(f)$

- If P' and P'' are two partitions of a rectangle $R \subseteq \mathbb{R}^n$,

  then $L_{P'}(f) \le U_{P''}(f)$

  - $L_{P'}(f) \le L_P(f) \le U_P(f) \le U_{P''}(f)$ for some

    common refinement P of P' and P''

- Let $P = \{R_i\}_i$ be a partition of a rectangle $R \subset \mathbb{R}^n$.

  Let $f : R \to \mathbb{R}$ and $g : R \to \mathbb{R}$ be bdd. Then:

  1. $U_P(f+g) \le U_P(f) + U_P(g)$

② $U_P(\lambda f) = \lambda U_P(f)$ for any $\lambda > 0$

③ $U_P(-f) = -L_P(f)$

④ If $f \leq g$ on R, then $U_P(f) \leq U_P(g)$

- Lower sum lemmas are similar

Definition of Riemann Sums

- Let R be a rectangle in $\mathbb{R}^n$. Let $f: R \to \mathbb{R}$ be bdd. Let $P = \{R_i\}_{i \in I}$ be a partition of R where I is a finite set of multi-indices. For each $i \in I$, choose a sample point $x_i^* \in R_i$. Then

  $$S_P^*(f) = \sum_{i \in I} f(x_i^*) \operatorname{vol}(R_i)$$ is the Riemann sum for f w/ P and these sample points

  - Tagged partitions: partitions and the associated sample points
    - Each sample point is a "tag" for the corresponding subrectangle
- We don't know whether the Riemann sum is an overestimate or underestimate

Let R be a rectangle in $\mathbb{R}^n$. Let $f: R \to \mathbb{R}$ and $g: R \to \mathbb{R}$

be bdd. Let $P = \{R_i\}_{i \in I}$ be a partition of R where I is a finite set of multi-indices. For each $i \in I$, choose a sample point $x_i^* \in R_i$. Then

① $S_P^*(f+\lambda g) = S_P^*(f) + \lambda S_P^*(g)$ for any $\lambda \in \mathbb{R}$

② If $f \leq g$ on R, then $S_P^*(f) \leq S_P^*(g)$

6.3 Integration Over a Rectangle

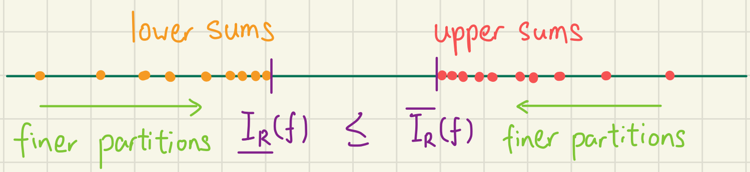
Definition of the Integral

- Let R be a rectangle in $\mathbb{R}^n$. Let $f: R \to \mathbb{R}$ be bdd.

  The lower integral of f on R and the upper integral of f on R are defined by

  $\underline{I_R}(f) = \sup_P L_P(f)$ and $\overline{I_R}(f) = \inf_P U_P(f)$,

  where the supremum and infimum are over all partitions P of the rectangle R



- Let R be a rectangle in $\mathbb{R}^n$. If $f: R \to \mathbb{R}$ is bdd, then both $\underline{I_R}(f)$ and $\overline{I_R}(f)$ exist, and $\underline{I_R}(f) \leq \overline{I_R}(f)$

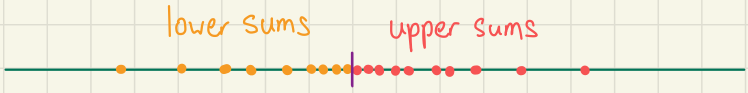
- Let R be a rectangle in $\mathbb{R}^n$. Let $f: R \to \mathbb{R}$ be bdd

  If $\underline{I_R}(f) = \overline{I_R}(f)$, then f is integrable on R and the integral of f on R is defined by

  $\int_R f\,dV := \underline{I_R}(f) = \overline{I_R}(f)$.

  If $\underline{I_R}(f) < \overline{I_R}$, then f is non-integrable

- dV stands for "volume element"
- Integrand is f, not $f(x)$
- This integral is the Darboux integral



$$\underline{I}_R(f) = \overline{I}_R(f)$$

- ε-characterization of integrability: Let R be a rectangle in $\mathbb{R}^n$. Let $f: R \to \mathbb{R}$ be bdd. f is integrable on R iff $\forall \varepsilon > 0$, $\exists$ a partition P of R s.t. $U_P(f) - L_P(f) < \varepsilon$
- Other notations: $\int \cdots \int_R f\,dV$ or $\int_R f$
  - For rectangles in $\mathbb{R}^3$: $\iiint_R f\,dV$
  - For rectangles in $\mathbb{R}^2$: $\int_R f\,dA$ or $\iint_R f\,dA$
    - dA stands for "area element"
  - Do not write $dx\,dy$ instead of dA or $dx_1 \cdots dx_n$ instead of dV

Properties of Integrals Over Rectangles

- Let R be a rectangle in $\mathbb{R}^n$. The identity function 1 is integrable on R and $\int_R 1\,dV = \mathrm{vol}(R)$
- Linearity: Let R be a rectangle in $\mathbb{R}^n$. Let f and g be

bdd functions on R. Let $\lambda \in \mathbb{R}$. If f and g are integrable on R, then $f+\lambda g$ is integrable on R and

$$\int_R (f+\lambda g)\, dV = \int_R f\, dV + \lambda \int_R g\, dV$$

- Monotonicity: Let R be a rectangle in $\mathbb{R}^n$. Let f and g be bdd functions on R. If f and g are integrable on R and $f \leq g$ on R, then $\int_R f\, dV \leq \int_R g\, dV$
- Triangle inequality: Let R be a rectangle in $\mathbb{R}^n$. Let f be a bdd function on R. If f is integrable on R, then $|f|$ is integrable on R and $\left|\int_R f\, dV\right| \leq \int_R |f|\, dV$
- Cauchy-Schwarz inequality: Let R be a rectangle in $\mathbb{R}^n$. Let f and g be bdd functions on R. If f and g are integrable on R, then fg is integrable on R and

$$\int_R fg\, dV \leq \left(\int_R f^2\, dV\right)^{1/2} \left(\int_R g^2\, dV\right)^{1/2}$$

- Additivity: Let R be a rectangle in $\mathbb{R}^n$. Let f be a bdd function on R. Sps $R = R' + R''$ is a union of two subrectangles R' and R'' w/ disjoint interiors. f is integrable on R iff f is integrable on both R' and R'', in which case

$$\int_R f\,dV = \int_{R'} f\,dV + \int_{R''} f\,dV$$

- Let R be a rectangle in $\mathbb{R}^n$. Let $f: R \to \mathbb{R}$ be bdd. Let $P_1, P_2, \dots$ be a sequence of partitions of R s.t. $\|P_N\| \to 0$ as $N \to \infty$. For each partition $P = \{R_i\}_{i \in I}$ in the sequence, pick a sample point $x_i^* \in R_i$ for every $i \in I$. If f is integrable on R, then

$$\int_R f\,dV = \lim_{N \to \infty} S^*_{P_N}(f)$$

6.4 Uniform Continuity and Integration

Uniform Continuity

- $\forall x \in A$, f cts at $x \iff \forall x \in A, \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall y \in A, \|x-y\| < \delta \Rightarrow \|f(x) - f(y)\| < \delta$
  - δ may depend on x
- Let $A \subseteq \mathbb{R}^n$ be a set. A function $f: A \to \mathbb{R}^m$ is uniformly continuous if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in A, \|x-y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$
  - For a subset $S \subseteq A$, f is uniformly continuous on S if $f|_S : S \to \mathbb{R}^m$ is uniformly continuous
- Uniform continuity is a global property of a function
  - Does not depend on any particular point in the domain
- If $f: A \to \mathbb{R}^m$ is uniformly cts, then f is cts
- Let $A \subseteq \mathbb{R}^n$. Let $f: A \to \mathbb{R}^m$. If A is cpt and f is cts, then f is uniformly cts

Continuous Functions are Integrable

- Let R be a rectangle in $\mathbb{R}^n$. Let $f: R \to \mathbb{R}$.

If f is cts on $\mathbb{R}$, then f is integrable on $\mathbb{R}$

- Let $a, b \in \mathbb{R}$ w/ $a < b$. Let $f : [a,b] \to \mathbb{R}$.

  If f is cts on $[a,b]$, then f is integrable on $[a,b]$

6.5 Sets With Zero Jordan Measure

Definition of Zero Volume

- A set $S \subseteq \mathbb{R}$ has zero Jordan measure (or zero volume) if for every $\varepsilon > 0$, there exists finitely many non-trivial intervals $I_1, \dots, I_N \subseteq \mathbb{R}$ s.t.
  $S \subseteq \bigcup_{i=1}^{N} I_i$ and $\sum_{i=1}^{N} \text{length}(I_i) < \varepsilon$
- A set $S \subseteq \mathbb{R}^n$ has zero Jordan measure (or zero volume) if for every $\varepsilon > 0$, there exists finitely many rectangles $R_1, \dots, R_N$ in $\mathbb{R}^n$ s.t.
  $S \subseteq \bigcup_{i=1}^{N} R_i$ and $\sum_{i=1}^{N} \text{vol}(R_i) < \varepsilon$
- Any unbounded set of $\mathbb{R}^n$ does not have zero Jordan measure
- Any set of $\mathbb{R}^n$ with non-empty interior does not have zero Jordan measure

Properties of Zero Volume Sets

- For sets in $\mathbb{R}^n$:
  - Any subset of a zero volume set has zero volume
  - Any finite union of zero volume sets has zero volume

- The closure of a zero volume set has zero volume

- Let $k < n$. Let R be a rectangle in $\mathbb{R}^k$.

  If f is a $\mathbb{R}^n$-valued function that is C^1 on an open set containing R, then $f(R) = \{f(x) : x \in R\}$ has zero Jordan measure in $\mathbb{R}^n$

6.6 Jordan Measurable Sets and Volume

Intro

- Let $S \subseteq \mathbb{R}^n$. The indicator function of S, denoted χ_S, is defined by $\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$

Jordan Measurable Sets

- The set $S \subseteq \mathbb{R}^n$ is Jordan measurable if S is bdd and ∂S has zero Jordan measure
- Let $S, T \subseteq \mathbb{R}^n$ be sets. Then:
  1. If S has zero Jordan measure, then S is Jordan measurable
  2. If S is Jordan measurable, then $\overline{S}$, S°, ∂S are Jordan measurable
  3. If S, T are Jordan measurable, then $S \cup T$ and $S \cap T$ are Jordan measurable
- Let $S \subseteq \mathbb{R}^n$. If S is Jordan measurable, then χ_S is integrable on any rectangle in $\mathbb{R}^n$ containing S

Definition of Volume

- Let S be a Jordan measurable set.

  Define the Jordan measure of S (or volume of S)

  to be $\text{vol}(S) := \int_R \chi_S \, dV$ for a rectangle R containing S

  - For $\mathbb{R}^2$, this can be referred to as the area of S and is denoted as $\text{area}(S)$

- Let $S \subseteq \mathbb{R}^n$ be a Jordan measurable set.

  If R and R' are rectangles each containing S, then χ_S is integrable on R and on R', and

  $\int_R \chi_S \, dV = \int_{R'} \chi_S \, dV$

  - Therefore the Jordan measure is well-defined
  - We can write $\text{vol}(S) = \int \chi_S \, dV = \int_S 1 \, dV$ and drop the rectangle notation

- Let $S, T \subseteq \mathbb{R}^n$ be Jordan measurable sets. Then

  1. If $S \subseteq T$ then $\text{vol}(S) \leq \text{vol}(T)$
  2. $\text{vol}(S \cup T) = \text{vol}(S) + \text{vol}(T) - \text{vol}(S \cap T)$

- Let $S \subseteq \mathbb{R}^n$. If S has zero Jordan measure,

  then $\text{vol}(S) = \int \chi_S \, dV = 0$

Integrability of Indicator Functions

- Let P be a partition of a rectangle $R \subseteq \mathbb{R}^n$.

  Let S be a rectangle lying inside R. Then P **subdivides** S if S can be written as a union of rectangles belonging to P

- Let R be a rectangle in $\mathbb{R}^n$. Let $S_1, \ldots, S_m$ be a finite collection of rectangles lying inside R. Then:

  1. There exists a partition of R subdividing every $S_1, \ldots, S_m$
  2. Let P be a partition of R. If P subdivides every $S_1, \ldots, S_m$ and P' is a refinement of P, then P' also subdivides every $S_1, \ldots, S_m$

6.7 Integration Over a Non-Rectangle

Definition of the Integral

- $\chi_S f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\chi_S f(x) = \begin{cases} f(x) & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

  - Even if f is not defined outside of S, this function is defined everywhere in $\mathbb{R}^n$

- Let $S \subseteq \mathbb{R}^n$ be bdd. Let $f : S \to \mathbb{R}$ be bdd.
  f is integrable on S if the function $\chi_S f : \mathbb{R}^n \to \mathbb{R}$ is integrable on a rectangle containing S.
  If so, the integral of f over S is $\int_S f\,dV := \int_R \chi_S f\,dV$ for a rectangle R containing S

Criteria For Integrability

- Let $S \subseteq \mathbb{R}$ be bdd. Let $f : S \to \mathbb{R}$ be bdd.
  If S is Jordan measurable and the set of discontinuities of f on S has zero Jordan measure, then f is integrable on S

- Let $S \subseteq \mathbb{R}^n$ be bdd. Let $f : S \to \mathbb{R}$ be bdd.
  1. If S has zero volume, then f is integrable on S and $\int_S f dV = 0$
     - Holds for any bdd function f
  2. If $f = 0$ on S except on a set of zero volume, then f is integrable on S and $\int_S f dV = 0$
     - Holds for any bdd set S

<u>Properties of Integrals Over the Set</u>

- Let $S \subseteq \mathbb{R}^n$ be Jordan measurable. Fix $\lambda \in \mathbb{R}$. The constant function λ is integrable on S and $\int_S \lambda dV = \lambda \operatorname{vol}(S)$
- Linearity: Let $S \subseteq \mathbb{R}^n$ be bdd. Let $f, g : S \to \mathbb{R}$ be bdd. Let $\lambda \in \mathbb{R}$. If f and g are integrable on S, then $f + \lambda g$ is integrable on S and $\int_S (f + \lambda g) dV = \int_S f dV + \lambda \int_S g dV$
- Monotonicity: Let $S \subseteq \mathbb{R}^n$ be bdd. Let $f, g : S \to \mathbb{R}$ be bdd. If f and g are integrable on S and $f \leq g$ on S, then $\int_S f dV \leq \int_S g dV$

- Triangle inequality: Let $S \subseteq \mathbb{R}^n$ be bdd.
  Let $f: S \to \mathbb{R}$ be bdd. If f is integrable on S,
  then $|f|$ is integrable on S and $\left| \int_S f\,dV \right| \leq \int_S |f|\,dV$
- Cauchy-Schwarz inequality: Let $S \subseteq \mathbb{R}^n$ be bdd.
  Let $f, g: S \to \mathbb{R}$ be bdd. If f and g are integrable on S,
  then fg is integrable on S and
  $\int_S fg\,dV \leq \left(\int_S f^2\,dV \right)^{1/2} \left(\int_S g^2\,dV \right)^{1/2}$
- Additivity: Let $S \subseteq \mathbb{R}^n$ be bdd. Let $f: S \to \mathbb{R}$ be bdd.
  Sps $S = S' \cup S''$ s.t. $S' \cap S''$ has zero Jordan measure.
  If f is integrable on both S' and S'', then
  f is integrable on S and $\int_S f\,dV = \int_{S'} f\,dV + \int_{S''} f\,dV$
- Let $S \subseteq \mathbb{R}^n$ be Jordan measurable. Let $f, g: S \to \mathbb{R}$ be bdd.
  If $f = g$ on S except on a set of zero volume.
  then f is integrable on S iff g is integrable on S.
  If so, $\int_S f\,dV = \int_S g\,dV$
  - An integral does not change value if it is
    modified on a set of zero volume

Integration on a Rectangle With Few Discontinuities

- Let $R \subseteq \mathbb{R}^n$ be a rectangle. Let $f: R \to \mathbb{R}$ be bdd.

  If the set of discontinuities of f inside R has

  zero Jordan measure, then f is integrable on R

6.8 Volumes, Averages, and Mass

Volume Under a Graph

- Let $S \subseteq \mathbb{R}^n$ be a cpt Jordan measurable set, Let $f: S \to [0, \infty)$ be cts. The $(n+1)$-dimensional set $T = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in S,\ 0 \leq y \leq f(x)\}$ is a cpt Jordan measurable set and satisfies $\mathrm{vol}(T) = \int_S f(x)\, dV$

Averages and Totals

- Let $S \subseteq \mathbb{R}^n$ be a Jordan measurable set with nonzero volume. Let f be integrable on S. The average value of f on S is $\frac{1}{\mathrm{vol}(S)} \int_S f\, dV$
- Total value of f over S $=$ (average value of f on S) $\times$ (volume of S)
- Integral mean value theorem: Let $S \subseteq \mathbb{R}^n$ be Jordan measurable. Let $f: S \to \mathbb{R}$ be cts. If S is cpt and path-connected, then $\exists p \in S$ s.t.
  $\int_S f\, dV = f(p)\, \mathrm{vol}(S)$
- Fix $p \in \mathbb{R}^n$. Let f be a real-valued function.

If f is cts on an open set containing p, then

$$f(p) = \lim_{\varepsilon \to 0^+} \frac{1}{\mathrm{vol}(B_\varepsilon(p))} \int_{B_\varepsilon(p)} f \, dV$$

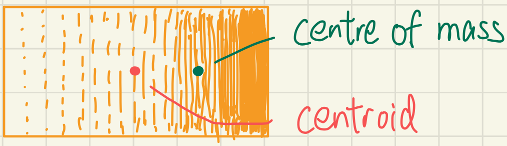
Mass

- Let $\delta : S \to [0, \infty)$ be the density function for an object $S \subseteq \mathbb{R}^n$. Assume S is bdd and δ is integrable on S. Its mass $m = \mathrm{mass}(S)$ and average density ρ are respectively

$$m = \int_S \delta \, dV \qquad \rho = \frac{1}{\mathrm{vol}(S)} \int_S \delta \, dV$$

- Let $\delta : S \to [0, \infty)$ be the density function for an object $S \subseteq \mathbb{R}^n$. Assume S is bdd and δ is integrable on S. Its centre of mass $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \mathbb{R}^n$ is $\bar{x} = \frac{1}{m} \int_S x \, \delta(x) \, dV$ where m is the mass of S.

  Its centroid is the centre of mass assuming density is uniform



- If $f : S \to \mathbb{R}^k$ w/ components $f_1, \ldots, f_k$, then

$$\int_S f \, dV := \left(\int_S f_1 \, dV, \ldots, \int_S f_k \, dV \right) \in \mathbb{R}^k$$

6.9 Probability

Sample Space and Events

- A sample space Ω is a non-empty set that contains all possible outcomes
- The event space Σ is a collection of subsets of Ω
  - Every element $A \in \Sigma$ is an event
- Let $\Omega \subseteq \mathbb{R}^n$ be Jordan measurable. Define the event space Σ to be

  $\Sigma = \{A \subseteq \Omega : A \text{ is Jordan measurable}\}$, then

  1. $\Omega \in \Sigma$
     - Some outcome must happen
  2. If $A \in \Sigma$, then $\Omega \setminus A \in \Sigma$
     - An event either occurs or does not occur
  3. If $A_1, \dots, A_N \in \Sigma$, then $A_1 \cup \cdots \cup A_N \in \Sigma$
     - Any finite union of events is an event

Probability Spaces and Densities

- A probability function $\mathbb{P} : \Sigma \to [0,1]$ is a function assigning each event in the event space a probability between 0 and 1

- If $A \in \Sigma$, $\mathbb{P}(A)$ is the probability that A occurs

- A probability space is the triple $(\Omega, \Sigma, \mathbb{P})$
- Let $\Omega \subseteq \mathbb{R}^n$ be Jordan measurable. Let Σ be all Jordan measurable subsets of Ω.

  Let $\phi : \Omega \to [0, \infty)$ be cts on S except for a set of zero Jordan measure. Assume $\int_\Omega \phi \, dV = 1$.

  For every $A \in \Sigma$, define $\mathbb{P}(A) = \int_A \phi \, dV$. Then:

  1. $\mathbb{P}(\Omega) = 1$
  2. $\mathbb{P}(A)$ exists and $0 \leq \mathbb{P}(A) \leq 1$ for every $A \in \Sigma$
  3. If $A_1, \ldots, A_n \in \Sigma$ are pairwise disjoint, then $\mathbb{P}(A_1 \cup \cdots \cup A_N) = \sum_{i=1}^{N} \mathbb{P}(A_i)$

  - $(\Omega, \Sigma, \mathbb{P})$ is a cts probability space in $\mathbb{R}^n$
  - ϕ is the probability density function of $\mathbb{P} : \Sigma \to [0,1]$

- Let $(\Omega, \Sigma, \mathbb{P})$ be a cts probability space in $\mathbb{R}^n$.

$\mathbb{P}$ is uniform if its PDF $\phi : \Omega \to [0, \infty)$ is constant.

That is, $\phi(x) = \dfrac{1}{\mathrm{vol}(\Omega)}$ for all $x \in \Omega$

Limitations of the Darboux Integral and the Jordan Measure

- Ω is Jordan measurable and hence bdd
- $\Omega \subseteq \mathbb{R}^n$ for a finite $n \in \mathbb{N}^+$
- The event space Σ of Jordan measurable subsets excludes many reasonable sets
- $\mathbb{P}$ satisfies finite additivity, but not countable additivity

7.1 Fubini's Theorem in 2D

Intro

- Let $f:[a,b]\to\mathbb{R}$ be bdd. If f is integrable on $[a,b]$, then $\int_{[a,b]} f\,dV = \int_a^b f(x)\,dx$

Integral of Slices

- Let $f:[a,b]\times[c,d]\to\mathbb{R}$.
  - A x-slice of f is the function $f^x:[c,d]\to\mathbb{R}$ of the form $f^x(y)=f(x,y)$ for some fixed $x\in[a,b]$
  - A y-slice of f is the function $f^y:[a,b]\to\mathbb{R}$ of the form $f^y(x)=f(x,y)$ for some fixed $y\in[c,d]$
- Let $f:[a,b]\times[c,d]\to\mathbb{R}$
  1. If f is bdd, then every slice of f is bdd
  2. If f is cts, then every slice of f is cts
- If $f:[a,b]\times[c,d]\to\mathbb{R}$ is cts, then both F and G defined by $F(x)=\int_c^d f(x,y)\,dy$ and $G(y)=\int_a^b f(x,y)\,dx$ is cts

Iterated Double Integrals

- Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be bdd. The quantities

  $$\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx \quad \text{and} \quad \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

  are iterated double integrals

- There is no obvious relationship between $\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$, $\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$, and $\int_{[a,b]\times[c,d]} f\,dA$

Fubini's Theorem in Two Dimensions

- Fubini's Theorem: Let $R=[a,b]\times[c,d]$ and let $f:R\to\mathbb{R}$ be bdd. For $x\in[a,b]$, define $f^x:[c,d]\to\mathbb{R}$ by $f^x(y)=f(x,y)$. Assume

  ① f^x is integrable on $[c,d]$ for every $x\in[a,b]$

  ② f is integrable on $[a,b]\times[c,d]$

  Then

  ① $\int_c^d f(x,y)\,dy$ exists for every $x\in[a,b]$

  ② $\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$ exists and equals $\iint_R f\,dA$

- Fubini's Corollary: If $f:[a,b]\times[c,d]\to\mathbb{R}$ is cts, then $\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$ and $\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$ both exist and are equal to $\iint_{[a,b]\times[c,d]} f\,dA$

7.2 Fubini's Theorem

Integrals of Slices in 3D

- Let $R = [a,b] \times [c,d] \times [e,f] \subseteq \mathbb{R}^3$. Let $\varphi : R \to \mathbb{R}$.
  A (x,y)-slice of φ is the function $\varphi^{x,y} : [e,f] \to \mathbb{R}$
  of the form $\varphi^{x,y}(z) = \varphi(x,y,z)$ for some fixed
  $x \in [a,b]$, $y \in [c,d]$
  - (y,z)-slice and (x,y)-slice are defined similarly
- Let $R = [a,b] \times [c,d] \times [e,f] \subseteq \mathbb{R}^3$. Let $\varphi : R \to \mathbb{R}$.
  A x-slice of φ is the function $\varphi^x : [c,d] \times [e,f] \to \mathbb{R}$
  of the form $\varphi^x(y,z) = \varphi(x,y,z)$ for some fixed
  $x \in [a,b]$
  - y-slice and z-slice are defined similarly

Iterated Triple Integrals and Fubini's Theorem in 3D

- Let $\varphi : [a,b] \times [c,d] \times [e,f] \to \mathbb{R}$ be bdd.
  The quantity $\int_a^b \int_c^d \int_e^f \varphi(x,y,z)\, dz\, dy\, dx$ is an
  iterated triple integral
- Fubini's Theorem: Let $R = [a,b] \times [c,d] \times [e,f]$.

Let $\varphi : R \to \mathbb{R}$ be bdd. If :

① For every $x \in [a,b]$, $y \in [c,d]$, the (x,y)-slice $\varphi^{(x,y)}$ is integrable on $[e,f]$

② For every $x \in [a,b]$, the x-slice φ^x is integrable on $[c,d] \times [e,f]$

③ φ is integrable on $R = [a,b] \times [c,d] \times [e,f]$

Then the iterated triple integral $\int_a^b \int_c^d \int_e^f \varphi(x,y,z)\,dz\,dy\,dx$ exists and is equal to $\iiint_R \varphi\, dV$

- Let $R = [a,b] \times [c,d] \times [e,f] \subseteq \mathbb{R}^3$.

  If $\varphi : R \to \mathbb{R}$ is cts, then every iterated triple integral of φ on R exists and equals $\iiint_R \varphi\, dV$

<u>Iterated Integrals and Fubini's Theorem in Any Dimension</u>

- Let R be a rectangle in $\mathbb{R}^n$. Let $f : R \to \mathbb{R}$. A function g is a slice of f on R if g is defined by fixing one or more coordinates of f in R

- Let R be a rectangle in $\mathbb{R}^n$. Let $f : R \to \mathbb{R}$.

  ① If f is bdd, then every slice of f is bdd

  ② If f is cts, then every slice of f is cts

- Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$. Let $\varphi : R \to \mathbb{R}$ be bdd.
  The quantity $\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} \varphi(x_1, \dots, x_n)\, dx_n \cdots dx_2\, dx_1$ is an
  iterated n-fold integral
- Fubini's Theorem: Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$.
  Let $f : R \to \mathbb{R}$ be bdd. If f is integrable on R and
  every slice of f on R is integrable on its domain,
  then every iterated integral of f on R exists and
  equal to the integral of f on R, i.e.
  $\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n)\, dx_n \cdots dx_1$ exists and all $n!$ orderings
  of this iterated integral exist and equal to $\int_R f\, dV$
- Let $R \subseteq \mathbb{R}^n$ be a rectangle. If $f : R \to \mathbb{R}$ is cts,
  then every iterated integral of f on R exists and
  equal to the integral of f on R
- Fubini's Theorem: Let $R \subseteq \mathbb{R}^n$ be a rectangle.
  Let $\varphi : R \times [a,b] \to \mathbb{R}$ be bdd. For every $t \in [a,b]$,
  define the slice $\varphi^t : R \to \mathbb{R}$ by $\varphi^t(x) = \varphi(x,t)$.
  If φ is integrable on $R \times [a,b]$, and for every $t \in [a,b]$
  the slice φ^t is integrable on R, then the function

$t \mapsto \int_R \varphi^t \, dV$ is integrable on $[a,b]$ and

$$\int_{R\times[a,b]} \varphi \, dV = \int_a^b \left(\int_R \varphi^t \, dV \right) dt$$

7.3 Double Integrals

- A set $S \subseteq \mathbb{R}^2$ is x-simple if there exists cts $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ s.t. $S = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\ f(x) \leq y \leq g(x)\}$
  - $\iint_S \varphi \, dA = \int_a^b \int_{f(x)}^{g(x)} \varphi(x,y) \, dy \, dx$
- A set $T \subseteq \mathbb{R}^2$ is y-simple if there exists cts $p : [c,d] \to \mathbb{R}$ and $q : [c,d] \to \mathbb{R}$ s.t. $T = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d,\ p(y) \leq x \leq q(y)\}$
  - $\iint_T \varphi \, dA = \int_c^d \int_{p(y)}^{q(y)} \varphi(x,y) \, dx \, dy$
- Can often break non-simple regions into a union of simple pieces
- Strategies for computing double integrals:
  - Sketch the region and describe in several ways
  - Directly calculate with FTC
  - Swap the order of integration w/ Fubini's theorem
  - Apply symmetries of the integrand or region
  - Break up region into smaller pieces
  - Interpret geometrically as a volume of classic object

7.4 Double Integrals in Polar Coordinates

Regions in Polar Coordinates

- The set $A = \{(r,\theta) \in \mathbb{R}^2 : 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$

  is not the same as the set $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

  - Instead, $B = g(A)$, where $g(r,\theta) = (r\cos\theta, r\sin\theta)$

    is the polar coordinate transformation
  - We could informally say "$0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi$

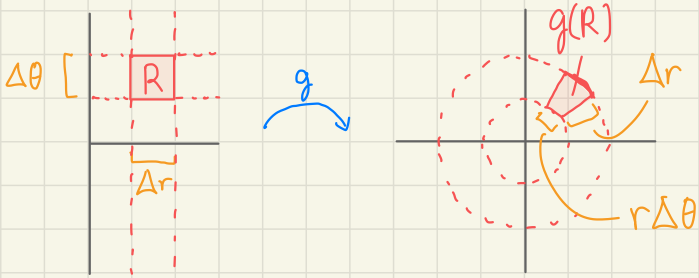
    is the unit disk $x^2 + y^2 \leq 1$ in polar form"
  - If $r < 0$, then the angle becomes $\theta + \pi$

Derivation of Integrals in Polar Coordinates

- For a fixed $r > 0$ and $\theta \in \mathbb{R}$, the small rectangle

  $R = [r, r+\Delta r] \times [\theta, \theta + \Delta\theta]$ in the (r,θ)-plane transforms

  under g to a piece of washer $g(R)$ in the (x,y)-plane



- Let $\Omega \subseteq \mathbb{R}^2$ be a Jordan measurable set s.t. the restricted polar coordinate transformation $g|_\Omega : \Omega \to g(\Omega)$ given by $g(r,\theta) = (r\cos\theta, r\sin\theta)$ is a bijection. If $f: g(\Omega) \to \mathbb{R}$ is integrable on $g(\Omega)$, then $F: \Omega \to \mathbb{R}$ given by $F(r,\theta) = (f \circ g)(r,\theta) \cdot r$ is integrable on Ω and $\iint_{g(\Omega)} f\,dA = \iint_\Omega F\,dA$

7.5 Triple Integrals

- Let $S \subseteq \mathbb{R}^3$ be the region of integration

Projection Method

1. Project S into a plane (e.g. (x,y)-plane) and call the resulting set $T \subseteq \mathbb{R}^2$
2. For any $(x,y) \in T$, $(x,y,z) \in \mathbb{R}^3$ lies in S iff (x,y,z) lies between $z = \gamma_1(x,y)$ and $z = \gamma_2(x,y)$.
   Then $\text{vol}(S) = \iint_T \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} dz\, dA$
3. Integrate over T

Slicing Method

1. Find a slice of S (e.g. z-slice), S_z, for $a \leq z \leq b$
   Then $\text{vol}(S) = \int_a^b \text{area}(S_z)\, dz$
2. Calculate $\text{area}(S_z)$

Using Geometry

- Observe characteristics of the volume, break down into easy-to-calculate pieces if necessary
- Sketch pictures

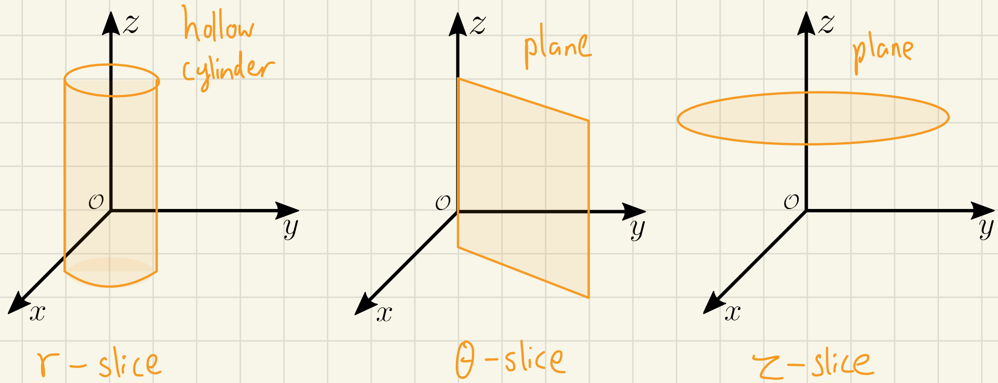
7.6 Triple Integrals in Cylindrical Coordinates

Intro

- Define $g: \mathbb{R}^3 \to \mathbb{R}^3$ as $g(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$
  - Domain: (r,θ,z)-space; codomain: (x,y,z)-space
- Want to integrate over regions in (x,y,z)-space by integrating in (r,θ,z)-space

Regions in Cylindrical Coordinates

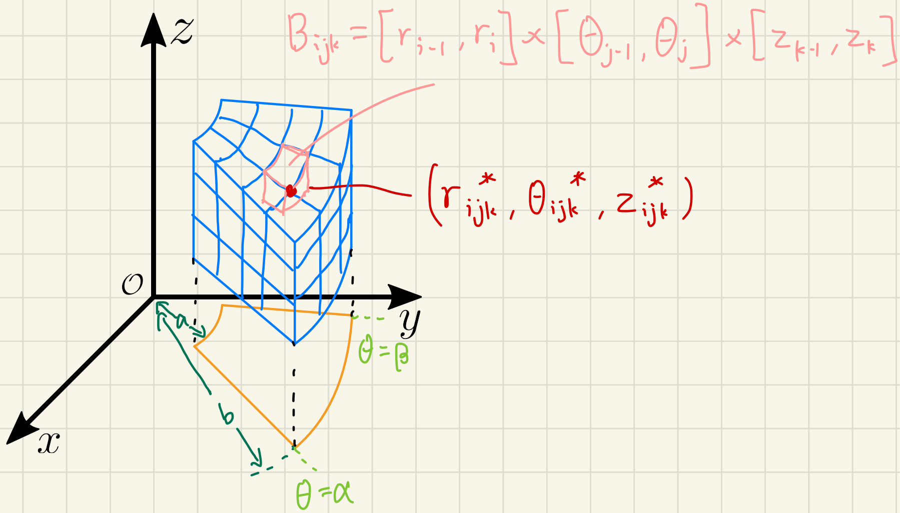
- Slices



Derivation of Integrals in Cylindrical Coordinates

- Let $R = [r, r+\Delta r] \times [\theta, \theta+\Delta\theta] \times [z, z+\Delta z]$
- $\text{vol}(g(R)) \approx r\Delta r \Delta\theta \Delta z = r\,\text{vol}(R)$
- For a rectangle $\Omega = \{(r,\theta,z) : a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$,

the transformed region $D = g(\Omega)$ is



- $$\iiint_{g(\Omega)} f\,dV = \int_c^d \int_\alpha^\beta \int_a^b (f \circ g)(r, \theta, z)\,|r|\,dr\,d\theta\,dz$$

- Let $\Omega \subseteq \mathbb{R}^3$ be Jordan measurable and the restricted cylindrical coordinate transformation $g|_\Omega : \Omega \to g(\Omega)$ is a bijection. If a function $f : g(\Omega) \to \mathbb{R}$ is integrable on $g(\Omega)$, then the function $F : \Omega \to \mathbb{R}$ given by $F(r, \theta, z) = (f \circ g)(r, \theta, z) \cdot |r|$ is integrable on Ω and
$$\iiint_{g(\Omega)} f\,dV = \iiint_\Omega F\,dV$$
    - $dx\,dy\,dz = dV = r\,dr\,d\theta\,dz$

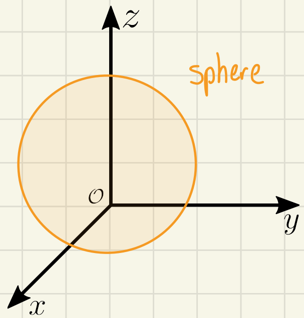
7.7 Triple Integrals in Spherical Coordinates

Intro

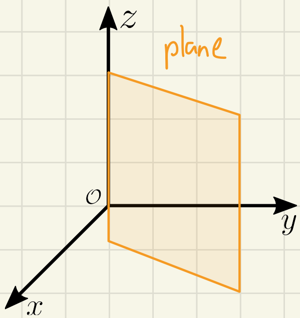
- Define $g: \mathbb{R}^3 \to \mathbb{R}^3$ as $g(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$
  - Domain: (ρ, θ, ϕ)-space; codomain: (x, y, z)-space
- Want to integrate over regions in (x,y,z)-space by integrating in (ρ, θ, ϕ)-space

Regions in Spherical Coordinates

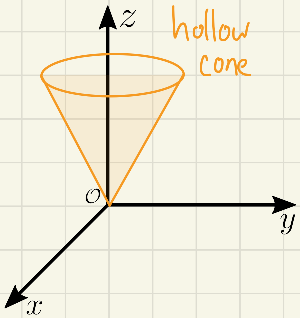
- Slices



ρ-slice



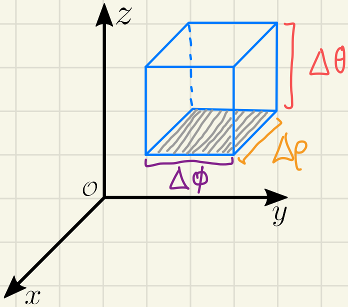
θ-slice

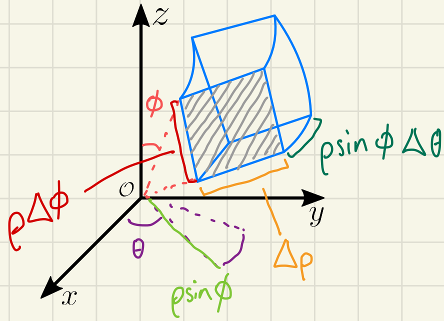


ϕ-slice

Derivation of Integrals in Spherical Coordinates

- Let $R = [\rho, \rho + \Delta\rho] \times [\theta, \theta + \Delta\theta] \times [\phi, \phi + \Delta\phi]$





- $\mathrm{vol}(g(R)) \approx \rho^2 \sin\phi\, \Delta\rho \Delta\theta \Delta\phi = \rho^2 \sin\phi\, \mathrm{vol}(R)$
- For a region $\Omega = [a,b] \times [\alpha,\beta] \times [\lambda,\mu]$,

  $$\iiint_{g(\Omega)} f\,dV = \int_\lambda^\mu \int_\alpha^\beta \int_\lambda^\mu (f \circ g)(\rho,\theta,\phi)\,|\rho^2 \sin\theta|\, d\rho\, d\theta\, d\phi$$
- Let $\Omega \subseteq \mathbb{R}^3$ be Jordan measurable and the restricted spherical coordinate transformation $g|_\Omega : \Omega \to g(\Omega)$ is a bijection. If a function $f : g(\Omega) \to \mathbb{R}$ is integrable on $g(\Omega)$, then the function $F : \Omega \to \mathbb{R}$ given by $F(\rho,\theta,\phi) = (f \circ g)(\rho,\theta,\phi) \cdot |\rho^2 \sin\phi|$ is integrable on Ω and $\iiint_{g(\Omega)} f\,dV = \iiint_\Omega F\,dV$
  - $dx\,dy\,dz = dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

7.8 Change of Variables

Derivation of Change of Variables

- Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear. For every rectangle $R \subseteq \mathbb{R}^n$,
  $\text{vol}(T(R)) = |\det(T)| \, \text{vol}(R)$
  - The "stretch factor" of a linear map is the absolute value of its determinant
- We can linearly approximate a transformed rectangle $g(R)$ where g is a nonlinear map
  - $\forall x \in \mathbb{R}^n, \ \ell_a(x) = g(a) + dg_a(x-a)$
  - $\ell_a(R)$ is a translated parallelogram
  - $\text{vol}(g(R)) \approx \text{vol}(dg_a(R)) = |\det(dg_a)| \, \text{vol}(R)$
    $= |\det Dg(a)| \, \text{vol}(R)$
  - The "stretch factor" of a nonlinear map is the absolute value of its linear approximation determinant

Statement and Consequences

- Change of Variables Theorem: Let $U, V \subseteq \mathbb{R}^n$.
  Let $g : U \to V$ be a diffeomorphism. Let $\Omega \subseteq U$

be cpt and Jordan measurable. The function f is integrable on $g(\Omega)$ iff $(f\circ g)|\det Dg|$ is integrable on Ω. If so, $\int_{g(\Omega)} f\,dV = \int_\Omega (f\circ g)|\det Dg|\,dV$.

- If Fubini's theorem is satisfied for both integrals, then

$$\int\cdots\int_{g(\Omega)} f(x)\,dx_1\cdots dx_n$$

$$= \int\cdots\int_\Omega (f\circ g)(u)\,|\det Dg(u)|\,du_1\cdots du_n$$

- Integration in $\mathbb{R}^n$ does not depend on the choice of coordinate system

- Let $U, V \subseteq \mathbb{R}$ be open sets w/ $[a,b] \subseteq U$.

  Let $g: U \to V$ be C^1 and increasing. If f is integrable on $[g(a), g(b)]$, then $f\circ g$ is integrable on $[a,b]$ and $\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u))\,g'(u)\,du$.

- Let $U, V \subseteq \mathbb{R}^n$ be open. Let $g: U \to V$ be a diffeomorphism.

  For any cpt Jordan measurable set $\Omega \subseteq U$,

  $\mathrm{vol}(g(\Omega)) = \int_\Omega |\det Dg|\,dV$

- Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be invertible and linear. For any cpt Jordan measurable set $\Omega \subseteq \mathbb{R}^n$, $\mathrm{vol}(T(\Omega)) = |\det(T)|\,\mathrm{vol}(\Omega)$.

- Let $U, V \subseteq \mathbb{R}^n$ be open. If $g: U \to V$ is bijective, and $\det Dg(x) \neq 0$ for all $x \in U$, then g is a diffeomorphism.

7.9 Improper Integrals

Local Integrability

- Let $\Omega \subseteq \mathbb{R}^n$. A real-valued function f is locally integrable on Ω if f is integrable on every cpt Jordan measurable subset of Ω.
    - Neither Ω nor f need to be bdd
- Let $\Omega \subseteq \mathbb{R}^n$. If a real-valued function f is cts on Ω, then f is locally integrable on Ω.
- Let $\Omega \subseteq \mathbb{R}^n$ be Jordan measurable. Let $f : \Omega \to \mathbb{R}$ be bdd. If f is integrable on Ω, then f is locally integrable on Ω.

Exhaustions

- Let $\Omega \subseteq \mathbb{R}^n$. A sequence of cpt Jordan measurable sets $\{\Omega_k\}_{k=1}^{\infty}$ is an exhaustion of Ω if $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ and $\forall k \geq 1$, $\Omega_k \subseteq \Omega_{k+1}^{\circ}$
- Let $\Omega \subseteq \mathbb{R}^n$. If there exists an exhaustion of Ω by cpt Jordan measurable sets $\{\Omega_k\}_{k=1}^{\infty}$, then Ω is open

Improper Integrals and Monotone Convergence

- Let $\Omega \subseteq \mathbb{R}^n$. Let $\{\Omega_k\}_{k=1}^{\infty}$ be an exhaustion of Ω by cpt Jordan measurable sets. Let $f : \Omega \to \mathbb{R}$ be locally integrable. The improper integral of f over Ω is given by $\int_\Omega f dV = \lim_{k\to\infty} \int_{\Omega_k} f dV$ provided that the limit does not depend on the choice of exhaustion. If so:
  - The improper integral converges when the limit exists
  - The improper integral diverges when the limit DNE
  - The improper integral diverges to ∞ $(-\infty)$ when the limit is ∞ $(-\infty)$

  If the limit depends on the choice of exhaustion, then the improper integral diverges
- Monotone Convergence Theorem: Let $\Omega \subseteq \mathbb{R}^n$ have an exhaustion by cpt Jordan measurable sets. Let f be a real-valued locally integrable function on Ω.

  If $f > 0$ on Ω then the improper integral $\int_\Omega f dV$ either converges or diverges to ∞
- Let $\Omega \subseteq \mathbb{R}^n$ have an exhaustion by cpt Jordan

measurable sets. If f is integrable on Ω, then the improper integral of f on Ω converges and its value is equal to the integral of f on Ω.

- Let $\Omega \subseteq \mathbb{R}^n$ have an exhaustion by cpt Jordan measurable sets. Let $f: \Omega \to \mathbb{R}$ and $g: \Omega \to \mathbb{R}$ be locally integrable. Fix $\lambda \in \mathbb{R}$. If the improper integrals $\int_\Omega f\,dV$ and $\int_\Omega g\,dV$ both converge, then the improper integral $\int_\Omega (f+\lambda g)\,dV$ converges and
$$\int_\Omega (f+\lambda g)\,dV = \int_\Omega f\,dV + \lambda \int_\Omega g\,dV$$

A Family of Improper Integrals

- Fix $p \in \mathbb{R}$. For the given improper integrals in $\mathbb{R}^n$, one has

$$\int_{\|x\|>1} \frac{1}{\|x\|^p}\,dV \begin{cases} \text{converges if } p > n \\ \text{diverges to } \infty \text{ if } p \leq n \end{cases}$$

$$\int_{0<\|x\|<1} \frac{1}{\|x\|^p}\,dV \begin{cases} \text{diverges to } \infty \text{ if } p \geq n \\ \text{converges if } p < n \end{cases}$$

Basic Comparison Test

- Basic comparison test: Let $\Omega \subseteq \mathbb{R}^n$ be open. Let f and g be real-valued locally integrable functions on Ω.

① If $0 \le f \le g$ on Ω and $\int_\Omega g\,dV$ converges

then $\int_\Omega f\,dV$ converges

② If $0 \le f \le g$ on Ω and $\int_\Omega f\,dV$ diverges

then $\int_\Omega g\,dV$ diverges

Absolute Convergence

- Let $\Omega \subseteq \mathbb{R}^n$. If f is locally integrable on Ω, then $|f|$ is locally integrable on Ω
- Let $\Omega \subseteq \mathbb{R}^n$ have an exhaustion by cpt Jordan measurable sets. Let f be a real-valued function locally integrable on Ω. The improper integral $\int_\Omega f\,dV$ absolutely converges if the improper integral $\int_\Omega |f|\,dV$ converges
- Let $\Omega \subseteq \mathbb{R}^n$ have an exhaustion by cpt Jordan measurable sets. Let f be a real-valued function locally integrable on Ω. If the improper integral $\int_\Omega |f|\,dV$ converges, then the improper integral $\int_\Omega f\,dV$ converges

Proof of the Monotone Convergence Theorem

- Heine-Borel Theorem: Let $A \subseteq \mathbb{R}^n$ be cpt.

  Let $\{V_j\}_{j=1}^{\infty}$ be a sequence of open sets s.t.

  $V_j \subseteq V_{j+1}$ for $j \geq 1$. If $A \subseteq \bigcup_{j=1}^{\infty} V_j$, then

  there exists $k \in \mathbb{N}^+$ s.t. $A \subseteq V_k$

8.1 Parametrized Curves

Simple Regular Parametrizations

- A map $\gamma : [a,b] \to \mathbb{R}^n$ is a parametrization of a set $C \subseteq \mathbb{R}^n$ if $C = \gamma([a,b])$ and γ is cts on $[a,b]$
  - If we momentarily stop ($\gamma'(t) = 0$), then we can rigidly change direction
- A map $\gamma : [a,b] \to \mathbb{R}^n$ is a regular parametrization of a set $C \subseteq \mathbb{R}^n$ when
  1. $C = \gamma([a,b])$ and γ is cts on $[a,b]$
  2. γ is C^1 on (a,b) and $\gamma'(t) \neq 0$ for any $t \in (a,b)$
  - If γ is not injective on its domain, then we could visit the same point multiple times
- A map $\gamma : [a,b] \to \mathbb{R}^n$ is a simple regular parametrization of a set $C \subseteq \mathbb{R}^n$ when
  1. $C = \gamma([a,b])$ and γ is cts on $[a,b]$
  2. γ is C^1 on (a,b) and $\gamma'(t) \neq 0$ for any $t \in (a,b)$
  3. γ is injective except possibly with $\gamma(a) = \gamma(b)$

- If additionally $\gamma(a) = \gamma(b)$, then γ is closed
- If γ satisfies only ① and ③, then γ is a simple parametrization

- If a map $\gamma : [a,b] \to \mathbb{R}^n$ is a simple regular parametrization of a set $C \subseteq \mathbb{R}^n$, then C is a 1-D regular surface at $\gamma(c)$ for every $c \in (a,b)$
  - Does not include endpoints $\{a,b\}$

Curves and Piecewise Curves

- A set $C \subseteq \mathbb{R}^n$ is a (parametrized simple regular) curve if there exists a simple regular parametrization of C
  - The curve is also closed if the parametrization is closed
- Every parametrized simple regular curve is a 1-D regular surface
- A set $C \subseteq \mathbb{R}^n$ is a piecewise (parametrized simple regular) curve if C can be written as a finite union of parametrized simple regular curves $C_1, \dots, C_k$ s.t. $C_i \cap C_j$ is finite for any pair of distinct curves C_i, C_j

Reparametrizations and Orientation

- Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be simple regular parametrizations of a set $C \subseteq \mathbb{R}^n$.
  γ_1 is a reparametrization of γ_2 if there exists a cts invertible $\varphi : [a,b] \to [c,d]$ s.t. φ is C^1 on (a,b) w/ φ' never zero, and $\gamma_1 = \gamma_2 \circ \varphi$
  - If $\varphi' > 0$ on (a,b), then γ_1 has the same orientation as γ_2
  - If $\varphi' < 0$ on (a,b), then γ_1 has the opposite orientation as γ_2
- Let $\gamma_1 : [a,b] \to \mathbb{R}^n$, $\gamma_2 : [c,d] \to \mathbb{R}^n$, $\gamma_3 : [e,f] \to \mathbb{R}^n$ be simple regular parametrizations of a set $C \subseteq \mathbb{R}^n$. Then
  1. Reflexive: γ_1 is a reparametrization of itself
  2. Symmetry: If γ_1 is a reparametrization of γ_2, then γ_2 is a reparametrization of γ_1
  3. Transitive: If γ_1 is a reparametrization of γ_2 and γ_2 is a reparametrization of γ_3, then γ_1 is a reparametrization of γ_3

8.2 Arc Length

Definition and Invariance

- The arc length (or length) of a piecewise curve $C \subseteq \mathbb{R}^n$ is $\ell(C) = \int_a^b \|\gamma'(t)\|\,dt$ where $\gamma : [a,b] \to \mathbb{R}^n$ is a parametrization of C
    - "The distance a particle travels is the integral of its speed"
- Invariance of arc length theorem: Let $C \subseteq \mathbb{R}^n$ be a curve. Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be parametrizations of C. If γ_1 is a reparametrization of γ_2, then $\int_a^b \|\gamma_1'(t)\|\,dt = \int_c^d \|\gamma_2'(t)\|\,dt$

Arc Length Parametrization

- Let $\gamma : [a,b] \to \mathbb{R}^n$ be a parametrization of a piecewise curve $C \subseteq \mathbb{R}^n$. The arc length parameter of γ is the function $s : [a,b] \to [0,\infty)$ given by $s(t) = \int_a^t \|\gamma'(u)\|\,du$, $a \le t \le b$
    - Depends on the parametrization γ

- "Length of γ on the interval $[a,t]$ for $a \le t \le b$"
- "Distance travelled by the particle from time a to time t"

- Let $\gamma : [a,b] \to \mathbb{R}^n$ be a parametrization of a piecewise curve in $\mathbb{R}^n$. The map γ is parametrized by arc length if $\|\gamma'(t)\| = 1$ for $a \le t \le b$
  - In this case we write $\gamma(s)$ instead of $\gamma(t)$

Derivation of Arc Length

- If $\gamma : [a,b] \to \mathbb{R}^n$ parametrizes a piecewise curve $C \subseteq \mathbb{R}^n$, then $\int_a^b \|\gamma'(t)\| \, dt = \sup_P \left\{ \sum_{i=1}^{N} \|\gamma(t_i) - \gamma(t_{i-1})\| \right\}$ where the supremum is over all partitions $P = \{t_0, t_1, \dots, t_N\}$ of $[a,b]$

Line Integrals of Scalar Functions

- Let $C \subseteq \mathbb{R}^n$ be a piecewise curve parametrized by $\gamma : [a,b] \to \mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be bdd. The line integral of f over C is

$$\int_C f \, ds := \int_a^b f(\gamma(t)) \|\gamma'(t)\| \, dt.$$

If this integral exists, then f is integrable on the curve C.

- ds: arc length element
  - "The infinitesimal length of the curve"
- "Net area of the surface traced out by f along C"

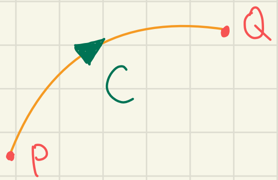
- Invariance of line integrals theorem: Let C be a piecewise curve in $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be bdd. Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be parametrizations of C. $(f \circ \gamma_1)\|\gamma_1'\|$ is integrable on $[a,b]$ iff $(f \circ \gamma_2)\|\gamma_2'\|$ is integrable on $[c,d]$. If so,
$$\int_a^b f(\gamma_1(t))\|\gamma_1'(t)\|\,dt = \int_c^d f(\gamma_2(t))\|\gamma_2'(t)\|\,dt$$
  - We can integrate on a curve in $\mathbb{R}^n$ independent of the choice of parametrization

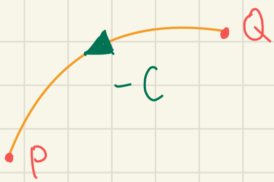
8.3 Line Integrals

Oriented Curves

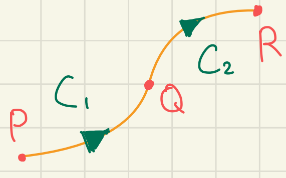
- Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be parametrizations of a curve C.
  1. If γ_1 is a reparam of γ_2 w/ the same orientation, then $\forall s \in (a,b), \forall t \in (c,d)$,
  $$\gamma_1(s) = \gamma_2(t) \Rightarrow \frac{\gamma_1'(s)}{\|\gamma_1'(s)\|} = \frac{\gamma_2'(t)}{\|\gamma_2'(t)\|}$$
  2. If γ_1 is a reparam of γ_2 w/ the opposite orientation, then $\forall s \in (a,b), \forall t \in (c,d)$,
  $$\gamma_1(s) = \gamma_2(t) \Rightarrow \frac{\gamma_1'(s)}{\|\gamma_1'(s)\|} = -\frac{\gamma_2'(t)}{\|\gamma_2'(t)\|}$$
  - "The unit tangent vector of two parametrizations w/ the same orientation remains the same at every point along the curve"
- An oriented curve C is a set of parametrizations that are reparams of each other w/ the same orientation
  - Formally a set of maps, informally a set traced out by a parametrization w/ a direction

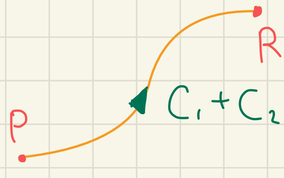
- Let C be an oriented curve parametrized by $\gamma : [a,b] \to \mathbb{R}^n$.
  The oppositely oriented curve $-C$ is the set of parametrizations that are reparams of γ w/ the opposite orientation





- Let C_1 and C_2 be oriented curves in $\mathbb{R}^n$.
  The concatenation of C_1 and C_2, denoted $C = C_1 + C_2$, is the set of cts maps $\gamma : [a,b] \to \mathbb{R}^2$ s.t. $\exists c \in (a,b)$ where $\gamma|_{[a,c]}$ and $\gamma|_{[c,b]}$ are parametrizations of C_1 and C_2 respectively





  - Can concatenate any number of curves
  - $C_1 - C_2$ is equivalent to $C_1 + (-C_2)$
- A piecewise closed curve in $\mathbb{R}^n$ is the concatenation of finitely many oriented curves in $\mathbb{R}^n$

Line Integrals of Vector Fields

- Let C be an oriented curve parametrized by $\gamma : [a,b] \to \mathbb{R}^n$ w/ unit tangent vector T. Let $F : C \to \mathbb{R}^n$. The line integral of F over C is given by
$$\int_C F \cdot T ds := \int_a^b F(\gamma(t)) \cdot T(t) \|\gamma'(t)\| dt$$
provided this integral exists
    - Equivalently, this is the work done by F along the curve C
    - $\int_C F \cdot T ds = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt$
    - $\int_C F \cdot T ds = \int_C F \cdot d\gamma$ or $\int_C F \cdot dr$
        - We can treat $d\gamma$ as $\gamma'(t) dt$
        $= (\gamma_1'(t)dt, \dots, \gamma_n'(t)dt)$
    - $\int_C F \cdot d\gamma = \int_C F_1 dx_1 + F_2 dx_2 + \cdots + F_n dx_n$
        - Think of $d\gamma$ as $(dx_1, \dots, dx_n)$ where $dx_j = \gamma_j'(t) dt$
- Let C, C_1, C_2 be oriented curves in $\mathbb{R}^n$. Let F, G be oriented curves in $\mathbb{R}^n$ defined on C, C_1, C_2. Then
    1. $\int_{-C} F \cdot T ds = - \int_C F \cdot T ds$

② $$\int_C (F + \lambda G) \cdot T\,ds = \int_C F \cdot T\,ds + \lambda \int_C G \cdot T\,ds$$

for $\lambda \in \mathbb{R}$

③ $$\int_{C_1 + C_2} F \cdot T\,ds = \int_{C_1} F \cdot T\,ds + \int_{C_2} F \cdot T\,ds$$

8.4 Fundamental Theorem of Line Integrals

Statement and Proof

- Fundamental theorem of line integrals: Let C be an oriented piecewise curve in $\mathbb{R}^n$ parametrized by $\gamma : [a, b] \to \mathbb{R}^n$. Let f be a real-valued function that is C^1 on an open set containing C. Then $\int_C \nabla f \cdot d\gamma = f(\gamma(b)) - f(\gamma(a))$
    - The line integral does not depend on the path
- The line integral of a gradient vector field over a closed curve evaluates to 0

Conservative Vector Fields and Potentials

- A vector field F is conservative on an open set $U \subseteq \mathbb{R}^n$ if there exists a function $f : U \to \mathbb{R}$ s.t. $F = \nabla f$ on U
    - f is the potential function or scalar potential of F
    - F is a gradient vector field

- If F is a conservative vector field,

  then $\int_C F \cdot dr$ only depends on the endpoints of C

- To determine whether F is conservative, assume that

  it has potential f, then solve the system of PDE

  $\frac{\partial f}{\partial x} = F_1(x,y)$, $\frac{\partial f}{\partial y} = F_2(x,y)$

<u>Irrotational Vector Fields</u>

- A C^1 vector field $F = (F_1, \dots, F_n)$ is irrotational on an

  open set $U \subseteq \mathbb{R}^n$ if $\forall 1 \leq i < j \leq n$, $\partial_i F_j = \partial_j F_i$ on U

- Let F be a vector field in $\mathbb{R}^n$ that is C^1 on an

  open set U. If F is conservative on U, then

  F is irrotational on U

8.5 Conservative Vector Fields

Physical Viewpoints

- If the work done by the vector field does not depend on the path taken by the particle, then the field is path-independent

Path Independence

- A vector field is conservative iff it satisfies path independence
- Let $F: \mathbb{R}^n \to \mathbb{R}^n$ that is cts on an open path-connected set $U \subseteq \mathbb{R}^n$. The following are equivalent:
  1. $\exists f: U \to \mathbb{R}$ s.t. f is C^1 and $F = \nabla f$
  2. $\int_{C_1} F \cdot d\gamma = \int_{C_2} F \cdot d\gamma$ for oriented piecewise curves $C_1, C_2 \subseteq U$ with the same endpoints
  3. $\int_C F \cdot d\gamma = 0$ for any closed piecewise curve $C \subseteq U$
  - Gradient, path-independent, and conservative vector fields are equivalent

Irrotational Vector Fields and Sets Without Holes

- Poincaré's lemma: Let $F: \mathbb{R}^n \to \mathbb{R}^n$ that is C^1 on the open set $U \subseteq \mathbb{R}^n$. If U is convex and F is irrotational on U, then F is conservative on U

- Jordan curve theorem: Let $C \subseteq \mathbb{R}^2$ be a simple closed curve. Then C divides $\mathbb{R}^2$ into two regions:

  ① An open bdd region Ω (the inside)

  ② An unbdd region $\mathbb{R}^2 \setminus \Omega$ (the outside)

  Moreover, Ω is Jordan measurable and $\partial\Omega = C$
- A set $D \subseteq \mathbb{R}^2$ is a simply connected domain if D is open, path-connected, and, for every simple closed curve lying in D, its inside is a subset of D
- A set $D \subseteq \mathbb{R}^3$ is a simply connected domain if D is open, path-connected, and any simple closed curve can shrink continuously to a point while staying entirely in D
- Let F be a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ that is C^1 on the open path-connected set D. If D is simply connected and F is irrotational on D, then F is conservative on D
  - "Irrotational vector fields are conservative if the vector field domain has no loop holes"

8.6 Circulation and Flux in 2D

Circulation and Curl in 2D

- Let $F : C \to \mathbb{R}^2$ where $C \subseteq \mathbb{R}^2$ is an oriented curve. Assume C is simple and closed. The circulation of F along C is $\int_C F \cdot T\, ds$
  - The work done by F along a closed curve C is identical to the circulation of F along C
- Let $F = (F_1, F_2)$ be a C^1 vector field in $\mathbb{R}^2$. The curl of F is the cts real-valued function $\operatorname{curl}(F) = \partial_1 F_2 - \partial_2 F_1$
  - F is irrotational iff $\operatorname{curl}(F) = 0$, such vector fields are curl-free
- Let $F : \mathbb{R}^2 \to \mathbb{R}^2$. Fix $p \in \mathbb{R}^2$. If F is C^1 on a neighbourhood of p, then

$$(\operatorname{curl} F)(p) = \lim_{\varepsilon \to 0^+} \frac{1}{\operatorname{area}(B_\varepsilon(p))} \oint_{\partial B_\varepsilon(p)} (F \cdot T)\, ds$$

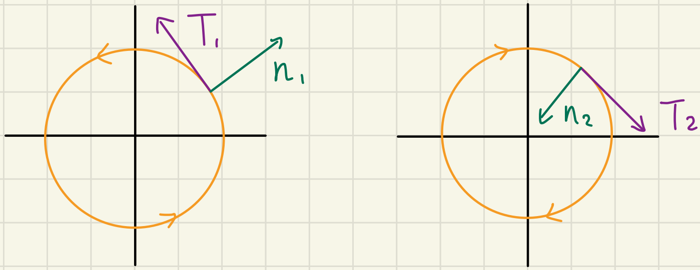
where $\partial B_\varepsilon(p)$ is the circle oriented counterclockwise

  - "Curl is infinitesimal circulation"

Unit Normal and Positive Orientation

- Let C be an oriented closed curve in $\mathbb{R}^2$ parametrized by $\gamma: [a,b] \to \mathbb{R}^n$ w/ unit tangent vector T.

  The unit normal of C is the cts function $n: [a,b] \to \mathbb{R}^2$ s.t. for every $t \in (a,b)$, $n(t)$ is a unit vector orthogonal to $T(t)$ and $\{n(t), T(t)\}$ is a positively oriented ordered basis for $\mathbb{R}^2$



- Let C be an oriented closed curve in $\mathbb{R}^2$

  1. C is positively oriented if the unit normal n of C points outward, or the unit tangent T traverses C counterclockwise

  2. C is negatively oriented if the unit normal n of C points inward, or the unit tangent T traverses C clockwise

Flux and Divergence in 2D

- Let $F : C \to \mathbb{R}^2$ where $C \subseteq \mathbb{R}^2$ is an oriented curve. Assume C is simple and closed. The flux of F across C is $\int_C F \cdot n\, ds$
  - Outward flux corresponds to a curve C oriented counterclockwise
  - Inward flux corresponds to a curve C oriented clockwise
  - Measures how much F aligns w/ the unit normal n of a curve C
- Let $F = (F_1, F_2)$ be a C^1 vector field in $\mathbb{R}^2$. The divergence of F is the cts real-valued function $\operatorname{div}(F) = \partial_1 F_1 + \partial_2 F_2$
  - Equivalent notation: $\nabla \cdot F$
  - If G is a C^1 vector field in $\mathbb{R}^n$, then $\operatorname{div}(G) = \partial_1 G_1 + \cdots + \partial_n G_n$
- A point $p \in \mathbb{R}^2$ is a source if $\operatorname{div}(F)(p) > 0$ and a sink if $\operatorname{div}(F)(p) < 0$

- A vector field is sourceless if $\operatorname{div}(F) = 0$
- Let $F: \mathbb{R}^2 \to \mathbb{R}^2$. Fix $p \in \mathbb{R}^2$. If F is C^1 on a neighbourhood of p, then

$$(\operatorname{div} F)(p) = \lim_{\varepsilon \to 0^+} \frac{1}{\operatorname{area}(B_\varepsilon(p))} \oint_{\partial B_\varepsilon(p)} (F \cdot n)\, ds$$

where $\partial B_\varepsilon(p)$ is a positively oriented circle

  - "Divergence is infinitesimal flux"

8.7 Green's Theorem and Curl

Regular Regions and Orienting the Boundary

- A cpt Jordan measurable set $R \subseteq \mathbb{R}^2$ is a regular region if $\overline{R^0} = R$
  - "A regular region is a cpt set whose boundary always touches the interior"
- Let $R \subseteq \mathbb{R}^2$ be a regular region whose boundary ∂R is a piecewise curve. The boundary ∂R is positively oriented (resp. negatively oriented) if the unit normal along the curve points outward away from R (resp. inward towards R). That is, the region always stays to the left (resp. right) as one traverses the boundary

Statement and Proof

- Green's theorem - curl form: Let F be a vector field in $\mathbb{R}^n$ that is C^1 on a regular region $R \subseteq \mathbb{R}^2$.
  If ∂R is a positively oriented piecewise curve, then
  $\oint_{\partial R} (F \cdot T)\, ds = \iint_R \operatorname{curl}(F)\, dA$

- If C is negatively oriented, then apply Green's theorem with $-C$
- " The total infinitesimal circulation over R is the circulation along its boundary ∂R "

8.8 Green's Theorem and Divergence

Regular Regions and Orienting the Boundary

- Right hand rule: thumb points out of the page,
  index finger: n, middle finger: T
- The boundary of a regular region is positively oriented if its unit normal points outward as we traverse the boundary

Statement and Proof

- Green's theorem - divergence form: Let F be a vector field in $\mathbb{R}^2$ that is C^1 on a regular region $R \subseteq \mathbb{R}^2$. If ∂R is a positively oriented piecewise curve, then
  $$\oint_{\partial R} (F \cdot n)\, ds = \iint_R \operatorname{div}(F)\, dA$$
  - "The total infinitesimal flux over R is the flux across ∂R"

9.1 Parametrized Surfaces

Simple Regular Parametrizations

- Let $S \subseteq \mathbb{R}^3$. Let $U \subseteq \mathbb{R}^2$ be cpt. A map $G : U \to \mathbb{R}^3$ is a (2-variable) parametrization of S if $\text{img}(G) = S$, and G is cts
- Let $U \subseteq \mathbb{R}^2$ be Jordan measurable and cpt. A map $G : U \to \mathbb{R}^3$ is regular if G is C^1 and $\{\partial_1 G, \partial_2 G\}$ is linearly independent at every point in U except for a set of zero Jordan measure in $\mathbb{R}^2$.
- Let $U \subseteq \mathbb{R}^2$ be Jordan measurable and cpt. A map $G : U \to \mathbb{R}^3$ is simple if G is injective on U except possibly along the boundary, i.e.

  $\forall x, y \in U,\ G(x) = G(y) \Rightarrow x = y$ or $x, y \in \partial U$
- Let $U \subseteq \mathbb{R}^2$. If a map $G : U \to \mathbb{R}^3$ is a simple regular parametrization of a set $S \subseteq \mathbb{R}^3$, then S is a 2D regular surface at $G(c)$ for every $c \in U^\circ$

Surfaces and Piecewise Surfaces

- A set $S \subseteq \mathbb{R}^3$ is a (parametrized simple regular) surface in $\mathbb{R}^3$ if there exists a simple regular two-variable parametrization of S
- A set $S \subseteq \mathbb{R}^3$ is a piecewise (parametrized simple regular) surface if S can be constructed by glueing together finitely many parametrized simple regular surfaces along their boundaries

Reparametrizations and Orientation

- Let $G: U \to \mathbb{R}^3$ and $H: V \to \mathbb{R}^3$ be simple regular parametrizations of a set $S \subseteq \mathbb{R}^3$. Define G to be a reparametrization of H if there exists a cts invertible $\varphi: U \to V$ s.t. φ is C^1 on U° w/ $\det D\varphi$ never zero, and $G = H \circ \varphi$
  1. If $\det D\varphi > 0$ on U°, then G has the same orientation as H
  2. If $\det D\varphi < 0$ on U°, then G has the opposite orientation as H

- Let $G_1 : U_1 \to \mathbb{R}^n$, $G_2 : U_2 \to \mathbb{R}^n$, $G_3 : U_3 \to \mathbb{R}^n$ be simple regular 2-variable parametrizations of a set $S \subseteq \mathbb{R}^3$. Then

  1. Reflexive: G_1 is a reparam of itself
  2. Symmetry: If G_1 is a reparam of G_2, then G_2 is a reparam of G_1
  3. Transitive: If G_1 is a reparam of G_2 and G_2 is a reparam of G_3, then G_1 is a reparam of G_3

9.2 Surface Area

Surface Area

- Let $S \subseteq \mathbb{R}^3$ be a surface parametrized by $G : U \to \mathbb{R}^3$.

  The surface area of S is defined as

  $A(S) = \iint_U \| \partial_1 G \times \partial_2 G \| \, dA$ provided the integral exists

  - The cross product can be computed by

    $$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

  - $\| a \times b \|$ represents the area of the parallelogram defined by a and b

  - Properties of the cross product: let $a, b, c \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$

    1. $a \times b = - b \times a$
    2. $a \times \lambda b = \lambda a \times b = \lambda (a \times b)$
    3. $(a \times b) \times c = a \times (b \times c)$
    4. $a \times (b + c) = (a \times b) + (a \times c)$

- Invariance of surface area theorem: Let $G : U \to S$ and

$H : V \to S$ be parametrizations of the surface $S \subseteq \mathbb{R}^3$.

Assume G is a reparam of H. The function $\|\partial_1 G \times \partial_2 G\|$ is integrable on U iff $\|\partial_1 H \times \partial_2 H\|$ is integrable on V.

If so, $\iint_U \|\partial_1 G \times \partial_2 G\| \, dA = \iint_V \|\partial_1 H \times \partial_2 H\| \, dA$

- The surface element is $dS = \|\partial_1 G \times \partial_2 G\| \, dA$ where $G : U \to \mathbb{R}^3$ is a parametrization of a surface $S \subseteq \mathbb{R}^3$
  - Represents the area of a small piece of S
  - $A(S) = \iint_S 1 \, dS$
  - The surface S and the S in dS have no relation

Surface Integrals of Scalar Functions

- Let $S \subseteq \mathbb{R}^3$ be a surface parametrized by $G : U \to \mathbb{R}^3$. Let $f : S \to \mathbb{R}$ be bdd. The (scalar) surface integral of f over S is given by

  $\iint_S f \, dS := \iint_U (f \circ G) \|\partial_1 G \times \partial_2 G\| \, dA$

  If this integral exists, then f is integrable on the surface S

- Invariance of scalar surface integrals theorem: Let S be a surface in $\mathbb{R}^3$. Let $G : U \to \mathbb{R}^3$ and $H : V \to \mathbb{R}^3$ be parametrizations of S. Let $f: S \to \mathbb{R}$ be bdd.

The function $(f \circ G) \| \partial_1 G \times \partial_2 G \|$ is integrable on U iff $(f \circ H) \| \partial_1 H \times \partial_2 H \|$ is integrable on V. If so

$$\iint_U (f \circ G) \| \partial_1 G \times \partial_2 G \| \, dA = \iint_V (f \circ H) \| \partial_1 H \times \partial_2 H \| \, dA$$

9.3 Orientation and Boundary of Surfaces

Unit Normal

- Let $G: U \to \mathbb{R}^3$ be a parametrization of a surface in $\mathbb{R}^3$.
  The unit normal (of the parametrization G) is given by
  $\dfrac{\partial_1 G \times \partial_2 G}{\|\partial_1 G \times \partial_2 G\|}$, which is a C^1 function defined on $U \subseteq \mathbb{R}^2$
  except for a set of zero Jordan measure

- Let $G: U \to \mathbb{R}^3$ and $H: V \to \mathbb{R}^3$ be parametrizations of a surface $S \subseteq \mathbb{R}^3$. Assume G is a reparam of H with $\varphi: U \to V$ satisfying $G = H \circ \varphi$.

  ① If G is a reparam of H with the same orientation,
  then $$\frac{(\partial_1 G \times \partial_2 G)(u,v)}{\|(\partial_1 G \times \partial_2 G)(u,v)\|} = \frac{(\partial_1 H \times \partial_2 H)(s,t)}{\|(\partial_1 H \times \partial_2 H)(s,t)\|}$$

  ② If G is a reparam of H with the opposite orientation,
  then $$\frac{(\partial_1 G \times \partial_2 G)(u,v)}{\|(\partial_1 G \times \partial_2 G)(u,v)\|} = -\frac{(\partial_1 H \times \partial_2 H)(s,t)}{\|(\partial_1 H \times \partial_2 H)(s,t)\|}$$

  for $(u,v) \in U$ and $(s,t) = \varphi(u,v) \in V$ except for a zero Jordan measure subset of U

Oriented Surfaces

- An oriented surface S is a set of (two-variable regular simple) parametrizations that are reparams of each other with the same orientation
  - May abuse notations by having S to also denote the set of points in $\mathbb{R}^3$
- Let S be an oriented surface in $\mathbb{R}^3$. A unit normal $n : S \to S^2$ is a cts function from $S \subseteq \mathbb{R}^3$ to the set of unit vectors S^2 in S^3 which, for any parametrization $G : U \to \mathbb{R}^3$ of the oriented surface S, is defined by
$$n(G(u,v)) = \frac{(\partial_1 G \times \partial_2 G)(u,v)}{\|(\partial_1 G \times \partial_2 G)(u,v)\|} \text{ for all } (u,v) \in U$$
aside from a zero Jordan measure subset of U
  - We could write $n(u,v)$ or n
- There are non-orientable surfaces, e.g. Mobius strip

Relative Boundary of a Surface

- Let S be a surface in $\mathbb{R}^3$.
  1. A point $p \in S$ is a (relative) boundary point of S if there exists an open set $V \subseteq \mathbb{R}^3$ containing p,

an open set $U \subseteq \mathbb{R}^2$, a cts invertible map $\varphi : U \cap \{(x,y) \in \mathbb{R}^2 : y \geq 0\} \to V \cap S$ s.t. φ^{-1} is cts and $\varphi^{-1}(p)$ lies on the x-axis

② The (relative) boundary of S, denoted ∂S, is the set of its relative boundary points

9.4 Surface Integrals

Surface Integrals and Vector Fields

- Let S be an oriented surface in $\mathbb{R}^3$ parametrized by $G: U \to \mathbb{R}^3$ w/ unit normal n. Let F be a vector field on S. The surface integral of F over S is given by $\iint_S F \cdot n \, dS := \iint_U (F \circ G) \cdot (\partial_1 G \times \partial_2 G) \, dA$ provided it exists
  - Equivalently, this is the flux of F across the surface S (in the n direction)

Basic Properties

- Invariance of flux theorem: Let S be an oriented surface in $\mathbb{R}^3$ w/ unit normal n. Let F be a vector field defined on S. Let $G: U \to \mathbb{R}^3$ and $H: V \to \mathbb{R}^3$ be parametrizations of S with the same orientation.

  The function $(F \circ G) \cdot (\partial_1 G \times \partial_2 G)$ is integrable on U iff $(F \circ H) \cdot (\partial_1 H \times \partial_2 H)$ is integrable on V. If so,

  $$\iint_U (F \circ G) \cdot (\partial_1 G \times \partial_2 G) \, dA = \iint_V (F \circ H) \cdot (\partial_1 H \times \partial_2 H) \, dA$$

- Let S be an oriented surface in $\mathbb{R}^3$. Its oppositely oriented surface $-S$ is the reparam of S w/ the opposite orientation
- Let S, T be oriented surfaces in $\mathbb{R}^3$. The concatenation of S and T is the piecewise oriented surface $S+T$ which may be formed by gluing together all or some of their relative boundaries
  - The unit normals do not need to appear "consistent"
- Let S, T be oriented surfaces in $\mathbb{R}^3$. Let F, G be cts vector fields in $\mathbb{R}^3$ defined on S and T. Then
  1. $\iint_{-S} F \cdot n \, dS = -\iint_S F \cdot n \, dS$
  2. For $\lambda \in \mathbb{R}$, $\iint_S (F + \lambda G) \cdot n \, dS$
     $= \iint_S F \cdot n \, dS + \lambda \iint_S G \cdot n \, dS$
  3. If $S+T$ is an oriented surface in $\mathbb{R}^3$, then $\iint_{S+T} F \cdot n \, dS = \iint_S F \cdot n \, dS + \iint_T F \cdot n \, dS$

9.5 Flux and Divergence in 3D

Flux Over Closed Surfaces

- A piecewise surface S in $\mathbb{R}^3$ is closed if its relative boundary ∂S is empty
  - A surface integral over a closed surface is denoted $\oiint_S$

Divergence

- Let $F = (F_1, F_2, F_3)$ be a C^1 vector field in $\mathbb{R}^3$. The divergence of F is the cts real-valued function $\operatorname{div}(F) = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$
  - $\operatorname{div}(F) = \nabla \cdot F = (\partial_1, \partial_2, \partial_3) \cdot (F_1, F_2, F_3)$
- Let F be a vector field in $\mathbb{R}^3$. Fix $p \in \mathbb{R}^3$ in its domain. If F is C^1 on an open set containing p, then
$(\operatorname{div} F)(p) = \lim_{\varepsilon \to 0^+} \frac{1}{\operatorname{vol}(B_\varepsilon(p))} \oiint_{\partial B_\varepsilon(p)} (F \cdot n)\, dS$
where $\partial B_\varepsilon(p)$ has outward unit normal
  - "Divergence is infinitesimal flux (or flux density)"
- Let F be a C^1 vector field in $\mathbb{R}^3$. A point $p \in \mathbb{R}^3$

is a source if $(\operatorname{div} F)(p) > 0$ and a sink if $(\operatorname{div})(p) < 0$.

  - F is sourceless if $\operatorname{div}(F) = 0$ everywhere

- Let F and G be C^1 vector fields in $\mathbb{R}^3$ w/ domain $U \subseteq \mathbb{R}^3$. Fix a C^1 real-valued function f on U and fix $\lambda \in \mathbb{R}$. All of the following hold on U:

  1. $\operatorname{div}(F + \lambda G) = \operatorname{div}(F) + \lambda \operatorname{div}(G)$
  2. $\operatorname{div}(fF) = (\nabla f) \cdot F + f \operatorname{div}(F)$
  3. $\operatorname{div}(\nabla f) = \partial_1^2 f + \partial_2^2 f + \partial_3^2 f$ provided f is C^2

- The Laplacian $\Delta f := \partial_1^2 f + \partial_2^2 f + \partial_3^2 f = \nabla \cdot \nabla f$

9.6 Divergence Theorem

Regular Regions and Orienting the Boundary

- Let $R \subseteq \mathbb{R}^3$ be a regular region whose boundary ∂R is a closed piecewise surface. ∂R is positively oriented (resp. negatively oriented) if the unit normal along the surface points outward (resp. inward) w.r.t. R
  - $\partial^2 R = \underbrace{\partial}_{\text{relative}}(\underbrace{\partial}_{\text{topological}} R) = \emptyset$
  - "Outward" is w.r.t a region

Divergence Theorem

- Divergence theorem: Let F be a vector field in $\mathbb{R}^3$ that is C^1 on a regular region $R \subseteq \mathbb{R}^3$. If ∂R is a closed piecewise surface and is positively oriented, then

$$\oiint_{\partial R} F \cdot n \, dS = \iiint_R \operatorname{div}(F) \, dV$$

  - "The total infinitesimal flux over R is the flux across ∂R"
  - "The total flow inside is the net flow across the edge"

9.7 Circulation and Curl in 3D

Circulation

- Let $F : C \to \mathbb{R}^3$ where $C \subseteq \mathbb{R}^3$ is an oriented curve. Assume C is simple and closed. The circulation of F along C is $\int_C F \cdot T \, ds$

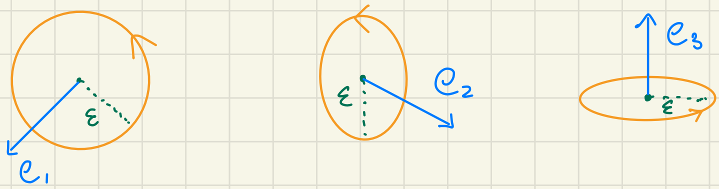
Definition of Curl

- Let F be a C^1 vector field on $\mathbb{R}^3$. The curl of F is the cts $\mathbb{R}^3$-valued function given by
  $\text{curl}(F) = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1)$
  - $\text{curl}(F) = \nabla \times F = (\partial_1, \partial_2, \partial_3) \times (F_1, F_2, F_3)$
- A C^1 vector field F in $\mathbb{R}^3$ iff $\text{curl}(F) = 0$ everywhere on its domain
- Irrotational vector fields in $\mathbb{R}^3$ are also called curl-free vector fields

Geometry of Curl

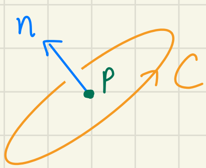
- When viewing the vector field from above, $(\text{curl}\, F) \cdot e_3 = \partial_1 F_2 - \partial_2 F_1$ looks like 2D curl

- The quantity $(\text{curl } F) \cdot e_3$ is the infinitesimal circulation of shrinking positively oriented ε-circles lying on the 2D plane orthogonal to e_3

- 

  $(\text{curl } F) \cdot e_1 = \partial_2 F_3 - \partial_3 F_2$ $\quad$ $(\text{curl } F) \cdot e_2 = \partial_3 F_1 - \partial_1 F_3$ $\quad$ $(\text{curl } F) \cdot e_3 = \partial_1 F_2 - \partial_2 F_1$

- Given a unit vector n and a point $p \in \mathbb{R}^3$, $(\text{curl } F)(p) \cdot n$ measures the infinitesimal circulation around the axis defined by n according to the right hand rule
  - Thumb: n, other fingers: tangential direction



- $|(\text{curl } F)(p) \cdot n|$ is the speed at which the fluid swirls around n
- Sign of $(\text{curl } F)(p) \cdot n$ is +ve if the fluid swirls counterclockwise and −ve if clockwise

Properties of Curl

- Let F be a C^1 vector field in $\mathbb{R}^3$. Then
  1. The maximum of $\text{curl}\, F(p) \cdot n$ over all unit vectors $n \in \mathbb{R}^3$ occurs when $n = + \frac{\nabla \times F(p)}{\|\nabla \times F(p)\|}$ and the max value is $\|\nabla \times F(p)\|$
  2. The minimum of $\text{curl}\, F(p) \cdot n$ over all unit vectors $n \in \mathbb{R}^3$ occurs when $n = - \frac{\nabla \times F(p)}{\|\nabla \times F(p)\|}$ and the min value is $-\|\nabla \times F(p)\|$
- The vector $\nabla \times F(p)$ points in the direction of fastest counterclockwise spin of F at p, and its norm $\|\nabla \times F(p)\|$ is the speed of the spin of F in this direction
- Let F and G be C^1 vector fields in $\mathbb{R}^3$ w/ domain $U \subseteq \mathbb{R}^3$. Fix a C^2 real-valued function on U and fix $\lambda \in \mathbb{R}$. All of the following hold on U:
  1. $\nabla \times (F + \lambda G) = \nabla \times F + \lambda \nabla \cdot G$
  2. $\nabla \times (fG) = f \nabla \times G + (\nabla f) \times G$
  3. $\nabla (F \times G) = (G \cdot \nabla) F + (\nabla \cdot G) F + (F \cdot \nabla) G + (\nabla \cdot F) G$

where $(G\cdot\nabla)F = \sum_{j=1}^{3} G_j \partial_j F$ and

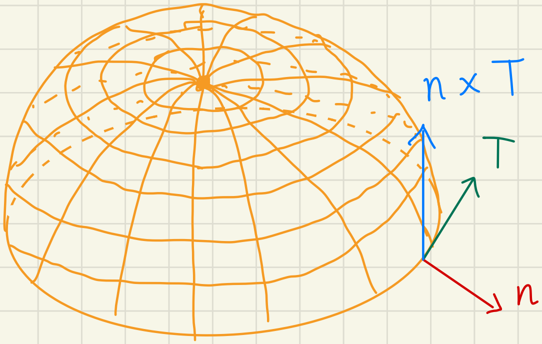
$$(F\cdot\nabla)G = \sum_{j=1}^{3} F_j \partial_j G$$

- If F is a C^2 vector field in $\mathbb{R}^3$ and f is a C^2 real-valued function, then $\operatorname{curl}(\nabla f) = (0,0,0)$ and $\operatorname{div}(\operatorname{curl}(F)) = 0$

9.8 Stokes' Theorem

Stokes' Orientation

- Given an oriented surface S, its relative boundary ∂S has the Stokes orientation if S is always on the left as one traverses the boundary ∂S as one's head pointing in the unit normal direction
- If n is the unit normal of the oriented surface S, and T is the unit tangent of the oriented boundary ∂S, then ∂S has the Stokes orientation provided $n \times T$ points towards S



Statement

- Stokes' theorem: Let $S \subseteq \mathbb{R}^3$ be a surface oriented w/ normal n and whose boundary ∂S is a closed piecewise curve. Let F be a vector field in $\mathbb{R}^3$ that is C^1 on an open set containing S. If S is endowed w/ the Stokes orientation, then

$$\oint_{\partial S} (F \cdot T)\, ds = \iint_S (\text{curl } F) \cdot n \, dS$$

- $\iint_S (\text{curl } F) \cdot n\, dS$ is the total circulation of F over S
- "The total infinitesimal circulation over S is the circulation along its boundary ∂S"

9.9 Div, Grad and Curl

Gradient and Curl

- Gradient vector fields are curl-free, i.e.

  $\mathrm{curl}(\mathrm{grad}(f)) = 0$

- Let F be a vector field in $\mathbb{R}^3$ that is C^1 on an open set $U \subseteq \mathbb{R}^3$. Assume U is convex.

  If F is curl-free on U, then F is a gradient vector field, i.e. there exists a scalar-valued C^2 function f on U s.t. $F = \mathrm{grad}(f)$ on U

Curl and Divergence

- Curl vector fields are divergence-free, i.e.

  $\mathrm{div}(\mathrm{curl}(G)) = 0$

- Let F be a vector field in $\mathbb{R}^3$ that is C^1 on an open set $U \subseteq \mathbb{R}^3$. Assume U is convex.

  If F is divergence-free on U, then F is a curl vector field, i.e. there exists a vector-valued C^2 function G on U s.t. $F = \mathrm{curl}(G)$ on U

A Unified View

- Let $U \subseteq \mathbb{R}^3$. Let $C^\infty(U)$ be the set of real-valued functions $f: U \to \mathbb{R}$ w/ infinitely many partial derivatives. Call the space of C^∞ scalar fields $V = C^\infty(U)$ and the space of C^∞ vector fields V^3
  - $\text{grad}: V \to V^3$ is a linear map
  - $\text{curl}: V^3 \to V^3$ is a linear map
  - $\text{div}: V^3 \to V$ is a linear map

$$V \xrightarrow{\text{grad}} V^3 \xrightarrow{\text{curl}} V^3 \xrightarrow{\text{div}} V$$