Math 3191
Applied Linear Algebra

Lecture 16: Change of Basis

Stephen Billups

University of Colorado at Denver

Rank

The **rank** of A is the dimension of the column space of A.

$\boxed{\text{rank } A = \dim \text{ Col } A = \text{\# of pivot columns of } A = \dim \text{ Row } A}$.

$$\underbrace{\text{rank } A}_{\updownarrow} + \underbrace{\text{dim Nul } A}_{\updownarrow} = \underbrace{n}_{\updownarrow}$$

$$\left\{\begin{array}{c}\text{\# of pivot}\\ \text{columns}\\ \text{of } A\end{array}\right\} \qquad \left\{\begin{array}{c}\text{\# of nonpivot}\\ \text{columns}\\ \text{of } A\end{array}\right\} \qquad \left\{\begin{array}{c}\text{\# of}\\ \text{columns}\\ \text{of } A\end{array}\right\}$$

THEOREM 14 THE RANK THEOREM

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{ Nul } A = n.$$

NOTE: Since Row $A =$ Col A^T,

$$\boxed{\text{rank } A = \text{rank } A^T}.$$

EXAMPLE:

Suppose that a 5×8 matrix A has rank 5. Find $\dim$ Nul A, $\dim$ Row A and rank A^T. Is Col $A = \mathbf{R}^5$?

Solution:

$$\underbrace{\text{rank } A}_{\updownarrow \atop 5} + \underbrace{\dim \text{ Nul } A}_{\downarrow \atop ?} = \underbrace{n}_{\updownarrow \atop 8}$$

$$5 + \dim \text{ Nul } A = 8 \qquad \Rightarrow \qquad \dim \text{ Nul } A = \underline{\hspace{6cm}}$$

$$\dim \text{ Row } A = \text{rank } A = \underline{\hspace{6cm}}$$

$$\Rightarrow \qquad \text{rank } A^T = \text{rank } \underline{\hspace{4cm}} = \underline{\hspace{4cm}}$$

Since rank $A = \#$ of pivots in $A = 5$, there is a pivot in every row. So the columns of A span $\mathbb{R}^5$ (by Theorem 4, page 43). Hence Col $A = \mathbb{R}^5$.

EXAMPLE: For a 9×12 matrix A, find the smallest possible value of $\dim$ Nul A.

Solution:

$$\text{rank } A + \dim \text{ Nul } A = 12$$

$$\dim \text{ Nul } A = 12 - \underbrace{\text{rank } A}$$

largest possible value=______

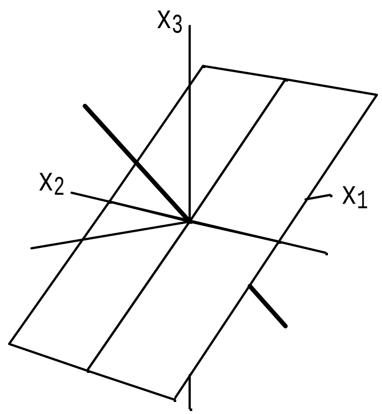
smallest possible value of $\dim$ Nul $A =$ ______

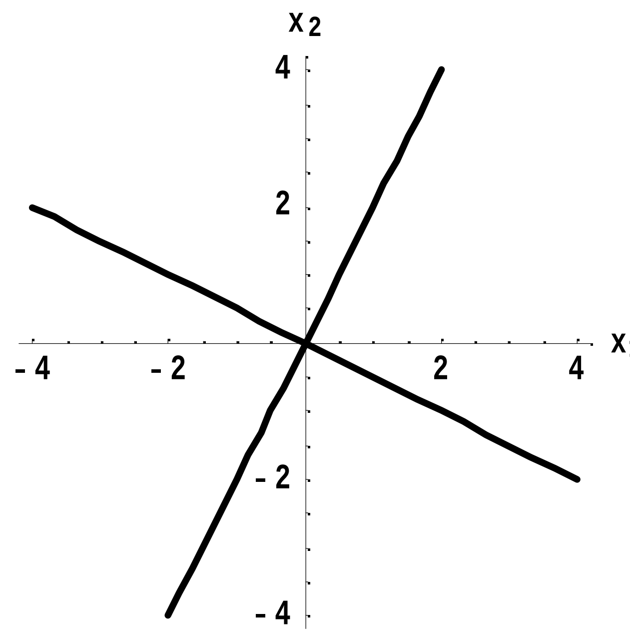
Visualizing Row A and Nul A

EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \end{bmatrix}$. One can easily verify the following:

- Basis for Nul $A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and therefore Nul A is a plane in $\mathbb{R}^3$.

- Basis for Row $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ and therefore Row A is a line in $\mathbb{R}^3$.
- Basis for Col $A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and therefore Col A is a line in $\mathbb{R}^2$.
- Basis for Nul $A^T = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and therefore Nul A^T is a line in $\mathbb{R}^2$.





Subspaces Nul A and Row A

Subspaces Nul A^T and Col A

The Rank Theorem provides us with a powerful tool for determining information about a system of equations.

EXAMPLE: A scientist solves a homogeneous system of 50 equations in 54 variables and finds that exactly 4 of the unknowns are free variables. Can the scientist be *certain* that any associated nonhomogeneous system (with the same coefficients) has a solution?

Solution:

Recall that

rank A = $\dim$ Col $A = \#$ of pivot columns of A

$\dim$ Nul $A =$ # of free variables

In this case $A\mathbf{x} = \mathbf{0}$ of where A is 50×54.
By the rank theorem,

$$\text{rank } A + \_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_$$

or

$$\text{rank } A = \_\_\_\_\_\_\_\_\_\_\_\_.$$

So any nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ has a solution because there is a pivot in every row.

THE INVERTIBLE MATRIX THEOREM (continued)

Let A be a square $n \times n$ matrix. The the following statements are equivalent:

m. The columns of A form a basis for $\mathbb{R}^n$

n. Col $A = \mathbb{R}^n$

o. dim Col $A = n$

p. rank $A = n$

q. Nul $A = \{\mathbf{0}\}$

r. dim Nul $A = 0$

Section 4.7: Change of Basis

- In Section 4.4, we introduced coordinates relative to a basis $\mathcal{B}$ and showed how to convert between coordinates relative to $\mathcal{B}$ and coordinates relative to the standard basis.
- We now look at how to change coordinates between two nonstandard bases $\mathcal{B}$ and $\mathcal{C}$.
- We begin by assuming that we know the coordinates of the basis vectors of $\mathcal{B}$ relative to the basis $\mathcal{C}$.
- Then we will show how to do it when you only know the coordinates of the two bases relative to the standard basis.

EXAMPLE

Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, and suppose that $\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$. Suppose that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{C}}$.

Solution:

$$
\begin{aligned}
\mathbf{x}]_{\mathcal{C}} &= [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} \\
&= 3[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}} \\
&= 3\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -6 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}
\end{aligned}
$$

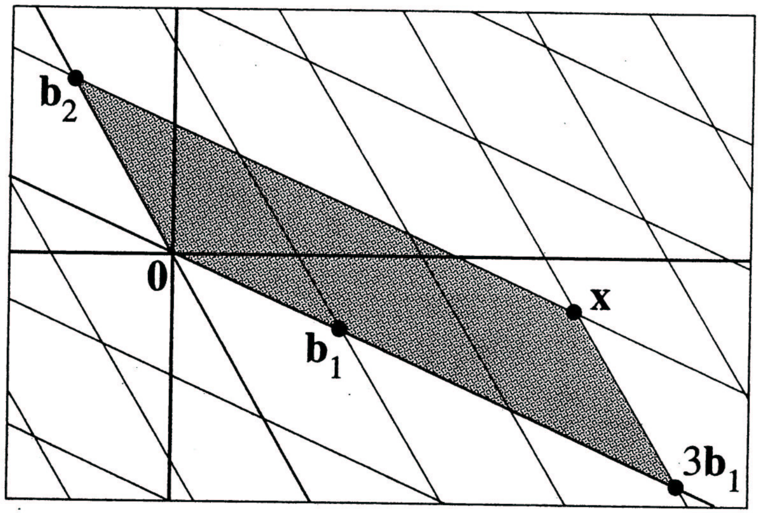
EXAMPLE

Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, and suppose that $\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$. Suppose that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{C}}$.

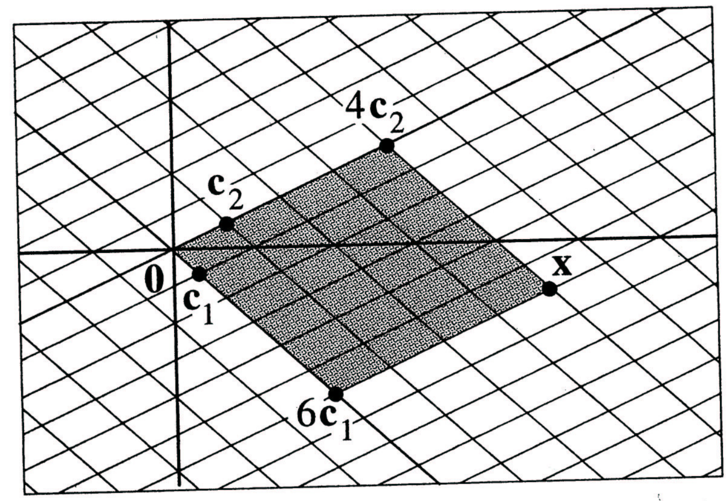
Solution:

$$
\begin{aligned}
\mathbf{x}]_{\mathcal{C}} &= [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} \\
&= 3[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}} \\
&= 3\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -6 \\ 1 \end{bmatrix} \\
&= \underset{\underset{\displaystyle \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}}{\uparrow}}{\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}} \quad \underset{\underset{\displaystyle [x]_{\mathcal{B}}}{\uparrow}}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} = \underset{\underset{\displaystyle [x]_{\mathcal{C}}}{\uparrow}}{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}
\end{aligned}
$$

Graphical Illustration



(a)



(b)

Coordinates relative to $\mathcal{B}$. Coordinates relative to $\mathcal{C}$.

Theorem 15

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be two bases of a vector space V. Then, there is a unique matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[x]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[x]_{\lfloor}.$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the the vectors in $\mathcal{B}$. That is

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

Graphical Illustration

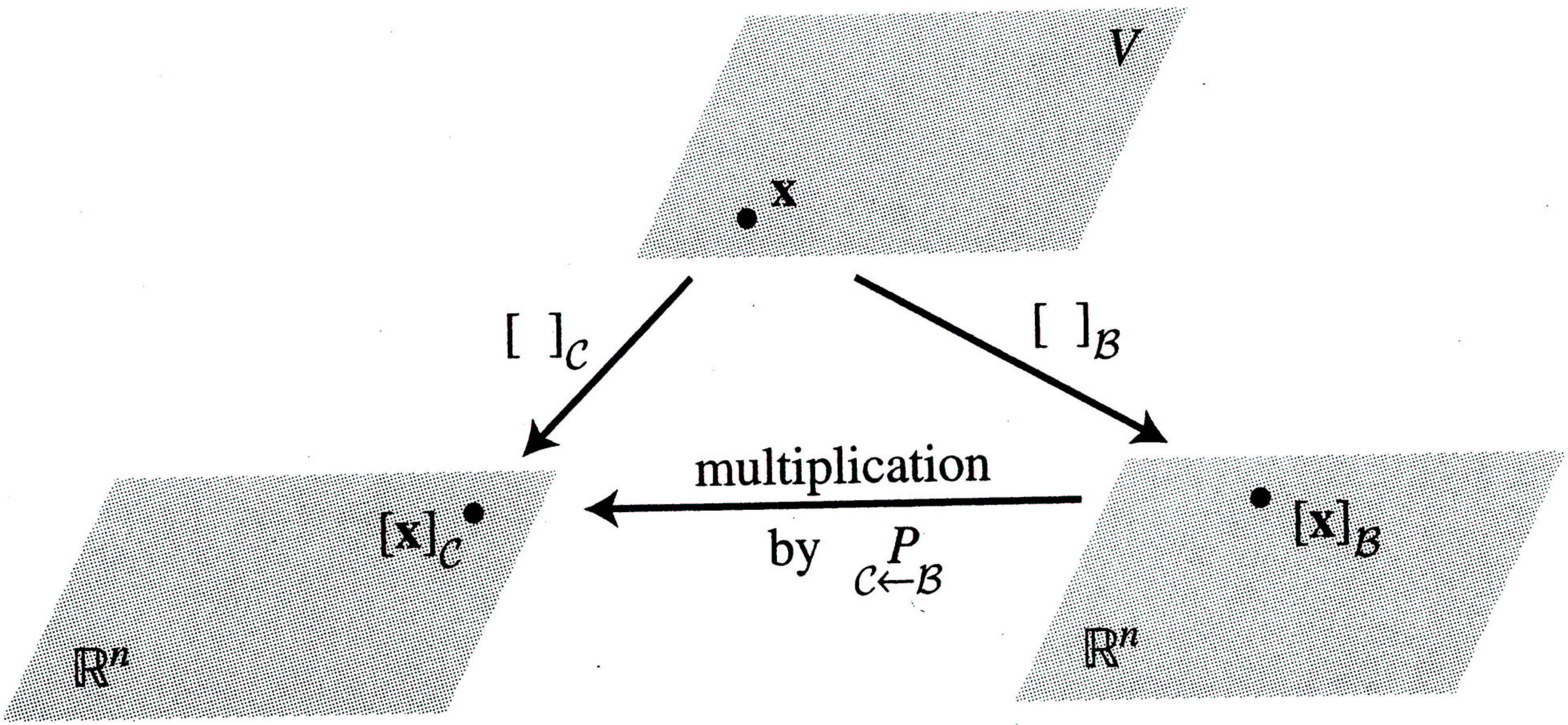


FIGURE 2 Two coordinate systems for V.

What if we don't know $[b_i]_{\mathcal{C}}$?

Let V be a 2 dimensional vector space with standard basis $\mathcal{E}$. Suppose $[\mathbf{b}_1]_{\mathcal{E}} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $[\mathbf{b}_2]_{\mathcal{E}} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $[\mathbf{c}_1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $[\mathbf{c}_2]_{\mathcal{E}} = \begin{bmatrix} -4 \\ -5 \end{bmatrix}$, and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Solution: We first find the coordinates of $\mathbf{b}_1$ and $\mathbf{b}_2$ relative to $\mathcal{C}$, by solving

$$\begin{bmatrix} [\mathbf{c}_1]_{\mathcal{E}} & [\mathbf{c}_2]_{\mathcal{E}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\mathbf{b}_1]_{\mathcal{E}} \quad \text{and} \quad \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{E}} & [\mathbf{c}_2]_{\mathcal{E}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\mathbf{b}_2]_{\mathcal{E}}.$$

cont.

Since both of these equations involved the same matrix, we can solve them simultaneously by row-reducing an expanded augmented matrix as follows:

$$\left[\begin{array}{cc|cc} [\mathbf{c}_1]_{\mathcal{E}} & [\mathbf{c}_2]_{\mathcal{E}} & [\mathbf{b}_1]_{\mathcal{E}} & [\mathbf{b}_2]_{\mathcal{E}} \end{array}\right] = \left[\begin{array}{rr|rr} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{rr|rr} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

Thus,

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \quad \text{and} \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

and

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

Forming the Change of Basis Matrix

Recap: To find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, row reduce the matrix

$$\left[\begin{array}{ccc|ccc} [\mathbf{c}_1]_{\mathcal{E}} & \cdots & [\mathbf{c}_n]_{\mathcal{E}} & [\mathbf{b}_1]_{\mathcal{E}} & \cdots & [\mathbf{b}_3]_{\mathcal{E}} \end{array}\right]$$

Resulting in the matrix

$$\left[\begin{array}{c|c} I & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{array}\right].$$

Example

Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$. Find the change of coordinates matrix from $\mathcal{B} = \{b_1, b_2\}$ to $\mathcal{C} = \{c_1, c_2\}$.

Solution:

$$\left[\begin{array}{ccc|ccc} [\mathbf{c}_1]_{\mathcal{E}} & \cdots & [\mathbf{c}_n]_{\mathcal{E}} & [\mathbf{b}_1]_{\mathcal{E}} & \cdots & [\mathbf{b}_3]_{\mathcal{E}} \end{array}\right] = \left[\begin{array}{rr|rr} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right].$$

So $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$.