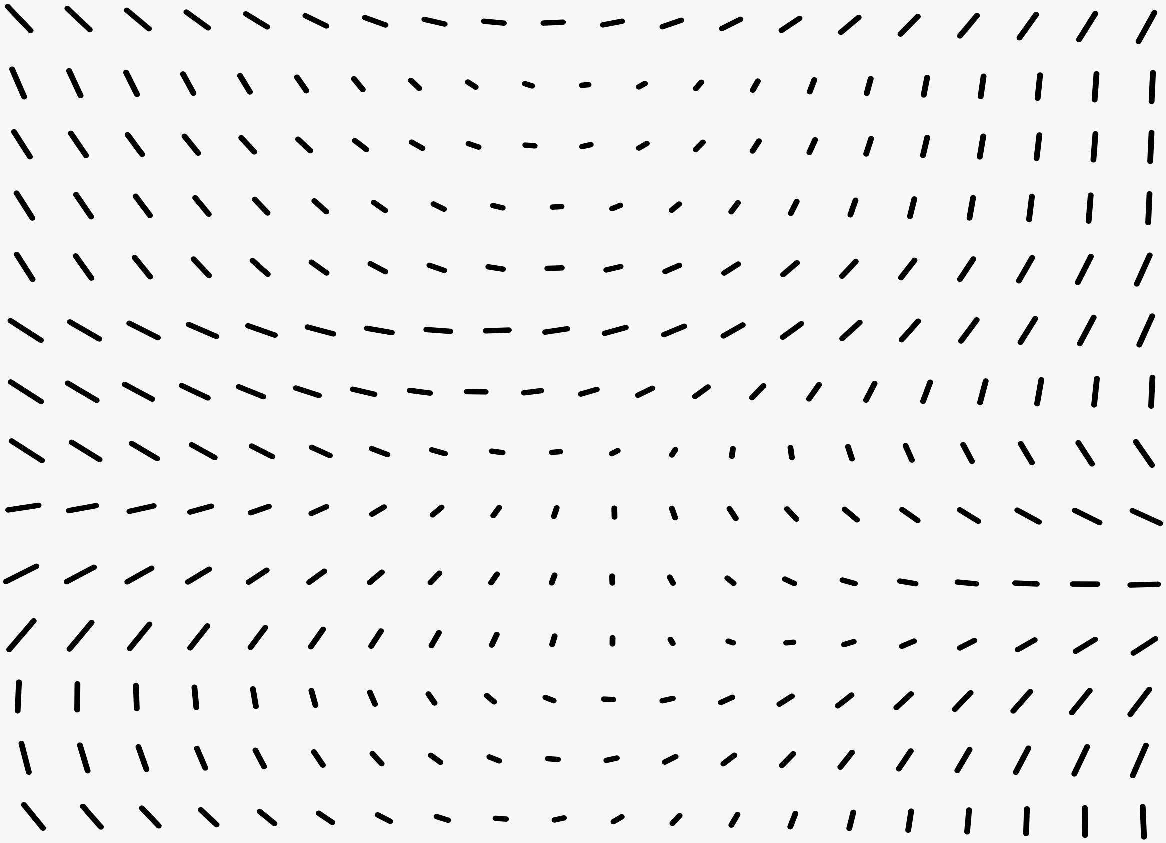
MAT244

Introduction to Ordinary Differential Equations



Midterm 1 Prep

First Order Equations

Integrating Factors

- For an ODE $y' + py = q$:
  - Integrating factor $\mu = e^{\int p\,dt}$
  - Then solve $(\mu y)' = \mu q$

Separable equations

- Take antiderivative on both sides

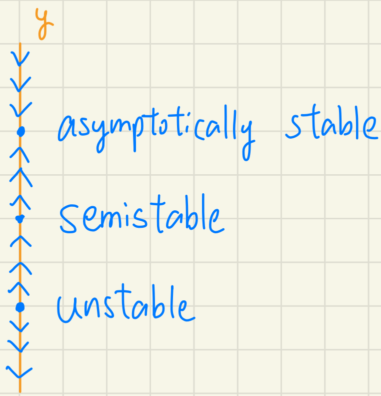
Exact Equations

- For an ODE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$
  - Exact iff $\partial_y M = \partial_x N$
  - Solve system $\begin{cases} \partial_x \psi = M \\ \partial_y \psi = N \end{cases}$ for ψ

Autonomous Equations $y' = f(y)$

- No dependence on t in f
- Solve like separable equations

- Equilibrium solution: $y' = 0$



Second Order Equations

Constant Coefficient Linear Equations

- For an ODE $ay'' + by' + cy = 0$
  - Let $p(r) = ar^2 + br + c$
  - $p(r)$ has real roots r_1, r_2

    $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
  - $p(r)$ has repeated real root r

    $y = c_1 e^{rt} + c_2 t e^{rt}$
  - $p(r)$ has complex roots $p \pm wi$

    $y = c_1 e^{pt} \cos(wt) + c_2 e^{pt} \sin(wt)$
- For an ODE $ay'' + by' + cy = g$

1. Find one particular solution q
2. Find general solution to homogenous equation
3. Add them up

- For particular solution:

Get	Try
e^{rt}	Ae^{rt}
$\cos(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$
$\sin(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$
$e^{rt}\cos(\omega t)$	$e^{rt}(A\sin(\omega t) + B\cos(\omega t))$
$e^{rt}\sin(\omega t)$	$e^{rt}(A\sin(\omega t) + B\cos(\omega t))$
polynomial of degree k	polynomial of degree k

  - If the RHS solves inhomogenous equation, need to multiply t^s, where s is multiplicity

Wronskian

- Given solutions y_1, y_2 of a 2^{nd} order homogeneous ODE:

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

- Solutions are fundamental set iff Wronskian is always 0

- Abel's identity: for an ODE $y''+py'+qy=0$
  - $W'=-pW$
  - $W=ke^{-\int p\,dt}$

Reduction of Order:

- For an ODE $y''+py'+qy=0$ given a solution y_1
  - Define u s.t. $y_1u''+(2y_1'+py_1)u'=0$
  - $y_2=uy_1$

Midterm 2 Prep

Higher-Order Systems (Constant Coefficients)

- Form: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y$
- Find characteristic polynomial

  $p(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$
- $y = e^{rt}$ solves the homogeneous equation iff

  r is a root of p
- Multiplicity $= 1$: solution is e^{rt}
- Multiplicity $m > 1$: solutions $e^{rt}, te^{rt}, \ldots, t^{m-1}e^{rt}$
  - m solutions
- Fundamental set : Wronskian

$$\begin{bmatrix} y_1(t) & \cdots & y_n(t) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix}$$ is never zero

Companion Matrices

- The nth order linear system

  $$y^{(n)}(t) + p_1 y^{(n-1)}(t) + \cdots + p_n y(t) = f(t)$$

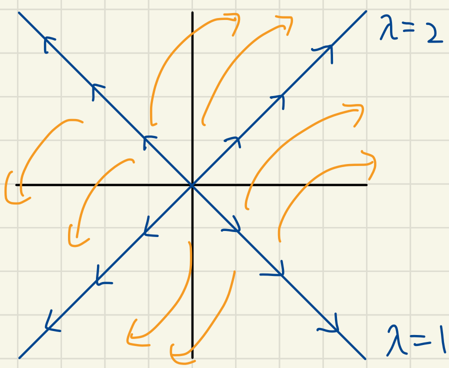
  is equivalent to $x' = Ax + g$, where

  - $x(t) = (y, y', y'', \dots, y^{(n-1)})$
  - $g = (0, 0, \dots, f(t))$
  - $$A = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -p_n & -p_{n-1} & \cdots & -p_1 \end{bmatrix}$$

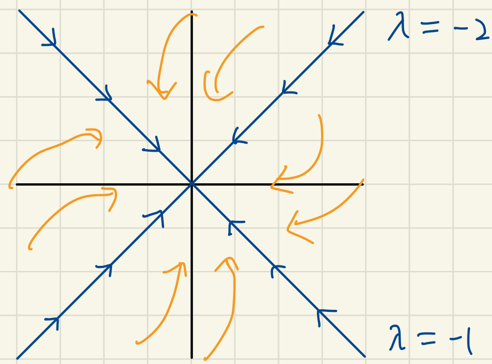
- A is the companion matrix

Direction Field & Phase Portrait

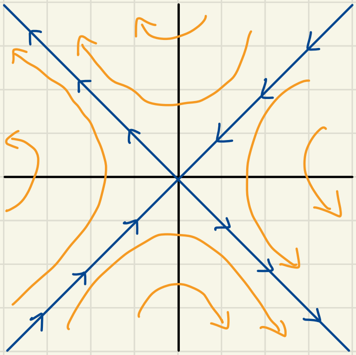
- Distinct real eigenvalues



"source"

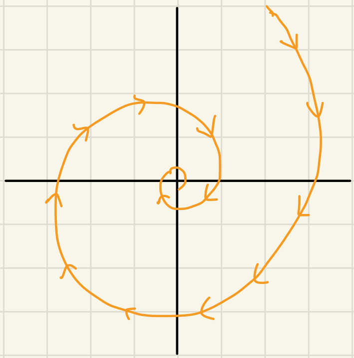


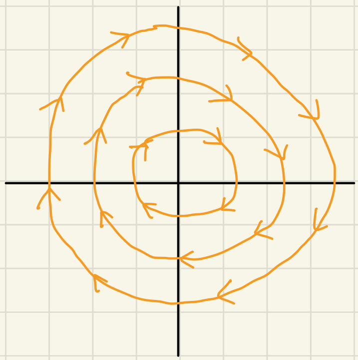
"sink"



- Complex eigenvalues

$\vec{x}' = A\vec{x}$: look at $A\vec{e}_1$ and $A\vec{e}_2$





real part $c_1 e^{rt}$ real part $c_1 e^{0t} = c_1$

Method of Undetermined Coefficients

- Characteristic polynomial: $p(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$

- RHS Try $y_p = \cdots$

RHS	Try $y_p = \cdots$
e^{rt}	Ae^{rt}
$\left.\begin{matrix}\cos(\omega t)\\ \sin(\omega t)\end{matrix}\right\}$	$A\cos(\omega t) + B\sin(\omega t)$

$$\left.\begin{array}{l} e^{at}\cos(bt) \\ e^{at}\sin(bt) \end{array}\right\} \qquad e^{at}(A\cos(bt) + B\sin(bt))$$

- If RHS solves homogeneous equation, multiply by t^s, where s is the order of root of $p(r)$

Variation of Parameter

- Calculate $y_p = u_1y_1 + \cdots + u_ny_n$
- $u_i'(t) = \dfrac{g(t)\,W_i(t)}{W(t)}$, where
  - $W(t)$ is the Wronskian
  - $W_i(t)$ is $W(t)$ with the ith column as $(0,0,\ldots,0,1)$

System of ODE

- Form: $\vec{x}' = A\vec{x}$ where A is $n\times n$ and diagonalizable
- Fundamental set of solutions: $\vec{x}_i(t) = e^{\lambda_i t}v_i$, where λ is an eigenvalue of A and v is its corresponding eigenvector
- Complex eigenvalues: use Euler's identity

- Fundamental matrix:

  Wronskian $W(t) = \det(\vec{x}_1 | \cdots | \vec{x}_n)$

  - The n solutions are a fundamental set iff the Wronskian is always nonzero

Matrix Exponentials

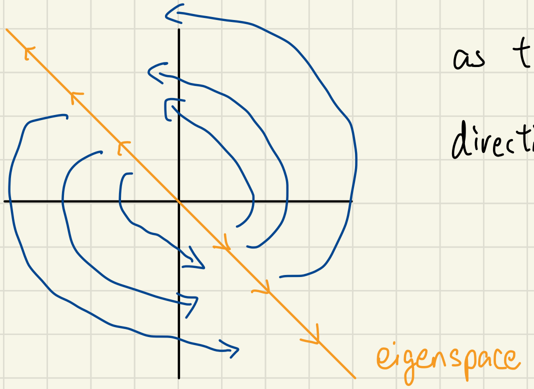
- If A is a matrix, then $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$

- If D is diagonal, then so is e^D and $(e^D)_{ii} = e^{D_{ii}}$

Final Prep

Non-Diagonalizable Matrices

- Find $Av = \lambda v$ (eigenvectors)
- Find $(A - \lambda)w = v$ (generalized eigenvectors)
- General Solution: $x(t) = c_1 e^{\lambda t} v + c_2 (t e^{\lambda t} v + e^{\lambda t} w)$
- Phase portrait:



as $t \to \infty$, approaches the direction of the eigenvector

Inhomogeneous Equations

- Form: $x' = Ax' + g(t)$
- If $x_p(t)$ is a particular solution, then every solution is of the form $x(t) = x_h(t) + x_p(t)$
  - $x_h(t) = c_1 x_1(t) + \cdots + C_n x_n(t)$ is the solution to the homogeneous system

- x_is are a fundamental set of solutions

Method of Undetermined Coefficients

- If $g = e^{rt}v$ where v is an eigenvector of A, with eigenvalue λ
  - If $r \neq \lambda$, try $x_p = ae^{rt}v$
  - If $r = \lambda$, try $x_p = ate^{rt}v$
- If $g = p(t)e^{rt}v$ where v is an eigenvector of A, with eigenvalue λ, and $p(t)$ is an nth degree polynomial
  - If $r \neq \lambda$, try $x_p = q(t)e^{rt}v$
  - If $r = \lambda$, try $x_p = tq(t)e^{rt}v$

  where q is the n^{th} degree polynomial we need to find
- If $g = e^{rt}p(t)$ where p is a vector whose entries are all polynomials in t of degree n or lower
  - If r is not an eigenvalue of A, try $x_p = e^{rt}q(t)$ where $q(t)$ is a vector whose entries are polynomials of t of degree n or lower
  - If r is an eigenvalue of A, same guess but $q(t)$ can have degree $n+1$

- If $g = p(t)e^{rt}w$ where w is a generalized eigenvector of A:

  $(A-\lambda)w = v$, $(A-\lambda)v = 0$

  - If $r \neq \lambda$, try $x_p = q_1(t)e^{rt}v + q_2(t)e^{rt}w$
  - If $r = \lambda$, try $x_p = tq_1(t)e^{rt}v + tq_2(t)e^{rt}w$

  where q is the nth degree polynomial we need to find

Variation of Parameters

- A matrix-valued function $\Psi(t)$ is a fundamental matrix iff

  $\Psi'(t) = P(t)\Psi(t)$, $\det(\Psi(t_0)) \neq 0$ for some t_0

  - Columns of Ψ are a fundamental set of solutions
  - $\Psi(t)$ is invertible for all times t
- If $x' = P(t)x + g$, try $x(t) = \Psi(t)u(t)$
  - $u'(t) = \Psi^{-1}(t)g(t)$

Non-Linear Systems

- Form: $x' = f(x)$ where $f: \mathbb{R}^n \to \mathbb{R}^n$
- A critical point of a nonlinear system $x' = f(x)$

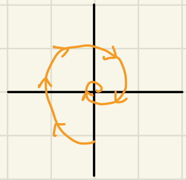
  is a point x_0 s.t. $x(t) = x_0$ is a constant solution

  of the ODE

  - Also called equilibrium solution

- A crit point x_0 of a non-linear ODE $x' = f(x)$ is called
  - Stable if for every small neighbourhood $N_1 := \{x : \|x - x_0\| < \varepsilon\}$ there exists a second small neighbourhood $N_2 := \{x : \|x - x_0\| < \delta\}$ so that all solutions that start in N_2 stay within N_1

  - Asymptotically stable if in addition to being stable, there is a small neighbourhood of x_0 s.t. all solutions $x(t)$ that start in that neighbourhood converge to x_0



  - Unstable if not stable
- Let x_0 be a crit point of $x' = f(x)$.

  The basin of attraction is all the points b near x_0 s.t. a solution starting at b tends to x_0 as $t \to \infty$
- A separatrix is a solution that bounds a domain of attraction
  - Boundary between two different asymptotic behaviour of solutions

Linearization of Non-linear Systems

- Let x_0 be an isolated crit point of an ODE $x' = f(x)$
- A system is locally linear if it is well-approximated by a linear ODE $x' = Ax$ near the critical point x_0
- A system is locally linear if for x near x_0 we can write $x' = A(x - x_0) + g(x)$ where $\lim_{x \to x_0} \frac{g(x)}{\|x - x_0\|} = 0$
- A 2D system $x' = F(x,y)$ $y' = G(x,y)$ is locally linear at a crit point if F and G are C^2
  - Under this condition, the system is well-approximated by

$$\frac{d}{dt}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{pmatrix}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

  - Called linearization of the system at the crit point (x_0, y_0)
  - Matrix of partial derivatives is called the Jacobian $J(x,y)$
- Consider an isolated crit point x_0 of a 2×2 system. Sps that the linearization does not have

1. Purely imaginary eigenvalues

2. One or two eigenvalues equal to 0

Then the stability of the crit point is the same as the linearized system

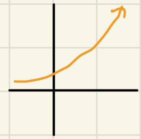
- Stability is indeterminate if the linearization has purely imaginary eigenvalues, or if there is a zero eigenvalue

Eigenvalues	Linearization	Non-Linear System
$0 < \lambda_1 < \lambda_2$	Unstable (Node)	Unstable (Node)
$\lambda_1 < \lambda_2 < 0$	Asymp Stable (Node)	Asymp Stable (Node)
$\lambda_1 < 0 < \lambda_2$	Unstable (Saddle)	Unstable (Saddle)
$0 < \lambda_1 = \lambda_2$	Unstable (Deg Node, Star Node)	Unstable (Spiral, Deg Node, Node, Star Node)
$\lambda_1 = \lambda_2 < 0$	Asymp Stable	Asymp Stable
$r \pm iu,\ r > 0$	Unstable (Spiral)	Unstable (Spiral)
$r \pm iu,\ r < 0$	Asymp Stable (Spiral)	Asymp Stable (Spiral)
$\pm iu$	Stable (Centre)	Indeterminate Stability (Spiral, Centre)

Competing Species Model

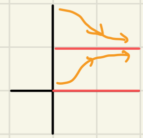
- Exponential growth:

  $x'(t) = \varepsilon_1 x(t)$



- Logistic growth:

  $x'(t) = x(t)\left[\varepsilon_1 - \sigma_1 x(t)\right]$



- Competing species model:

  $x'(t) = x(\varepsilon_1 - \sigma_1 x - \alpha_1 y)$

  $y'(t) = y(\varepsilon_2 - \sigma_2 y - \alpha_2 x)$

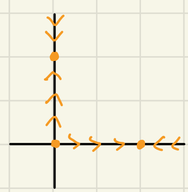
  - $x(t)$ and $y(t)$ are populations
  - Both species compete for the same resources

- Want to find whether two species can co-exist
  - If so, stable or unstable
  - Determined by whether $\alpha_1 \alpha_2 > \sigma_1 \sigma_2$

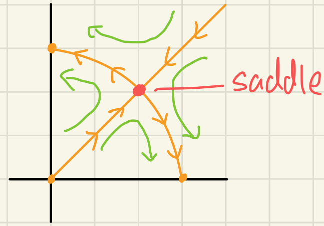
    or $\alpha_1 \alpha_2 < \sigma_1 \sigma_2$

- 3 crit points: $(0,0)$, $(0, \varepsilon_2/\sigma_2)$, $(\varepsilon_1/\sigma_1, 0)$
  - Solutions w/ one of the species

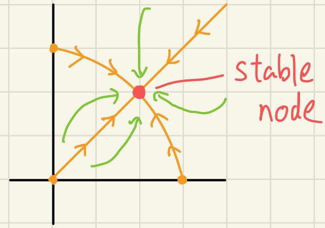
    as 0



- If there is a crit point (x_0, y_0) w/ $x_0, y_0 > 0$, and $\alpha_1 \alpha_2 > \sigma_1 \sigma_2$, then the phase portrait looks like



- If there is a crit point (x_0, y_0) w/ $x_0, y_0 > 0$, and $\alpha_1 \alpha_2 < \sigma_1 \sigma_2$, then the phase portrait looks like



Predator-Prey Model (Lotka-Volterra)

- Form: $\begin{cases} x'(t) = x(a - \alpha y) \\ y'(t) = y(-c + \gamma x) \end{cases}$

- Solutions on coordinate axes

- One crit point in 1^{st} quadrant

- General case that always have the same cyclic behaviour:

$$\alpha y - a \log y + \gamma x - c \log x = C$$