Numerical Methods

Introduction

How do we solve an engineering problem?

Problem Description

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Mathematical Model

↓

Solution of Mathematical Model

↓

Using the Solution

Introduction

- Consider the following equations.

■ **Solve**

$$\frac{dy}{dx}\left(x^2 - \sin(x)\right)$$

$$F = \int_0^{30} \left(\frac{\cos(z) + z}{5 + z} \right) e^{-2z/30} dz$$

$$\int x^3 + x - e^x$$

$$\frac{x}{1 + \sin(x)} + e^x = 0$$

$$ax^2 + bx + c = 0$$

Numerical Methods - Definitions

Numerical Methods

- Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithmetic operations.
- Numerical methods involve large numbers of tedious arithmetic calculations.
- These methods have gained popularity due to the advancements in efficient computational tools such as digital computers and calculators.

Numerical Methods

Noncomputer methods

- **Analytical** or exact methods
- **Graphical** solutions used to solve complex problems but the results are not very precise. They are extremely tedious without computers ➔ limited problems.
- **Calculators**

Analytical vs. Numerical methods

Examples: Analytical Methods

- Differentiation

$$\frac{dy}{dx}\left(x^2 - \sin(x)\right) = 2x - \cos(x)$$

- Integration

$$\int x^3 + x - e^x = \frac{x^4}{4} + \frac{x^2}{2} - e^x + c$$

- Root(s) of an Equation

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ax}}{2a}$$

Analytical vs. Numerical methods

Need for Numerical Methods

- In general, there are few analytical (closed-form) solutions for many practical engineering problems.
- Numerical methods can handle:
  - Large systems of equations
  - Non-linearity
  - Complicated geometries that are common in engineering practice and that are often impossible to solve analytically.

Examples:

$$F = \int_0^{30} \left(\frac{\cos(z) + z}{5 + z} \right) e^{-2z/30} dz$$

$$\frac{x}{1 + \sin(x)} + e^x = 0$$

Reasons to study numerical Analysis

- Powerful problem solving techniques and can be used to handle large systems of equations
- It enables you to intelligently use the commercial software packages as well as designing your own algorithm.
- Numerical Methods are efficient vehicles in learning to use computers
- It Reinforce your understanding of mathematics; where it reduces higher mathematics to basic arithmetic operation.

Mathematical Modeling

Mathematical Model

- A formulation or equation that expresses the essential features of a physical system or process in mathematical terms.
- Generally, it can be represented as a functional relationship of the form

$$\begin{matrix}\text{Dependent} \\ \text{variable}\end{matrix} = f\left(\begin{matrix}\text{independent} \\ \text{variable}\end{matrix}, \text{parameters}, \begin{matrix}\text{forcing} \\ \text{functions}\end{matrix}\right)$$

Mathematical Modeling

$$\begin{matrix}\text{Dependent} \\ \text{variable}\end{matrix} = f\left(\begin{matrix}\text{independent} \\ \text{variable}\end{matrix}, \text{parameters}, \begin{matrix}\text{forcing} \\ \text{functions}\end{matrix}\right)$$

Dependent variable =	A characteristic that usually reflects the behavior or state of the system
Independent variables =	Are usually dimensions, such as time and space
Parameters =	Are reflective of system's properties or compositions
Forcing functions =	Are external influences acting on the system

Simple Mathematical Model

Example: Newton's Second Law

(The time rate of change of momentum of a body is equal to the resultant force acting on it)

$$\boxed{F = ma \qquad \text{or} \qquad a = \frac{F}{m}}$$

- a = acceleration (m/s^2) ….**the dependent variable**
- m = mass of the object (kg) ….**the parameter** representing a property of the system.
- f = **force** acting on the body (N)

Typical characteristics of Math. model

- It describes a natural process or system in mathematical way
- It represents the idealization and simplification of reality.
- It yields reproducible results, and can be used for predictive purpose.

Complex Mathematical Model

Example: Newton's Second Law

$$m\frac{dv}{dt} = F$$

$F = F_D + F_U$

$F_D = mg$

$F_U = -cv$

F_D = downward force due to gravity

F_U = upward force due air resistance

Where:

c = drag coefficient (kg/s),

v = falling velocity (m/s)

Complex Mathematical Model

$$m\frac{dv}{dt} = F_D + F_U$$

$$= mg - cv$$

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

At rest: (v = 0 at t = 0),

Calculus can be used to solve the equation

$$\boxed{v(t) = \frac{gm}{c}\left[1 - e^{-(c/m)t}\right]}$$

Analytical solution to Newton's Second Law

A parachutist with a mass of 68.1 kg jumps out of a stationary hot-air balloon. Compute the velocities v in an increment of 2 seconds prior to the opening of the chute. Use a drag coefficient value of 12.5 kg/s, and tabulate your values.

$$\boxed{v(t) = \frac{gm}{c}\left[1 - e^{-(c/m)t}\right]}$$

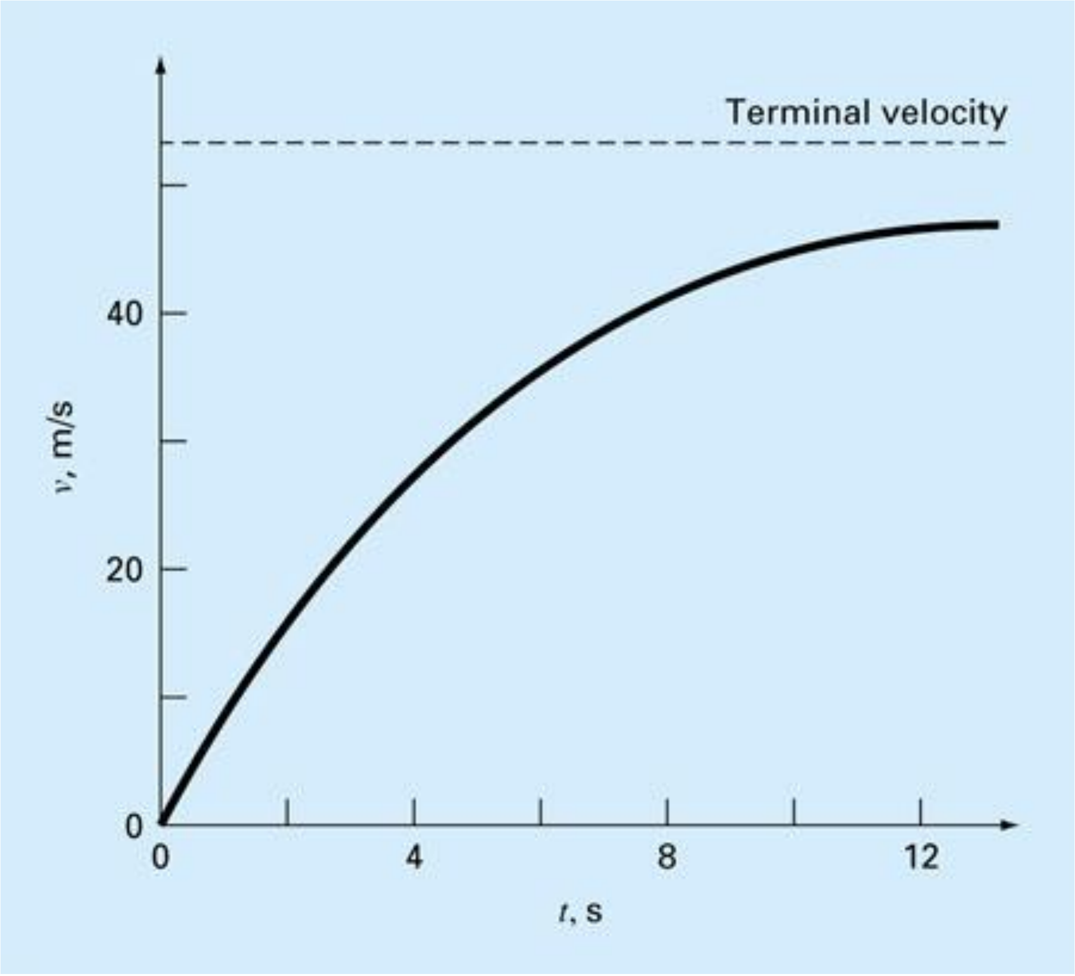
Analytical solution to Newton's Second Law

$$v = \frac{9.8(68.1)}{12.5}\left[1 - e^{-(12.5/68.1)t}\right]$$

$$= 53.3904\left(1 - e^{-0.18355t}\right)$$

t (s)	*v* (m/s)
0	0
2	16.405
4	27.7693
6	35.6418
8	41.0953
10	44.8731
12	47.4902
∞	53.3904

Analytical solution to Newton's Second Law



Numerical Solution to Newton's Second Law

- Numerical solution: approximates the exact solution by arithmetic operations.

- Suppose

$$\frac{dv}{dt} \approx \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

$$\frac{dv}{dt} = g - \frac{c}{m} v \quad \rightarrow \quad \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c}{m} v$$

$$\boxed{v(t_{i+1}) = v(t_i) + \left[g - \frac{c}{m} v(t_i) \right] (t_{i+1} - t_i) \qquad (4)}$$

New value = old value + slope X step size

Numerical Solution to Newton's Second Law

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$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c}{m} v(t_i)\right](t_{i+1} - t_i)$$

At $t_i = 0$, the velocity of the parachutist is zero. Using this information and the parameter values from Example 1, Eq. 4 can be used to compute the velocity at $t_{i+1} = 2$ seconds, that is

$$v = 0 + \left[9.8 - \frac{12.5}{68.1}(0)\right](2 - 0) = 19.60 \text{ m/s}$$

Numerical Solution to Newton's Second Law

For the next interval (from t = 2 to 4 s), the computation is repeated, with the following result

$$v = 19.60 + \left[9.8 - \frac{12.5}{68.1}(19.60)\right](4-2) = 32.00 \text{ m/s}$$

The calculation is continued in a similar fashion to obtain additional values as shown the next viewgraph:

$$v(t_{i+1}) = v(t_i) + \left[9.8 - \frac{12.5}{68.1}v(t_i)\right](t_{i+1} - t_i)$$

t (s)	v (m/s)
0	0.000
2	19.600
4	32.005
6	39.856
8	44.824
10	47.969
12	49.959
∞	53.390

Comparison between Analytical vs. Numerical Solution