Differential Equation

- Differential equations are very important to solve many science and engineering problems.

- To describe various numerical methods for the solution of ordinary differential equation, we consider the general first order differential equation

$$\frac{dy}{dx} = y' = f(x, y)$$

$$\textit{with the initial condition} \quad y(x_0) = y_0$$

- This equations are not in closed form hence they are described in terms of x and y.

- This method can be applied to the solution of systems of first order differencial equations.

Differential Equation

- This will yield the solution in one of the two forms:
  - A series for y in terms of power of x, from which the value of y can be obtained by direct substitution.
    - Example: Methods of Taylor and Picard.
  - A set of tabulated values of x and y.
    - Example: Methods of Euler and Runge-Kutta.
    - These methods are called step-by-step methods or matching methods.
    - This is because the values of y are computed by short steps ahead for equal intervals h of the independent variable.
    - In the methods of Euler and Range-Kutta, h should be kept small.
    - These methods can be applied for tabulating y over a limited range only.

Solution by Taylor's Series

- Let us consider the differential equation

$$\left.\begin{aligned} y' &= f(x, y) \\ \textit{with the initial condition} \quad y(x_0) &= y_0 \end{aligned}\right\} \quad (1.1)$$

- If $y(x)$ is the exact solution of (1.1), then the Taylor's series for $y(x)$ gives us,

$$f(x) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

- Around $x = x_0$ is we get

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \cdots$$

- If the values of x_0, y_0', y_0'' ... are known then equation (1.2) gives a power series for y in terms of x.

Example

From the Taylor series for $y(x)$, find $y(0.1)$ correct to four decimal places if $y(x)$ satisfies $y' = x - y^2$ and $y(0) = 1$

The Taylor series for $y(x)$ is given by

$$y(x) = 1 + xy_0' + \frac{x^2}{2} y_0'' + \frac{x^3}{6} y_0^{\,lll} + \frac{x^4}{24} y_0^{\,iv} + \frac{x^5}{120} y_0^{\,v} + \cdots$$

The derivatives y'_0, y''_0, ...etc are obtained by

$$y'(x) = x - y^2 \qquad y_0' = -1$$

$$y''(x) = 1 - 2yy' \qquad y_0'' = 3$$

$$y^{lll}(x) = \; -2yy'' - 2y'^2 \qquad y^{lll}{}_0 = -8$$

$$y^{iv}(x) = -2yy^{lll} - 6y'y'' \qquad y^{iv}{}_0 = 34$$

$$y^{v}(x) = -2yy^{iv} - 8y'y^{lll} - 6y''^2 \qquad y^{v}{}_0 = -186$$

Example: Cont.

Using these values, the Taylor series becomes

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 - \frac{17}{12}x^4 - \frac{31}{20}x^5 + $$

To obtain the value of $y(0.1)$, it is found that the terms upto x^4 should be considered.

Then, $y(0.1) = 0.9138$

Class Work

Given $\dfrac{dy}{dx} - 1 = xy$ and $y(0)=1$, obtain the Taylor series for $y(x)$ and compute $y(0.1)$ correct to four decimal places.

Picard's Method of Successive Approximation

- Let us consider the differential equation

$$\left.\begin{aligned} y' &= f(x,y) \\ \textit{with the initial condition}\quad y(x_0) &= y_0 \end{aligned}\right\} \quad (1.1)$$

- Therefore,

$$\frac{dy}{dx} = f(x,y)$$

$$\Rightarrow dy = f(x,y)dx = \left(\frac{dy}{dx}\right)dx$$

- Integrating this between corresponding limits for x and y, we have

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)dx = \int_{x_0}^{x} \left(\frac{dy}{dx}\right)dx$$

Picard's Method of Successive Approximation

- We know that

$$\left.\begin{array}{r} y' = f(x, y) \\ \text{with the initial condition } y(x_0) = y_0 \end{array}\right\} \quad (1.1)$$

- Integrating this equation (1.1), we have

$$\int_{y_0}^{y} dy = y - y_0 = \int_{x_0}^{x} f(x, y)\,dx,$$

$$Therefore \quad y = y_0 + \int_{x_0}^{x} f(x, y)\,dx = y_0 + \int_{x_0}^{x} \left(\frac{dy}{dx}\right) dx \quad (1.2)$$

- Equation (1.2) has y under the integral sign as well as outside it.

Picard's Method of Successive Approximation

We have seen equation (1.2) is like

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx$$

- An equation of this kind is called an integral equation.
- They can be solved by a process of successive approximations, or iteration, if the indicated integration can be performed in the successive steps.
- To solve $dy/dx = f(x, y)$ by Picard's method of successive approximations of $f(x, y)$.

Picard's Method of Successive Approximation

- In Picard's method, the initial condition is $y^{(0)} = y_0$
- The first approximation is obtained by using the value of y_0

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

- The integrand is now a function of x alone (since y_0 is known), so the indicated integration can be performed.
- Having now a first approximation to y, we substitute it for y in the integrand of (1.2) and integrate again.

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

- Thus, obtaining a second approximation of y.

Picard's Method of Successive Approximation

- The process is repeated in this way as many times as may be necessary or desirable.
- The n^{th} approximation is given by the equation

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\,dx$$

Example

Solve the equation $y' = x + y^2$ subject to the condition $y = 1$ when $x = 0$ using Picard's method (evaluate up to the second approximation).

Solution

Initial condition is $y^{(0)} = y_0 = 1$

First approximation is,

$$y^{(1)} = y_0 + \int_{x_0}^{x} f\left(x, y_0\right) dx$$

$$= y_0 + \int_{x_0}^{x} \left(\frac{dy}{dx}\right) dx$$

$$= y_0 + \int_{0}^{x} \left(x + (y_0)^2\right) dx = 1 + \int_{0}^{x} \left(x + (1)^2\right) dx = 1 + \frac{1}{2}x^2 + x$$

Example (Cont.)

Second approximation is,

$$y^{(2)} = y_0 + \int_0^x f\left(x, y^{(1)}\right)dx$$

$$= y_0 + \int_{x_0}^x \left(\frac{dy}{dx}\right)dx$$

$$= y_0 + \int_0^x \left(x + (y^{(1)})^2\right)dx$$

$$= 1 + \int_0^x \left(x + \left(1 + x + \frac{1}{2}x^2\right)^2 \right)dx$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5$$

Class Work

Solve the equation $$\frac{dy}{dx} = x + y$$

subject to the condition $y = 1$ when $x = 0$ using Picard's method. Find the value of y when $x = 0.1$ correct to four decimal places. Find y up to the 3rd approximation.

Answer: $y(0.1)=1.1103$

Home Work

Given the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2+1}$

with the initial condition $y = 0$ when $x = 0$, use Picard's method to obtain y for $x = 0.25$, 0.5 and 1.0 correct to three decimal places and up to the 2^{nd} approximation of y.

Answer:

$y(0.25)=0.005$

$y(0.5)=0.042$

$y(1.0)=0.321$

Home Work

Using Picard's method, obtain the solution of

$$\frac{dy}{dx} = x(1 + x^3 y), \quad y(0) = 3$$

Home Work

Given $\dfrac{dy}{dx} - 1 = xy$ and y(0)=1, compute y(0.1) correct to four decimal places using Picard's method.

Euler's Formula

- Suppose that we wish to solve the equation

$$\left.\begin{aligned} y' &= f(x, y) \\ \textit{with the initial condition} \quad y(x_0) &= y_0 \end{aligned}\right\} \quad (1.1)$$

- We have to solve the equation for values of y at x (where $x = x_r = x_0 + rh$ ($r = 1, 2, \ldots$)).

- Integrating (1.1) at $x = x_1$, we obtain (seen in Picard's method)

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\,dx$$

- Assuming that $f(x, y) = f(x_0, y_0)$ in $x_0 \le x \le x_1$

- This give Euler's formula $y_1 = y_0 + hf(x_0, y_0)$

Euler's Formula

- Similarly for the range $x_1 \le x \le x_2$, we have $y_2 = y_1 + \int_{x_1}^{x_2} f(x, y)dx$
- Substituting $f(x_1, y_1)$ for $f(x, y)$ in $x_1 \le x \le x_2$ we obtain

$$y_2 = y_1 + hf(x_1, y_1)$$

- Proceeding in this way, we obtain the general formula

$$y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0, 1, 2, \ldots$$

- This process is very slow.
- To obtain reasonable accuracy with Euler's method, we need to take a smaller value for h.

Example

Consider the Euler's method, we consider the differential equation $y' = -y$ with the condition $y(0) = 1$ and $h = .01$. Find y_1, y_2, y_3 and y_4.

Solution

Successive application of the equation is

$$y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0,1,2,\cdots$$

With $h = 0.01$ gives

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.01(-1) = 0.99$$

$$y_2 = y_1 + hf(x_1, y_1) = 0.99 + 0.01(-0.99) = 0.9801$$

$$y_3 = y_2 + hf(x_2, y_2) = 0.9801 + 0.01(-0.9801) = 0.9703$$

$$y_4 = y_3 + hf(x_3, y_3) = 0.9703 + 0.01(-0.9703) = 0.9606$$

Class Work

Using Euler's Formula solve the following differential equation

(i) $$\frac{dy}{dx} + 2y = 0, \quad y(0) = 1$$

(ii) $$\frac{dy}{dx} - 1 = y^2, \quad y(0) = 0$$

In each case take $h = 0.1$ and obtain $y(0.1)$, $y(0.2)$ and $y(0.3)$

Home Work

Solve by Euler's method the equation

$$\frac{dy}{dx} = x + y, \quad y(0) = 0$$

Choose h = 0.2 and compute y(0.4) and y(0.6)

Modified Euler's Formula

- Since the Euler's method is slow and not so helpful for practical uses, there exists a modified Euler's Formula.

- We will use Euler's formula to start our initial guess of y_n taking

$$y_{n+1}^{(0)} = y_n + hf(x_n, y_n)$$

- For example, we start guessing y_1 by the initial guess of $y_1^{(0)}$ as

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

- To calculate the successive approximations of y_1, instead of approximating $f(x, y)$ by $f(x_0, y_0)$ we now approximate the integral

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\,dx$$

by means of trapezoidal rule to obtain y_1, where

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

Modified Euler's Formula

- Thus obtain the iteration formula for the $(m+1)$th approximation of y_1 as

$$y_1^{(m+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(m)}\right)\right] \quad ,m = 0,1,2,\cdots$$

- Similarly, the general formula for the $(m+1)$th approximation of y_{n+1} becomes

$$y_{n+1}^{(m+1)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y_{n+1}^{(m)}\right)\right]$$

Example

Determine the value of y when $x = 0.1$ given that $y(0) = 1$,
$y' = x + y$ and $h = 0.05$

Solution

$y_{n+1}^{(0)} = y_n + hf(x_n, y_n)$ and $y_{n+1}^{(m+1)} = y_n + h[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(m)})]/2$

Given, $f(x_0, y_0) = y_0' = x_0 + y_0 = 0 + 1 = 1$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.05)(1) = 1.05$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(0)})\right)$$

$$= y_0 + \frac{h}{2}\left((x_0 + y_0) + (x_1 + y_1^{(0)})\right)$$

$$= 1 + \frac{(0.05)}{2}\left([0+1] + [0.05 + 1.05]\right) = 1.0525$$

Example (Cont.)

$$y_1^{(2)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(1)})\right)$$

$$= 1 + \frac{(0.05)}{2}\left([0+1] + [0.05 + 1.0525]\right) = 1.05256$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(3)})\right)$$

$$= 1 + \frac{(0.05)}{2}\left([0+1] + [0.05 + 1.05256]\right) = 1.05256$$

Since $y_1^{(3)}$ is same as $y_1^{(2)}$ we can get no further change in y by continuing the approximations.

We therefore take $y_1 = 1.0526$ $\left(\dfrac{dy}{dx}\right)_1 = f(x_1, y_1) = x_1 + y_1 = 1.1026$

Example (Cont.)

We get, y_1 = 1.0526, therefore,

$$f(x_1, y_1) = \left(\frac{dy}{dx}\right)_1 = x_1 + y_1 = 0.05 + 1.0526 = 1.1026$$

$$y_2^{(0)} = y_1 + hf(x_1, y_1) = 1.0526 + (0.05)(1.1026) = 1.1103$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y_2^{(0)})\right)$$

$$= 1.0526 + \frac{(0.05)}{2}\left(1.1026 + [0.1 + 1.1103]\right) = 1.1104$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y_2^{(1)})\right)$$

$$= 1.0526 + \frac{(0.05)}{2}\left(1.1026 + [0.1 + 1.1104]\right) = 1.1104$$

Therefore, $y = 1.1104$

Home Work

Determine the value of y when $x = 0.1$ given that $y(0)=1$

and $y' = x^2 + y$ and $h = 0.05$

Answer: 1.1055