Question 01:

Domain : The domain of a function is the complete set of possible values of the independent variable. Simply, the domain is the set of all possible x-values which will make the function work.

Range : The range of a function is the complete set of all possible resulting values of the dependent variable after we have substituted the domain.

Given,

$$f(x) = \frac{x-3}{2x+1}$$

Here,

$$2x + 1 = 0$$

$$\therefore x = -\frac{1}{2}$$

The function will be undefined if $x = -\frac{1}{2}$

$\therefore$ Domain of the function, $D_{f=R-\left\{-\frac{1}{2}\right\}}$

Let,

$y = f(x) = \frac{x-3}{2x+1}$

$\Rightarrow 2xy + y = x - 3$

$\Rightarrow x = \frac{y+3}{1-2y}$

$\therefore f^{-1}(y) = \frac{y+3}{1-2y}$

$\therefore f^{-1}(x) = \frac{x+3}{1-2x}$

Here,

$$1 - 2x = 0$$

$$\therefore x = \frac{1}{2}$$

The function will be undefined if $x = \frac{1}{2}$

$\therefore$ Range of the function, $R_{f=R-\left\{\frac{1}{2}\right\}}$

Question 02:

(a)

Given,

$f(x) = \frac{x-3}{2x+1}$

The function will be undefined if

$2x + 1 = 0$

$\therefore x = -\frac{1}{2}$

$\therefore$ Domain of the function, $D_{f=R-\left\{-\frac{1}{2}\right\}}$

Let,

$y = f(x) = \frac{x-3}{2x+1}$

$\Rightarrow 2xy + y = x - 3$

$\Rightarrow x = \frac{y+3}{1-2y}$

$\therefore f^{-1}(y) = \frac{y+3}{1-2y}$

The function will be undefined if

$1 - 2y = 0$

$\therefore y = \frac{1}{2}$

$\therefore$ Range of the function, $R_{f=R-\left\{\frac{1}{2}\right\}}$

From the process of finding range we get the inverse function,

$\therefore f^{-1}(y) = \frac{y+3}{1-2y}$

$\therefore f^{-1}(x) = \frac{x+3}{1-2x}$

(b)

f(x)is defined as follows:

f(x) = 0 , x = 0

= x, x> 0

= −x, x < 0

Draw the graph of the function. Does f(x)exists at x = 0?Justify your answer.

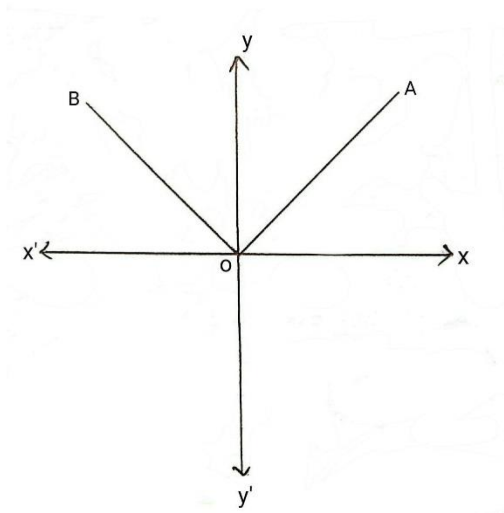
<u>Solution:</u>

We have,

f(x) = 0, x = 0

= x, x> 0

= −x, x < 0



The graph as shown is consists of two lines $\overrightarrow{OA}$ and $\overrightarrow{OB}$ which is bisect the angles $< xoy$ and $< yox'$ respectively. This is also the graph of the function

f(x)=|x|

f(x) is define for x=0 and positive values of x for graph is a x axis corue in the first quadaunt.

Question 03:

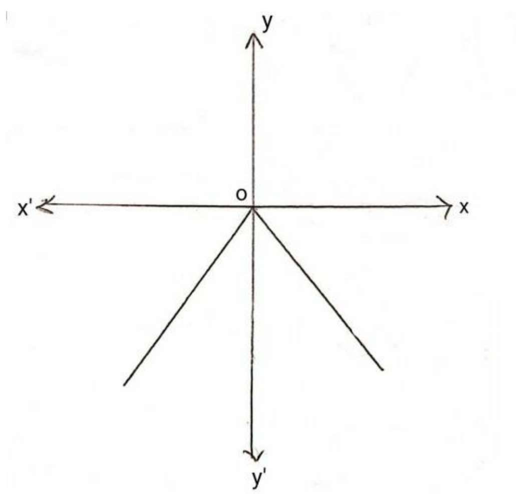
(a)

Draw the graph of y = x − |x| , where |x| Denotes the greatest integer not greater than x.

Solution:

The function is y=x-$|x|$, where $|x|$ denotes the greatest integer not greater than x. So, we can cosider x value.

Considering the x now the function is y= -|x|



(b)

1. Find the domain and range of the function$x\frac{x^2-1}{x-1}$. Also sketch the graph.

$$f(x) = \frac{x^2-1}{x-1} = x + 1$$

$$x + 1 > 0$$

$$x < 1$$

$The\ Domain\ is$

$So\ that\ Df\ = (-\propto, 1) \cup (1, \propto)$

$There,$

$y = \frac{x^2-1}{x-1} = y = x + 1$

$x = y - 1$

$y - 1 \geq 0$

$y \leq 1$

$The\ Range\ Rf = R - \{1\}\ or\ \{x : X\epsilon R\ and\ y \leq 1\}$

Question 4:

(i)

Find the domain and range of the function (i) (x) = $\frac{x^2+1}{x^2-5x+6}$

$f(x) = \frac{x^2+1}{x^2-5x+6}$

$= \frac{x^2+1}{(x-3)(x-2)}$-28

$(x-3)\ (x-2) \neq 0$

$x \neq 3{,}2$

$The\ Funcation\ Domain,$

$\{ X \in R : X\ < 2\ \ Or\ 2\ < X\ \ < 3\ Or\ X > 3$

$Interval : (-\propto, 2\ ) \cup (2\ , 3) \cup (3\ \propto)$

$Range:\ y = \frac{x^2+1}{x^2-5x+6}$

$\rightarrow yx^2 - 5xy + 6y = x^2 + 1$

$\rightarrow \quad x^2(y-1) - 5xy + (6y-1) = 0$

$Here \;=\; b^2 - 4ac$

$\rightarrow \quad (-5y) \quad ^2 - 4\,(y-1).(6y-1) \geq 0$

$\rightarrow 25y^2 - \{24y^2 - 4y - 24y + 4\} \geq 0$

$\rightarrow \quad 25y^2 - 24y^2 + 4y - 24y - 4 \geq 0$

$\rightarrow \quad y^2 + 28y - 4 \geq 0$

$\rightarrow \quad y = \frac{-28 \pm \sqrt{28^2 - 4.1.(-4)}}{2.1}$

$\rightarrow \quad y = \frac{-28 \pm \sqrt{800}}{2}$

$= \frac{-28 \pm 20\sqrt{2}}{2}$

$= -14 \pm 10\sqrt{2}$

$= 10\sqrt{2-14,} - 10\sqrt{2-14}$

$\therefore The\ Range\ \{Y\ \epsilon\ R : Y \leq\ -10\sqrt{2-14\quad}\quad Or\ \ Y\ \geq 10\sqrt{2-14\quad}\ \}$

(ii)

Given,

$$f(x) = \frac{|x|}{x}$$

Domain, $D_f = \mathbb{R}$

We know,

$$f(x) = 1, x > 0$$

$$f(x) = -1, x > 0$$

$$f(x) = 0, x = 0 \quad (undefined)$$

$\therefore$ Range, $R_f = \{-1,1\}$

(iii)

$f(x) = |x + 1| + |x|$

The function is defined for all the real numbers.

Domain, $D_f = \mathbb{R}$

In the function,

$$x < -1, f(x) = -x - x - 1 = -2x - 1$$

$-1 \leq x \leq o, f(x) = x + 1 - x = 1$

$x \geq 0, f(x) = x + x + 1 = 2x + 1$

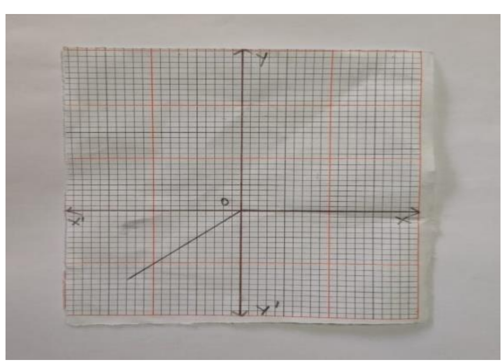
$\therefore$ Range, $R_f = [1, \infty]$

(iv) Given,

$$f(x) = x - |x|$$

The function is defined for all the real number

Domain, $D_f = \mathbb{R}$
Range, $R_f = (-\infty, \infty)$

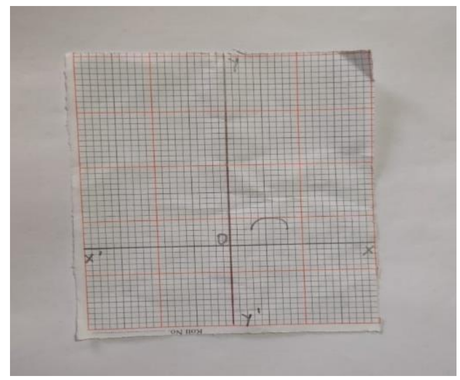


(v)

Given,

$f(x) = \sqrt{x} - 1 + \sqrt{5} - x$

Since, $f(x)$ is real, the values of x must be such that both $\sqrt{x} - 1$ & $\sqrt{x} - 5$ are real quantities, which requires that $(x-1) \geq 0$ & $(5 - x) \geq 0$



$x \geq 1, x \leq 5$

Domain, D_f = $[1,5]$

$x = 1, f(x) = 0 + \sqrt{5} - 1 = 2$ (minimum value)

$x = 5, f(x) = \sqrt{4} + 0 = 2$ (minimum value)

$x = 3, f(x) = \sqrt{(3} - 1) + \sqrt{5} - 3 = 2\sqrt{2}$ (maximum value)

∴Range, R_f =$[2,2\sqrt{2}]$

(vi)

$f(x) = \frac{1}{\sqrt{|x| - x}}$

f (x) is difined, when $| x | - x > 0$

$$| x | > x$$

And this inequality is satisfied for all values of $x < 0$

So the domain of definition of f (x) is

$$(- \alpha,0) \text{ or } (- \alpha < x < 0)$$

$$y = \sqrt{| x | - x}$$

$$\Rightarrow y^2 = | x | - x$$

$$y \geq 0 \qquad |y| \geq 0$$

$$y \leq 0 \qquad y \leq 1$$

Reng of $R_f \ \{ y \epsilon R : y \geq 1\}$

Question 05:

A function f (x) defined as follows $f(x) = \begin{cases} 3 + 2x & -\frac{3}{2} \leq x \leq 0 \\ 3 - 2x \ \text{ for } & 0 \leq x \leq \frac{3}{2} \\ -3 - 2x & x \geq \frac{3}{2} \end{cases}$

Discuss continuiting of f (x) at $x = 0$ and $x = \frac{3}{2}$

Hence,

$$\underset{x \to 0^-}{Lt} f(x) = \underset{x \to 0^-}{Lt} (3 + 2x) = 3$$

$$\underset{x \to 0^+}{Lt} \text{f}(x) = \underset{x \to 0^+}{Lt} (3 - 2x) = 3$$

$$\underset{x \to 0^-}{Lt} \text{f}(x) = \underset{x \to 0^+}{Lt} \text{f}(x)$$

= f (0), f (x) is continuous at x = 0

Again,

$$\underset{x \to \frac{3}{2}}{Lt}^{-0}$$

$$= \underset{x \to \frac{3}{2}}{Lt}^{+0} \text{f}(x) = \text{f}\left(\frac{3}{2}\right) = -6$$

Continuous at x = $\frac{3}{2}$.

Question 6:

(a)

Let $f(x) be\ a\ function\ of\ x\ which\ is\ differeatiable\ at\ x = c$

$then,$

$\frac{f(c-h)-f(c)}{h} = \frac{f(c-h)-f(c)}{h} = f\,'(c)$

$Now,$

$f(c+h) - f(c) = \frac{f(c-h)-f(c)}{h} \times h$

$Taking\ Limit\ Sides\ On\ \ h \to 0$

$\lim\limits_{h\to 0} \{f(c+h) - f(c)\} = \lim\limits_{h\to 0} \frac{\{f(c+h)-f(c)}{h} \times h\}$

$\lim\limits_{h\to 0} \{f(c+h) - f(c)\} = \lim\limits_{h\to 0} \frac{f(c+h)-f(c)}{h} \lim\limits_{h\to 0}$ h

$\lim\limits_{h\to 0} f(c+h) - f(c) = f'(c) \times 0$

$\lim_{h \to 0} f(c+h) - f(c) = 0$

$\lim_{h \to 0} f\ (c+h) = f(c) \dots\dots\dots\dots\dots\dots\dots.(1)$

$Again\ Semilerly\ ,$

$\lim_{h \to 0} f(c-h) = f(c) \dots\dots\dots\dots\dots\dots\dots\dots(2)$

$from\ eq^n\ \ (1)\ and\ (2) \quad \to$

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$

$f(x) is\ continous\ at\ x = c$

$Every\ Defferentiable\ Function\ is\ \ continous\ .$

$For\ Converse\ Part$

$Let,$

$$f(x) = |x| = -x \quad if\ x < 0$$
$$= 0 \qquad if \quad x = 0$$
$$= x \qquad if\ x > 0$$

$First\ \ We\ Examine\ That\ Continuity\ of$

$f\ (x) \quad at\ x = 0$

$L.H.S\ = \lim_{h \to 0} f\ (o-h)$

$= \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} -(0-h)$

$= \lim_{h \to 0} \quad h = 0 \dots\dots\dots\dots..(ii)$

$R.H.S \quad = \lim_{h \to 0} \quad f(0+h)$

$= \lim_{h \to 0} \quad (0+h)$

$= \lim_{h \to 0}\ h = 0 \dots\dots\dots\dots\dots(iv)$

$Also\ We\ Have$,

$f(x) = 0, at\ x = 0$

$\lim_{h \to 0} h = 0$ ………………………….(v)

$From\ eq^n$ (iii),(iv) and (v)

$L.H.S = R.H.S = f(0)$

$f(x) \quad is\ Continous\ at\ x = 0$

$For\ differentiability\ at\ x = 0.$

$L.H.D = \lim_{h \to 0} \frac{f(0-h)-f(0)}{-h}$

$=\lim_{h \to 0} \frac{(0-h)-0}{-h}$

$= \lim_{h \to 0} \frac{-(0-h)-0}{-h}$

$= \lim_{h \to 0} \frac{h-0}{-h}$

$\lim_{h \to 0} -1 = -1$ ………………………………..(vi)

$R.H.D = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h \to 0} \frac{(0+h)-0}{h}$

$= \lim_{h \to 0} \frac{(0+h)-0}{h}$

$= \lim_{h \to 0} 1 = 1$ ………………(vii)

$From\ eq^n\ (vi) and\ (vii)$

$L.H.D \neq R.H.D$

(b)

$$f(x) = \begin{cases} 1 & for\ x < 0 \\ 1 + sinx & for\ 0 \le x \le \frac{\pi}{2} \\ 2 + \left(x - \frac{\pi}{2}\right)^2 & for \frac{\pi}{2} \le x \end{cases}$$

Here, at x= $\frac{\pi}{2}$ when,

$\lim_{x \to \frac{\pi}{2}^-} f(x)$ = $1 + sinx$ = 2

$\lim_{x \to \frac{\pi}{2}^+} f(x) = 2 + \left(x - \frac{\pi}{2}\right)^2$ = 2

$\therefore \lim_{x \to \frac{\pi}{2}^-} \quad = \lim_{x \to \frac{\pi}{2}^+}$

So, the function $f(x)$ is exist at $x = \frac{\pi}{2}$

Again, at $x = 0$ when,

$\lim_{x \to 0^-} f(x) = 1 + sinx = 1$

$\lim_{x \to 0^+} f(x) = 2 + \left(x - \frac{\pi}{2}\right)^2 = 2 + \left(-\frac{\pi}{2}\right)^2$

$\therefore \lim_{x \to 0^-} \quad \neq \lim_{x \to 0^+}$

So, the function $f(x)$ is not exist at $x = 0$.

***Question* 07**: *Define continuity of a function.Show that* $f(x) = |x|$ *is continuous at* $x0$ *but* $f'(x)$ *doesn't exist.*

The formal definition of continuity at a point has three conditions.

1. The Function is defined at $x = a$, that is, $f(x)$ equals a real number.
2. The limit of the function as x approaches a exists.
3. The limit of the function as x approaches a is equal to the function value at $x = a$.

Again,

$$f(x) = |x| = \{x \quad if \quad x \geq 0 - x \quad if \quad x < 0$$

$$|x| = n = 0 \; and \; |x| = (-x) = 0$$

$$\lim_{x \to 0^-} |x| = 0 \; which \; is \; of \; course \; equal \; to \; f(0)$$

To show that $f(x) = |x| \; is \; not \; differentiative$

$$f'(0) = \lim_{x \to 0} \frac{f(0 + x) - f(0)}{x} \; doesn't \; exists$$

$$\Rightarrow \lim_{x \to 0} \frac{|0 + x| - |0|}{x}$$

$$\Rightarrow \lim_{x \to 0} \frac{|x|}{x} \Rightarrow \frac{|x|}{x} = \{1 \quad if \quad x > 0 - 1 \quad if \quad x < 0$$

So, the limit from the right is 1. While the limit from the left =-1.

(Ans:)

***Question* 08**: $Examine \; the \; continuity \; of \; the \; function \; f(x) \; at \; x = \frac{3}{2} \; where \quad f(x) = \{3 - 2x, \quad 0 \leq x \leq \frac{3}{2} - 3 - 2x, \quad x \geq \frac{3}{2}$

$$f(x) = lt_{x \to \frac{3}{2}^{-0}} (3 - 2x) = 0$$

$$f(x) = lt_{x \to \frac{3}{2}^{+0}} (-3 - 2x) = -6$$

$$f\left(\frac{3}{2}\right) = -3 - 2.\frac{3}{2} = -6$$

$$since\ f(\ x)\ doesn't\ exist$$

$here\ f(x)\ is\ dis\ continous\ at\ x = \frac{3}{2}$ (Ans:)

Question 9:

In particular, any differentiable function must be continuous at every point is its domains. The convers down not hold: a continuous functions need not be differentiable for example.

Question 10:

If(x) - f(0) = | x sin($\frac{1}{x}$)- 0|

= |x sin $\frac{1}{x}$| = |x| |sin$\frac{1}{x}$|

$\leq x$. sinx |sin$\frac{1}{x}$| ≤ 1

<∈for |x-0|<∈

The relations are satisfied, if f = ∈

So, f(x) is continuous at x=0.

Question 11:

Define continuity at a point, If the function

$f(x) = \begin{cases} \frac{-x^2-16}{x-4} & \text{if } x \neq 4 \\ & x = 4 \end{cases}$ is

a if $x = 4$

Continuous at point 4, What is the value of a

$\lim_{x\to 4} \frac{x^2-16}{x-4}$

$\lim_{x\to 4}(x+4)$

$= 4+4$

$= 8$

Hence, In order that $f(x)$ may be continuous at $x = 0$, $f(4)$ may be continuous at, $f(4)$ must be 8

Question 12:

A function is defined as $f(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 1 - x & \frac{1}{2} \leq x < 1 \end{cases}$

Discuss the differentiability and continuity of $f(x) = \frac{1}{2}$

Now $| f(x) - f(x) | = | x - (1 - x) |$

$= | x |.| 1 - x |$

$\leq x$. since $| 1 - x | \leq 1$

$< \varepsilon$ for $| \frac{1}{2} - 0 | < \varepsilon$

The relation are satisfaied if $\partial = \varepsilon$

So $f(x)$ is continuous at $x = \frac{1}{2}$

Question 13 :

$$\lim_{x \to o-} f(x) = \lim_{x \to 0-} (-x) = 0$$

$$\lim_{x \to o-} f(x) = \lim_{x \to 0+} ((x) = 0$$

F(0) = 0

F(x) is continuous at x = 0

To show that f(x) = |x| is not differentiable

f*(0) = $\lim_{x \to 0} f(0th) - f(0)/x$ does not exist

$$\lim_{x \to 0} |x|/x$$

So x=0 is not differentiable but f(x) = |x| is continuous

Question 14 :

(a)

$$\lim_{x \to a} f(x) and\ f(a) is\ different\ because\ when\ we\ say\ x \to a,$$

It implies x is a approaching clause to a and it doesn't implies x=a

In straight term though in many calculations x can be replaced with a where it is allowable

Given,

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x > 0 \\ & x = 0 \end{cases}$$

$\lim_{x \to o} f(x)$ = $\lim_{x \to 0} x$ = 0

$\lim_{x \to o} f(x)$ doesn't exists.

(b)

Here,

$$\underset{x \to 0-0}{\mathrm{lt}} f(x) = \underset{x \to 0-0}{\mathrm{lt}} (-x)$$

$$= 0$$

$$\underset{x \to 0+0}{\mathrm{lt}} f(x) = \underset{x \to 0+0}{\mathrm{lt}} (x)$$

$$= 0$$

f (0) = 0, Since left hand limit = right hand limit = functional value x = 0,

So f (x) is continuous at x = 0.

Question 15:

(a)

To show that f (x) = |x| is continuous at 0, Show that $\lim\limits_{x \to 0} |x|$ = $|0|$ = 0

Use ε – δ if required, or use the piecewise definition of absolute value.

f (x) = |x| = $\begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}$

So, $\lim\limits_{x \to 0^+} |x|$ = $\lim\limits_{x \to 0^+} x$ = 0

And $\lim\limits_{x \to 0^-} |x|$ = $\lim\limits_{x \to 0^-} (-x)$ = 0

Therefore,

$\lim\limits_{x \to 0} |x|$ = 0 which is, of course equal to f (0)

So f(x) is continuous at x = 0.

Now, To show that f (x) = |x| is not differentiable

So that f' (0) = $\lim\limits_{h \to 0} \frac{f(o+h) - f(0)}{h}$ does not exists.

Observe that, $\lim\limits_{h \to 0} \frac{|o+h| - |0|}{h}$ = $\lim\limits_{h \to 0} \frac{|h|}{h}$

But $\lim\limits_{h \to 0} \frac{|h|}{h}$ = $\begin{cases} 1, & if\ h > 0 \\ -1, & if\ h < 0 \end{cases}$

So the limit from the right is 1, while the limit from the left is −1

So the two sided limit does not exist.

That is, the derivative does not exist at x = 0.

(b)

$$\underset{x \to 0}{Lt} (\cos x)^{\cot^2 x}$$

Let,

$$y = (\cos x)^{\cot^2 x}$$

Taking log on both side ,

$$\log y = log\ (\cos x)^{\cot^2 x}$$

$$= \cot^2 x \,. \log(\cos x)$$

$$= \frac{\log (\cos x)}{\tan^2 x}$$

$$\log y = \underset{x \to 0}{Lt} \frac{\log (\cos x)}{\tan^2 x}$$

$$= \underset{x \to 0}{Lt} \frac{-\frac{sinx}{cosx}}{2tan\,x.\ \sec^2 x}$$

$$= \underset{x \to 0}{Lt} \frac{-\,tanx}{2tanx} \,. \cos^2 x$$

$$= -\ \frac{1}{2}$$

$$y = \ e^{-\frac{1}{2}}\ \text{(Ans)}$$

Question 16:

$$\lim_{x\to 0}(1+\left(\frac{\tan x}{x}\right))^{-\frac{1}{x}}$$

$$\Rightarrow \lim_{x\to 0}\left(1+\left(\frac{\tan x - x}{x}\right)\right)^{\frac{1}{x}}$$

$$\Rightarrow e\lim_{x\to 0}\left(\frac{\tan x - x}{x}\right).\frac{1}{x}$$

$$\Rightarrow e\lim_{x\to 0}\left(\frac{\tan x - x}{x^2}\right)$$

$$\Rightarrow e\lim_{x\to 0}\left(\frac{\sec^2 x - 1}{2x}\right) = e\lim_{x\to 0}\left(\frac{2secx.secx.tanx}{2}\right)$$

$$\Rightarrow e\lim_{x\to 0}\left(\frac{2\sec^2 x - tanx}{2}\right)$$

$$\Rightarrow 1$$

Differentiate $x^{\sin^{-1}x}$ with respect to $\sin^{-1} x$

Let,

$$\text{v} = x^{\sin -1}x \text{ , u } = \sin^{-1} x$$

$$\Rightarrow \text{ v } = x^{\sin -1}x$$

$$\Rightarrow \ln \text{v} = \sin^{-1} x\, .\, lnx$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{\sin^{-1} x}{x} + \ln \text{x} . \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dv}{dx} = x^{\sin -1}x \left[\frac{\sin^{-1} x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right]$$

Again,

$$\frac{du}{dx} = \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dv}{dx} \div \frac{du}{dx} = x^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right] \div \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{\mathrm{dv}}{\mathrm{dx}} \cdot \frac{dx}{du} = x^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right] \cdot \frac{\sqrt{1-x^2}}{1}$$

$$\Rightarrow \frac{\mathrm{dv}}{\mathrm{du}} = x^{\sin^{-1}x}\left(\frac{\sqrt{(1-x^2)}\sin^{-1}x + xlnx}{x.\sqrt{1-x^2}}\right) . \sqrt{1-x^2}$$

$$\Rightarrow \frac{\mathrm{dv}}{\mathrm{du}} = x^{\sin^{-1}x}\left(\frac{\sqrt{(1-x^2)}\sin^{-1}x + xlnx}{x}\right)$$

$$\therefore \frac{\mathrm{dv}}{\mathrm{du}} = x^{(\sin^{-1}x-1)}\left[\sqrt{1-x^2}\,(\sin^{-1}x) + xlnx\right] \text{ (Ans)}$$

Question 17:

(i) $\lim\limits_{x\to 0}\frac{e^x-e^{-x}-2x}{x-sinx}$

$= \lim\limits_{x\to 0}\frac{e^x+e^{-x}-2}{1-cosx}$

$= \lim\limits_{x\to 0}\frac{e^x-e^{-x}}{sinx}$

$= \lim\limits_{x\to 0}\frac{e^x+e^{-x}}{cosx}$

= 2.

(ii) $\lim_{x\to 0}\left(\frac{sinx}{x}\right)^{\frac{1}{x^2}}$

$=\lim_{x\to 0}\left(1+\frac{sinx}{x}-1\right)^{\frac{1}{x^2}}$

$=\lim_{x\to 0}\left(1+\frac{sinx-x}{x}\right)^{\frac{1}{x^2}}$

$=e^{\lim_{x\to 0}\left(\frac{sinx-x}{x}\times\frac{1}{x^2}\right)}$

$=e^{\lim_{x\to 0}\left(\frac{sinx-x}{x^3}\right)}$

$=e^{\lim_{x\to 0}\left(\frac{cosx-1}{3x^2}\right)}$

$=e^{\lim_{x\to 0}\left(-\frac{sinx}{6x}\right)}$

$=e^{\lim_{x\to 0}\left(-\frac{cosx}{6}\right)}$

$=e^{-\frac{1}{6}}.$

(iii)

$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

$=\lim_{x\to 0}\left(1+\frac{\sin x-x}{x}-1\right)^{\frac{1}{x}}$

$=\lim_{x\to 0}\left(1+\frac{\sin x-x}{x}\right)^{\frac{1}{x}}$

$=e^{\lim_{x\to 0}\left(\frac{sinx-x}{x^2}\right)}$

$=e^{\lim_{x\to 0}\left(\frac{cosx-1}{2x}\right)}$

$=e^{\lim\limits_{x\to 0}\frac{-sinx}{2}}$

$=e^{\lim\limits_{x\to 0} -cosx}$

$=e^{-1}$ (Ans.)

(iv)

$$\lim_{x\to\frac{\pi}{2}} \frac{e^{\tan x-1}}{e^{\tan x+1}}$$

$$= \lim_{x\to\frac{\pi}{2}-0} \frac{1-\frac{1}{e^{\tan x}}}{1+\frac{1}{e^{\tan x}}} \qquad = \frac{1-0}{1+0} \qquad =1 \qquad \text{(Ans.)}$$

$$= \lim_{x\to\frac{\pi}{2}+0} \frac{e^{\tan x-1}}{e^{\tan x+1}} \qquad = \frac{0-1}{0+1} \qquad = -1 \qquad \text{(Ans.)}$$

Question 18:

(i)

$$\underset{x\to 0}{Lt} (\cos x)^{\cot^2 x}$$

Let,

$y = (\cos x)^{\cot^2 x}$

Taking log on both side ,

$\log y = log\ (\cos x)^{\cot^2 x}$

$= \cot^2 x \,.\log(\cos x)$

$$= \frac{\log(\cos x)}{\tan^2 x}$$

$$\log y = \underset{x \to 0}{Lt} \frac{\log(\cos x)}{\tan^2 x}$$

$$= \underset{x \to 0}{Lt} \frac{-\frac{sinx}{cosx}}{2tan\, x.\ \sec^2 x}$$

$$= \underset{x \to 0}{Lt} \frac{-\,tanx}{2tanx}\, .\cos^2 x$$

$$= -\ \frac{1}{2}$$

$$y = e^{-\frac{1}{2}} \text{ (Ans)}$$

(ii)

$$\lim_{x \to \frac{\pi}{2}} \frac{e^{\tan x - 1}}{e^{\tan x + 1}}$$

$$= \lim_{x \to \frac{\pi}{2} - 0} \frac{1 - \frac{1}{e^{\tan x}}}{1 + \frac{1}{e^{\tan x}}} \qquad = \frac{1-0}{1+0} \qquad = 1 \qquad \text{(Ans.)}$$

$$= \lim_{x \to \frac{\pi}{2} + 0} \frac{e^{\tan x - 1}}{e^{\tan x + 1}} \qquad = \frac{0-1}{0+1} \qquad = -1 \qquad \text{(Ans.)}$$

(iv)

$$\lim_{x\to 0}(1+\left(\frac{\tan x}{x}\right))^{-\frac{1}{x}}$$

$$\Rightarrow \lim_{x\to 0}\left(1+\left(\frac{\tan x - x}{x}\right)\right)^{\frac{1}{x}}$$

$$\Rightarrow e\lim_{x\to 0}\left(\frac{\tan x - x}{x}\right).\frac{1}{x}$$

$$\Rightarrow e\lim_{x\to 0}\left(\frac{\tan x - x}{x^2}\right)$$

$$\Rightarrow e\lim_{x\to 0}\left(\frac{\sec^2 x - 1}{2x}\right) = e\lim_{x\to 0}\left(\frac{2secx.secx.tanx}{2}\right)$$

$$\Rightarrow e\lim_{x\to 0}\left(\frac{2\sec^2 x - tanx}{2}\right)$$

$$\Rightarrow 1$$

(v)

$$\lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x}$$

$$\lim_{x\to 0} \frac{a\sin x - 2\sin x\cos x}{\tan^3 x}$$

$$\lim_{x\to 0} \frac{\sin x(a - 2\cos x)}{\tan^3 x}$$

$$\lim_{x\to 0} \frac{\sin x[a - 2(1 - 2!x2 + \ldots..)]}{\tan^3 x}$$

$$\lim_{x\to 0} \frac{\boldsymbol{sinx}[(\boldsymbol{a} - 2) + \boldsymbol{x}2(1 + \boldsymbol{higher\ power\ of\ x})]}{\boldsymbol{tan}^3\boldsymbol{x}}$$

Hence for given limit to exist a−2=0⇒a=2, and corresponding limit is 1.

ANS:2

(vii)

$$\lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{sinx}$$

$$= \lim_{x\to 0}\frac{x}{sinx}\times x\times\sin\left(\frac{1}{x}\right)$$

$$=\lim_{x\to 0}\frac{1}{\frac{sinx}{x}}\times\lim_{x\to 0}x\times\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$$

$$=1\times 0\times\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$$

$$=0\ \left[-1\le\sin\left(\frac{1}{x}\right)\le 1\right]$$

$$\therefore\ \lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{sinx}=0.$$

(viii)

$$\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{1+\sqrt{\frac{x}{x^2}+\sqrt{\frac{x}{x^4}}}}}$$

$= \lim_{x\to\infty} \frac{1}{\left(\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}\right)}$

$= \lim_{y\to 0} \frac{1}{\sqrt{1+\sqrt{y+\sqrt{y^3}}}}$ Where, $y = \frac{1}{x}$ and as $x \to \infty, y \to 0$

$=\frac{1}{1}$

=1.

(ix)

$\lim_{x\to 0}(1-\cos^3x)/\tan^2x$

applied limit

$=(1-\cos^3 0)/\tan^2 0$

=(1-1)/0

=0/0

when we get 0/0 form we apply LA' Hopital rule

$\lim_{x\to 0} (1-\cos^3x)/\tan^2x$

$=\lim_{x\to 0}(0+3\cos^2x\sin x)/2\tan x\sec^2x$

$=\lim_{x\to 0}3\cos^2x\sin x/2(\sin x)/(\cos x).\sec^2x$

$=\lim_{x\to 0} 3\cos^2x/2\sec^3x$

$=3\cos^2 0/2\sec^3 0$

$=3/2$

(ans).

Question 19:

(i)

$(sinx)$^(cosx)

Let $y=(\sin x)^{\cos x}$

getting ln in both side

$\ln y= \cos x. \ln(\sin x)$

Differentiate both side w.r.t x

$1/y.dy/dx=\cos^2x/\sin x-\sin x.\ln(\sin x)$

$dy/dx=y(\cos^2x-\sin^2x.\ln(\sin x))/\sin x$

$=(\sin x)^{\cos x}.(\cos^2x-\sin^2x\ln(\sin x))/\sin x$

(ans)

(ii)

Let ,$y=x\sin^{-1}$ $z=\sin^{-1}x$

$y=(\sin z)^2$

$\log y=z \log \sin z$

both side differentiating

$\frac{1}{y}.\frac{dy}{dx}$=(log sinz+z $\frac{cosz}{sinz}$)

$$\frac{dy}{dx} = y(log\ log\, sinz + z\ cotz)$$

=xsin^{-1}(log sin^{-1}x $\frac{\sqrt{1-x^2}}{x}$)

(iii)

$y=(\sin x)^{\cos x} + (\cos x)^{sins}$

Where $u = (\sin x)^{\cos x}$, $v = (\cos x)^{sins}$

$\text{Log } u = \cos x \log \sin x$

$\frac{1}{u}. \frac{du}{dx} = \cos x \frac{\cos x}{\sin x} - \sin x \log x$

$\frac{du}{dz} = u(\cot x \cos x - \sin \log \sin x)$

$=(\sin x)^{\cos x}(\cot x \cos x - \sin x \log \sin x)$

Similarly, $\frac{dy}{dz} = \frac{du}{dx} + \frac{dv}{dx}$

$= (\sin x)^{\cos x}(\cot x \cos x - \sin x \log \sin x) + (\cos x)^{\sin x}(\cos x \log \cos x - \sin x \tan x)$

(iv)

x^x^x

Let, y=x^x^x

or, log y= log(x^x^x)

or, log y= x^x log x

or, log(log y)=log(x^x log(x))

or, log(log y)= log x^x+log(log x)

$or, \log(\log y)=x \log x+\log(\log x)$

Differentiated on both sides,

$or,(1/\log y)(1/y)(dy/dx)= x(1/x)+(\log x)1+(1/\log x)\ (1/x)$

$or,dy/dx=y \log y[1+\log x+1/x \log x]$

$or,dy/dx=(x^x^x)(x^x)\log x[1+\log x+1/x \log x]$

$or,dy/dx=(x^x^x)(x^x)[(\log x)^2+\log x+½]$ ***Ans.***

(v)

y log(xy)

Let, $x=y \log(xy)$

$or,x=y(\log x+\log y)$

$or,x=y \log x+$y log y

Differentiated on both sides,

$or,1=y\ d/dx \log x+dy/dx \log x+y(d/dx \log y) + dy/dx \log y$

$or,1=y(1/x)+dy/dx \log x+y(1/y)dy/dx+dy/dx \log y$

$or,1=y/x+dy/dx \log x+dy/dx+dy/dx \log y$

$or, 1-y/x=dy/dx(\log x+1+\log y)$

$or, (x-y)/x=dy/dx(\log xy+1)$

$or, dy/dx=(x-y)/x(\log xy+1)$ ***Ans***.

(vi)

Let $u = \tan^{-1}\frac{\sqrt{1+x^2-1}}{x}$

Substitute $x=\tan\vartheta$

$u=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$

$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$\Rightarrow u=\frac{\theta}{2}$

$\Rightarrow u=\frac{1}{2}(\tan^{-1}x)$ ....(1)

Let $v=(\tan^{-1}x)$ .....(2)

From (1) and (2),it follows

$\Rightarrow u=\frac{v}{2}$

$\Rightarrow \frac{du}{dv}=\frac{1}{2}$

ANS: $\frac{1}{2}$

(vii)

$\tan^{-1}\frac{1}{\sqrt{x^2-1}}$

$$y = \tan^{-1}\frac{1}{\sqrt{x^2-1}}$$

- $\frac{dy}{dx} = \frac{1}{(\frac{1}{\sqrt{x^2-1}})^2+1}\frac{-1}{2}(x^2-1)^{3/2}\,2x$
- $\frac{dy}{dx} = \frac{1}{\frac{1}{x^2-1}+1}\left(\frac{(1-x)}{(x^2-1)^{3/2}}\right)$
- $\frac{dy}{dx} = \frac{x^2-1}{x^2+1-1}\left(\frac{-x}{(x^2-1)^{3/2}}\right)$
- $\frac{dy}{dx} = \left(\frac{-x}{x^2(x^2-1)^{1/2}}\right)$
- $\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$

ANS: $\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$

(ix)

sin(logx)

Let y=sin(logx)

Differentiate both side with w.r.t x

dy/dx=d(sin(logx))/dx

=cos(logx).d(logx)/dx

=cos(logx)/xln10.

d(sin(logx))/dx=cos(logx)/xln10.

(ans).

(xi)

$$\tan^{-1}\frac{\cos}{1+\sin x}$$

$$=\frac{1}{1+\left(\frac{\cos x}{1+\sin x}\right)^2}\cdot\frac{(1+\sin x)\cdot(-\sin x)-\cos^2 x}{(1+\sin x)^2}$$

$$=\frac{(1+\sin x)^2}{(1+\sin^2 x)^2+\cos^2 x}.\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+sinx)^2}$$

$$=\frac{-(\sin x+\sin^2 x+\cos^2 x)}{(1+sinx)^2+\cos^2 x}$$

$$=\frac{-sinx}{1+2sinx+\sin^2 x+\cos^2 x}$$

$$=\frac{-(\sin x+1)}{2(1+\sin)}$$

$$=-\frac{1}{2}$$

(Ans.)

(xii)

$\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Let,

$$Y=\tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

$$\tan y=\sqrt{\frac{1-x}{1+x}}$$

$$\sec^2 x\ \frac{dy}{dx}=\frac{1}{2\sqrt{\frac{1-x}{1+x}}}\ [\frac{(1+x)(-1)-(1-x)}{(1+x)^2}]$$

$$\sec^2 x\ \frac{dy}{dx}=\frac{1}{2\sqrt{\frac{1-x}{1+x}}}\ \frac{-2}{(1+x)^2}$$

$$(1+\tan^2 x)\ \frac{dy}{dx}=\frac{-1}{(1+x)^2\sqrt{\frac{1-x}{1+x}}}$$

$$[1+(\frac{1-x}{1+x})\hat{}2\ ]\ \ )\ \frac{dy}{dx}=\frac{-1}{(1+x)^2\sqrt{\frac{1-x}{1+x}}}$$

$$\therefore\ \frac{dy}{dx}=\ \frac{-1}{2}\ \frac{1}{(1+x)^2\sqrt{\frac{1-x}{1+x}}}\qquad \text{(Ans)}$$

(xiii)

$\sin\sqrt{x}$

Solution : $\sin\sqrt{x}$

$=\frac{\cos\sqrt{x}}{2\sqrt{x}}$

$= ½\ \frac{\cos\sqrt{x}}{\sqrt{x}}$ (Ans)

(xiv)

$\sin(logx)$

$=\frac{\cos(logx)}{x}$ (Ans)

Question 20:

(a).

$Here,$

$y = e^{asin^{-1}x}$........................(i)

$\Rightarrow y_1 = \dfrac{a}{\sqrt{1-x^2}}\ \ e^{asin^{-1}x}$

$= \dfrac{y}{\sqrt{1-x^2}} \dots\dots\dots\dots\dots\dots\dots..(ii)$

$\Rightarrow (1-x^2){y_1}^2 = a^2y^2$

$Differentiating\ above\ eq^n\ w.r.t.x$

$(1-x^2)2y_1y_2 + {y_1}^2(-2x) = a^2 2yy_1$

$\Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0$

$\Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0$.....................$(iii)$

$Differentiating\ above\ eq^n\ w.r.t.x\ using\ leibnitz's\ theorem, we\ get -$

$[y_{n+2}(1-x^2) + n_{c_1}y_{n+1}(-2x) + n_{c_2}y_n(-2)] - \left(y_{n+1}x + n_{c_1}y_n.1 - a^2y_n\right) = 0$

$\Rightarrow y_{n+2}(1-x^2) + y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + xy_n) - a^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2x-1)^x y_{n+1} - (n^2+a^2)y_n = 0$

(Showed.)

(b).

If $\sin y = x\sin(a+y)$, *prove that*

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Given, $\sin y = x\sin(a+y)$

$$X = \frac{\sin y}{\sin(a+y)}$$

$$\frac{dx}{dy} = \frac{d}{dy}\left\{\frac{\sin y}{\sin(a+y)}\right\}$$

$$\frac{dx}{dy} = \frac{\sin(a+y)\frac{d}{dx}(\sin y) - \sin y\frac{d}{dx}\sin(a+y)}{\{\sin(a+y)\}^2}$$

$$\frac{dx}{dy} = \frac{\sin(a+y) - \sin y\cos(a+y)}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \cos(a+y)\sin y}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

[PROVED]

(c)

We have y= e^{ax} sinbx

dy/dx=ae^{ax} sinbx + e^{ax} bcosbx

$d^2y/dx^2 = a^2e^{ax}$ sinbx+abe^{ax} cosbx+ abe^{ax} cosbx-b^2e^{ax}sinbx y^2 2ay +($a^{2+}b^2$)y

=a^2e^{ax}sinbx+ abe^{ax} cosbx+ abe^{ax} cosbx-b^2e^{ax}sinbx-2a(ae^{ax} sinbx+be^{ax}cosbx)+

($a^{2+}b^2$) e^{ax}sinbx

=e^{ax}(a^2sinbx+2ab cosbx-b^2sinbx-2 a^2sinbx-2ab cosbx +($a^{2+}b^2$)sinbx)

=0

So, Y_2***-2a***y_1***+***($a^{2+}b^2$)y=0

[PROVED]

(d)

y=sin (a$\sin^{-1} x$) Then prove that $y_{n+2}(1-x^2)$-(2n+1) x y_{n+1}

we have ,

Y=sin($a\sin^{-1}x$)

Y1=cos ($a\sin^{-1}x$) $\frac{a}{\sqrt{1-x^2}}$

(1-x^2)$y_1^2 = a^2\cos 2\,(a\sin^{-1}x)$........(1)

Differenting avobe equation w.r.t.x

(1-x^2)$2y1y2 + y1(-2x) = a^2 2cos\,(a\sin^{-1}x)\;\; a\sin(a\sin^{-1}x)$

(1-x^2)$2y_2y_1 - 2xy_1^2 = 2y\cos(a\sin^{-1}x)\frac{1}{\sqrt{(1-x^2)}}$

(1-x^2)$2y_2y_1 - 2xy_1^2 = 2yy_1$

(1-x^2)$y_2 - xy_1 - y = 0 \ldots\ldots\ldots\ldots..(2)$

Using Leibnitz's theorem, We get----

$\left[y_{n+2}(1-x^2) + n_{c_1}y_{n+1}(-2x) + n_{c_2}y_n(-2)\right] - [y_{n+1}x + n_{c_1}y_n.1 -$ y_n]=0

$[y_{n+2}(1-x^2) - y_{n+1}\,2xn - 2n(n-1)y_n] - [y_{n+1}\,x + ny_n] - y_n = 0$

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+1)y_n = 0$

(showed)

(e).

Here,

$\sin^{-1}y = m\sin^{-1}x$

$\Rightarrow y = \sin(m\sin^{-1}x)$...................$(i)$

$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}}\cos(m\sin^{-1}x)$..............$(ii)$

$\Rightarrow (1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$

$\Rightarrow (1-x^2)y_1^2 = m^2[1-\sin^2(m\sin^{-1}x)]$

$\Rightarrow (1-x^2)y_1^2 = m(1-y^2)$………………$(iii)$

$differenting\ with\ respect\ to\ x, we\ get -$

$(1-x^2)2y_1y_2 + y_1^2(-2x) + m^2 2yy_1 = 0$

$\Rightarrow (1-x^2)y_2 - y_1x + m^2y = 0$

$using\ libnitz's\ theorem, we\ get -$

$[y_{n+2}(1-x^2) + n_{c_1}y_{n+1}(-2x) + n_{c_2}y_n(-2)] - \left(y_{n+1}x + n_{c_1}y_n.1 - m^2y_n\right) = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n+1)y_n - (y_{n+1}x + xy_n) + m^2y_n = 0$

$\Rightarrow y_{n+2}(1-x^2)y_{n+2} - (2x-1)^x y_{n+1} - (m^2-n^2)y_n = 0$

(Showed.)

(g).

$$\Rightarrow y_1 = e^{cos^{-1}} \frac{1}{-\sqrt{1-x^2}}$$

$$\Rightarrow y_1 = e^{cos^{-1}} \frac{y}{-\sqrt{1-x^2}}$$

$\Rightarrow (\sqrt{1-x^2})y_1 = -y$

$\Rightarrow (1-x^2)y_1 = -y^2$

$Differentiating\ above\ eq^n\ w.r.t\ x\ using\ leibnitz's\ theorem, we\ get -$

$[y_{n+2}(1-x^2) + n_{c_1}y_{n+1}(-2x) + n_{c_2}y_n(-2)] - \left(y_{n+1}x + n_{c_1}y_n.1 - y_n\right) = 0$

$\Rightarrow (y_{n+2}(1-x^2) - y_{n+1}2xn - 2n(n-1)y_n) - (y_{n+1}x + y_nn) - y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+1)y_n = 0$

(Showed.)

(h).

$y = \sin^{-1} x$

$y^{-1} = \dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)y_1^2 = 1$

$\Rightarrow (1-x^2)y_1^2 - 1 = 0$..................$(i)$

$Differentiating\ above\ eq^n\ w.r.t\ x$

$(1-x^2)2y_1y_2 + y_1^2(-2x) = 0$

$\Rightarrow (1-x^2)y_1 - xy_1 = 0$

$Applying\ Libnitz's\ theorem,$

$[y_{n+2}(1-x^2) + n_{c_1}y_{n+1}(-2x) + n_{c_2}y_n(-2)] - \left(y_{n+1}x + n_{c_1}y_n.1 - y_n\right) = 0$

$\Rightarrow (y_{n+2}(1-x^2) - y_{n+1}2xn - 2n(n-1)y_n) - (y_{n+1}x + y_n n) = 0$

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$