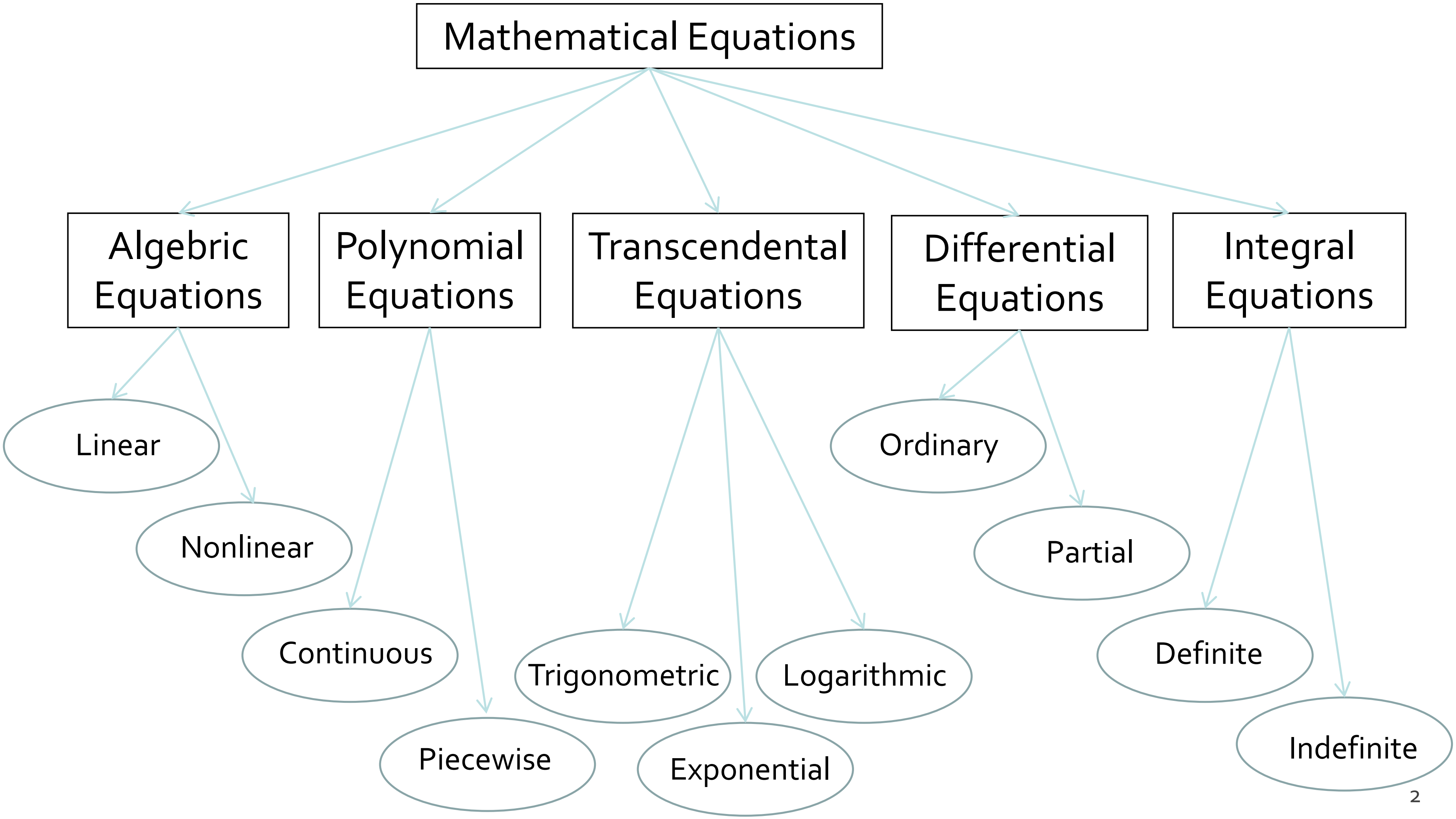
# Lecture 3

## Solution of Nonlinear Equations

## ( Root Finding Problems )

- Definitions
- Classification of Methods
  - Analytical Solutions
  - Graphical Methods
  - Numerical Methods
    - Bracketing Methods
    - Open Methods
- Convergence Notations

# Different forms of mathematical equations



# What is numerical computing?

- Numerical computing is an approach for solving complex mathematical problems using only simple arithmatic operations
- The appoarch involves, in most of the cases, formulation of mathematical models of physical situations that can be solved with arithmatic operations
- It requires development, analysis and use of algorithm
- Algorithm is a systematic procedure that solves a problem or a number of problems
- Its efficiency may be measured by the number of steps in the algoritm, the computer time, and the amount of memory (of the computing instrument) that is required

# Advantage of Numerical Methods

- The major advantage of numerical methods is that a numerical value can be obtained even when the problem has no “analytical” solution
- The mathematical operations required are essentially addition, subtraction, multiplication, and division plus making comparisons
- It is important to realize that solution by numerical analysis is always numerical
- Analytical methods, on the other hand, usually give a result in terms of mathematical functions that can then be evaluated for specific instances

# Scope of Numerical Analysis

- Finding roots of equations
- Solving systems of linear algebric equations
- Interpolation and regression analysis
- Numerical differentiation
- Numerical Integration
- Solution of ordinary differential equations
- Boundary value problems
- Solution of matrix problem

# Steps of Solving a Practical Problem

Step #1:

- State the problem clearly, including any simplifying assumptions.

Step #2:

- Develop a mathematical statement of the problem in a form that can be solved for a numerical answer
- This process may involve the use of calculus.
- In some situations, other mathematical procedures may be employed.
- When this statement is a differential equation, appropiate initial conditions and/or boundary conditions must be specified

# Steps of Solving a Practical Problem

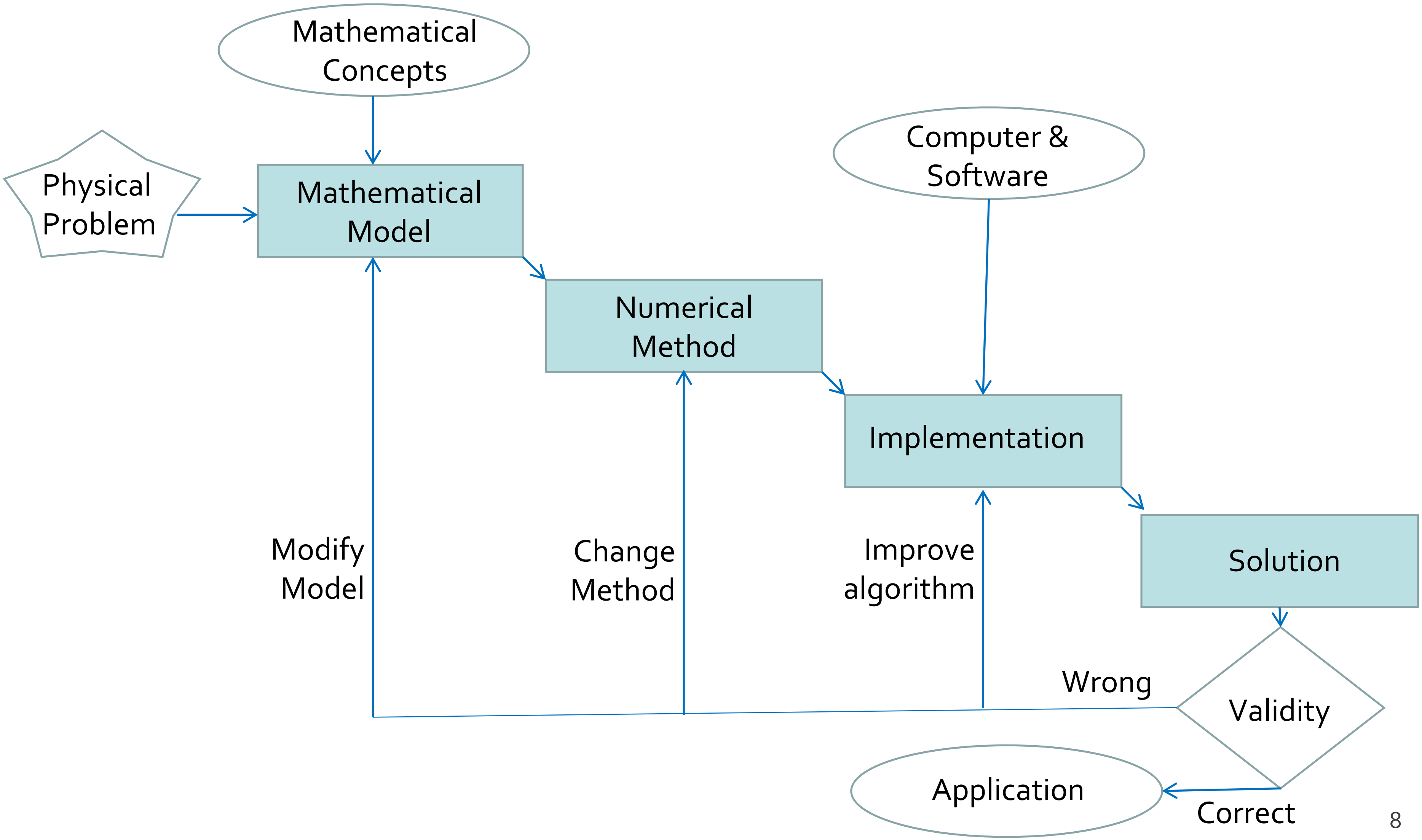
Step #3:

- Solve the equations that are obtained from step #2
- Sometimes the method will be algebric
- But frequently more advanced methods will be needed
- The result of this step is a numerical answer or set of answers

Step #4:

- Interpret the numerical result to arrive at a decision
- This will require experience and understanding of the situation in which the problem is embedded

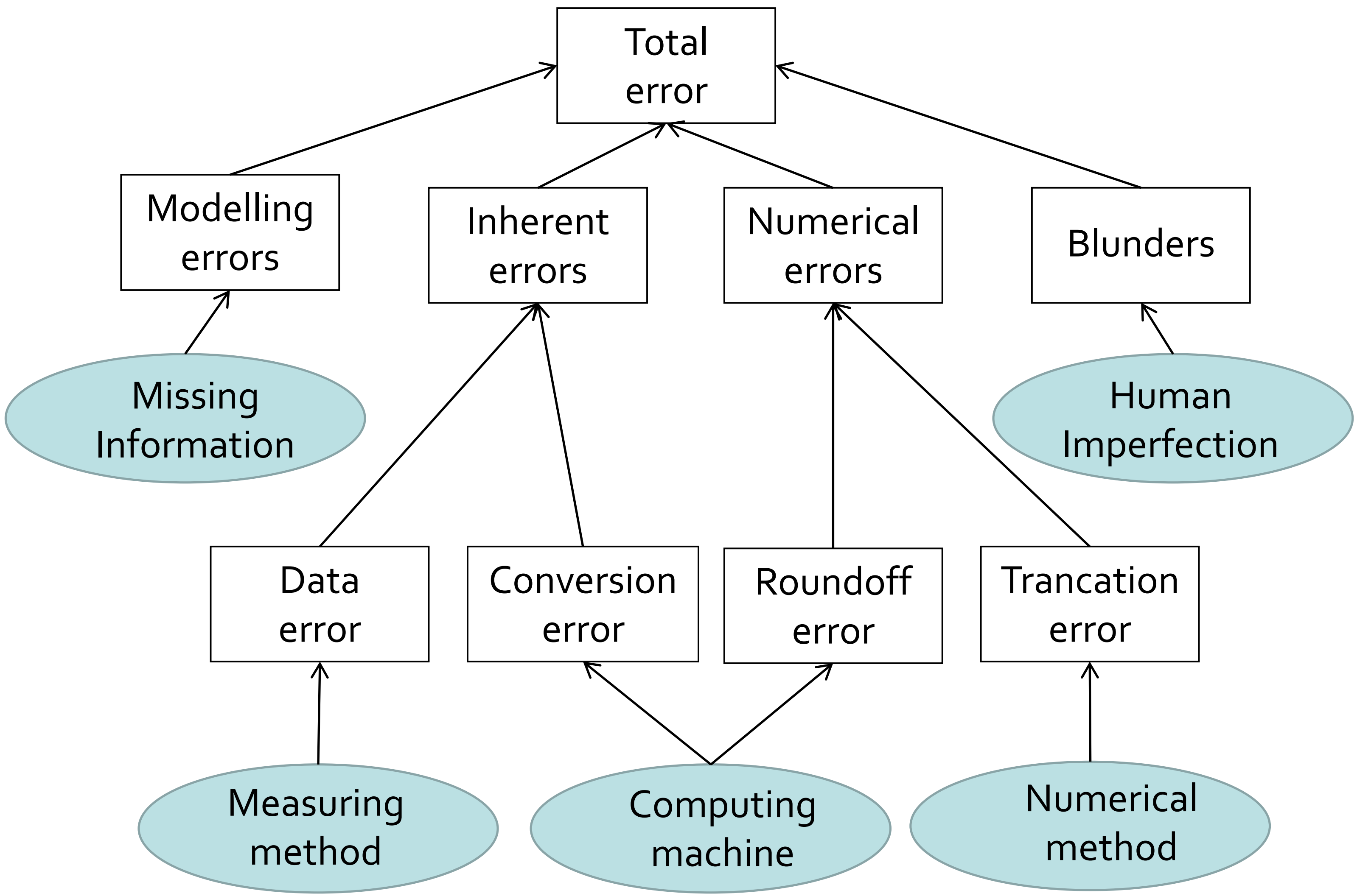
# Numerical Computing Process



# Accuracy in Numerical Analysis

- Numerical analysis is an approximation, but results can be made as accurately as desired.
- Errors come in a variety forms and sizes; some are avoidable and some are not
- For example, data conversion and roundoff errors can not be avoided, but human errors can be eliminated
- Although certain errors can not be eliminated completely, we must atleast know the bounds of these errors to make use of our final selection
- It is therefore essential to know that how errors arise, how they grow during numerical process and how they affect the accuracy of a solution

# Taxonomy of errors



# Modelling errors

- In many situations it is impractical to model each of the components accurately and so certain simplifying assumptions are made
- For example, while developing a model for calculaing the force acting on a falling body, we may not be able to estimate the air resistance coefficient (drag coefficient) properly or determine the direction and magnitude of wind force acting on the body and so on
- Since the model is the basic input to the numerical process, no numerical method will provide adequate results if the model is erroneously conceived and formulated

# Inherent errors

- Inherent error (also known as input error) contain two components, namely, data errors and conversion errors

**Data error**

- Data error (also known as emperical error) arises when data for a problem are obtained by some experimental means and are, therefore, of limited accuracy and precision

**Conversion error**

- Conversion error (also known as representational error) arise due to the limitations of the computer to store the data exactly

# Numerical Errors

- Numerical errors (also known as procedural error) are introduced during the process of implementation of a numerical method

**Roundoff error**

- Roundoff error occur when a fixed number of digits are used to represent exact numbers
- 42.7893 will be rounded off upto 2 decimal digits as 42.79

**Trancation error**

- Trancation error arise from using an approximation in place of an exact mathematical procedure
- Typically it is the error resulting from the trancation of numerical process
- We often use finite number of terms to estimate the sum of infinite series

# Blunders

- Blunders are the errors that are caused due to human imperfection
- Some common type of this error are:
  - Lack of understanding of the problem
  - Wrong assumption
  - Overlooking of some basic assumptions required for formulating the model
  - Error in deriving the mathematical equation or using a model that does not describe adequately the physical system under study
  - Selecting a wrong numerical method for solving the mathematical model
  - Selecting a wrong algorithm for implementing the numerical method
  - Making mistakes in the computer program
  - Mistake in data input
  - Wrong guessing the initial value

# Root Finding Problems

Many problems in Science and Engineering are expressed as:

$$\boxed{\begin{array}{l}\text{Given a continuous function } f(x), \\ \text{find the value } r \text{ such that } f(r) = 0\end{array}}$$

These problems are called root finding problems.

# Roots of Equations

A number *r* that satisfies an equation is called a root of the equation.

$$\text{The equation: } x^4 - 3x^3 - 7x^2 + 15x = -18$$

$$\text{has four roots} : -2,\ 3,\ 3,\ and\ -1\,.$$

$$\text{i.e., } x^4 - 3x^3 - 7x^2 + 15x + 18 = (x+2)(x-3)^2(x+1)$$

The equation has two simple roots $(-1 \text{ and } -2)$ and a repeated root $(3)$ with multiplicity $= 2$.

# Zeros of a Function

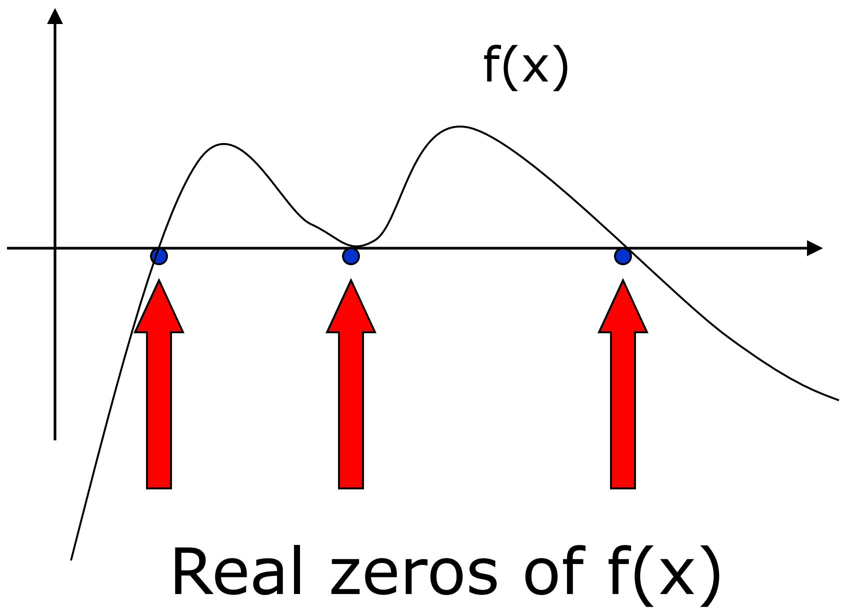
Let $f(x)$ be a real-valued function of a real variable. Any number $r$ for which $f(r)=0$ is called a zero of the function.

*Examples:*

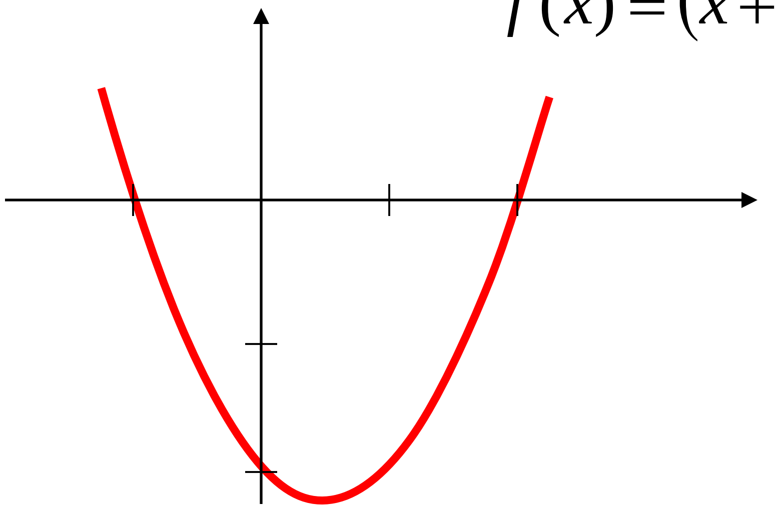
*2 and 3 are zeros of the function* $f(x) = (x-2)(x-3)$.

# Graphical Interpretation of Zeros

- The real zeros of a function ***f(x)*** are the values of ***x*** at which the graph of the function crosses (or touches) the x-axis.



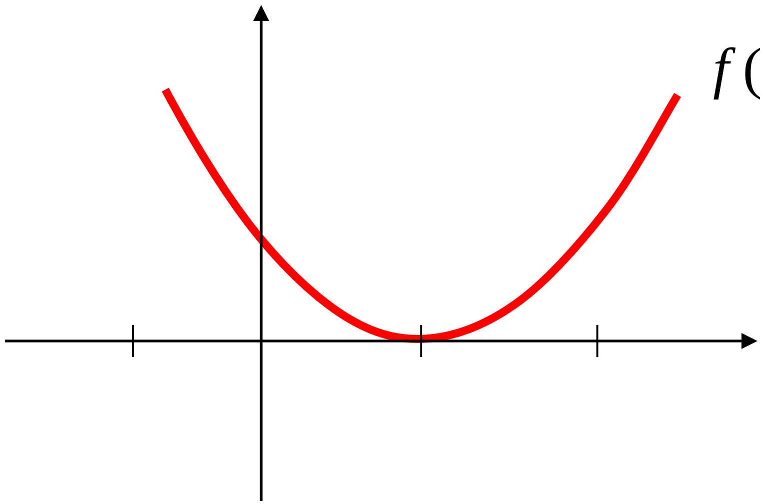
# Simple Zeros




$$f(x) = (x+1)(x-2) = x^2 - x - 2$$

has two simple zeros (one at $x = 2$ and one at $x = -1$)

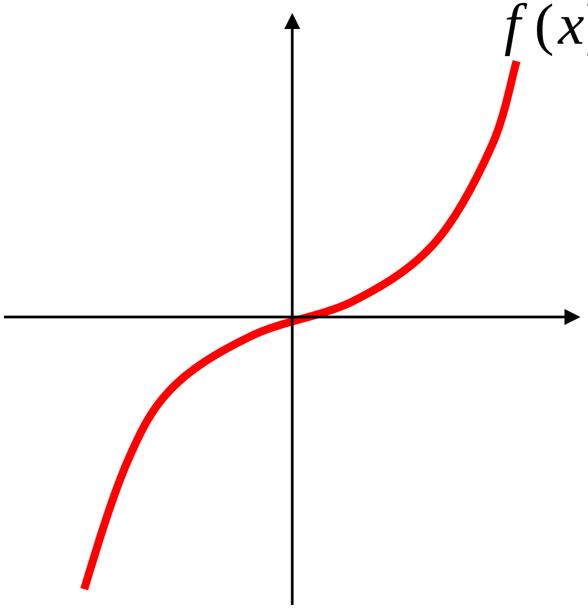
# Multiple Zeros




$$f(x) = (x-1)^2 = x^2 - 2x + 1$$

has double zeros (zero with muliplicity = 2) at x = 1

# Multiple Zeros




$$f(x) = x^3$$

$$\text{has a zero with muliplicity} = 3 \ \text{at } x = 0$$

# Facts

- Any $n^{th}$ order polynomial has exactly n zeros (counting real and complex zeros with their multiplicities).
- Any polynomial with an odd order has at least one real zero.
- If a function has a zero at ***x=r*** with multiplicity ***m*** then the function and its first ***(m-1)*** derivatives are zero at ***x=r*** *and the* ***m****th derivative at* ***r*** *is not zero*.

# Roots of Equations & Zeros of Function

Given the equation:

$$x^4 - 3x^3 - 7x^2 + 15x = -18$$

Move all terms to one side of the equation:

$$x^4 - 3x^3 - 7x^2 + 15x + 18 = 0$$

Define $f(x)$ as:

$$f(x) = x^4 - 3x^3 - 7x^2 + 15x + 18$$

The zeros of $f(x)$ are the same as the roots of the equation $f(x) = 0$

(Which are $-2$, 3, 3, and $-1$)

# Solution Methods

Several ways to solve nonlinear equations are possible:

- Analytical Solutions
  - Possible for special equations only
- Graphical Solutions
  - Useful for providing initial guesses for other methods
- Numerical Solutions
  - Open methods
  - Bracketing methods

# Analytical Methods

Analytical Solutions are available for special equations only.

$$\text{Analytical solution of}: \; a\,x^2 + b\,x + c = 0$$

$$roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{No analytical solution is available for}: \quad x - e^{-x} = 0$$

# Graphical Methods

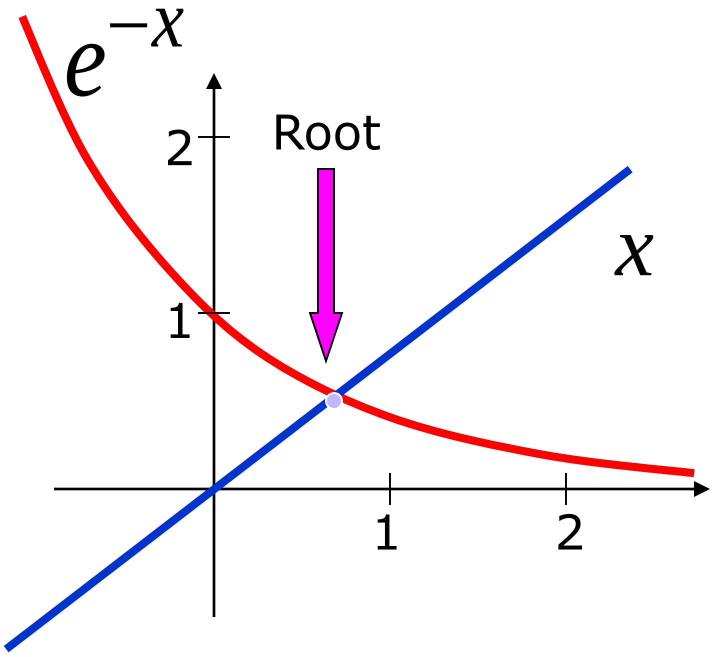
- Graphical methods are useful to provide an initial guess to be used by other methods.

$$Solve$$

$$x = e^{-x}$$

$$The \;\; root \in [0,1]$$

$$root \approx 0.6$$



# Numerical Iterative Methods

- An iterative technique usually begins with an approximate value of the root, known as the initial guess, which is then successively corrected iteration by iteration under a certain mathematical basis
- The process of iteration stops when the desired level of accuracy is obtained
- Since ,in many cases, the iterative method needs a large number of iterations and arithmatic opeartion to reach a solution, the use of computers has become inevitable to make the task simple and efficient

# Iterative Methods (continued)

- Iterative methods, based on the number of guesses they use, can be categorized into two categories:
  - Bracketing methods (Interpolation methods)
  - Open end methods (Extrapolation methods)
- Bracketing methods starts with two initial guesses that 'bracket' the root and then systematically reduce the width of the bracket until the solution is reached
- Two popular methods under Bracketing category are
  - Bisection method
  - False position method
- These methods are based on the assumption that the function changes sign in the vicinity of a root

# Iterative Methods (continued)

- Open end methods use a single starting value or two values that do not necessarily bracket the root
- The following iterative methods fall under this category:
  - Newton-Raphson method
  - Secant method
  - Muller's method
  - Fixed-point method
  - Bairstow's method

# Starting an iterative process

- Before an iterative process is initaited, we have to determine either an approximate value of root or a ‘search’ interval that contains a root
- One simple method is to plot the function
- Graphical representation will not only provide us rough estimate of the root but also help us in understanding the properties of the function

**Largest possible root**

- For a polynomial represented by

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ........ + a_1 x \;\; + a_0 = 0$$

the largest possible root is given by

$$x_1^* = -\frac{a_{n-1}}{a_n}$$

# Starting an iterative process

**Search Bracket**

- Another relationship that might be useful for determining the search intervals that contain the real roots of a polynomial is

$$\left|x^*\right| \le \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

  where x is the root of the polynomial. This will be the maximum absolute value of the roots
- That means that no roots exceed $x_{max}$ in absolute magnitude and thus, all real roots lie within the interval

$$(-|x^*_{max}|,|x^*_{max}|)$$

# Starting an iterative process (continued)

- There is another relationship that suggests an interval for roots.
- All roots x satisfy the inequality

$$\left|x^*\right| \leq 1 + \frac{1}{\left|a_n\right|} \max\left\{\left|a_0\right|, \left|a_1\right|, \left|a_2\right|, \ldots\ldots\ldots \left|a_{n-1}\right|\right\}$$

where the 'max' denotes the maximum of the absolute values of $|a_0|$, $|a_1|$, $|a_2|$,...... $|a_{n-2}|$, $|a_{n-1}|$

# Example #1

- Consider the polynomial equation $2x^3 - 8x^2 + 2x + 12 = 0$
- Estimate the possible initial guess value

------------------------------------------

- The largest possible root is $x_1^* = -\frac{a_{n-1}}{a_n}$

- That is, no root can be larger than the value 4
- All roots must satisfy the relation $\left|x^*\right| \leq \sqrt{\left(\frac{-8}{2}\right)^2 - 2\left(\frac{2}{2}\right)} = \sqrt{14}$

- Therefore, all real roots lie in the interval $\left(-\sqrt{14}, \sqrt{14}\right)$ .
- We can use these two points as initial guesses for the bracketing methods and one of them for the open end methods

# Iteration Stopping Criterion

- We must have an objective criterion for deciding when to stop the process
- We may use one of the following tests
  - $\left|x_{i+1} - x_i\right| \le E_a$ (absolute error in x)
  - $\left|\dfrac{x_{i+1} - x_i}{x_i}\right| \le E_r$ (relative error in x) x<>0
  - $\left|f(x_{i+1})\right| \le E$ (value of function at root)

- There may be the situations where these tests may fail
- In cases where we do not know whether the process converges or not, we must have a limit on the number of iterations, like

  Iterations $\ge$ N (limit on iterations)

# Bracketing Methods

- In bracketing methods, the method starts with an interval that contains the root and a procedure is used to obtain a smaller interval containing the root.

- Examples of bracketing methods:
  - Bisection method
  - False position method

# Open Methods

- In the open methods, the method starts with one or more initial guess points. In each iteration, a new guess of the root is obtained.
- Open methods are usually more efficient than bracketing methods.
- They may not converge to a root.

# Open Methods

- **Characteristics:**
  - Initial estimates need not bracket the root
  - Generally converge faster
  - **NOT** guaranteed to converge
- **Open Methods Considered**:
  - Fixed-point Methods
  - Newton-Raphson Iteration
  - Secant Method

# Convergence Notation

A sequence $x_1, x_2, ..., x_n, ...$ is said to **converge** to $x$ if to every $\varepsilon > 0$ there exists $N$ such that:

$$|x_n - x| < \varepsilon \quad \forall n > N$$

# Convergence Notation

Let $x_1, x_2, \ldots,$ converge to $x$.

Linear Convergence: $$\frac{\left|x_{n+1} - x\right|}{\left|x_n - x\right|} \leq C$$

Quadratic Convergence: $$\frac{\left|x_{n+1} - x\right|}{\left|x_n - x\right|^2} \leq C$$

Convergence of order $P$: $$\frac{\left|x_{n+1} - x\right|}{\left|x_n - x\right|^p} \leq C$$

# Speed of Convergence

- We can compare different methods in terms of their convergence rate.
- Quadratic convergence is faster than linear convergence.
- A method with convergence order $\boldsymbol{q}$ converges faster than a method with convergence order $\boldsymbol{p}$ if $\boldsymbol{q>p}$.
- Methods of convergence order $\boldsymbol{p>1}$ are said to have super linear convergence.

# Bisection Method

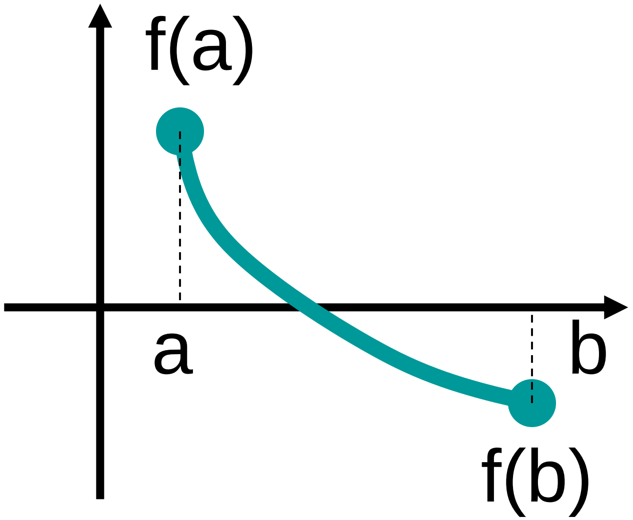
- The **Bisection method** is one of the simplest methods to find a zero of a nonlinear function.
- It is also called **interval halving** method.
- To use the Bisection method, one needs an initial interval that is known to contain a zero of the function.
- The method systematically reduces the interval. It does this by dividing the interval into two equal parts, performs a simple test and based on the result of the test, half of the interval is thrown away.
- The procedure is repeated until the desired interval size is obtained.

# Intermediate Value Theorem

- Let f(x) be defined on the interval [a,b].

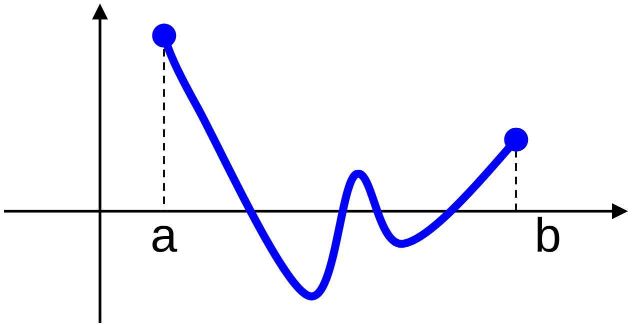
- Intermediate value theorem:

  if a function is continuous and f(a) and f(b) have different signs then the function has at least one zero in the interval [a,b].



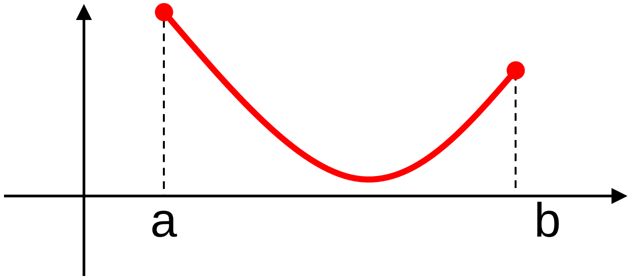
# Examples

- If f(a) and f(b) have the same sign, the function may have an even number of real zeros or no real zeros in the interval [a, b].




The function has four real zeros

- Bisection method can not be used in these cases.

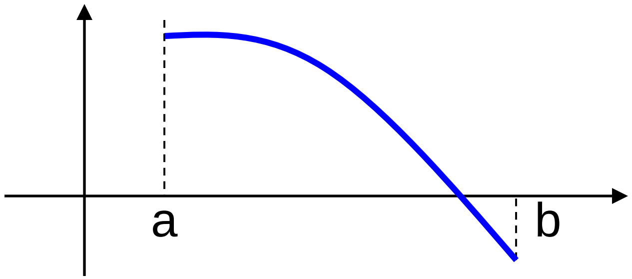



The function has no real zeros

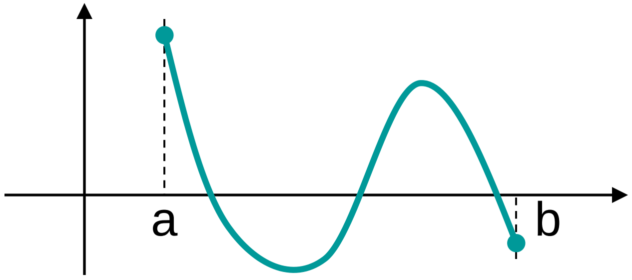
# Two More Examples

- If f(a) and f(b) have different signs, the function has at least one real zero.

- Bisection method can be used to find one of the zeros.




The function has one real zero




The function has three real zeros

# Bisection Method

- If the function is continuous on [a,b] and f(a) and f(b) have different signs, Bisection method obtains a new interval that is half of the current interval and the sign of the function at the end points of the interval are different.

- This allows us to repeat the Bisection procedure to further reduce the size of the interval.

# Bisection Method

**Assumptions:**

Given an interval [a,b]

f(x) is continuous on [a,b]

f(a) and f(b) have opposite signs.

These assumptions ensure the existence of at least one zero in the interval [a,b] and the bisection method can be used to obtain a smaller interval that contains the zero.

# Bisection Algorithm

**Assumptions:**

- f(x) is continuous on [a,b]
- f(a) f(b) < 0

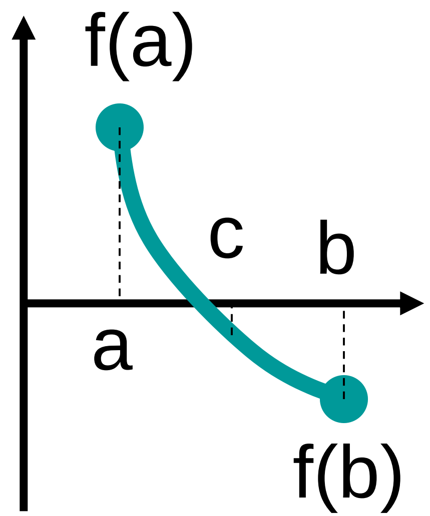
**Algorithm:**

**Loop**

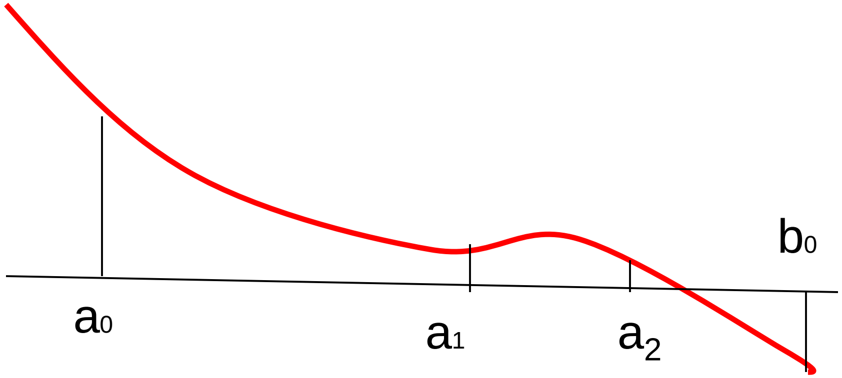
1. Compute the mid point c=(a+b)/2
2. Find $xc = (a+b)/2$.
3. If $f(xc) = 0$, then $xc$ is the root of the equation.
3. Otherwise, If f(a) f(c) < 0 then new interval [a, c]
   If f(a) f(c) > 0 then new interval [c, b]

**End loop**

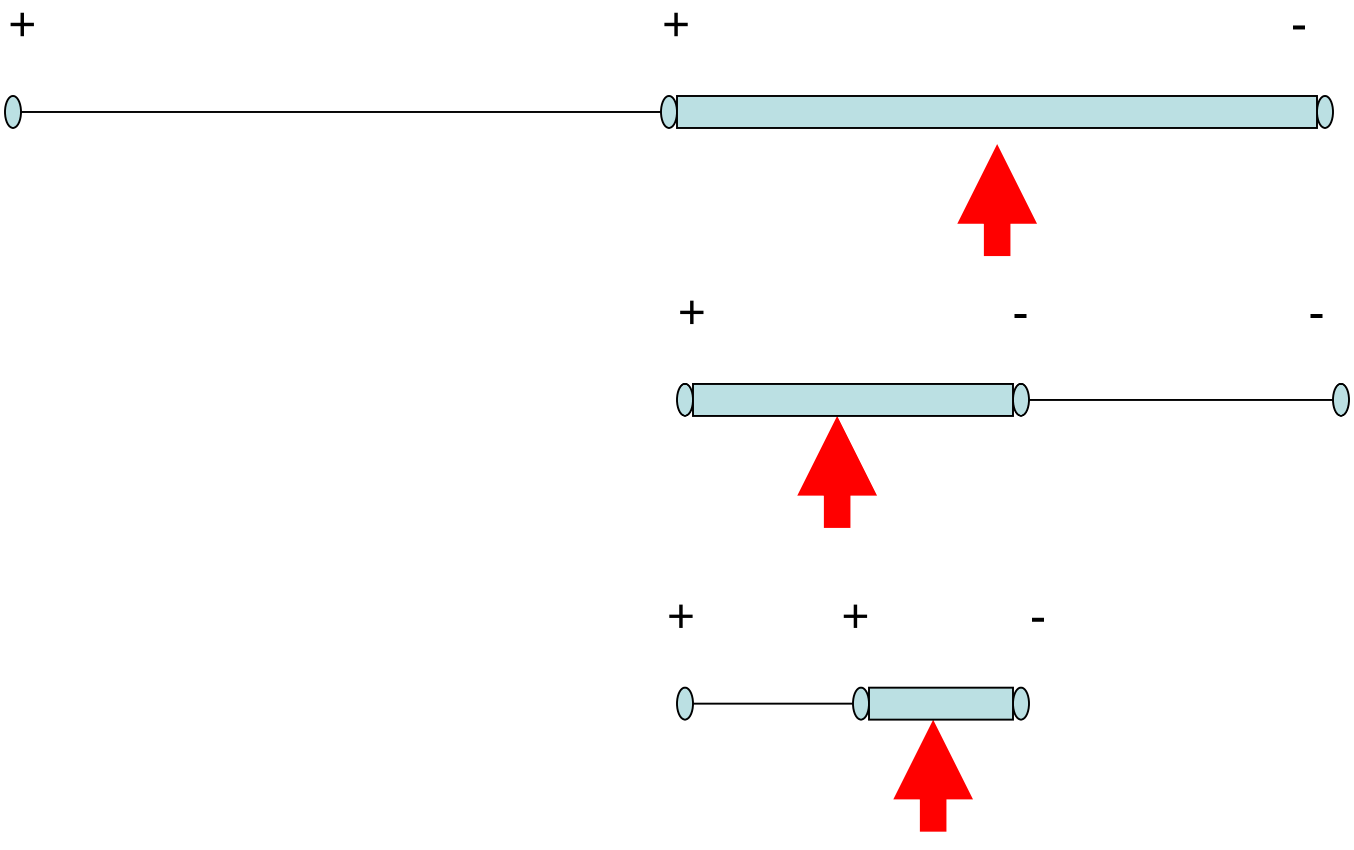
We continue this process until we find the root (i.e., $f(x_c)$= 0), or the latest interval is smaller than some specified tolerance.



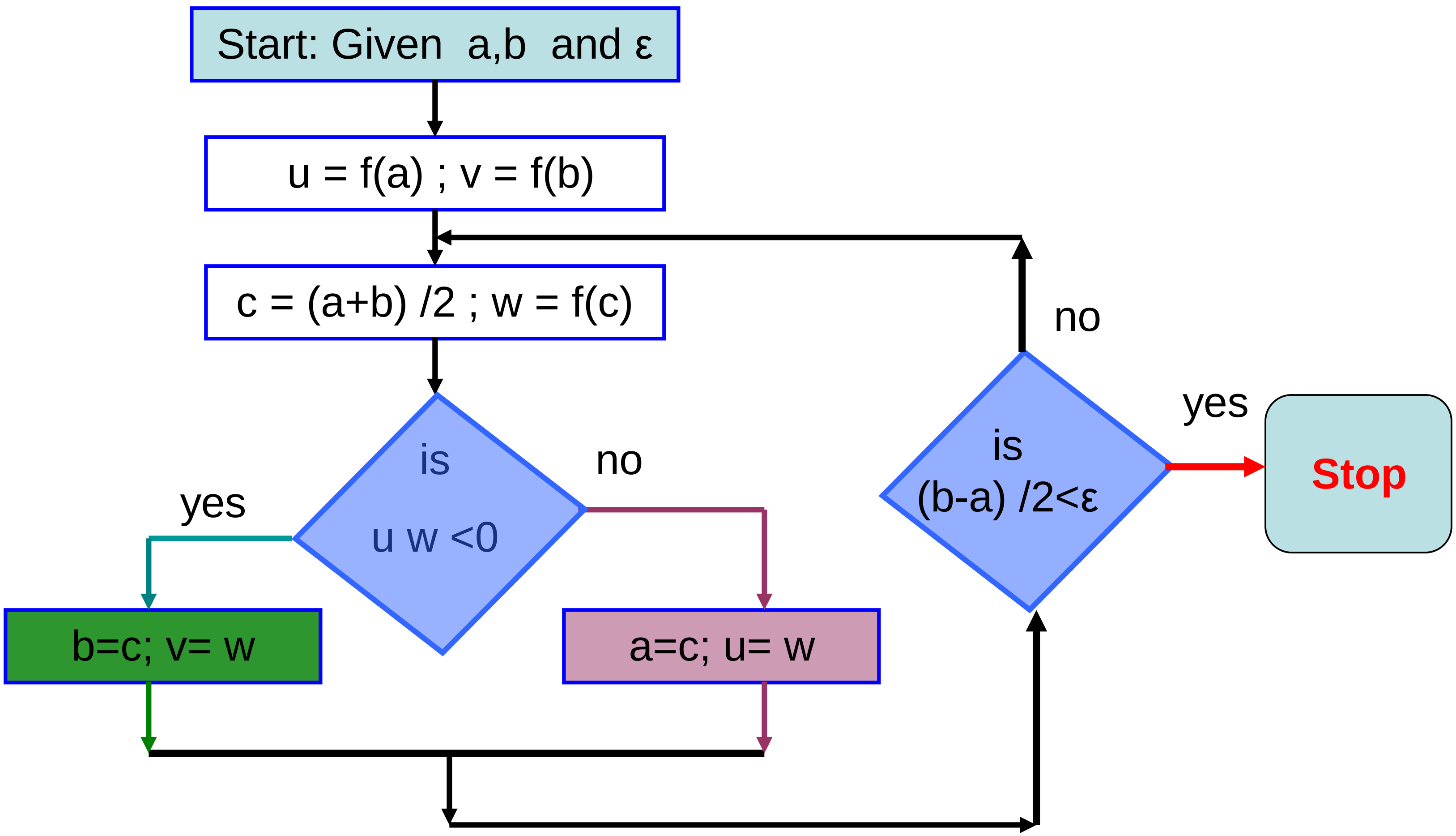
# Bisection Method



# Example



# Flow Chart of Bisection Method



# Bisection Method: Example 1

Find the root of the equation $x^3 + 4x^2 - 1 = 0$.

## Solution

Let, $a = 0$ and $b = 1$.

Now, $f(0) = (0)^3 + 4(0)^2 - 1 = -1 < 0$ and
$f(1) = (1)^3 + 4(1)^2 - 1 = 4 > 0$.
i.e., $f(a)$ and $f(b)$ has opposite signs.

Therefore, $f(x)$ has a root in the interval $[a, b] = [0, 1]$

$x_c = (0 + 1)/2 = 0.5$,
$f(0.5) = 0.125$. Now $f(a)$ and $f(x_c)$ has opposite signs

So, the next interval is $[0, 0.5]$

# Bisection Method: Example 1

Find the root of the equation $x^3 + 4x^2 - 1 = 0$.

Solution

| ***a*** | ***b*** | **$x_c$ = (a+b)/2** | ***f*(*a*)** | ***f*(*b*)** | ***f*($x_c$)** |
|---|---|---|---|---|---|
| 0 | 1 | 0.5 | **-1** | 4 | **0.125** |
| 0 | 0.5 | 0.25 | -1 | **0.125** | **- 0.73438** |
| 0.25 | 0.5 | 0.375 | -0.73438 | **0.125** | **- 0.38477** |
| 0.375 | 0.5 | 0.4375 | -0.38477 | **0.125** | **- 0.15063** |
| 0.4375 | 0.5 | 0.46875 | -0.15063 | **0.125** | **- 0.0181** |
| 0.46875 | 0.5 | 0.484375 | **-0.0181** | 0.125 | **0.05212** |
| 0.46875 | 0.484375 | 0.476563 | **- 0.0181** | 0.05212 | **0.01668** |

... and so we approach the root 0.472834.

# Example

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval [0,2]?

**Answer:**

$f(x)$ is continuous on $[0,2]$

and $\mathrm{f}(0) * \mathrm{f}(2) = (1)(3) = 3 > 0$

$\Rightarrow$ Assumptions are not satisfied

$\Rightarrow$ Bisection method can not be used

# Example

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval [0,1]?

**Answer:**

$f(x)$ is continuous on [0,1]

and $f(0) * f(1) = (1)(-1) = -1 < 0$

$\Rightarrow$ Assumptions are satisfied

$\Rightarrow$ Bisection method can be used

# Best Estimate and Error Level

Bisection method obtains an interval that is guaranteed to contain a zero of the function.

Questions:

- What is the best estimate of the zero of ***f(x)***?
- What is the error level in the obtained estimate?

# Best Estimate and Error Level

The best estimate of the zero of the function ***f(x)*** after the first iteration of the Bisection method is the mid point of the initial interval:

$$Estimate\ of\ the\ zero:\ \ r = \frac{b+a}{2}$$

$$Error \leq \frac{b-a}{2}$$

# Stopping Criteria

Two common stopping criteria

1. Stop after a fixed number of iterations
2. Stop when the absolute error is less than a specified value

How are these criteria related?

# Stopping Criteria

$c_n$ : is the midpoint of the interval at the $n^{\text{th}}$ iteration

( $c_n$ is usually used as the estimate of the root).

r : is the zero of the function.

After $n$ iterations:

$$|error| = |r - c_n| \leq E_a^n = \frac{b-a}{2^n} = \frac{\Delta x^0}{2^n}$$

# Convergence Analysis

*Given* $f(x)$, $a$, $b$, *and* $\varepsilon$

How many iterations are needed such that: $|x - r| \le \varepsilon$

where $r$ is the zero of $f(x)$ and $x$ is the bisection estimate (i.e., $x = c_k$)?

$$n \ge \frac{\log(b-a) - \log(\varepsilon)}{\log(2)}$$

# Convergence Analysis – Alternative Form

$$n \geq \frac{\log(b-a)-\log(\varepsilon)}{\log(2)}$$

$$n \geq \log_2\left(\frac{\text{width of initial interval}}{\text{desired error}}\right) = \log_2\left(\frac{b-a}{\varepsilon}\right)$$

# Example

$a = 6,\ \ b = 7,\ \varepsilon = 0.0005$

How many iterations are needed such that: $|x - r| \le \varepsilon$?

$$n \ge \frac{\log(b-a) - \log(\varepsilon)}{\log(2)} = \frac{\log(1) - \log(0.0005)}{\log(2)} = 10.9658$$

$$\Rightarrow n \ge 11$$

# Example

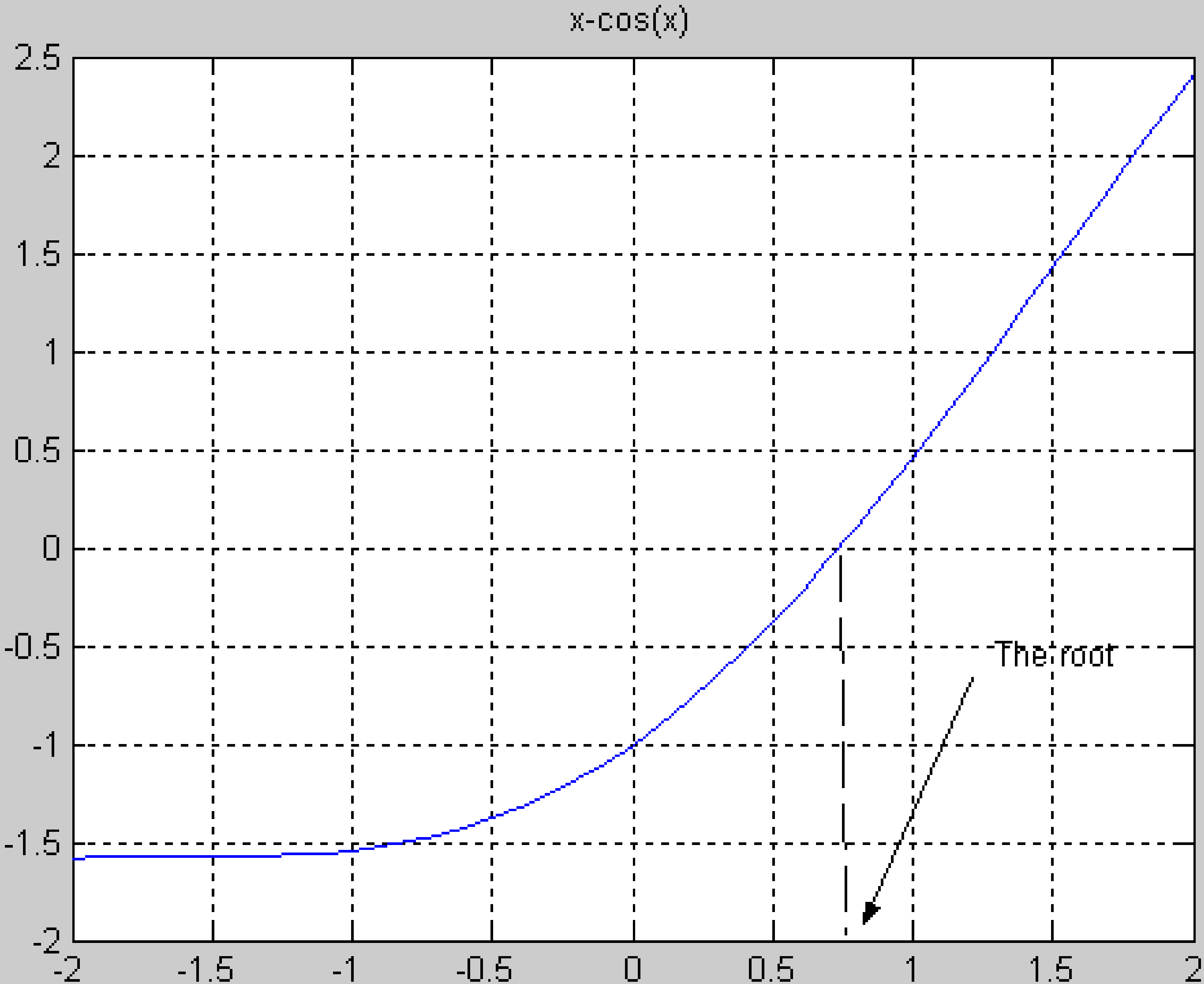
- Use Bisection method to find a root of the equation $x = \cos(x)$ with absolute error $<0.02$
  (assume the initial interval [0.5, 0.9])

Question 1: What is $f(x)$ ?

Question 2: Are the assumptions satisfied ?

Question 3: How many iterations are needed ?

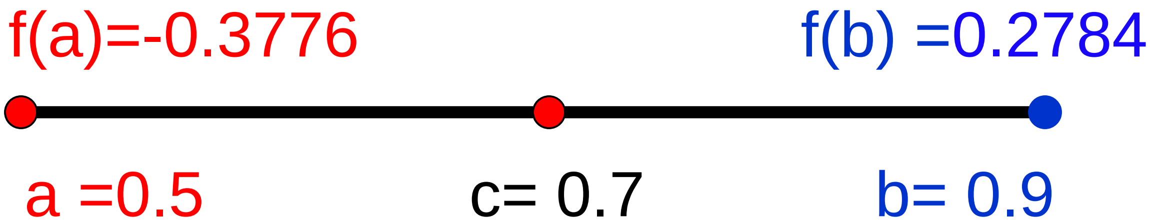
Question 4: How to compute the new estimate ?



x-cos(x)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
The root

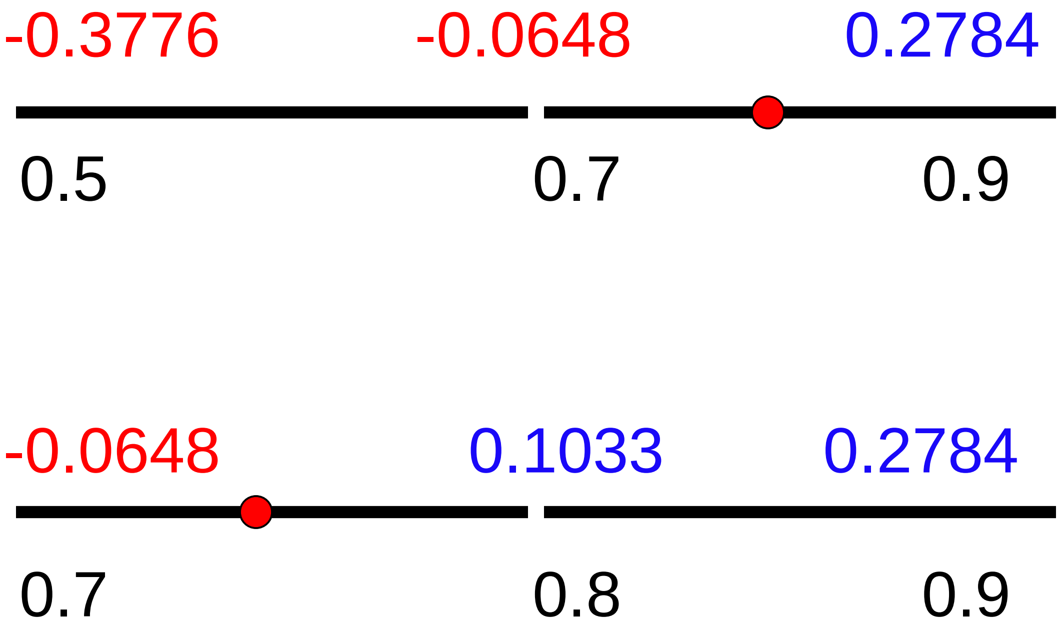
# Bisection Method

## Initial Interval




Error < 0.2

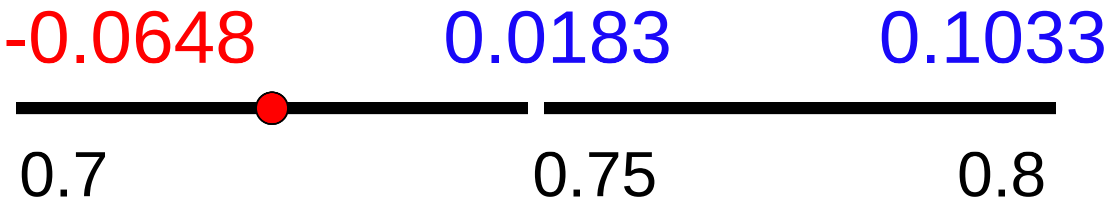
# Bisection Method



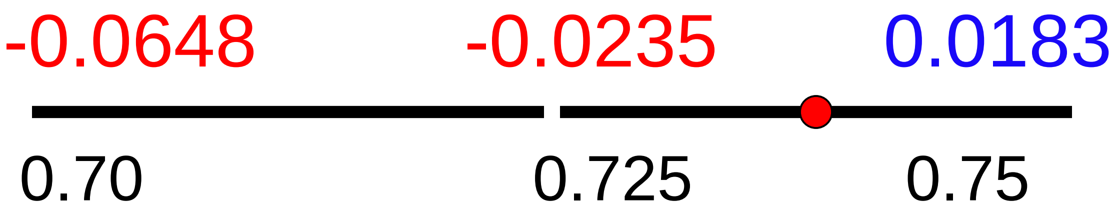

Error < 0.1

Error < 0.05

# Bisection Method




Error < 0.025




Error < .0125

# Summary

- Initial interval containing the root: [0.5,0.9]

- After 5 iterations:
  - Interval containing the root: [0.725, 0.75]
  - Best estimate of the root is 0.7375
  - | Error | < 0.0125

# A Matlab Program of Bisection Method

```
a=.5; b=.9;
u=a-cos(a);
v=b-cos(b);
   for i=1:5
        c=(a+b)/2
        fc=c-cos(c)
        if u*fc<0
          b=c ; v=fc;
        else
          a=c; u=fc;
        end
   end
```

```
c =
    0.7000
fc =
   -0.0648
c =
    0.8000
fc =
    0.1033
c =
    0.7500
fc =
    0.0183
c =
    0.7250
fc =
  -0.0235
```

# Example

Find the root of:

$$f(x) = x^3 - 3x + 1 \text{ in the interval:}[0,1]$$

* $f(x)$ is continuous

* $f(0) = 1, \; f(1) = -1 \Rightarrow f(a)\, f(b) < 0$

$\Rightarrow$ Bisection method can be used to find the root

# Example

| Iteration | a | b | c= $\frac{(a+b)}{2}$ | f(c) | $\frac{(b-a)}{2}$ |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0.5 | -0.375 | 0.5 |
| 2 | 0 | 0.5 | 0.25 | 0.266 | 0.25 |
| 3 | 0.25 | 0.5 | .375 | -7.23E-3 | 0.125 |
| 4 | 0.25 | 0.375 | 0.3125 | 9.30E-2 | 0.0625 |
| 5 | 0.3125 | 0.375 | 0.34375 | 9.37E-3 | 0.03125 |

# Bisection Method

**Advantages**

- Simple and easy to implement
- One function evaluation per iteration
- The size of the interval containing the zero is reduced by 50% after each iteration
- The number of iterations can be determined a priori
- No knowledge of the derivative is needed
- The function does not have to be differentiable

**Disadvantage**

- Slow to converge
- Good intermediate approximations may be discarded
- We need two initial guesses *a* and *b* which bracket the root.
- It is among the *slowest* methods to find the root.
- When an interval contains more than one root, the bisection method can find *only* one of them.

# Bisection Method: Class Work

- Find the real root of the equation $f(x)=x3 - x - 1= 0$ correct to 2 decimal places.($\varepsilon$=0.01).
- Answer: 1.328125
- Find the real root of the equation $f(x)=x4 - cos(x) + x = 0$ correct to 2 decimal places.($\varepsilon$=0.01).
- Answer: 0.637695

# Iteration Method

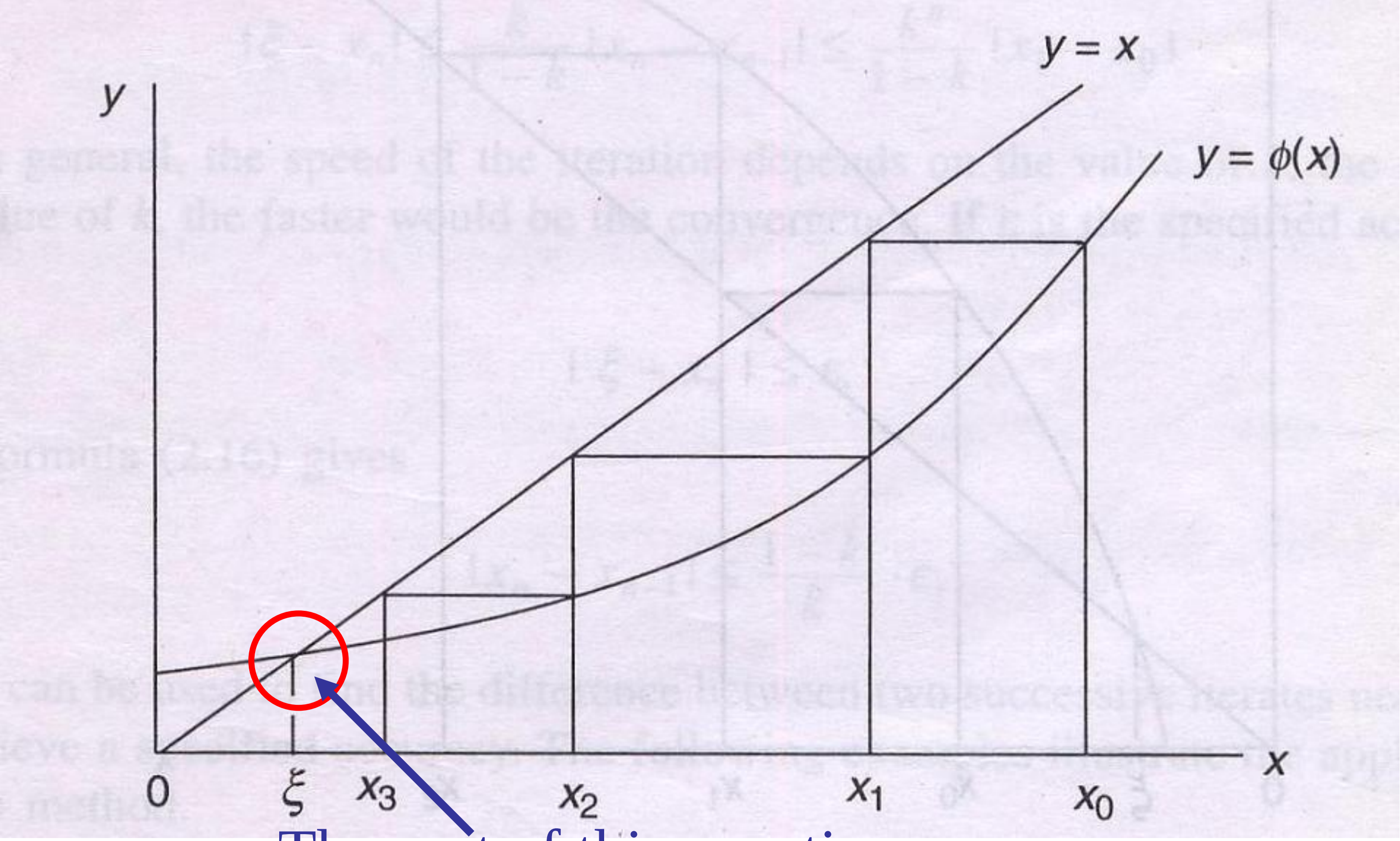
- Suppose we have an equation in the form $g(x) = 0$
- Rewrite the equation in the form $x = f(x)$.
- Start with an initial guess $x_0$, which is an ***approximation*** of the root.
- Calculate $x_1, \ldots, x_n, \ldots$ such that
  - $x_1 = f(x_0)$
  - $x_2 = f(x_1)$
  - $x_3 = f(x_2)$ ...
- Iterate the same process until $(x_n - x_{n-1})$ smaller than some specified tolerance.
- Geometrically, where the two graphs $x$ and $f(x)$ intersects, that is the real root of the equation.

# Iteration Method: Convergence Conditions

- Any arbitrary approximation $x_0$, $x_1$, $x_2$ does not assure that it will converge to the actual root $x$ of the equation.
  - E.g. $x = 10^x + 1$,
  - if $x_0 = 0$, $x_1 = 2$, $x_2 = 101$, …. that does not converge to the actual root $x$
  - As $n$ increase, $x_n$ increases without limit!
- The equation $x = f(x)$ converges to the real root $x$,
  - if $f(x)$ is continuous
  - If $|f'(x)| < 1$
- The equation $x = f(x)$ does not converges to the real root $x$ if $|f'(x)| > 1$
- Therefore, $g(x) = 0$ has to be re-written as $x = f(x)$ in such a way that $|f'(x)| < 1$

# Iteration Method: Convergence Conditions

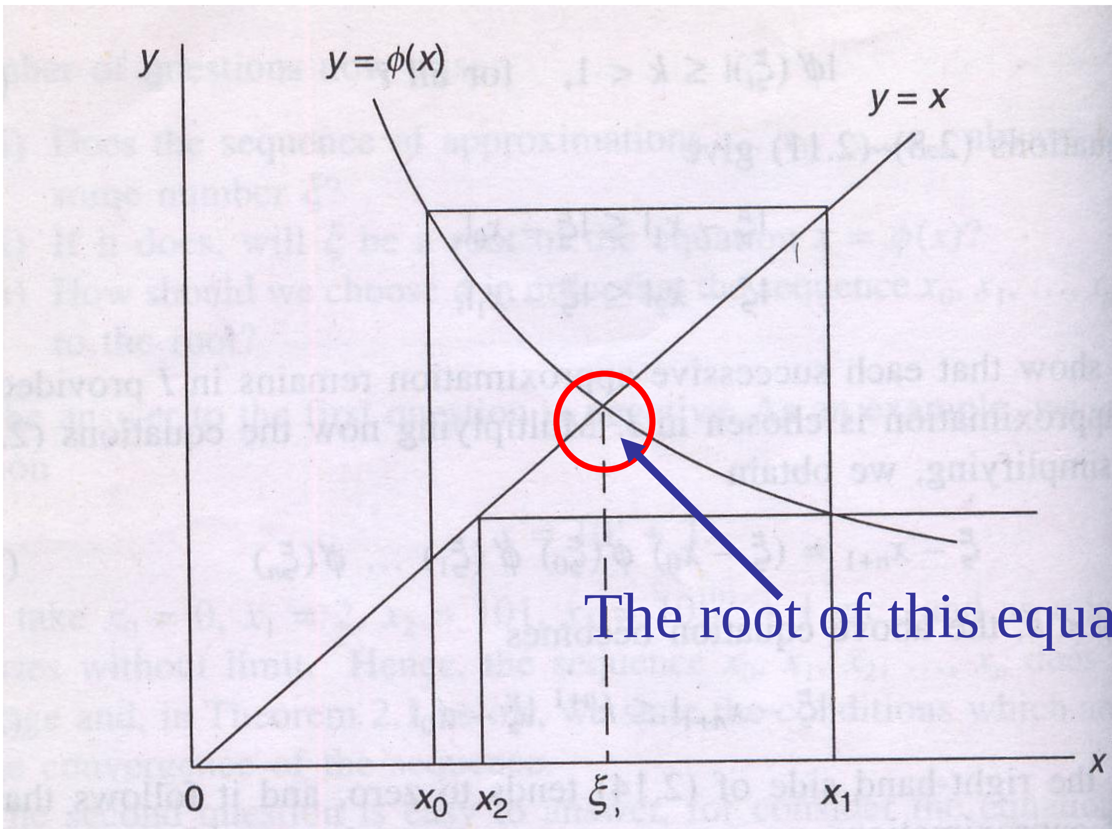
Convergence of $x_{n+1} = f(x_n)$, when $| f'(x) | < 1$




The root of this equation

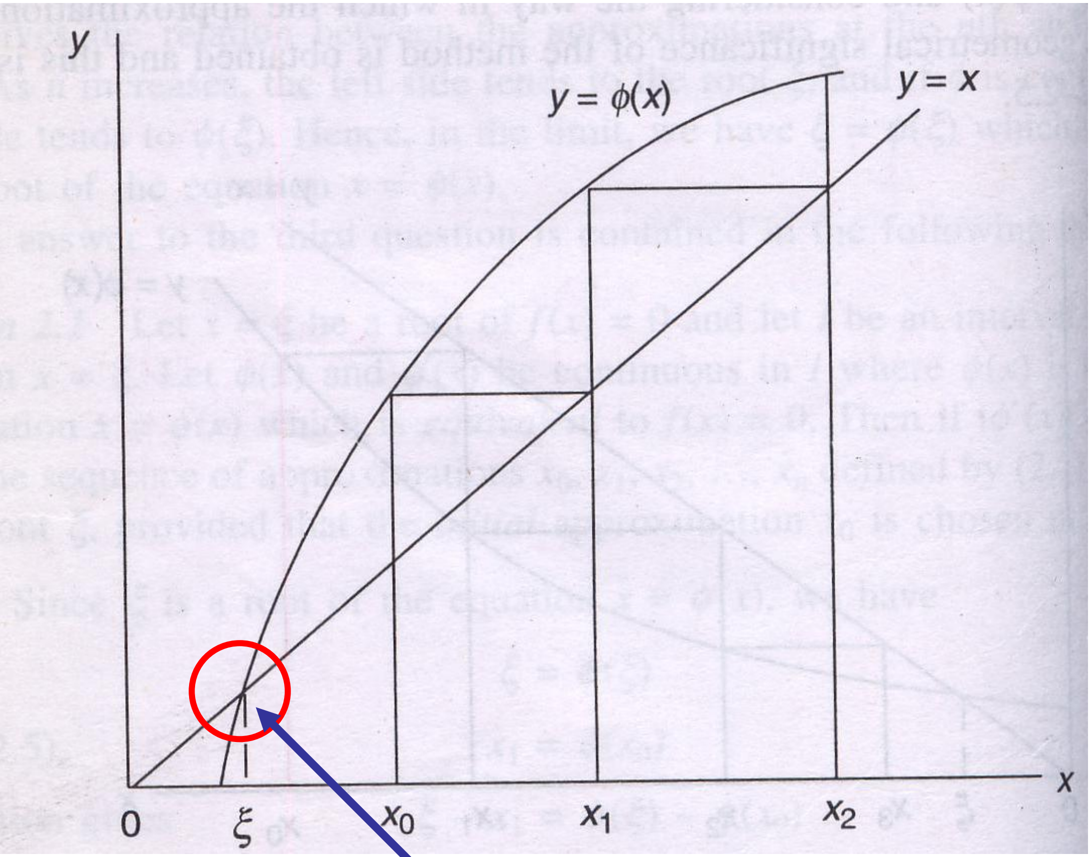
# Iteration Method: Convergence Conditions

$x_{n+1} = f(x_n)$ oscillates but ultimately converges, when $| f'(x) | < 1$, but $f'(x) < 0$



# Iteration Method: Convergence Conditions

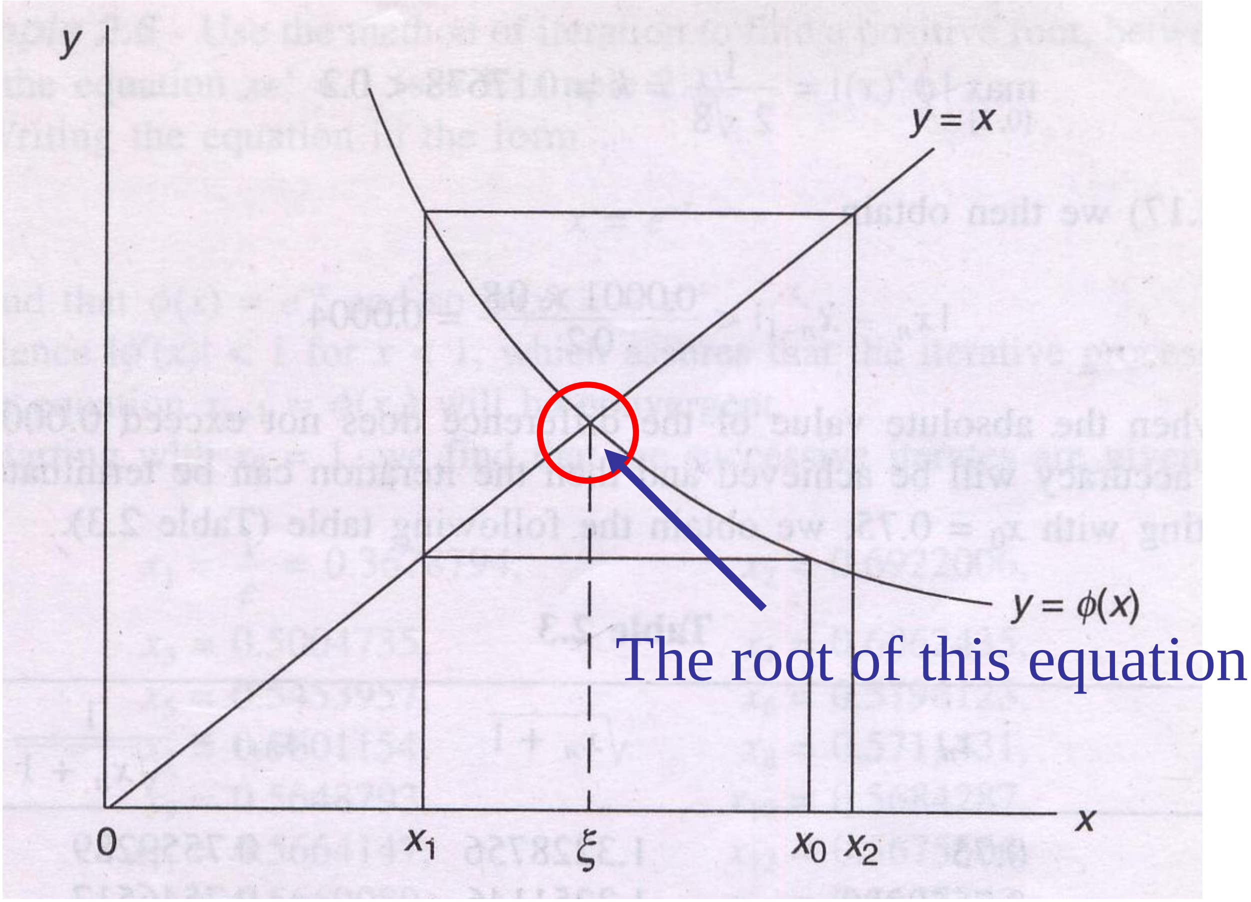
$x_{n+1} = f(x_n)$ diverges, when $f'(x) > 1$




The root of this equation

# Iteration Method: Convergence Conditions

$x_{n+1} = f(x_n)$ diverges, when $f'(x) > 1$



# Iteration Method: Example

Solve $x = 2 + sin(x)/2$

Solution

Here $f(x) = 2 + sin(x)/2$

Starting with $x_0 = 2$ we calculate $x_1$, $x_2$,...

| | |
|---|---|
| $x_0$ | 2 |
| $x_1 = f(x_0)$ | 2.454648713 |
| $x_2 = f(x_1)$ | 2.31708862 |
| $x_3 = f(x_2)$ | 2.367105575 |
| $x_4 = f(x_3)$ | 2.349674771 |
| $x_5 = f(x_4)$ | 2.355850929 |
| $x_6 = f(x_5)$ | 2.353674837 |
| $x_7 = f(x_6)$ | 2.354443099 |
| $x_8 = f(x_7)$ | 2.354172058 |
| $x_9 = f(x_8)$ | 2.354267705 |
| $x_{10} = f(x_9)$ | 2.354233955 |
| $x_{11} = f(x_{10})$ | 2.354245864 |

# Iteration Method: Example

Find the real root of the equation

$$g(x) = x^3 + x^2 - 1 = 0$$

Rewrite $g(x)$

$x^3 + x^2 - 1 = 0$

or, $x^3 + x^2 = 1$

or, $x^2(x+1) = 1$

or, $x^2 = 1/(x+1)$

or, $x = 1/\sqrt{(x+1)}$

Let, $x_0 = 0.75$

| |
|---|
| $x_0 = 0.7500000$ |
| $x_1 = 0.7559289$ |
| $x_2 = 0.7546517$ |
| $x_3 = 0.7549263$ |
| $x_4 = 0.7548672$ |
| $x_5 = 0.7548799$ |
| $x_6 = 0.7548772$ |
| $x_7 = 0.7548778$ |
| $x_8 = 0.7548776$ |
| $x_9 = 0.7548777$ |
| $x_{10} = 0.7548777$ |

# Iteration Method: Class Work

Find the real root of the equation using iterative method (till 4 decimal places).

$$e^{-x} = 10x$$

Answer:
0.091276527

# Iterative Method: Drawbacks

- We need an approximate initial guesses $x_0$.
- It is also a *slower* method to find the root.
- If the equation has more than one roots, then this method can find *only* one of them.

# Convergence Criteria of Iteration Method

***Proof*** Since $\xi$ is a root of the equation $x = \phi(x)$, we have

$$\xi = \phi(\xi) \tag{2.10}$$

From (2.9),

$$x_1 = \phi(x_0) \tag{2.11}$$

Subtraction gives

$$\xi - x_1 = \phi(\xi) - \phi(x_0)$$

By using the mean value theorem (see Theorem 1.3), the right-hand side can be written as $(\xi - x_0)\phi'(\xi_0)$, $x_0 < \xi_0 < \xi$. Hence we obtain

$$\xi - x_1 = (\xi - x_0)\phi'(\xi_0), \quad x_0 < \xi_0 < \xi \tag{2.12}$$

Similarly we obtain

$$\xi - x_2 = (\xi - x_1)\phi'(\xi_1), \quad x_1 < \xi_1 < \xi \tag{2.13}$$

$$\xi - x_3 = (\xi - x_2)\phi'(\xi_2), \quad x_2 < \xi_2 < \xi \tag{2.14}$$

$$\vdots$$

$$\xi - x_{n+1} = (\xi - x_n)\phi'(\xi_n), \qquad x_n < \xi_n < \xi \tag{2.15}$$

If we let

$$|\phi'(\xi_i)| \leq k < 1, \quad \text{for all } i \tag{2.16}$$

then Eqs. (2.12)–(2.15) give

$$|\xi - x_1| \le |\xi - x_0|, \qquad |\xi - x_2| \le |\xi - x_1|, \ldots,$$

which show that each successive approximation remains in $I$ provided that the initial approximation is chosen in $I$. Now, multiplying Eqs. (2.12) to (2.15) and simplifying, we obtain

$$\xi - x_{n+1} = (\xi - x_0)\,\phi'(\xi_0)\,\phi'(\xi_1)\ldots\phi'(\xi_n), \tag{2.17}$$

Since $|\phi'(\xi_i)| < k$, the above equation becomes

$$|\xi - x_{n+1}| \le k^{n+1}\,|\xi - x_0| \tag{2.18}$$

As $n \to \infty$, the right-hand side of (2.18) tends to zero, and it follows that the sequence of approximations $x_0, x_1, \ldots,$ converges to the root $\xi$ if $k < 1$.

# Acceleration of Convergence : Aitken's $\Delta^2$ Process

From the relation

$$|\xi - x_{n+1}| = |\phi(\xi) - \phi(x_n)| \le k\,|\xi - x_n|, \qquad k < 1$$

it is clear that the iteration method is linearly convergent. This slow rate of convergence can be accelerated by using Aitken's method, which is described below.

Let $x_{i-1}, x_i, x_{i+1}$ be three successive approximations to the desired root $x = \xi$ of the equation $x = \phi(x)$. From Section 2.4, we know that

$$\xi - x_i = k(\xi - x_{i-1}), \qquad \xi - x_{i+1} = k(\xi - x_i)$$

Dividing, we obtain

$$\frac{\xi - x_i}{\xi - x_{i+1}} = \frac{\xi - x_{i-1}}{\xi - x_i},$$

which gives on simplification

$$\xi = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}}. \qquad (2.22)$$

# Acceleration of Convergence : Aitken's $\Delta^2$ Process

If we now define $\Delta x_i$ and $\Delta^2 x_i$ by the relations

$$\Delta x_i = x_{i+1} - x_i \quad \text{and} \quad \Delta^2 x_i = \Delta(\Delta x_i),$$

then

$$\begin{aligned}
\Delta^2 x_{i-1} &= \Delta(\Delta x_{i-1}) \\
&= \Delta(x_i - x_{i-1}) \\
&= \Delta x_i - \Delta x_{i-1} \\
&= x_{i+1} - x_i - (x_i - x_{i-1}) \\
&= x_{i+1} - 2x_i + x_{i-1}.
\end{aligned}$$

Hence (2.22) can be written in the simpler form

$$\xi = x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}. \tag{2.23}$$

which explains the term $\Delta^2$-process.

# Acceleration of Convergence : Aitken's $\Delta^2$ Process

In any numerical application, the values of the following underlined quantities must be obtained.

$$
\begin{array}{lll}
\hline
x_{i-1} & & \\
 & \Delta x_{i-1} & \\
x_i & & \Delta^2 x_{i-1} \\
 & \Delta x_i & \\
x_{i+1} & & \\
\hline
\end{array}
$$

# Problems

***Example 2.7*** Find the root of the equation $2x = \cos x + 3$ correct to three decimal places.

We rewrite the equation in the form

$$x = \frac{1}{2}(\cos x + 3) \tag{i}$$

so that

$$\phi(x) = \frac{1}{2}(\cos x + 3),$$

and

$$|\phi'(x)| = \left|\frac{\sin x}{2}\right| < 1.$$

Hence the iteration method can be applied to the eq. (i) and we start with $x_0 = \pi/2$. The successive iterates are

$$x_1 = 1.5, \qquad x_2 = 1.535, \qquad x_3 = 1.518,$$

$$x_4 = 1.526, \qquad x_5 = 1.522, \qquad x_6 = 1.524,$$

# Problems

Hence the iteration method can be applied to the eq. (i) and we start with $x_0 = \pi/2$. The successive iterates are

$$x_1 = 1.5, \quad x_2 = 1.535, \quad x_3 = 1.518,$$

$$x_4 = 1.526, \quad x_5 = 1.522, \quad x_6 = 1.524,$$

$$x_7 = 1.523, \quad x_8 = 1.524.$$

Hence we take the solution as 1.524 correct to three decimal places.

# According to Aitken's

As before,

| | | |
|---|---|---|
| $x_1 = 1.5$ | | |
| | 0.035 | |
| $x_2 = 1.535$ | | $-0.052$ |
| | $-0.017$ | |
| $x_3 = 1.518$ | | |

Hence we obtain from Eq. (2.23)

$$x_4 = 1.518 - \frac{(-0.017)^2}{-0.052} = 1.524,$$

which corresponds to six normal iterations.

# The Method of False Position Or Regula Falsi

- Like the bisection method, Method of False Position requires two initial guesses $x_a$ and $x_b$ such that $f(x) = 0$ and $f(x_a)$ and $f(x_b)$ has opposite signs.

- Since the graph of $y = f(x)$ crosses the $x$-axis between these two points, a root must lie in between these points.

- The difference between these two methods is, instead of simply dividing the region in two, it obtains a new point $x_1$ which is (hopefully, but not necessarily) closer to the root.

- If $f(x_a)$ and $f(x_1)$ has opposite signs, then the new interval to be explored is $[x_a, x_1]$.

- Otherwise, the new interval is $[x_b, x_1]$.

- The procedure is repeated till the root is obtained to the desired accuracy.

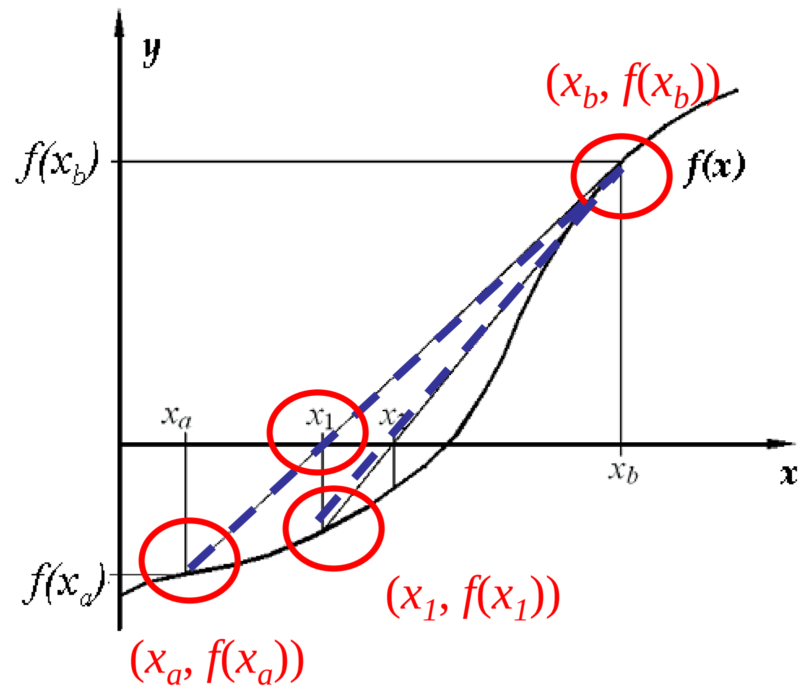
# The Method of False Position

- This method consists in replacing the part of the curve between the points $(x_a, f(x_a))$ and $(x_b, f(x_b))$.

- The equation of the chord joining the two points $(x_a, f(x_a))$ and $(x_b, f(x_b))$ is

$$\frac{y - f(x_a)}{x - x_a} = \frac{f(x_b) - f(x_a)}{x_b - x_a}$$

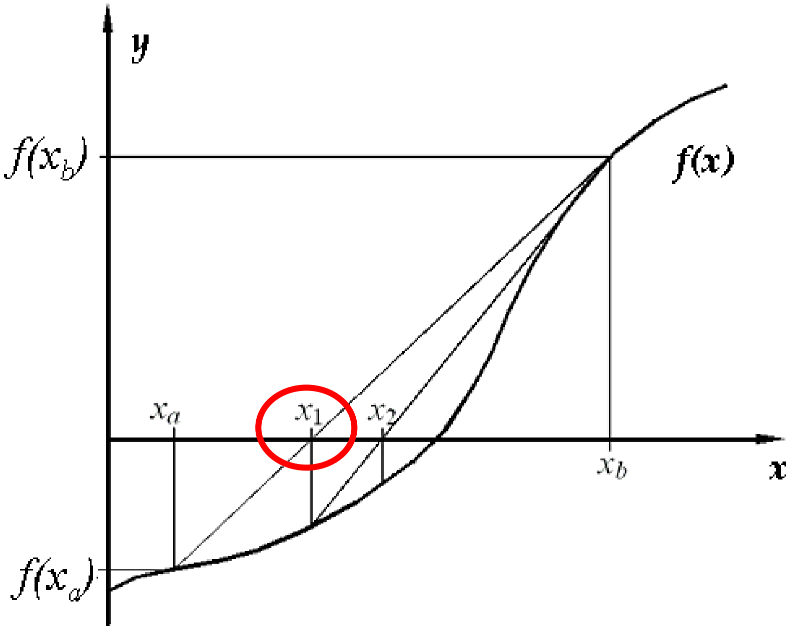
- It takes the point of intersection of the chord with the $x$-axis as an approximation to the root (here, $x_1$).



# The Method of False Position

- The point of intersection in the present case is given by putting $y = 0$ in the equation

$$\frac{y - f(x_a)}{x - x_a} = \frac{f(x_b) - f(x_a)}{x_b - x_a}$$




- Thus we obtain

$$x = x_a - \frac{f(x_a)}{f(x_b) - f(x_a)}(x_b - x_a)$$

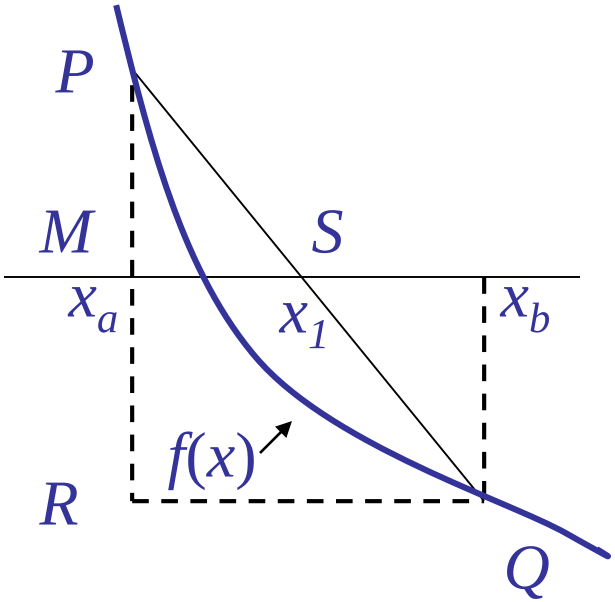
- Hence, the approximate root is

$$x_1 = x_a - \frac{f(x_a)}{f(x_b) - f(x_a)}(x_b - x_a)$$

# The Method of False Position : Geometric Significance

Here, for Δ PMS and Δ PRQ

- MS/MP = RQ/RP
- $(x_1 – x_a)/f(x_a) = (x_b – x_a) /(f(x_b)+- f(x_a))$
- $x_1 – x_a = f(x_a)(x_b – x_a) /(f(x_a)+- f(x_b))$
- $x_1 = x_a - f(x_a)(x_b – x_a) /(f(x_b) - f(x_a))$



# The Method of False Position: Example

Find the real root of the equation till 2 decimal place

$$f(x) = x^3 - 2x - 5 = 0$$

We observe that $f(2) = -1$ and $f(3) = 16$

And hence a root lies between 2 and 3. Then

| *x0* | *x1* | *x2* | *f(x0)* | *f(x1)* | *f(x2)* |
|---|---|---|---|---|---|
| 2 | 3 | 2.058824 | -1 | 16 | -0.3908 |
| 2.058824 | 3 | 2.081264 | -0.3908 | 16 | -0.1472 |
| 2.081264 | 3 | 2.089639 | -0.1472 | 16 | -0.05468 |
| 2.089639 | 3 | 2.09274 | -0.05468 | 16 | -0.0202 |
| 2.09274 | 3 | 2.093884 | -0.0202 | 16 | -0.00745 |

| | |
|---|---|
| *x1* | 2.059 |
| *x2* | 2.081 |
| *x3* | 2.090 |
| *x4* | 2.093 |

$x_4$ is correct to 2 decimal places.

# The Method of False Position: Example

Class Work

Find the real root of the equation till 2 decimal place
$x^3 - 2x^2 + 3x = 5$ between the points 1 and 2.

Result 1.843734

# The Method of False Position: Example

Class Work

Find the real root of the equation till 2 decimal place

$$sin\, x + x - 1 = 0.$$

Result 0.510973

# Pitfalls of the False Position Method

- Although a method such as false position is often superior to bisection, there are some cases (when function has significant curvature) that violate this general conclusion.

- In such cases, the approximate error might be misleading and the results should always be checked by substituting the root estimate into the original equation and determining whether the result is close to zero.

- major weakness of the false-position method: its one sidedness That is, as iterations are proceeding, one of the bracketing points will tend stay fixed which lead to poor convergence.

***Advantages:***

1. Simple
2. Brackets the Root

***Disadvantages:***

1. Can be VERY slow
2. Like Bisection, need an initial interval around the root.

# Newton-Raphson Method

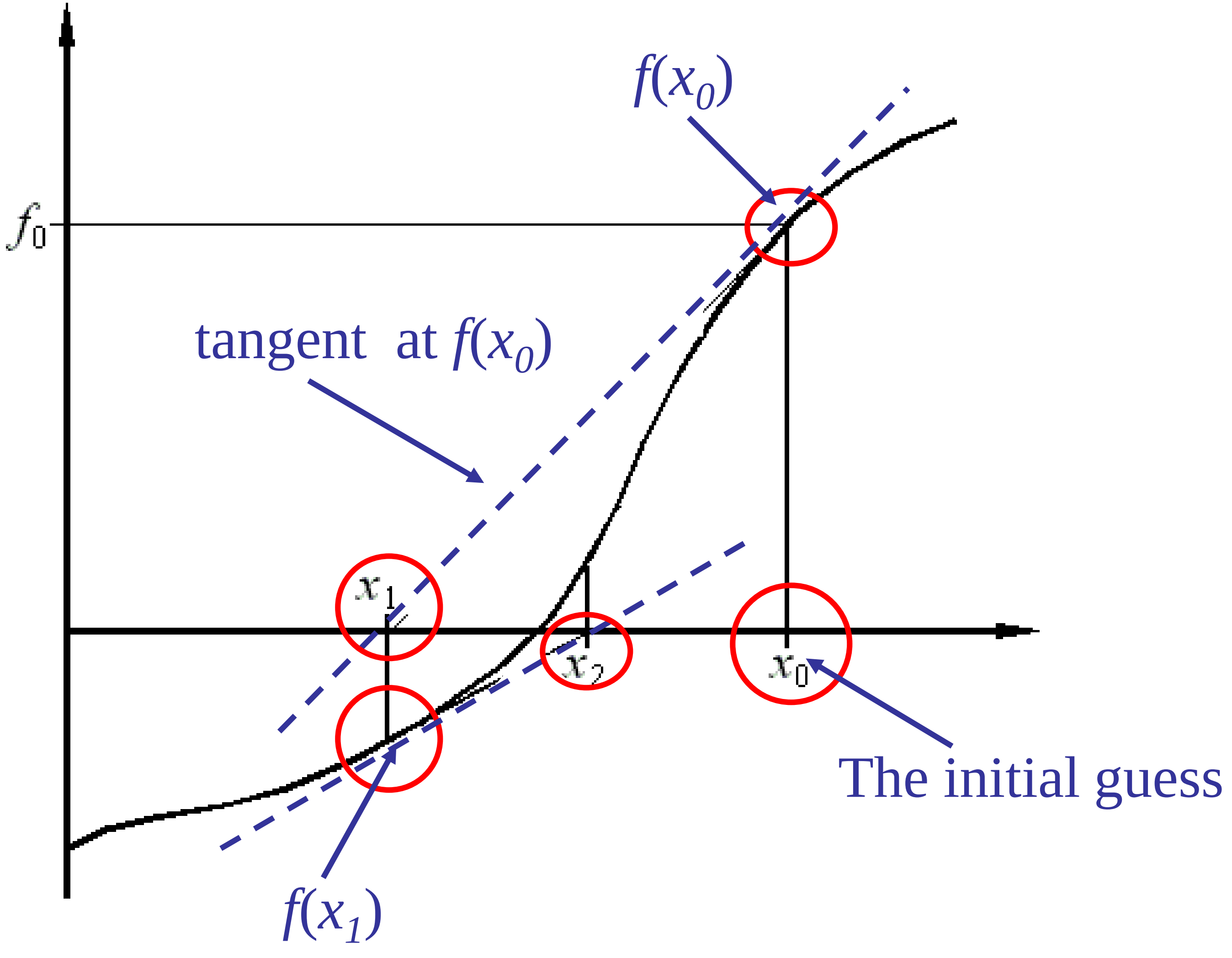
- This method is more efficient than the Bisection and Iteration methods.

If

- $x$ is the real root and $x_0$ is an initial approximation of the real root of an equation $f(x) = 0$,
- $f'(x_0) \neq 0$,
- $f(x)$ has the same sign between $x_0$ and $x$,

Then, the tangent at $f(x_0)$ can lead to the real root $x$.

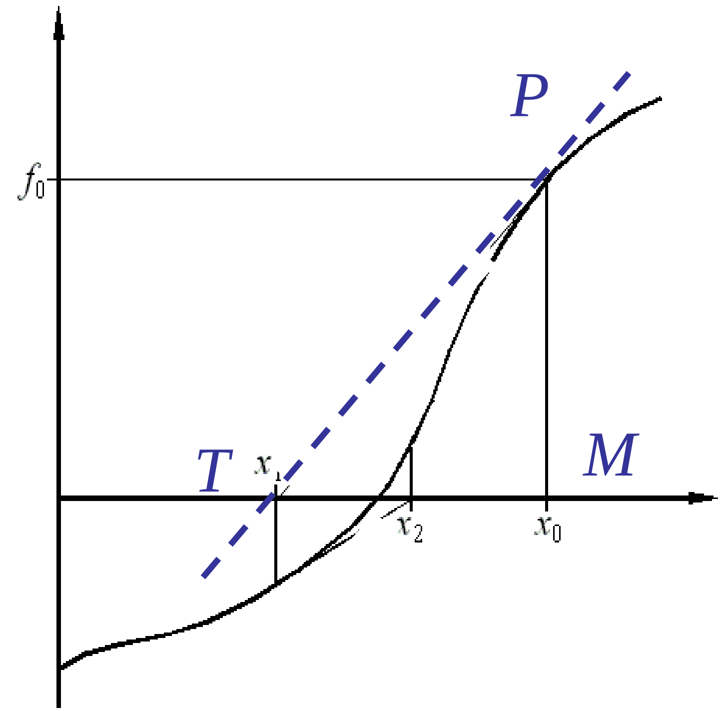
# Newton-Raphson Method: Geometric Significance



# Newton-Raphson Method: Geometric Significance

Here,

- The slope at $x_1$ is $tan\ (PTM)$
- $tan\ (PTM) = PM/\mathrm{TM}$
- $tan\ (PTM) = f\,(x_0)/h$
- Again, $tan\ (PTM) = f\,'(x_0)$
- Therefore, $f\,'(x_0) = f\,(x_0)/h$
- Or, $h = f\,(x_0)/\,f\,'(x_0)$
- $x_1 = x_0 - h$
- Therefore, $x_1 = x_0 - f\,(x_0)/\,f\,'(x_0)$
- Similarly, $x_2 = x_1 - f\,(x_1)/\,f\,'(x_1)$



# Geometric Significance

**Newton-Raphson Method:**

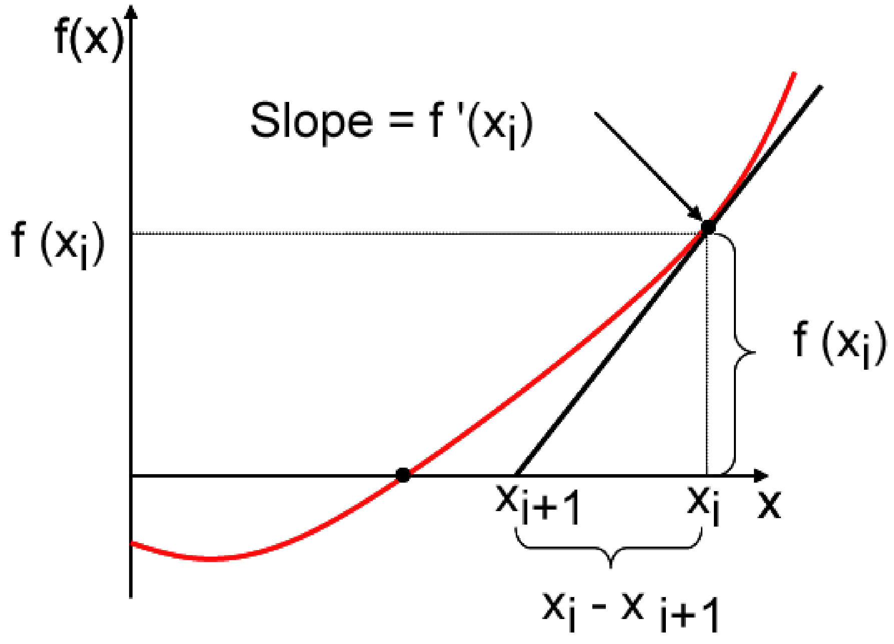
**Geometrical Derivation:**

Slope of tangent at $x_i$ is

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

Solve for $x_{i+1}$:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$




[Note that this is the same form as the generalized one-point iteration, $x_{i+1} = g(x_i)$]

# Newton-Raphson Method

Methodology

- Let $x_0$ be an approximate root of $f(x) = 0$ and
- Let, $x_1$ is the correct root such that $x_1 = x_0 + h$ and $f(x_1) = 0$.
- Expanding $f(x_0+h)$ by Taylor's series, we obtain,

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots = 0$$

- Neglecting the second and higher order derivatives, we have

$$f(x_0) + hf'(x_0) = 0$$

- Which gives

$$h = -\frac{f(x_0)}{f'(x_0)}$$

# Newton-Raphson Method (Cont'd.)

- A better approximation than $x_0$ is therefore given by $x_1$ where

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Successive approximation are given by $x_2$, $x_3$, ….., $x_n$, $x_{n+1}$ where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- This formula is known as the Newton-Raphson formula.

# Newton-Raphson Method: Example

Find the real root of the equation using Newton-Raphson's Method

$f(x) = x^3 + 4x^2 - 1 = 0,\ f'(x) = 3x^2 + 4*2x - 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\ 3} + 4x_n^{\ 2} - 1}{3x_n^{\ 2} + 8x}$$

| | |
|---|---|
| $x_0$ | 0.5 |
| $x_1$ | 0.473684211 |
| $x_2$ | 0.472834787 |
| $x_3$ | 0.472833909 |
| $x_4$ | 0.472833909 |

# Newton-Raphson Method: Example

## Class work

Use Newton-Raphson's Method to find a root of the equation correct to 2 decimal places.($\varepsilon$=0.01)

$$x^3 - 2x - 5 = 0$$

$f(x) = x^3 - 2x - 5$

$f'(x) = 3x^2 - 2$

Result 2.094551482

# Newton-Raphson Method: Example

## Class work

Use Newton-Raphson's Method to find a root of the equation correct to 2 decimal places.($\varepsilon$=0.01)

$$x \sin x = - \cos x$$

$f(x) = x \sin x + \cos x$

$f'(x) = x \cos x$

Result 2.798386046

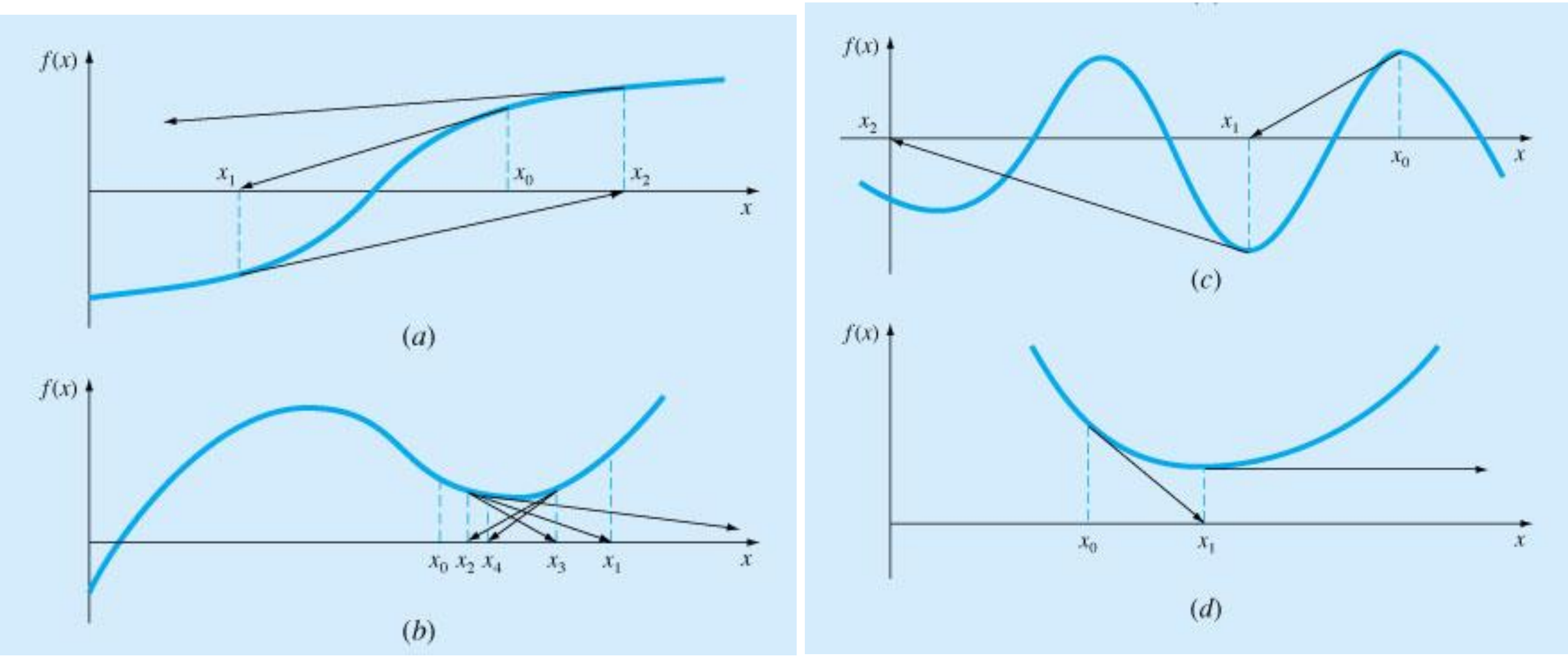
# Advantages

- Converges fast (quadratic convergence), if it converges.
- Requires only one guess

# Newton-Raphson Method: Drawbacks

- The Newton-Raphson method requires the calculation of the *derivative* of a function, which is not always easy.
- If $f'$ vanishes at an iteration point, then the method will fail to converge.
- When the step is too large or the value is oscillating, other more conservative methods should take over the case.

# Pitfalls of The Newton Raphson Method




**Cases where Newton Raphson method diverges or exhibit poor convergence.**

**a) Reflection point** **b) oscillating around a local optimum**

**c) Near zero slop , and** **d) zero slop**

# Newton's Method

$$Given \quad f(x),\ f'(x),\ x_0$$

$$Assumpution \quad f'(x_0) \neq 0$$

---

$$for \quad i = 0{:}n$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$end$$

```
C     FORTRAN PROGRAM
        F(X) = X**3-3*X**2+1
        FP(X) = 3*X**2-6*X
        X = 4
        DO 10 I = 1,5
         X = X - F(X)/FP(X)
         PRINT *, X
10       CONTINUE
        STOP
        END
```

# Newton's Method

$$Given \quad f(x),\ f'(x),\ x_0$$

$$Assumpution \quad f'(x_0) \neq 0$$

---

$$for \quad i = 0:n$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$end$$

F.m

$$function \quad [F] = F(X)$$

$$F = X^\wedge 3 - 3 * X^\wedge 2 + 1$$

FP.m

$$function \quad [FP] = FP(X)$$

$$FP = 3 * X^\wedge 2 - 6 * X$$

$$\% \quad MATLAB \ PROGRAM$$

$$X = 4$$

$$for \ i = 1:5$$

$$X = X - F(X) / FP(X)$$

$$end$$

# Example

Find a zero of the function $f(x) = x^3 - 2x^2 + x - 3$ , $x_0 = 4$

$f'(x) = 3x^2 - 4x + 1$

Iteration 1: $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 4 - \dfrac{33}{33} = 3$

Iteration 2: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 3 - \dfrac{9}{16} = 2.4375$

Iteration 3: $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 2.4375 - \dfrac{2.0369}{9.0742} = 2.2130$

# Example

| k (Iteration) | $x_k$ | $f(x_k)$ | $f'(x_k)$ | $x_{k+1}$ | $\lvert x_{k+1} - x_k \rvert$ |
|---|---|---|---|---|---|
| 0 | **4** | **33** | **33** | **3** | 1 |
| 1 | **3** | **9** | **16** | **2.4375** | 0.5625 |
| 2 | **2.4375** | **2.0369** | **9.0742** | **2.2130** | 0.2245 |
| 3 | **2.2130** | **0.2564** | **6.8404** | **2.1756** | 0.0384 |
| 4 | **2.1756** | **0.0065** | **6.4969** | **2.1746** | 0.0010 |

# Convergence Analysis

Theorem:

Let $f(x)$, $f'(x)$ and $f''(x)$ be continuous at $x \approx r$ where $f(r) = 0$. If $f'(r) \neq 0$ then there exists $\delta > 0$ such that $\left|x_0 - r\right| \leq \delta \Rightarrow \dfrac{\left|x_{k+1} - r\right|}{\left|x_k - r\right|^2} \leq C$

$$C = \frac{1}{2} \frac{\max\limits_{\left|x_0 - r\right| \leq \delta} \left|f''(x)\right|}{\min\limits_{\left|x_0 - r\right| \leq \delta} \left|f'(x)\right|}$$

# Convergence Analysis
## Remarks

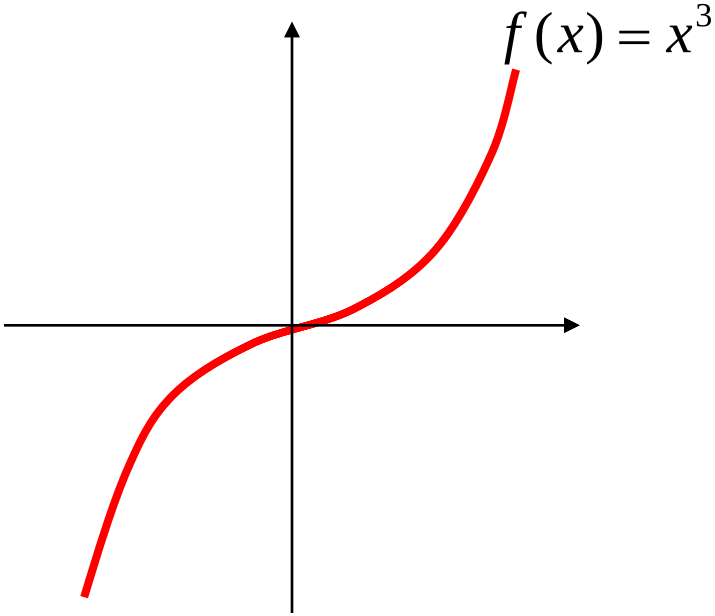
When the guess is close enough to a simple root of the function then Newton's method is guaranteed to converge quadratically.

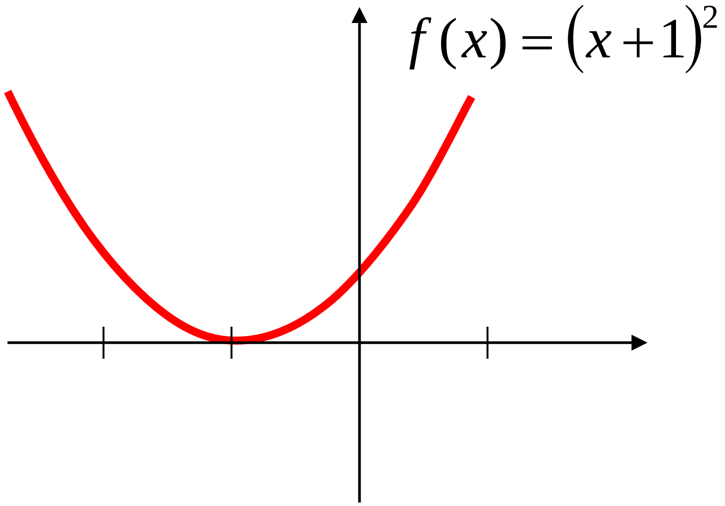
Quadratic convergence means that the number of correct digits is nearly doubled at each iteration.

# Problems with Newton's Method

- If the initial guess of the root is far from the root the method may not converge.
- Newton's method converges linearly near multiple zeros { $f(r) = f'(r) = 0$ }. In such a case, modified algorithms can be used to regain the quadratic convergence.

# Multiple Roots

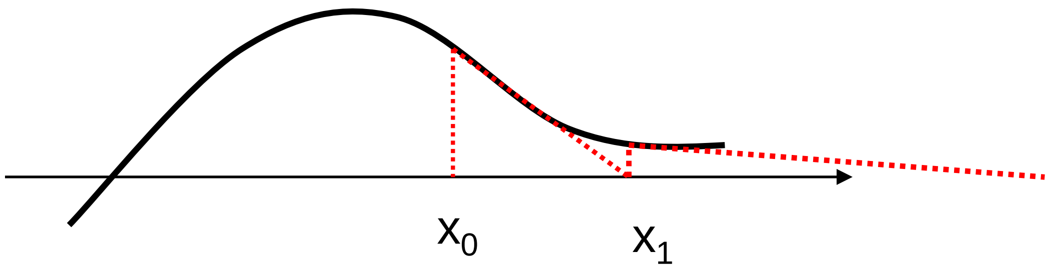



$f(x)$ has three zeros at $x = 0$

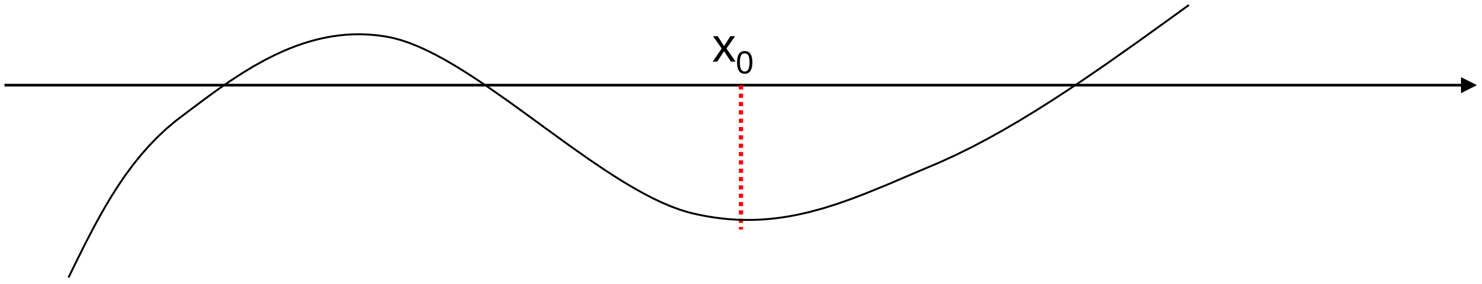
$f(x)$ has two zeros at $x = -1$

# Problems with Newton's Method
## - Runaway -




The estimates of the root is going away from the root.

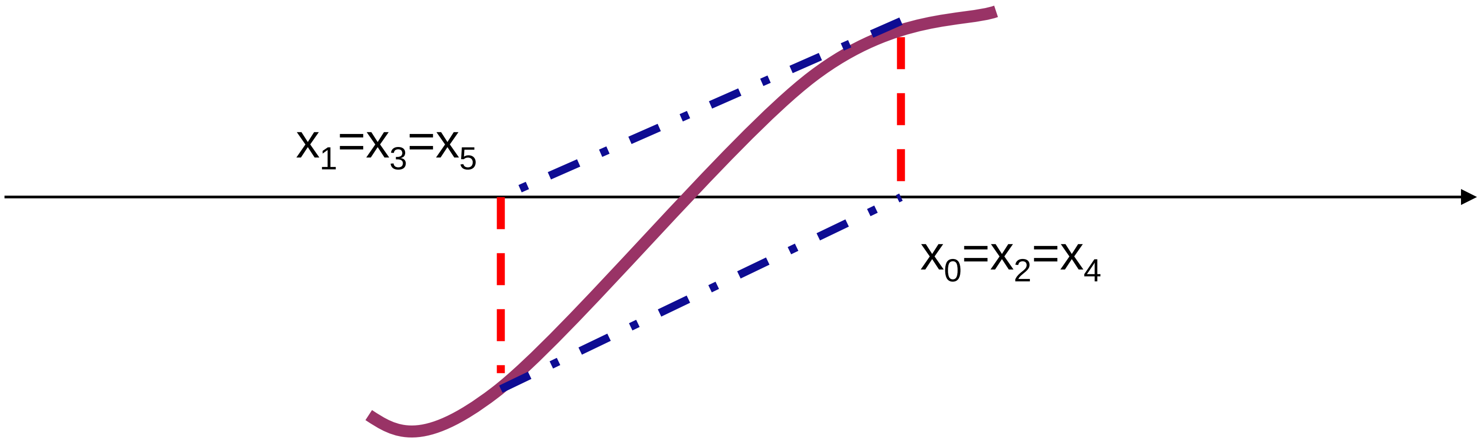
# Problems with Newton's Method
# - Flat Spot -




The value of ***f'(x)*** is zero, the algorithm fails.

If ***f '(x)*** is very small then $\boldsymbol{x_1}$ will be very far from $\boldsymbol{x_0}$.

# Problems with Newton's Method
## - Cycle -




The algorithm cycles between two values $x_0$ and $x_1$

# Generalized Newton's Method

- If $\xi$ is a root of $f(x)=0$ with multiplicity $p$, then the generalized Newton's formula is

$$x_{n+1} = x_n - p\frac{f(x)}{f'(x)}$$

- Since $\xi$ is a root of $f(x) = 0$ with multiplicity $p$, it follows that $\xi$ is a root of $f'(x) = 0$ with multiplicity $(p - 1)$, of $f''(x) = 0$ with multiplicity $(p - 2)$, and so on.

# Generalized Newton's Method

- Hence the expressions

$$x_0 - p\frac{f(x_0)}{f'(x_0)}, x_0 - (p-1)\frac{f'(x_0)}{f''(x_0)}, x_0 - (p-2)\frac{f''(x_0)}{f'''(x_0)}$$

- must have the same value if there is a root with multiplicity *p*, provided that the initial approximation $x_0$ is chosen sufficiently close to the root.

# Example

***Example*** Find a double root of the equation

$$f(x) = x^3 - x^2 - x + 1 = 0.$$

Here $f'(x) = 3x^2 - 2x - 1$, and $f''(x) = 6x - 2$. With $x_0 = 0.8$, we obtain

$$x_0 - 2\frac{f(x_0)}{f'(x_0)} = 0.8 - 2\frac{0.072}{-(0.68)} = 1.012,$$

and

$$x_0 - \frac{f'(x_0)}{f''(x_0)} = 0.8 - \frac{-(0.68)}{2.8} = 1.043,$$

# Example

The closeness of these values indicates that there is a doublel root near to unity. For the next approximation, we choose $x_1 = 1.01$ and obtain

$$x_1 - 2\frac{f(x_1)}{f'(x_1)} = 1.01 - 0.0099 = 1.0001,$$

and $$x_1 - \frac{f'(x_1)}{f''(x_1)} = 1.01 - 0.0099 = 1.0001,$$

Hence we conclude that there is a double root at $x = 1.0001$ which is sufficiently close to the actual root unity.

On the other hand, if we apply Newton-Raphson method with $x_0 = 0.8,$ we obtain $x_1 = 0.8 + 0.106 \approx 0.91,$ and $x_2 = 0.91 + 0.046 \approx 0.96.$

# Newton's Method for Systems of Non Linear Equations

$Given$: $X_0$ an initial guess of the root of $F(x)=0$

*Newton's Iteration*

$$X_{k+1} = X_k - \left[F'(X_k)\right]^{-1} F(X_k)$$

$$F(X) = \begin{bmatrix} f_1(x_1, x_2, \ldots) \\ f_2(x_1, x_2, \ldots) \\ \vdots \end{bmatrix}, \quad F'(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \vdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \\ \vdots & & \end{bmatrix}$$

# Example

- Solve the following system of equations:

$$y + x^2 - 0.5 - x = 0$$

$$x^2 - 5xy - y = 0$$

$$\text{Initial guess } \; x = 1, \;\; y = 0$$

$$F = \begin{bmatrix} y + x^2 - 0.5 - x \\ x^2 - 5xy - y \end{bmatrix}, \;\; F' = \begin{bmatrix} 2x - 1 & 1 \\ 2x - 5y & -5x - 1 \end{bmatrix}, \; X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

# Solution Using Newton's Method

Iteration1:

$$F=\begin{bmatrix} y+x^2-0.5-x \\ x^2-5xy-y \end{bmatrix}=\begin{bmatrix} -0.5 \\ 1 \end{bmatrix}=,\ \ F'=\begin{bmatrix} 2x-1 & 1 \\ 2x-5y & -5x-1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}$$

$$X_1=\begin{bmatrix} 1 \\ 0 \end{bmatrix}-\begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}^{-1}\begin{bmatrix} -0.5 \\ 1 \end{bmatrix}=\begin{bmatrix} 1.25 \\ 0.25 \end{bmatrix}$$

Iteration 2:

$$F=\begin{bmatrix} 0.0625 \\ -0.25 \end{bmatrix}=,\ \ F'=\begin{bmatrix} 1.5 & 1 \\ 1.25 & -7.25 \end{bmatrix}$$

$$X_2=\begin{bmatrix} 1.25 \\ 0.25 \end{bmatrix}-\begin{bmatrix} 1.5 & 1 \\ 1.25 & -7.25 \end{bmatrix}^{-1}\begin{bmatrix} 0.0625 \\ -0.25 \end{bmatrix}=\begin{bmatrix} 1.2332 \\ 0.2126 \end{bmatrix}$$

# Example

## Try this

- Solve the following system of equations:

$$y + x^2 - 1 - x = 0$$

$$x^2 - 2y^2 - y = 0$$

$$\text{Initial guess } \; x = 0, \;\; y = 0$$

$$F = \begin{bmatrix} y + x^2 - 1 - x \\ x^2 - 2y^2 - y \end{bmatrix}, \;\; F' = \begin{bmatrix} 2x - 1 & 1 \\ 2x & -4y - 1 \end{bmatrix}, \;\; X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

# Example

## Solution

| $Iteration$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $X_k$ | $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ | $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ | $\begin{bmatrix} -0.6 \\ 0.2 \end{bmatrix}$ | $\begin{bmatrix} -0.5287 \\ 0.1969 \end{bmatrix}$ | $\begin{bmatrix} -0.5257 \\ 0.1980 \end{bmatrix}$ | $\begin{bmatrix} -0.5257 \\ 0.1980 \end{bmatrix}$ |

# Secant Method

- Secant Method
- Examples
- Convergence Analysis

# Newton's Method (Review)

$$Assumptions: \quad f(x),\ f'(x),\ x_0 \quad are\ available,$$

$$f'(x_0) \neq 0$$

$$Newton's\ Method\ new\ estimate:$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Problem:

$f'(x_i)$ is not available,

or difficult to obtain analytically.

# Secant Method

- We have seen that the Newton-Raphson method requires the evaluation of derivatives of the function and this is not always possible, particularly in the case of functions arising in practical problems.
- In the secant method, the derivative at $x_n$ is approximated by the formula

# Secant Method

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

if $x_i$ *and* $x_{i-1}$ are two initial points:

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{\dfrac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}} = \quad x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}}$$

# Secant Method

Assumptions:

Two initial points $x_i$ *and* $x_{i-1}$

*such that* $f(x_i) \neq f(x_{i-1})$

New estimate (Secant Method):

$$x_{i+1} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

# Secant Method

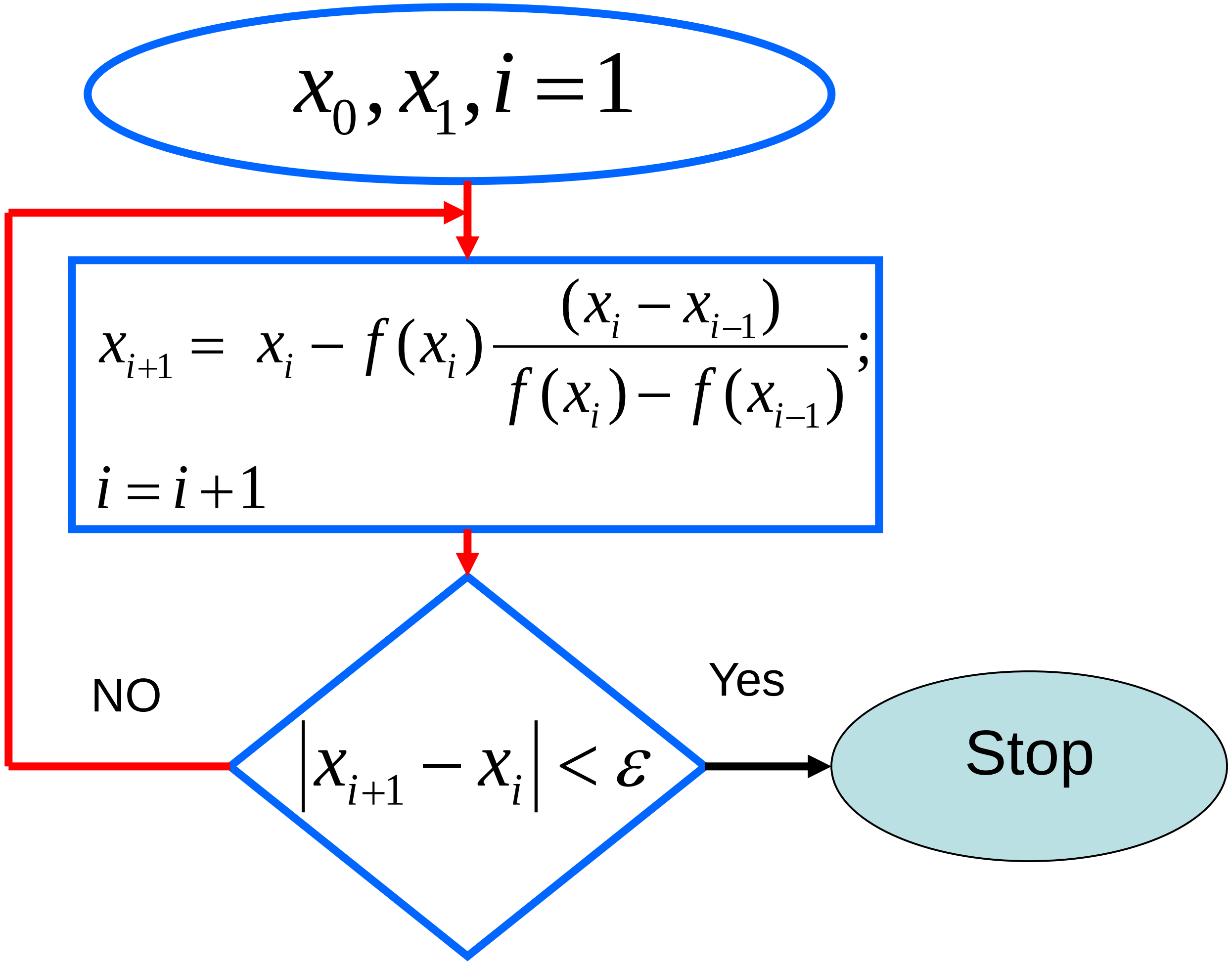
$$f(x) = x^2 - 2x + 0.5$$

$$x_0 = 0$$

$$x_1 = 1$$

$$x_{i+1} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

# Secant Method - Flowchart



# Modified Secant Method

In this modified Secant method, only one initial guess is needed:

$$f'(x_i) \approx \frac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\dfrac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}} = x_i - \frac{\delta x_i f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$$

Problem: How to select $\delta$?

If not selected properly, the method may diverge.

# Example

*Example* Find a real root of the equation $x^3 - 2x - 5 = 0$ using secant method.

Let the two initial approximations be given by $x_{-1} = 2$ and $x_0 = 3$.

We have

$$f(x_{-1}) = f_1 = 8 - 9 = -1, \quad \text{and} \quad f(x_0) = f_0 = 27 - 11 = 16.$$

$$x_1 = \frac{2(16) - 3(-1)}{17} = \frac{35}{17} = 2.058823529.$$

Also,

$$f(x_1) = f_1 = -0.390799923.$$

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{3(-0.390799923) - 2.058823529(16)}{-16.390799923} = 2.08126366.$$

Again

$$f(x_2) = f_2 = -0.147204057.$$

$$x_3 = 2.094824145.$$

# Example

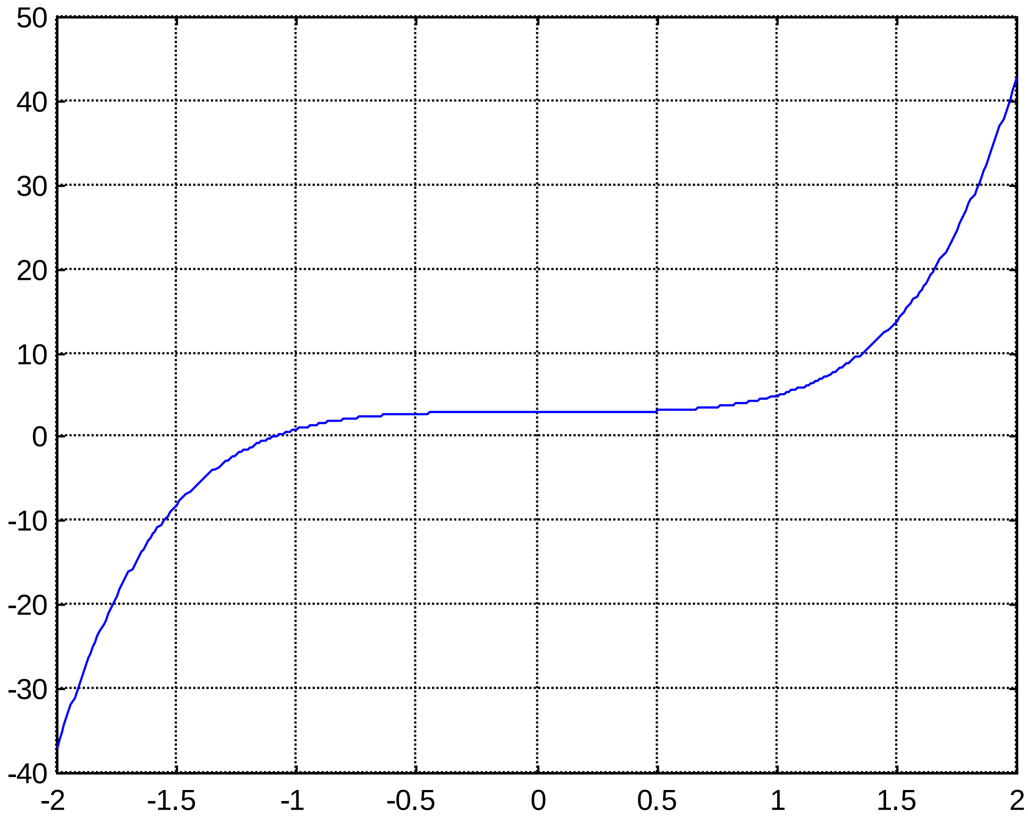
Find the roots of :

$$f(x) = x^5 + x^3 + 3$$

Initial points

$$x_0 = -1 \; and \; x_1 = -1.1$$

$$with \;\; error < 0.001$$



# Example

| x(i) | f(x(i)) | x(i+1) | \|x(i+1)-x(i)\| |
|---|---|---|---|
| -1.0000 | 1.0000 | -1.1000 | 0.1000 |
| -1.1000 | 0.0585 | -1.1062 | 0. 0062 |
| -1.1062 | 0.0102 | -1.1052 | 0.0009 |
| -1.1052 | 0.0001 | -1.1052 | 0.0000 |

# Convergence Analysis

- The rate of convergence of the Secant method is super linear:

$$\frac{\left|x_{i+1}-r\right|}{\left|x_i-r\right|^{\alpha}} \le C, \qquad \alpha \approx 1.62$$

$r: root \qquad x_i:$ estimate of the root at the i$^{\text{th}}$ iteration.

- It is better than Bisection method but not as good as Newton's method.

# Comparison of Root Finding Methods

- Advantages/disadvantages
- Examples

# Summary

| Method | Advantages | Disadvantages |
|---|---|---|
| Bisection | - Easy, Reliable, Convergent<br>- One function evaluation per iteration<br>- No knowledge of derivative is needed | - Slow<br>- Needs an interval [a,b] containing the root, i.e., $f(a)f(b)<0$ |
| Newton | - Fast (if near the root)<br>- Two function evaluations per iteration | - May diverge<br>- Needs derivative and an initial guess $x_0$ such that $f'(x_0)$ is nonzero |
| Secant | - Fast (slower than Newton)<br>- One function evaluation per iteration<br>- No knowledge of derivative is needed | - May diverge<br>- Needs two initial points guess $x_0$, $x_1$ such that $f(x_0)- f(x_1)$ is nonzero |

# Example

Use Secant method to find the root of :

$$f(x) = x^6 - x - 1$$

Two initial points $x_0 = 1$ *and* $x_1 = 1.5$

$$x_{i+1} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

# Solution

| k | $x_k$ | $f(x_k)$ |
|---|---|---|
| 0 | 1.0000 | -1.0000 |
| 1 | 1.5000 | 8.8906 |
| 2 | 1.0506 | -0.7062 |
| 3 | 1.0836 | -0.4645 |
| 4 | 1.1472 | 0.1321 |
| 5 | 1.1331 | -0.0165 |
| 6 | 1.1347 | -0.0005 |

# Example

Use Newton's Method to find a root of :

$f(x) = x^3 - x - 1$

Use the initial point : $x_0 = 1$.

Stop after three iterations, or

if $\left| x_{k+1} - x_k \right| < 0.001$, or

if $\left| f(x_k) \right| < 0.0001$.

# Five Iterations of the Solution

| k | $x_k$ | $f(x_k)$ | $f'(x_k)$ | ERROR |
|---|---|---|---|---|
| 0 | 1.0000 | -1.0000 | 2.0000 | |
| 1 | 1.5000 | 0.8750 | 5.7500 | 0.1522 |
| 2 | 1.3478 | 0.1007 | 4.4499 | 0.0226 |
| 3 | 1.3252 | 0.0021 | 4.2685 | 0.0005 |
| 4 | 1.3247 | 0.0000 | 4.2646 | 0.0000 |
| 5 | 1.3247 | 0.0000 | 4.2646 | 0.0000 |

# Example

Use Newton's Method to find a root of:

$$f(x) = e^{-x} - x$$

Use the initial point: $x_0 = 1.$

Stop after three iterations, or

if $\left| x_{k+1} - x_k \right| < 0.001,$ or

if $\left| f(x_k) \right| < 0.0001.$

# Example

Use Newton's Method to find a root of :

$$f(x) = e^{-x} - x, \qquad f'(x) = -e^{-x} - 1$$

| $x_k$ | $f(x_k)$ | $f'(x_k)$ | $\frac{f(x_k)}{f'(x_k)}$ |
|---|---|---|---|
| 1.0000 | -0.6321 | -1.3679 | 0.4621 |
| 0.5379 | 0.0461 | -1.5840 | -0.0291 |
| 0.5670 | 0.0002 | -1.5672 | -0.0002 |
| 0.5671 | 0.0000 | -1.5671 | -0.0000 |

# Example

Estimates of the root of: $x - \cos(x) = 0$.

| | | |
|---|---|---|
| 0.60000000000000 | | Initial guess |
| 0.74401731944598 | 1 | correct digit |
| 0.73909047688624 | 4 | correct digits |
| 0.73908513322147 | 10 | correct digits |
| 0.73908513321516 | 14 | correct digits |

# Example

In estimating the root of: ***x-cos(x)=0***, to get more than 13 correct digits:

- 4 iterations of Newton ($x_0$=0.8)
- 43 iterations of Bisection method (initial interval [0.6, 0.8])
- 5 iterations of Secant method ( $x_0$=0.6, $x_1$=0.8)