ROMBERG INTEGRATION

Assume the following formulation to compute the true value of the integral

true value, true error with step size h

$$\underset{\text{true value}}{I} = \underset{\text{approx with Traperoidal with step size equal to } h}{I(h)} + \underset{\text{true error with step size } h}{E(h)} \qquad ---(1)$$

Assume that two integrals with step sizes of h_1 and h_2 are available, where the Integral with h_2 is more accurate than the one with h_1

$$\underbrace{I(h_1)}_{\text{less accurate}} + E(h_1) = \underbrace{I(h_2)}_{\text{more accurate}} + E(h_2) \qquad ---(2)$$

It can be shown that the true error could be written, in general as:

$$E = A_1 h^2 + A_2 h^4 + A_3 h^6 + \dots \qquad ---(3)$$

for small h, we can.

$$E = A_1 h^2 + O(h^4)$$

$$E \approx A_1 h^2 \qquad ---(4)$$

For two different step sizes:

$$\frac{E(h_1)}{E(h_2)} \approx \frac{A_1 h_1^2}{A_1 h_2^2} \Rightarrow E(h_1) = E(h_2)\left(\frac{h_1}{h_2}\right)^2 \quad --(5)$$

(2)

By (5) in (2)

$$I(h_1) + E(h_2)\left(\frac{h_1}{h_2}\right)^2 = I(h_2) + E(h_2) \quad ---(6)$$

Solve by $E(h_2)$

$$E(h_2) = \Longrightarrow = \frac{I(h_2) - I(h_1)}{(h_1/h_2)^2 - 1} \quad --(7)$$

Expressing (1) for h_2 :

$$I = I(h_2) + E(h_2) \quad ---(8)$$

By (7) in (8)

$$I = I(h_2) + \frac{I(h_2) - I(h_1)}{(h_1/h_2)^2 - 1} \quad --(9)$$

Where $I(h_2)$ is the approximation more accurate and $I(h_1)$ is the approximation less accurate

for the special case of halving the step size for each new approximation

$$h_2 = \frac{h_1}{2} \Longrightarrow \frac{h_1}{h_2} = 2$$

$$I = I(h_2) + \frac{I(h_2) - I(h_1)}{2^2 - 1}$$

$$I = I(h_2) + \frac{I(h_2) - I(h_1)}{3} + O(h^4) \quad --(10)$$

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The method can be applied again but know

$$E \approx A_2 h^4 + \Theta(h^6) \quad --- (11)$$

for two different step sizes:

$$\frac{E(h_1)}{E(h_2)} \approx \frac{A_2 h_1^4}{A_2 h_2^4} \equiv \left(\frac{h_1}{h_2}\right)^4 \quad --- (12)$$

Plug in (12) in (6).

approximation with Romberg Integration

$$I(h_1) + E(h_2)\left(\frac{h_1}{h_2}\right)^4 = I(h_2) + E(h_2) \quad --- (13)$$

approx w/Romberg

$$E(h_2) = \frac{I(h_2) - I(h_1)}{\left(\frac{h_1}{h_2}\right)^4 - 1} \quad --- (14)$$

Plug in (14) in (8):

$$I = I(h_2) + \frac{I(h_2) - I(h_1)}{\left(\frac{h_1}{h_2}\right)^4 - 1} \quad --- (15)$$

* For the special case of halving the new step size $h_2 = h_1/2 \Rightarrow h_1/h_2 = 2$

$$I = I(h_2) + \frac{I(h_2) - I(h_1)}{2^4 - 1} \quad --- (16)$$

$$I = I(h_2) + \frac{I(h_2) - I(h_1)}{15} \quad --- 17$$

$\Rightarrow \Theta(h^6)$

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TRAPEZOIDAL ONLY	First Order Romberg	2^{nd} order	3^{rd} order
Trapezoidal(h_1)	$I^1 + \mathcal{O}(h^4)$	$I^2 + \mathcal{O}(h^6)$	$I^3 + \mathcal{O}(h^8)$
Trapezoidal(h_2)	$I^1 + \mathcal{O}(h^4)$	$I^2 + \mathcal{O}(h^6)$	$I^3 + \mathcal{O}(h^8)$
Trapezoidal(h_3)		$I^2 + \mathcal{O}(h^6)$	
$\vdots$			
Trap(h_n)	$I^1 + \mathcal{O}(h^4)$		

$$I^2 = I^1(h_i) + \frac{I^1(h_{i+1}) - I^1(h_i)}{3} + \mathcal{O}(h^4)$$

$$I^3 = I^2(h_i) + \frac{I^2(h_{i+1}) - I^2(h_i)}{15} + \mathcal{O}(h^6)$$

$$I^4 = I^3(h_i) + \frac{I^3(h_{i+1}) - I^3(h_i)}{63} + \mathcal{O}(h^8)$$

In general

$$I_k^{i \leftarrow \text{order}} = \frac{4^i I_k^{i-1} - I_{k-1}^{i-1}}{4^i - 1}$$

more accurate

less accurate

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In general

$$T_k^i = \frac{4^i T_k^{i-1} - T_{k-1}^{i-1}}{4^i - 1}$$

for the special case of $\frac{h_1}{h_2} = 2$

or $\frac{h_j}{h_{j-1}} = 2$

$i = 1$,

$$T_k^1 = \frac{4 T_k^0 - T_{k-1}^0}{4^1 - 1} = \frac{4 T_k^0 - T_{k-1}^0}{3}$$

$i = 2$,

$$T_k^2 = \frac{4^2 T_k^1 - T_{k-1}^1}{4^2 - 1} = \frac{16 T_k^0 - T_{k-1}^0}{15}$$

$i = 3$

$$T_k^3 = \frac{4^3 T_k^2 - T_{k-1}^2}{4^3 - 1} = \frac{64 T_k^0 - T_{k-1}^0}{63}$$

$i = 4$

$$T_k^4 = \frac{4^4 T_k^3 - T_{k-1}^3}{4^4 - 1} = \frac{256 T_k^0 - T_{k-1}^0}{255}$$

$i = 5$

$$T_k^5 = \frac{4^5 T_k^4 - T_{k-1}^4}{4^5 - 1} = \frac{1024 T_k^0 - T_{k-1}^0}{1023}$$

where $k \equiv$ more accurate representation

$k-1 \equiv$ less accurate representation

One Example

Consider

$$\int_1^2 \frac{1}{x} dx = \ln 2$$

We will use this integral to illustrate how Romberg integration works. First, compute the trapezoid approximations starting with *n* = 1 and doubling *n* each time:

$$
\begin{aligned}
&n = 1: \ T_1^0 = \left(1 + \tfrac{1}{2}\right)\tfrac{1}{2} = 0.75; \\
&n = 2: \ T_2^0 = 0.5\left(\tfrac{1}{1.5}\right) + \tfrac{0.5}{2}(1 + \tfrac{1}{2}) = 0.708333333 \\
&n = 4: \ T_3^0 = 0.25\left(\tfrac{1}{1.25} + \tfrac{1}{1.5} + \tfrac{1}{1.75}\right) + \tfrac{0.25}{2}(1 + \tfrac{1}{2}) = 0.69702380952 \\
&n = 8: \ T_4^0 = 0.69412185037 \\
&n = 16: T_5^0 = 0.69314718191.
\end{aligned}
$$

Next we use the formula (for i=1,2,3…., don't use for i=0):

more accurate, k

less accurate, k-1

Romberg order

$$T_k^i = \frac{4^i T_k^{i-1} - T_{k-1}^{i-1}}{4^i - 1}$$

where i is the order of the extrapolation. Initially, order i=0 is calculation of the Integrals with the regular Trapezoidal rule alone for different n's. Then, i=1 is the first iteration with Romberg integration method, and so on. The index k is the more accurate approximation of the integral and k-1 is the less accurate.

NOTE: Develop the formulas for first, second, third and fourth order Romberg integration.

The easiest way is to keep track of computations is to build a table of the form:

Trapezoidal only with different n's	First Order	Second Order	Fourth Order	Fifth Order
T_1^0				
T_2^0	T_2^1			
T_3^0	T_3^1	T_3^2		
T_4^0	T_4^1	T_4^2	T_4^3	
T_5^0	T_5^1	T_5^2	T_5^3	T_5^4

Starting with the first column (which we just computed), all other entries can be easily computed. For example starting with $T_1^0,\ T_2^0$ we find

$$T_2^1 = \frac{4T_2^0 - T_1^0}{3} = 0.694444$$

$$T_3^1 = \frac{4T_3^0 - T_2^0}{3} = 0.693253; \quad T_3^2 = \frac{16T_3^1 - T_2^1}{15} = 0.69317460$$

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and so on. Every entry depends only on its left and left-top neighbor. Continuing in this way, we get the following table:

Trapezoidal only with different n's	First Order	Second Order	Fourth Order	Fifth Order
0.75000000000				
0.70833333333	0.69444444444			
0.69702380952	0.69325396825	0.69317460317		
0.69412185037	0.69315453065	0.69314790148	0.69314747764	
0.69339120220	0.69314765281	0.69314719429	0.69314718307	0.69314718191

The correct digits are shown in bold (the exact answer to 15 digits is given by ln2 = 0.693147180559945). Here is the table listing error. $T_i^k - \ln 2$

5.7e-02
1.5e-02 1.3e-03
3.9e-03 1.1e-04 2.7e-05

9.7e-04 7.4e-06 7.2e-07 3.0e-07
2.4e-04 4.7e-07 1.4e-08 2.5e-09 1.4e-09

Note that each successive iteration yields around two extra digits. The final iteration only required n = 16 function evaluations, plus $O(\ln n)$ arithmetic operations to build the table.

Exercise. Use four iterations of Romberg integration to estimate $\pi = \int_0^1 \frac{4}{1+x^2} dx.$ Comment on the accuracy of your result.