Vector spaces

Let, F be a field of scalars and V be a non empty set of vectors. If V contains the following rules of vector addition and scalar multiplication then V is called vector space over F.

In vector addition:

I.) $\forall u, v \in V \Rightarrow u+v \in V$

II.) $\forall u, v, w \in V \Rightarrow (u+v)+w = u+(v+w)$.

III.) $\exists\, 0 \in V \Rightarrow u+0 = 0+u = u, \forall u \in V$, Here 0 is null vect.

IV.) $u \in V \Rightarrow \exists -u \in V$ where $u+(-u) = (-u)+u = 0$.

(V) $\forall u, v \in V \Rightarrow u+v = v+u$.

In scalar multiplication.

(VI) $\forall a \in F$ and $\forall u \in V \Rightarrow au \in V$

(VII) $\forall a \in F$ and $\forall u, v \in V \in V \Rightarrow a(u+v) = au+av$

(VIII) $\forall a, b \in F$ and $\forall u \in V \Rightarrow (a+b)u = au+bu$

(IX) $\forall a, b \in F$ and $\forall u \in V \Rightarrow (ab)u = a(bu)$.

(X) $\exists\, 1 \in F (1 \neq 0) \Rightarrow 1u = u.1 = u, \forall u \in V$.

The vector space V in field F is denoted by $V(F)$

$A = 1\hat{i} - 2\hat{j} - 3\hat{k}$

$B = 2\hat{i} + 3\hat{j} + 5\hat{k}$.

$\overrightarrow{AB} = 1\hat{i} + 5\hat{j} + 8\hat{k}$.

$|\overrightarrow{AB}| = \sqrt{1^2 + 5^2 + 8^2}$

$= \sqrt{1 + 25 + 64}$

$= \sqrt{90}$

$= 3\sqrt{10}$.

Vector in $\mathbb{R}^2$:-

The set of all ordered pair of real numbers is called two dimensional real space and is denoted by $\mathbb{R}^2$.

$\therefore \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ where

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

vector in $\mathbb{R}^3$:

The set of all ordered triplets of real numbers is called three dimentional real space and is denoted by $\mathbb{R}^3$.

$\therefore \mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ where

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

n-tuple :-

If n is positive integer then a sequence of n real numbers $(a_1, a_2, \ldots a_n)$ is called real ordered n-tuple.

when $n = 2$ it is called ordered pair.

" $n = 3$ " " " triple or triplet.

Vector in $\mathbb{R}^n$:-

If n is a positive integer then the set of all ordered n-tuples of real numbers is called real n-space and is denoted by $\mathbb{R}^n$.

and if $u \in \mathbb{R}^n$, $u = (u_1, u_2, \ldots u_n)$ then u is called the n-dimentional vector in $\mathbb{R}^n$.

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Addition of two vector in $\mathbb{R}^n$.

Let, $u, v \in \mathbb{R}^n$ where,

$u = (u_1, u_2 \ldots u_n)$ and.

$v = (v_1, v_2 \ldots v_n)$.

Then the sum of u and v, denoted by $u+v$ and is defined by.

$$u+v = (u_1+v_1, u_2+v_2, \ldots u_n+v_n)$$

For Example :-

$u = (1, 2, 5)$

$v = (0, -3, 5)$

then, $u+v = (1+0, 2-3, 5+5)$

$= (1, -1, 10)$

Difference of two vector in $\mathbb{R}^n$:

Let, $u, v \in \mathbb{R}^n$ where,

$u = (u_1, u_2 \ldots u_n)$

$v = (v_1, v_2 \ldots v_n)$.

$u - v = (u_1 - v_1, u_2 - v_2 \ldots u_n - v_n)$:

Example :-

$u = (1, 2, 5)$

$v = (0, -3, 5)$

Then, $u - v) = (1 - 0, 2 + 3, 5 - 5)$

$= (1, 5, 0)$.

Scalar multiplication $\mathbb{R}^n$

Let, $u \in \mathbb{R}^n$, where, $u = (u_1, u_2 \ldots u_n)$ then scalar multiplication of u by K is denoted by Ku and is defined by

$Ku = (Ku_1, Ku_2, \ldots Ku_n)$, where $K \in R$.

Zero Vector :-

The vector whose entries (components) are all zero is called the zero vector.

$0 = (0, 0, \ldots 0)$

Dot or inner product of non-zero vector

Let,

u and v be two non zero vectors and θ be the angle between them then the dot product of u and v is denoted by $u.v$ and defined by.

$$u.v = \|u\| \|v\| \cos\theta$$

Dot or inner product in $\mathbb{R}^2$

Let, $(u,v) \in \mathbb{R}^2$ where,

$$u = (u_1, u_2), \quad v = (v_1, v_2).$$

Then the dot or inner product of u and v denoted by $u.v$ and defined by.

$$u.v = u_1v_1 + u_2v_2$$

Dot or inner product in $\mathbb{R}^n$

Let, $(u,v) \in \mathbb{R}^n$ where.

$$u = (u_1, u_2 \cdots u_n) \text{ and } v = (v_1, v_2 \cdots v_n),$$

Then the dot or inner product of u and v denoted by $u.v$ and defined by

$$u.v = u_1v_1 + u_2v_2 \cdots + u_nv_n.$$

Distance in $\mathbb{R}^n$:-

Let, $u, v \in \mathbb{R}^n$, where,

$$u = (u_1, u_2, \dots u_n) \text{ and } v = (v_1, v_2 \dots v_n).$$

Then the distance of u and v is denoted by $d(u,v)$ or defined by.

$$d(u,v) = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \dots (u_n-v_n)^2}$$

Norm or length in $\mathbb{R}^n$:-

let, $u \in \mathbb{R}^n$, where,

$$u = (u_1, u_2 \dots u_n).$$

Then the norm/length of u is denoted by $\|u\|$ and define by.

$$\|u\| = \sqrt{u \cdot u}$$

$$= \sqrt{u_1^2 + u_2^2 + u_3^2 \dots + u_n^2}.$$

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Unit vector in $\mathbb{R}^n$:-

A vector $\bar{e} \in \mathbb{R}^n$ is called a unit vector if $\|\bar{e}\| = 1$.

Let $\bar{u}$ be the any non-zero vector in $\mathbb{R}$ then the unit vector of $\bar{u}$ is denoted by $\bar{e}_u$ and is defined by $\bar{e}_u = \dfrac{\bar{u}}{\|\bar{u}\|}$.

If $\underline{u} = (1, 2, 3)$ then find unit vector

$$\|\underline{u}\| = \sqrt{1^2 + 2^2 + 3^2}$$

$$= \sqrt{1+4+9}$$

$$= \sqrt{14}.$$

$$\text{unit vector} = \frac{\underline{u}}{\|\underline{u}\|}$$

$$= \frac{1\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}}$$

$$= \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}.$$

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n-dimensional complex space :-

If n is the positive integer the set of all tuplets of complex number is called complex n space. The n dimensional complex space is denoted by $\mathbb{C}^n$, which is defined by

$$\mathbb{C}^n = \{(z_1, z_2 \ldots z_n) : z_i \in \mathbb{C},\ i = 1, 2\}.$$

Dot or inner product in c^n :-

Let, $\bar{u}$ and $\bar{v} \in c^n$ where

$\bar{u} = (u_1, u_2 \ldots u_n)$ and

$\bar{v} = (v_1, v_2 \ldots v_n)$ then

the dot (or inner) product of $\bar{u}$ and $\bar{v}$ is defined by followings.

$$\bar{u} \cdot \bar{v} = u_1 \cdot \bar{v}_1 + u_2 \bar{v}_2 \rightarrow \ldots u_n \bar{v}_n.$$

since, $\bar{u} \cdot \bar{v} \neq \bar{v} \cdot \bar{u}$

$$\bar{v} \cdot u = v_1 \bar{u}_1 + \ldots v_n \bar{u}_n$$

Norm in C^n:

Let, $\underline{u} \in C^n$ where

$\underline{u} = (u_1, u_2 \ldots u_n)$.

then the ~~vector~~ norm of $\underline{u}$ is denoted by

$$\|\underline{u}\| = \sqrt{u_1\bar{u}_1 + u_2\bar{u}_2 + \ldots u_n\bar{u}_n}.$$

$$= \sqrt{(|u_1|)^2 + (|u_2|)^2 + (|u_3|)^2 \ldots + (|u_n|)^2}.$$

vv.g

Theorem:

Cauchy-schwarz inequality:-

$\forall \underline{u}, \underline{v} \in \mathbb{R}^n$, prove that.

$$|\underline{u} \cdot \underline{v}| \leq \|\underline{u}\| \cdot \|\underline{v}\|$$

proof:-

If $\underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$

then $\|\underline{u}\| = 0$ and $\|\underline{v}\| = 0$.

$$|\underline{u}\cdot\underline{v}| \le \|\underline{u}\|\cdot\|\underline{v}\| .$$

$$\Rightarrow \quad 0 \le 0$$

$$\therefore \quad |\underline{u}\cdot\underline{v}| \le \|\underline{u}\|\cdot\|\underline{v}\|$$

Let, $\underline{u} = (u_1, u_2 \ldots u_n)$ and

$\underline{v} = (v_1, v_2 \ldots v_n)$

$$\underline{u}\cdot\underline{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

$$\le |u_1v_1| + |u_2v_2| + \cdots + |u_nv_n|$$

modulus inequality

$$\therefore \quad |\underline{u}\cdot\underline{v}| \le \sum_{i=1}^{n} |u_iv_i| \quad \text{———} \quad (1)$$

Now, $(a+b)^2 \ge 0 \quad \forall a, b \in \mathbb{R}$

$$\Rightarrow 2ab \le a^2+b^2 \quad \text{———} \quad (2)$$

Let, $a = \dfrac{|u_1|}{\|u_1\|}$, $b = \dfrac{|v_1|}{\|v_1\|}$ in eqn (2) &

we get,

$$2\cdot\frac{|u_1|}{\|u_1\|}\cdot\frac{|v_1|}{\|v_1\|} \le \frac{|u_1|^2}{\|u_1\|^2} + \frac{|v_1|^2}{\|v_1\|^2} \quad \text{———} \quad (3)$$

By the derivation of the norm of a vector in $\mathbb{R}^n$, we get.

$$\|u\|^2 = \sum_{i=1}^{n} u_i^2 = \sum_{i=1}^{n} |u_i|^2 \text{ and}$$

$$\|v\|^2 = \sum_{i=1}^{n} v_i^2 = \sum_{i=1}^{n} |v_i|^2$$

from eq^n (3) $\Rightarrow$

$$2\,\frac{\sum_{i=1}^{n} |u_i||v_i|}{\|\underline{u}\|\,\|\underline{v}\|} \leq \frac{\sum_{i=1}^{n} |u_i|^2}{\|\underline{u}\|^2} + \frac{\sum_{i=1}^{n} |v_i|^2}{\|\underline{v}\|^2}$$

$$\Rightarrow 2\,\frac{\sum_{i=1}^{n} |u_i v_i|}{\|\underline{u}\|\,\|\underline{v}\|} \leq \frac{\|\underline{u}\|^2}{\|\underline{u}\|^2} + \frac{\|\underline{v}\|^2}{\|\underline{v}\|^2}$$

$$\Rightarrow \sum_{i=1}^{n} |u_i v_i| \leq \|\underline{u}\|\,\|\underline{v}\|$$

$$\therefore |uv| \leq \|\underline{u}\|.\|\underline{v}\|$$

<u>proved</u>.

If $\underline{u}, \underline{v} \in \mathbb{R}^2$: $\underline{u} = (5, -2)$ and $\underline{v} = (3, 4)$

then prove that,

$$\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$$

$$\underline{u} + \underline{v} = (5-2) + (3, 4)$$

$$= (8, 2).$$

$$\|\underline{u} + \underline{v}\| = \sqrt{8^2 + 2^2}$$

$$= \sqrt{68}$$

$$= 2\sqrt{17}.$$

Again $\|\underline{u}\| = \sqrt{5^2 + (-2)^2}$

$$= \sqrt{29}.$$

$$\|\underline{v}\| = \sqrt{3^2 + 4^2}$$

$$= 5.$$

$\therefore$ $\|\underline{u}\| + \|\underline{v}\| = 5 + \sqrt{29}$:

Since, $2\sqrt{17} < 5 + \sqrt{29}$

$\therefore$ $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$

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M2 Find, $\underline{u}\cdot\underline{v}$ and $\underline{v}\cdot\underline{u}$ where.

(i) $\underline{u} = (3-2i,\ 4i,\ 1+6i)$, $\underline{v} = (5+i,\ 2-3i,\ 7+2i)$

(ii) $\underline{u} = (1+i,\ 2,\ 3-i)$, $\underline{v} = (-i,\ 4,\ 2+i)$.

i.) $\underline{u}\cdot\underline{v} = (3-2i)(5-i) + (4i)(2+3i) + (1+6i)(7-2i)$

$= 15 - 3i - 10i + 2i^2 + 8i + 12i^2 + 7 - 2i + 42i - 12i^2$

$= 15 - \cancel{13i} - 2 + \cancel{8i} - \cancel{12} + 7 + 40i + \cancel{12}$

$= 20 + 40i$

(i) $\underline{v} \cdot \underline{u} = (5+i) \cdot (3+2i) + (2-3i) \cdot (0-4i)$
$+ (7+2i)(1-6i)$.

$= 15 + 10i + 3i + 2i^2 + 0 - 8i - 0 + 12i^2$
$+ 7 - 42i + 2i - 12i^2$

$= 15 + 13i - 2 - 8i - 12 + 7 - 40i + 12$.

$= 20 - 35i$

$= 5(4 - 7i)$.

ii-) $\underline{u} \cdot \underline{v} = (1+i)(0+i) + (2+0i)(4-0i)$
$+ (3-i)(2-i)$

$= i + i^2 + 8 + 6 - 3i - 2i + i^2$

$= i - 1 + 14 - 5i - 1$

$= 12 - 4i$

$= 4(3 - i)$

ii.)

$\underline{v} \cdot \underline{u} = (0-i)(1-i) + (9+0i)(3-0i) + (2+i)(3+i)$

$= -i + i^2 + 8 + 6 + 2i + 3i + i^2$

$= -i - 1 + 19 + 5i - 1$

$= 12 + 4i$

$= 4(3+i)$.

Q.

If $\underline{u} = (3-7i, 2i, -1+i)$ and

$\underline{v} = (9-i, 11+2i, 8-3i)$.

Find $\|\underline{u} - \underline{v}\|$ and $\underline{u} \cdot \underline{v}$.

$\underline{u} - \underline{v} = (3-7i, 2i, -1+i) - (9-i, 11+2i, 8-3i)$

$= (-1-6i, -11, -9+4i)$.

$\therefore \|\underline{u} - \underline{v}\| = \sqrt{(-1-6i)^2 + (-11)^2 + (-9+4i)^2}$

$$= \sqrt{(-1)^2 - 2(-1)(6i) + (6i)^2 + 121 + (-9)^2 + 2(-9).4i + (4i)^2}$$

$$= \sqrt{1 + 12i + 36i^2 + 121 + 81 - 72i + 16i^2}$$

$$= \sqrt{1 + 12i - 36 + 121 + 81 - 72i - 16}$$

$$||\underline{u} - \underline{v}|| = \sqrt{1 + 36 + 121 + 81 + 16}$$

$$= \sqrt{255}$$

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ii.) From the definition of field we know.

$\exists\, 0 \in F \Rightarrow 0+k = k+0 = k.$

$\forall k \in F.$

If $k=0$. then $0+0=0$

$\Rightarrow (0+0)\underline{u} = 0\underline{u}$

$\Rightarrow 0\underline{u}+0\underline{u} = 0\underline{u}$ $[\because (a+b)\underline{u} = a\underline{u}+b\underline{u}]$

$\Rightarrow 0\underline{u}+0\underline{u}+(-0\underline{u}) = 0\underline{u}+(-0\underline{u})$

$\Rightarrow 0\underline{u}+\underline{0} = \underline{0}$ $[\because \underline{u}+(-\underline{u}) = \underline{0}]$.

$\Rightarrow 0\underline{u} = \underline{0}$ $[\because \underline{u}+\underline{0} = \underline{u}]$.

proved.

iii.) Let,

$k\underline{u} = \underline{0}$ and $k \neq 0 \Rightarrow k^{-1} \in F$ and $kk^{-1}=1$

Now, $\underline{u} = 1\underline{u} = (kk^{-1})\underline{u} = k^{-1}(k\underline{u})$ $[\because ab(\underline{u}) = a(b\underline{u})]$

$\Rightarrow \underline{u} = k^{-1}\underline{0}$.

$\underline{u} = \underline{0}$ $[\because k\underline{0} = \underline{0}]$.

Again Let,

$k\underline{u} = \underline{0}$ and $\underline{u} \neq \underline{0}$.

Now, if possible $k \neq 0$ then $kk^{-1}=1$, $k^{-1} \in F$

where, $\underline{u} = 1\underline{u} = (kk^{-1})\underline{u}$

$\Rightarrow \underline{u} = k^{-1}(k\underline{u})$

$= k^{-1}\underline{0}$.

$\therefore \underline{u} = \underline{0}$.

but if it is possible becomes $\underline{u} \neq 0$.

i.e $k \neq 0$ is not true

$\therefore k = 0$ proved.

iv.) $k\underline{0} = \underline{0}$

$\Rightarrow k(\underline{u} + (-\underline{u})) = \underline{0}$ $[\because \underline{u} + (-\underline{u}) = \underline{0}]$.

$\Rightarrow k\underline{u} + k(-\underline{u}) = \underline{0}$ $[\because a(\underline{u} + \underline{v}) = a\underline{u} + a\underline{v}]$.

$\Rightarrow k\underline{u} + k(-\underline{u}) = k\underline{u} + (-k\underline{u})$.

$\Rightarrow k(-\underline{u}) = (-k\underline{u})$. —— (7)

again,

$0\underline{u} = \underline{0}$

$\Rightarrow (k + (-k))\underline{u} = \underline{0}$ $[\because k + (-k) = 0]$.

$\Rightarrow k1 + (-k)\underline{u} = \underline{0}$

$\Rightarrow k\underline{u} + (-k)\underline{u} = k\underline{u} + (-k\underline{u})$

$\Rightarrow (-k)\underline{u} = -k\underline{u}$ —— (8)

From (i) and (ii) we get.

$$k(-\underline{u}) = k(-\underline{u}) = -k\underline{u}$$

proved

v.) We know,

$$\underline{u} - \underline{v} = \underline{u} + (-\underline{v}).$$

and $k(-\underline{v}) = -k\underline{v}$.

Now,

$$k(\underline{u} - \underline{v}) = k(\underline{u} + (-\underline{v}))$$

$$= k\underline{u} + k(-\underline{v}) \quad [\because a(\underline{u}+\underline{v}) = a\underline{u} + a\underline{v}]$$

$$= k\underline{u} - k\underline{v}$$

$$\therefore k(\underline{u} - \underline{v}) = k\underline{u} - k\underline{v}$$

proved.

If S and T are two subspaces of a vector space V(F) then show that $S \cap T$ is a subspace of V(F).

Soln:- Since S and T are two subspaces.

So, $\underline{0} \in S$ and $\underline{0} \in T$, it implies that $\underline{0} \in S \cap T$.

i.e $S \cap T \neq \phi$

let,

$\underline{u}, \underline{v} \in S \cap T$ and $\alpha, \beta \in F$.

$\Rightarrow \underline{u}, \underline{v} \in S$ and $\underline{u}, \underline{v} \in T$.

$\Rightarrow \alpha \underline{u} + \beta \underline{v} \in S$ and $\alpha \underline{u} + \beta \underline{v} \in T$ (since S and T are subsp.)

$\Rightarrow \alpha \underline{u} + \beta \underline{v} \in S \cap T$.

So that $S \cap T$ is a sub space of $V(F)$ (proved)

If S and T are two subspaces of a vector space $V(F)$ then show that $S+T$ is a subspace of $V(F)$.

Soln:- Since S and T are subspaces.

So that,

$\underline{0} \in S$ and $\underline{0} \in T$.

$\Rightarrow \underline{0} + \underline{0} \in S+T$.

$\Rightarrow \underline{0} \in S+T$.

i.e $S+T \neq \phi$.

let, $\underline{u}, \underline{v} \in S+T$, where, $\underline{u} = \underline{u}_1 + \underline{u}_2$, $\underline{v} = \underline{v}_1 + \underline{v}_2$.

and

$u_1, v_1 \in S$, $u_2, v_2 \in T$.

Now, $\alpha, \beta \in F$.

$\alpha \underline{u}_1 + \beta \underline{v}_1 \in S$ and $\alpha u_2 + \beta v_2 \in T$.

$\Rightarrow (\alpha \underline{u}_1 + \beta v_1) + (\alpha u_2 + \beta v_2) \in S+T$.

$\Rightarrow \alpha (\underline{u} + u_2) + \beta (v_1 + v_2) \in S+T$.

$\Rightarrow \alpha \underline{u} + \beta \underline{v} \in S+T$.

so that $S+T$ is a subspace of $V(F)$

proved

Prove that a subset W of a vector space $V(F)$ is called a subspace of $V(F)$ if and only if.

i.) W is non-empty i.e. $W \neq \phi$

ii.) W is closed under vector addition i.e $\forall \underline{u}, \underline{v} \in W \Rightarrow \underline{u} + \underline{v} \in W$.

iii.) W is closed under scalar multiplication i.e $\forall k \in F$ and $\underline{u} \in W$ $k\underline{u} \in W$.

<u>Soln:</u> - let,

the subset satisfies the following three

i.) $W \neq \phi$

ii.) $\forall \underline{u}, \underline{v} \in W \Rightarrow \underline{u} + \underline{v} \in W$.

iii.) $\forall \underline{u} \in W$ and $k \in F \Rightarrow k\underline{u} \in W$.

In this case, we have to show that W is subspace that is satisfies all laws of vector space since W is a subset of V, so W satifies the following laws:-

A_1 : $\forall \underline{u}, \underline{v}, \underline{w} \in W \Rightarrow (\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$

A_2 : $\forall \underline{u}, \underline{v} \in W \Rightarrow \underline{u} + \underline{v} = \underline{v} + \underline{u}$

M_1 : $\forall k \in F$ and $\underline{u}, \underline{v} \in W \Rightarrow k(\underline{u} + \underline{v}) = k\underline{u} + k\underline{v}$.

M_2: $\forall\ k_1, k_2 \in F$ and $\forall\ \underline{u} \in W \Rightarrow (k_1 + k_2)\underline{u} = k_1\underline{u} + k_2\underline{u}$

M_3: $\forall\ k_1, k_2 \in F$ and $\forall\ \underline{u} \in W \Rightarrow (k_1 k_2)\underline{u} = k_1(k_2\underline{u})$

M_4: $1 \in F$ and $\forall\ \underline{u} \in W \Rightarrow 1 \cdot \underline{u} = \underline{u}$.

Again, $\underline{u} \in W$ and $0\underline{u} \in W$ (where 0 is scalar)

$\Rightarrow \underline{0} \in W$ where $\underline{u} + \underline{0} = \underline{u}$, $\forall \underline{u} \in W \Rightarrow (-1)\underline{u} \in W$

$\Rightarrow -\underline{u} \in W$ where $\underline{u} + (-\underline{u}) = \underline{0}$

Here, W satifies all laws of vector space. So W is a subset of $V(F)$.

Conversely, if W is a subspace of $V(F)$ then W satifies the all laws of vector space. that is.

i.) $W \neq \emptyset$, ii) $\underline{u}, \underline{v} \in W \Rightarrow \underline{u} + \underline{v} \in W$.

iii.) $\forall\ k \in F$ and $\forall\ \underline{u} \in W \Rightarrow k\underline{u} \in W$

(proved)

Determine wheather the following sets are subspace $\mathbb{R}^3$.

i.) $\{(a, b, 1) : a, b \in \mathbb{R}\}$

ii.) $\{(a, b, c) : a+b+c = 0;\ a, b, c \in \mathbb{R}\}$.

iii.) $\{(a, b, c) : a^2+b^2+c^2 \leq 1;\ a, b, c \in \mathbb{R}\}$.

iv.) $\{(a, b, c) : a \geq 0\}$.

v.) $\{(a, b, c) : 3a+2b-c = 0\}$.

Soln: (i)

Given,

$$W = \{(a, b, 1) : a, b \in \mathbb{R}\}.$$

Here,

$$(0, 0, 0) \notin W.$$

So that W is not vector space.

ii.) Given,

$$W = \{(a, b, c) : a+b+c = 0;\ a, b, c \in \mathbb{R}\}.$$

$\underline{0} = (0, 0, 0) \in W$ where $(0+0+0 = 0)$

let,

$\underline{u} = (a_1, b_1, c_1) \in W$, and $\underline{v} = (a_2, b_2, c_2) \in W$.

$\therefore\ a_1+b_1+c_1 = 0$ and $a_2+b_2+c_2 = 0$.

for any k_1 and k_2.

$$k_1\underline{v} + k_2\underline{w} = k_1(a_1 + b_1, c_1) + k_2(a_2 + b_2 + c_2)$$

$$\cancel{= k_1a_1 + k_1b_1 + k_1c_1 + k_2a_2 + k_2b_2 +}$$

$$= (k_1a_1 + k_2a_2,\ k_1b_1 + k_2b_2,\ k_1c_1 + k_2c_2)$$

Now,

$$(k_1a_1 + k_2a_2) + (k_1b_1 + k_2b_2) + (k_1c_1 + k_2c_2)$$

$$= k_1(a_1 + b_1 + c_1) + k_2(a_1 + b_1 + c_1)$$

$$= k_1(0) + k_2(0)$$

$$= 0$$

$\therefore\ k\underline{v} + k_2\underline{w} \in W.$

so W is a vector space of $\mathbb{R}^3$.

(iii) let,

$\underline{v} \in W$, where $\underline{v} = (1,0,0)$.

If $k=2$ then, $k\underline{v} = (1,0,0) = (2,0,0)$

Here,

$$2^2 + 0^2 + 0^2 = 4 \nleq 1$$

$\therefore k\underline{v} \notin W$

So that W is not a subspace of $\mathbb{R}^3$.

(iv) Let,

$\underline{u} = (2,0,0) \in W$ and $-1 \in F$.

Here,

$$-1\underline{u} = -1(2,0,0)$$

$$= (-2,0,0) \in W.$$

$\therefore -2 \ngeq 0$.

$\therefore$ W is not subspace

(v) Let,

$\underline{u} \in W$, Here, $\underline{u} = (0,0,0) \in W$.

$\therefore 3.0 + 2.0 - 0 = 0 \in W$.

W is not empty.

$\therefore \underline{u} = (a_1, b_1, c_1) \in W$ and $\underline{v} = (a_2, b_2, c_2) \in W$.

when, $3a_1 + 2b_1 - c_1 = 0$.

$3a_2 + 2b_2 - c_2 = 0$.

$\alpha, \beta \in F$.

$\therefore \alpha \underline{u} + \beta \underline{v} = \alpha(a_1, b_1, c_1) + \beta(a_2, b_2, c_2)$

$= (\alpha a_1, \alpha b_1, \alpha c_1) + (\beta a_2, \beta b_2, \beta c_2)$

$= ((\alpha a_1 + \beta a_2), (\alpha b_1 + \beta b_2), (\alpha c_1 + \beta c_2))$

Now,

$3(\alpha a_1 + \beta a_2) + 2(\alpha b_1 + \beta b_2) - \alpha c_1 + \beta c_2 = 0$.

$= \alpha(3a_1 + 2b_1 - c_1) + \beta(3a_2 + 2b_2 - c_2)$

$= \alpha 0 + \beta = 0$.

$\therefore \alpha \underline{u} + \beta \underline{v} \in W$.

If S and T are two subspaces of $V_3(\mathbb{R})$ where $S = \{(a,b,c) : a=b=c\}$ and $T = \{(a,b,c)\}$ then show that $V_3(\mathbb{R}) = S \oplus T$.

<u>Sol^n</u>

Let,

$$(a,b,c) \in V_3(\mathbb{R})$$

Now,

$$(a,b,c) = (a,a,a) + (0, b-a, c-a)$$

$(a,a,a) \in S$ and $(0, b-a, c-a) \in T$.

$\therefore V_3(\mathbb{R}) = S + T$

Again let,

$$(a,b,c) \in S \cap T$$

$\Rightarrow (a,b,c) \in S$ and $(a,b,c) \in T$.

$\Rightarrow a=b=c$ and $a=0 \Rightarrow a=b=c=0$.

$\therefore S \cap T = \{0\}$.

$\therefore$ let $V_3(\mathbb{R}) = S \oplus T$.

Linear combination (dependance and independa

Linear combination :-

Let, $v(F)$ be a vector space where

$(\underline{v}_1, \underline{v}_2 \cdots \underline{v}_n) \in \underline{v}$ and $(\alpha_1, \alpha_2 \cdots \alpha_n) \in F$.

$\therefore \quad (\underline{v}_1\alpha_1 + \underline{v}_2\alpha_2 + \underline{v}_3\alpha_3 + \cdots + \underline{v}_n\alpha_n) = \underline{u}$

where, $\underline{u} \in \underline{v}$ then $\underline{u}$ is called the linear combination of $v_1, v_2, v_3 \cdots v_n$.

Linear dependence of vector :-

Let $v(F)$ be a vector space where $(v_1, v_2 \cdots v_n) \in v$ and $(\alpha_1, \alpha_2 \cdots \alpha_n) \in F$ If $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots \alpha_n v_n = 0$ and at least one of the elements of the set $\{\alpha_1, \alpha_2 \cdots \alpha_n\}$ is not zero, then the vectors $v_1, v_2 \cdots v_n$ be called linear dependent

Linear indepedence of vector :-

let $V(F)$ be a vector space where $v_1, v_2 \cdots v_n \in V$ and $(\alpha_1, \alpha_2 \cdots \alpha_n) \in F$ If $\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = \underline{0}$

and the elements of the set $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ are zero then the vectors be called linear independent.

Prove that the non zero vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ of vector space $V(F)$ are linearly independent if and only if one of the vectors $\underline{v}_i$ is a linear ~~independent~~ combination of the proceeding vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3 \ldots \underline{v}_n$.

Let,

$$\overline{v}_i = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 \ldots\ldots + \alpha_n \underline{v}_n \quad \text{(where } \alpha_1, \alpha_2 \ldots \alpha_n \text{ are scallai)}$$

$$\Rightarrow \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \alpha_{i-1} \underline{v}_{i-1} + (-1) \underline{v}_i = \underline{0}.$$

$$\Rightarrow \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \ldots + \alpha_{i-1} \underline{v}_{i-1} + (-1) \underline{v}_i + 0 \underline{v}_{i+1} + \ldots + 0 \underline{v}_n = \underline{0}$$

Since the co-efficient of $\underline{v}_i$ is $(-1) \neq 0$.

So that,

$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ vectors are linearly dependent

Conversely,

let the vectors are linearly dependent
Now we prove that a vector $\underline{v_i}$ can be expressed as a linear combination of the preceding vector of it.

if vectors are linearly dependent, then

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \underline{0}.$$

where, any scalar $\alpha_1, \alpha_2 \ldots \alpha_n$ is not zero.

let,

the last non zero scalar be α_i.

$$\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \cdots + \alpha_i \underline{v}_i = \underline{0} \quad \text{—(1)}$$

if $i = 1$, then,

$$\alpha_1 \underline{v}_1 = 0 \Rightarrow \alpha_1 = 0 \quad [\because \underline{v}_1 \neq 0]$$

but $\alpha_i \neq 0$ [when, $i = 1$] $\therefore i > 1$

From eqn ① we get.

$$\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \cdots + \alpha_{i-1} \underline{v}_{i-1} = -\alpha_i \underline{v}_i$$

$$\Rightarrow \underline{v}_i = \left(-\frac{\alpha_1}{\alpha_i}\right) \underline{v}_1 + \left(\frac{\alpha_2}{-\alpha_i}\right) \underline{v}_2 + \cdots + \left(\frac{\alpha_{i-1}}{\alpha_i}\right) \underline{v}_{i-1}$$

Proved.

Find the vectors $\underline{u}$ as a linear combination of the vectors $\underline{u}_1$, $\underline{u}_2$ and $\underline{u}_3$ where, $\underline{u} = (1,-2,5)$, $\underline{u}_1 = (1,1,1)$, $\underline{u}_2 = (1,2,3)$, $\underline{u}_3 = (2,-1,1)$.

Soln:-

Given,

$$\underline{u} = (1,-2,5)$$
$$\underline{u}_1 = (1,1,1)$$
$$\underline{u}_2 = (1,2,3)$$
$$\underline{u}_3 = (2,-1,1).$$

Let,

$$\underline{u} = \alpha u_1 + \beta u_2 + \gamma u_3. \quad \text{—— ①}$$

where, α, β and γ are scalar.

Now,

$$(1,-2,5) = \alpha(1,1,1) + \beta(1,2,3) + \gamma(2,-1,1).$$

$$= (\alpha+\beta+2\gamma,\ \alpha+2\beta-\gamma,\ \alpha+3\beta+\gamma)$$

Now

$$\alpha + \beta + 2\gamma = 1$$
$$\alpha + 2\beta - \gamma = -2$$
$$\alpha + 3\beta + \gamma = 5$$

$\Rightarrow$

$$\alpha + \beta + 2\gamma = 1$$
$$\beta - 3\gamma = -3 \qquad R_2 \leftarrow R_2 - R_1$$
$$2\beta - \gamma = 4 \qquad R_3 \leftarrow R_3 - R_1$$

$\Rightarrow$

$$\alpha + \beta + 2\gamma = 1$$
$$\beta - 3\gamma = -3 \qquad R_3 \leftarrow R_3 - 2R_1$$
$$5\gamma = 10$$

From, $R_3 \Rightarrow$

$$5\gamma = 10$$
$$\Rightarrow \gamma = 2$$

Putting $\gamma = 2$ in $R_2 \Rightarrow$

$$\beta = -3 + 3 \times 2 = 3.$$

Putting β and γ in $R_1 \Rightarrow$

$$\alpha = 1 - 3 - 4$$
$$= -6.$$

From eq^n ① we get.

$$\underline{u} = -6\underline{u}_1 + 3\underline{u}_2 + 2\underline{u}_3 \quad \underline{Ans}.$$

Express

Generator Basis and dimension

Generator

Let $V(F)$ is a vector space and $S = \{v_1, v_2, v_3 \ldots v_n\}$ is a finite subset of it. If $\underline{u} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \ldots + \alpha_n \underline{v}_n, \forall \underline{u} \in V$ and $\alpha_1, \alpha_2 \ldots \alpha_n \in F$, then is called generator or space of $V(F)$.

If ~~s is called~~ is denoted by $V(F) = L(S)$ or $\langle S \rangle$

Basis and dimension

Let $V(F)$ is vector space and $S = \{v_1, v_2, v_3 \ldots v_n\}$ is finite subset of it. If S is a generator of $V(F)$ and S be linear indepedent then S is a called basis of $V(F)$.

If S is basis of vector space $V(F)$ then the numbers of vectors in S is called dimension of $V(F)$.