Unit-2 Roots of Equation

Humanities & Science Department

Introduction

- An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$, where $a_0, a_1, a_2 \ldots a_n$ are constants and n is a positive integer , is called an algebraic polynomial of degree n if $a_0 \neq 0$.
- The equation $f(x) = 0$ is called an algebraic equation if $f(x)$ is an algebraic polynomial, e.g. $x^3 - 4x - 9 = 0$.
- If $f(x)$ contains functions such as trigonometric, logarithmic, exponential, etc. , then $f(x)$ is called a transcendental equation, e.g. , $2x^3 - \log(x + 3)\, tanx + e^x = 0$.

- ✓ In general, an equation is solved by factorization. But in many cases, the method of factorization fails. In such cases, numerical methods are used. There are some methods to solve the equation $\boldsymbol{f(x) = 0}$, which are given as follows.

What does Equation mean ?

- An *Equation* is a statement of an *equality* containing one or more variables.

(Ref. Wikipedia)

- In simple words , If in mathematical expression “ = ” appears , than it is called *EQUATION* .
- Otherwise, It is known as *Inequality*.

What does “ROOT” mean ?

- A values of x that satisfies mathematical expression $f(x) = K$.

- Example :

1. $x^2 + 2x + 1 = 0$

$\Longrightarrow x = -1\,,-1$

Here, -1 is called Root.

2. $x^2 + 2x + 1 = 1$

$\Longrightarrow x = 0\,,-2$

Here, $0, -2$ are called Root.

Types of Equation

- There are two types of Equation.

  1. Linear Equation

     A mathematical equation that geometrically represents a “Line”, is known as Linear Equation.

  2. Non Linear Equation

     If not Linear, than it is known as Non-Linear Equation.

- In this chapter, we solve system of Non-Linear equations.

Methods for Solution

There are Two types of method.

- Bracketing Method

1. Bisection Method
2. Regula Falsi Method / False Position Method

- Iterative Method

1. Secant Method
2. Newton-Raphson Method (NR Method)

Intermediate Value Property

- If a function f(x) is continuous on a closed interval [a, b], and if K is a number between f(a) and f(b), then there must be a point c in the interval [a, b] such that f(c) = K.
- In Bracketing Method, We find an interval [a, b] such that, $f(a) < 0 < f(b)$. So, we can find $c \in [a, b]$ such that $f(c) = 0$.
- Here, c is called ROOT of Non – Linear Equation.

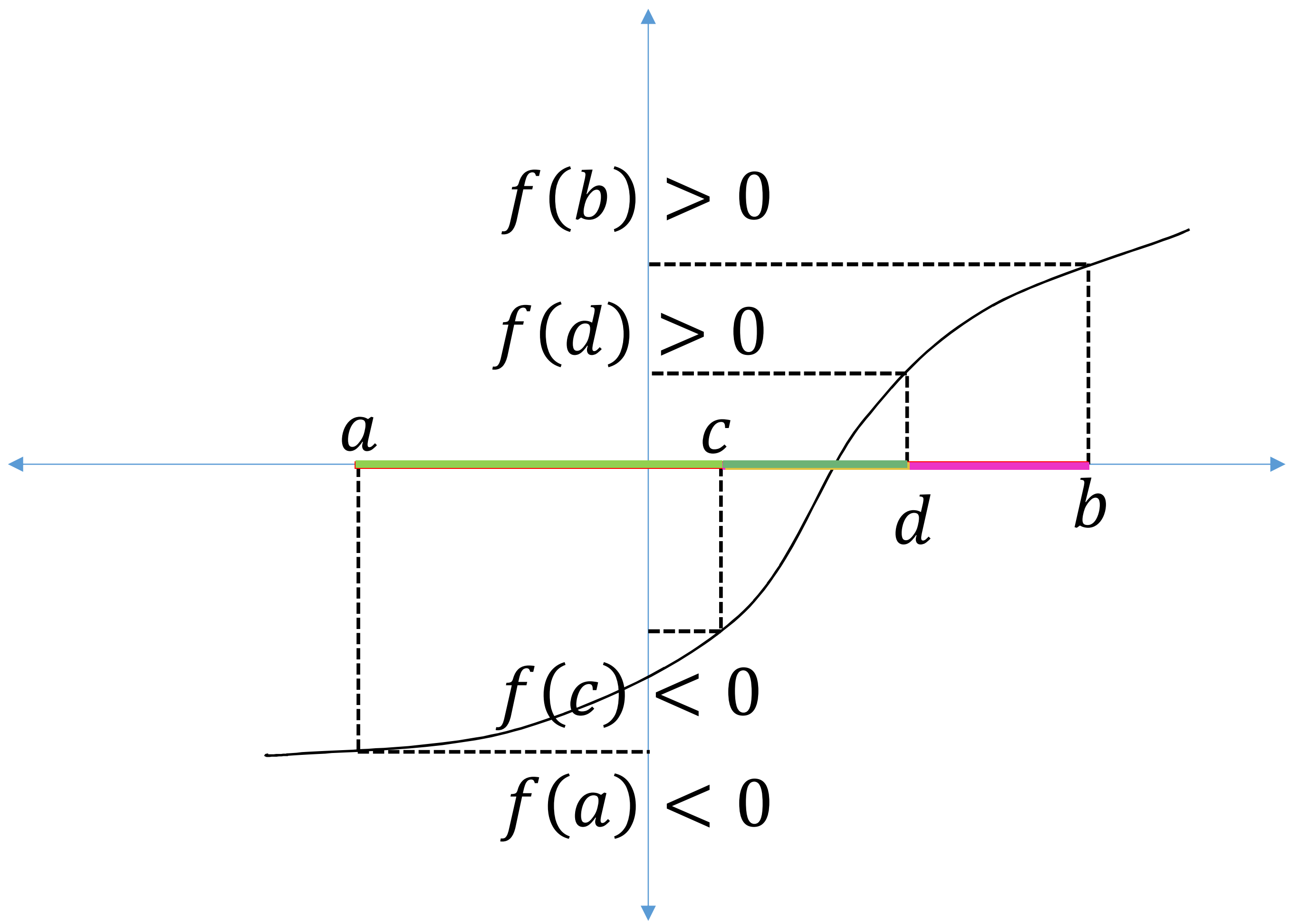
- Methods to find Roots of equations
- ✓ Bisection Method
- ✓ Regula-Falsi Method
- ✓ Secant Method
- ✓ Newton-Raphson Method
- ✓ Succesive Approximation method
- ✓ Budan’s Theorem
- ✓ Bairstow’s Method

Bisection method

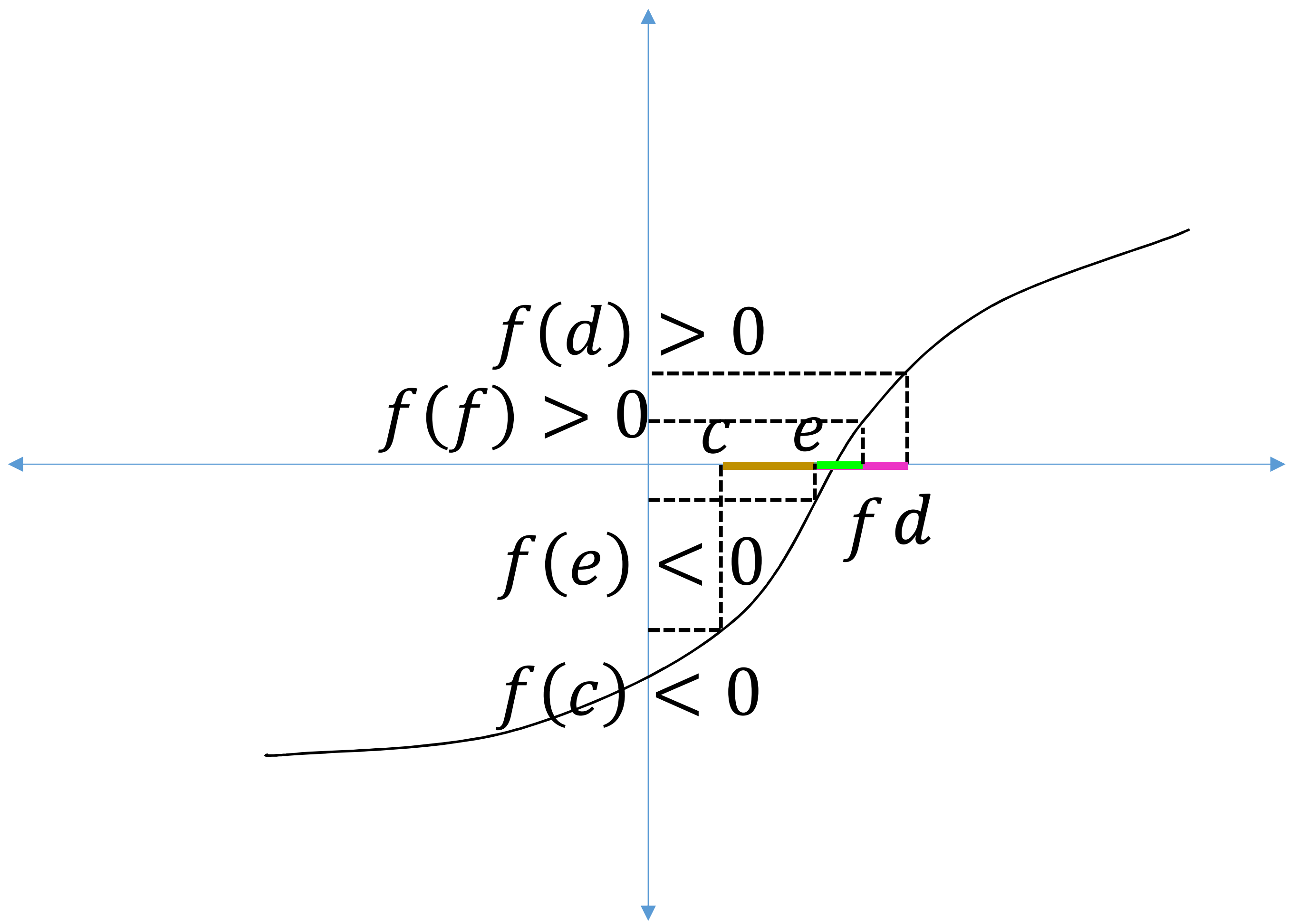
Bisection Method

- Find a and b such that $f(a) < 0$ & $f(b) > 0$.
- Root lies in interval $[a, b]$.
- Find c such that $c = \frac{a+b}{2}$. Check $f(c)$.
- If $f(c) < 0$, then replace a. New Interval will be $[c, b]$.
- If $f(c) < 0$, then replace a. New Interval will be $[a, c]$.
- Repeat same steps till $f(c)$ is equal or nearer to zero.

Geometrical Meaning



Geometrical Meaning



Steps to solve Bisection method

Step-I Choose two approximation a & b Such that $f(a).f(b) < 0$

Step-II Evaluate the first approximation to the root is

$$x_1 = \frac{a+b}{2} \text{ (Mid point)}$$

Step-III The root lies between a & x_1 OR x_1& b according as $f(x_1)$ is positive or negative

Evaluate the second approximation to the root is

$$x_2 = \frac{a+x_1}{2} \text{ OR } x_2 = \frac{x_1+b}{2}$$

Step-IV Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

Example

Find the positive root of $\mathbf{x} = \boldsymbol{cos\, x}$ correct up to three decimal places by bisection method.

Here, $f(x) = x - \cos x$=0

$f(0) = 0 - \cos 0 = -1 < 0$ and

$f(1) = 1 - \cos 1 = 0.4597 > 0$

Since $f(0) < 0$ and $f(1) > 0$

∴ root lies between 0 and 1.

First approximation to the root is

$$\therefore x_1 = \frac{0+1}{2} = 0.5$$

n	$a(<)$	$b(>)$	$x_n = \frac{a+b}{2}$	$f(x_n)$
0	0	1	$x_0 = 0.5$	$f(x_0) < 0$
1	0.5	1	$x_1 = 0.75$	$f(x_1) > 0$
2	0.5	0.75	$x_2 = 0.6250$	$f(x_2) < 0$
3	0.625	0.75	$x_3 = 0.6875$	$f(x_3) < 0$
4	0.6875	0.75	$x_4 = 0.7188$	$f(x_4) < 0$
5	0.7188	0.75	$x_5 = 0.7344$	$f(x_5) < 0$
6	0.7344	0.75	$x_6 = 0.7422$	$f(x_6) > 0$
7	0.7344	0.7422	$x_7 = 0.7383$	$f(x_7) < 0$
8	0.7383	0.7422	$x_8 = 0.7403$	$f(x_8) > 0$
9	0.7383	0.7403	$x_9 = 0.7393$	$f(x_9) > 0$
10	0.7383	0.7393	$x_{10} = 0.7388$	$f(x_{10}) < 0$
11	0.7388	0.7393	$x_{11} = 0.7391$	$f(x_{11}) < 0$
12	0.7391	0.7393	$x_{12} = 0.7392$	$f(x_{12}) < 0$
13	0.7391	0.7392	$x_{13} = 0.7392$	

Example

Since x_{12} and x_{13} are same up to four decimal places, The positive root is 0.7392.

Example

Explain bisection method for solution of equation. Using this method find the approximate solution $\boldsymbol{x^3 + x - 1 = 0}$ of correct up to three decimal points.

Here $f(x) = x^3 + x - 1 = 0$

$\therefore f(0) = 0 + 0 - 1 = -1 < 0$

$and\ f(1) = 1 + 1 - 1 = 1 > 0$

$\therefore$ root lies between 0 and 1.

$$\therefore x_1 = \frac{0+1}{2} = 0.5$$

n	$a(<)$	$b(>)$	$x_n = \frac{a+b}{2}$	$f(x_n)$
0	0	1	$x_0 = 0.5$	$f(x_0) < 0$
1	0.5	1	$x_1 = 0.75$	$f(x_1) > 0$
2	0.5	0.75	$x_2 = 0.625$	$f(x_2) < 0$
3	0.625	0.75	$x_3 = 0.688$	$f(x_3) > 0$
4	0.625	0.688	$x_4 = 0.657$	$f(x_4) < 0$
5	0.657	0.688	$x_5 = 0.673$	$f(x_5) < 0$
6	0.673	0.688	$x_6 = 0.681$	$f(x_6) < 0$
7	0.681	0.688	$x_7 = 0.685$	$f(x_7) > 0$
8	0.681	0.685	$x_8 = 0.683$	$f(x_8) > 0$
9	0.681	0.683	$x_9 = 0.682$	$f(x_9) < 0$
10	0.682	0.683	$x_{10} = 0.683$	$f(x_{10}) > 0$
11	0.682	0.683	$x_{11} = 0.683$	

Since x_{10} and x_{11} are same up to three decimal places, The positive root is 0.683.

Example

Find the negative root of $x^3 - 7\mathbf{x} + 3 = 0$ bisection method up to three decimal place.

Here $f(x) = x^3 - 7x + 3 = 0$

$\therefore f(0) = 0 - 0 + 3 = 3 > 0$

$f(-1) = -1 + 7 + 3 = 9 > 0$

$f(-2) = -8 + 14 + 3 = 9 > 0$

$f(-3) = -27 + 21 + 3 = -3 < 0$

$\therefore$ root lies between -3 and -2.

$$\therefore x_1 = \frac{-3 - 2}{2} = -2.5$$

n	$a(<)$	$b(>)$	$x_n = \frac{a+b}{2}$	$f(x_n)$
0	-3	-2	$x_0 = -2.5$	$f(x_0) < 0$
1	-3	-2.5	$x_1 = -2.75$	$f(x_1) > 0$
2	-3	-2.75	$x_2 = -2.875$	$f(x_2) < 0$
3	-2.875	-2.75	$x_3 = -2.813$	$f(x_3) > 0$
4	-2.875	-2.813	$x_4 = -2.844$	$f(x_4) < 0$
5	-2.844	-2.813	$x_5 = -2.829$	$f(x_5) < 0$
6	-2.844	-2.829	$x_6 = -2.837$	$f(x_6) < 0$
7	-2.844	-2.837	$x_7 = -2.841$	$f(x_7) > 0$
8	-2.841	-2.837	$x_8 = -2.839$	$f(x_8) < 0$
9	-2.839	-2.837	$x_9 = -2.838$	$f(x_9) > 0$
10	-2.839	-2.838	$x_{10} = -2.839$	$f(x_{10}) < 0$
11	-2.839	-2.838	$x_{11} = -2.839$	

Iteration method or Successive approximation method

Steps to solve Iteration method or Successive approximation method

Step-I To find root of $f(x) = 0$.

Rewrite in the form $x = \phi(x)$

Step-II Check $|\phi'(x)| < 1$ for all x .

where x lies in the the root containing interval.

Step-III Let $x = x_0$ (initial) then $x_1 = \phi(x_0)$

$$x_2 = \phi(x_1) \dots \dots$$

Proceeding in this way

$$x_n = \phi(x_{n-1})$$
$$x_{n+1} = \phi(x_n); n = 0,1,2,3, \dots$$

Example

Using method of successive approximation solve the equation $2x - \log_{10} x = 7$ correct up to four decimal places. (GTU-DEC-2015)

Solution:

Here, $f(x) = 2x - \log_{10} x - 7$

$\therefore f(2) = 4 - \log_{10} 2 - 7 = -3.3010 < 0$

$\therefore f(3) = 6 - \log_{10} 3 - 7 = -1.4471 < 0$

$\therefore f(4) = 8 - \log_{10} 4 - 7 = 0.398 > 0$

Since, $f(3) < 0$ & $f(4) > 0$

$\therefore$Root lies between $(3,4)$.

Now, rewrite equation $2x - \log_{10} x = 7$

$$\Rightarrow 2x - \log_{10} x - 7 = 0 \Rightarrow x = \frac{7 + \log_{10} x}{2} = \phi(x)$$

$$\therefore \phi(x) = \frac{7 + \log_{10} x}{2}$$

Now,

$$\phi'(x) = \frac{1}{2}\left(\frac{1}{x}\log_{10} e\right)$$

$$\left[\begin{array}{c} \because \log_{10} x = \dfrac{\log_e x}{\log_e 10} \\ \dfrac{d}{dx}(\log_{10} x) = \dfrac{1}{x}\log_{10} e \end{array}\right]$$

$$\therefore |\phi'(x)| < 1 \; for \; 3 < x < 4$$

$$\left[\because \phi'(3.5) = \frac{1}{2}\left(\frac{1}{3.5}\log_{10} e\right) = 0.062 < 1\right]$$

Hence, the iteration method can be applied.

By the iteration method, taking $x_0 = 3$

$$x_1 = \phi(x_0) = \frac{1}{2}(\log_{10} 3 + 7) = 3.7386$$

$$x_2 = \phi(x_1) = 3.7864$$

$$x_3 = \phi(x_2) = 3.7891$$

$$x_4 = \phi(x_3) = 3.7893$$

$$x_5 = \phi(x_4) \approx 3.7893$$

Example

Using method of successive approximation solve the equation $\mathbf{e^{-x} - 10x = 0}$ correct up to four two places.
(GTU-DEC-2015)

Solution:

Here, $f(x) = e^{-x} - 10x$.

$f(0) = 1 > 0$

$f(1) = -9.63 < 0$

Since, $f(1) < 0$ & $f(0) > 0$

$\therefore$Root lies between $(0,1)$.

Now, rewrite equation $e^{-x} - 10x = 0$

$\Rightarrow 10x = e^{-x}$

$$\Rightarrow x = \frac{e^{-x}}{10} = \phi(x)$$

$$\therefore \phi'(x) = \frac{(-e^{-x})}{10}$$

$$\therefore |\phi'(x)| < 1 \; for \; 0 < x < 1$$

$$[\because \phi'(0.5) = 0.0606 < 1]$$

Hence the iteration method can be applied,

By the iteration method, taking $x_0 = 0$

$$x_1 = \phi(x_0) = \frac{e^0}{10} = 0.1$$

$$x_2 = \phi(x_1) = \frac{e^{-0.1}}{10} = 0.0905$$

$$x_3 = \phi(x_2) = \frac{e^{-0.0905}}{10} = 0.0913$$

$$x_4 = \phi(x_3) = \frac{e^{-0.0913}}{10} = 0.0913$$

Budan's Theorem

- ✓Let, $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ be a polynomial function with real coefficients $a_0 , a_1 , a_2 , \dots \ , a_n$. Let $v(c)$ be the number of variations of signs in the sequence $f(x), f'(x), f''(x), \dots . f^n(x)\ when\ x = c$, where c is any real number.

- ✓The number of roots of $f(x)$in the interval $[a , b]$, counted with their order of multiplicity is equal to $|v(a) - v(b)| - 2m, for\ some\ m \in N$.

- ✓i.e. the number of roots of $f(x)$ is equal to $v(a) - (b)\ OR$ $v(a) - v(b)$ decreased by an even integer.

Steps to find number of roots

✓Step 1: Obtain all the derivatives of given polynomial.

✓Step 2: Evaluate $f(x)$ and all it's derivatives at the end points and write the signs and find $v(x)$.

✓Step 3: find number of roots by Budan's theorem.

Example

$f(x) = x^4 - 4x^3 - 5x^2 + 3x + 2$ Apply **Budan's** theorem on $[-\mathbf{1}, \mathbf{1}]$ to estimate the number of roots in the intervals$[-\mathbf{1}, \mathbf{0}]$ and $[\mathbf{0}, \mathbf{1}]$.

Solution:

Step 1: Obtain all the derivatives of given polynomial.

$f(x) = \mathrm{x}^4 - 4\mathrm{x}^3 - 5\mathrm{x}^2 + 3\mathrm{x} + 2$

$f'(x) = 4x^3 - 12x^2 - 10x + 3$

$f''(x) = 12x^2 - 24x - 10$

$f'''(x) = 24x - 24$

$f^{iv}(x) = 24$

✓ **Step 2: Evaluate $f(x)$ and all its derivatives at the end points, write the signs and find $v(x)$.**

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{iv}(x)$	$v(x)$
-1	-1	-3	26	-48	24	**3**
0	2	3	-10	-24	24	**2**
1	-3	-15	-22	0	24	**1**

✓ **Step 3: Find number of roots by Budan's theorem.**

Number of roots in $[a,b] = [-1,0] = |v(a) - v(b)|$

$$= v(-1) - v(0) = 3 - 2 = 1.$$

Number of roots in $[a,b] = [0,1] = |v(a) - v(b)|$

$$= v(0) - v(1) = 2 - 1 = 1.$$

So,

Number of roots in $[a, b] = [-1,0] = 1$

Number of roots in $[a, b] = [0,1] = 1$

Hence,

there is one root in the interval $[-1,0]$

there is one root in the interval $[0,1]$.

Example

Apply Budan's theorem to the equation $x^4 - 7x^2 + 6x - 1 = 0$ to estimate the number of roots in the interval $[-2, -1]$.

Solution :

✓**Step 1: Obtain all the derivatives of given polynomial.**

$f(x) = \mathrm{x}^4 - 7\mathrm{x}^2 + 6\mathrm{x} - 1$

$f'(x) = 4x^3 - 14x + 6$

$f''(x) = 12x^2 - 14$

$f'''(x) = 24x$

$f^{iv}(x) = 24$

✓**Step 2: Evaluate $f(x)$ and all its derivatives at the end points, write the signs and find $v(x)$.**

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{iv}(x)$	$v(x)$
-2	-25	2	34	-48	24	3
-1	-13	16	-2	-24	24	3

✓**Step 3: Find number of roots by Budan's theorem.**

Number of roots in $[a, b] = [-2, -1] = v(a) - v(b)$

$$= v(-2) - v(-1) = 3 - 3 = 0.$$

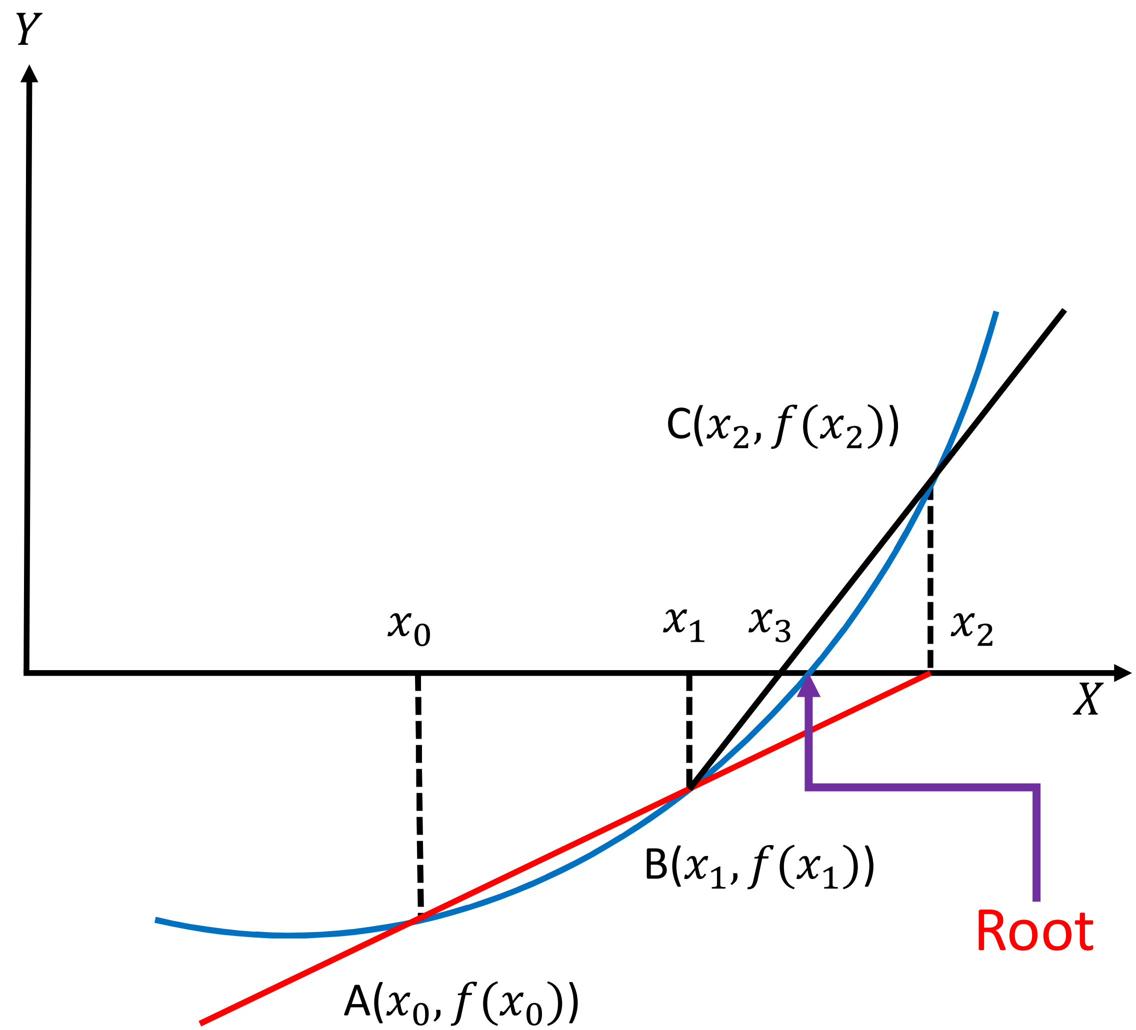
So, there is no root in the interval $[-2, -1]$

Secant method

Here we choose two points x_0 and x_1

Equation of the chord joining the points $A(x_0, f(x_0))$ & $B(x_1, f(x_1))$ is

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$



Explanation :

We have equation of two point joining $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1)$$

$$\begin{bmatrix} \because y - y_1 = m(x - x_1) \\ \because m = \dfrac{y_1 - y_0}{x_1 - x_0} \end{bmatrix}$$

Now, We prove at $(x_2, 0)$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

Now, the point where curve crosses the x-axis (y=0) is given by

$$0 - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1)$$

$$\therefore -f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)} = (x - x_1)$$

$$\therefore f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)} = x_1 - x$$

$$\therefore x = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$\therefore x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)} \qquad (\because \text{New point } x = x_2)$$

Secant method

The General formula for Secant is ,Given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)\,; \quad n \geq 1$$

Note: If the secant method once converges, its rate of convergence is 1.6 which is faster than that method of false position.

Secant method

Step-I Choose two approximation x_0 & x_1

Step-II Evaluate the first approximation to the root is

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

Step-III General formula for Secant method is ,given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n); n \geq 1$$

Step-IV This Procedure is repeated till the root is found to be the desired accuracy.

Example

Use secant method to find the roots of $\cos x - xe^{x} = 0$ correct up to three decimal places.

Solution:

We have $f(x) = \cos x - xe^{x}$ and consider $x_0 = 0$ & $x_1 = 1$ for the starting points.

By secant method

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n); n \geq 1.$$

$$\text{Let } n = 1 \Rightarrow x_2 = x_1 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) \cdot f(x_1)$$

We have $x_2 = x_1 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) \cdot f(x_1)$ and

$x_0 = 0, x_1 = 1, f(x_0) = 1, f(x_1) = -2.1780$

1st Iteration:

$$\therefore x_2 = 1 - \left(\frac{1-0}{f(1) - f(0)}\right). f(1)$$

$$\therefore \mathrm{x}_2 = 1 - \left(\frac{1-0}{-2.1780 - 1}\right)(-2.1780)$$

$\therefore x_2 = 0.315$

$f(x_2) = f(0.315) = 0.519$

Let, $n = 2$

2nd Iteration:

$$\therefore x_3 = x_2 - \left(\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right) . f(x_2)$$

$$\therefore x_3 = 0.315 - \left(\frac{0.315 - 1}{0.519 - (-2.178)}\right)(0.519)$$

$$\therefore x_3 = 0.447$$

$$\therefore f(x_3) = 0.203$$

Similarly we can progress,

Let, $n = 3$

3rd Iteration:

$x_4 = 0.532$

$\therefore f(x_4) = -0.043$

Let, $n = 4$

4^{th} Iteration:

$x_5 = 0.517$

$\therefore f(x_5) = 0.002$

Let, $n = 5$

5^{th} Iteration:

$x_6 = 0.518$

$\therefore f(x_6) = 0.001$

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n); n \geq 1.$$

$f(x) = \cos x - xe^x$

n	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	$x_0 = 0$	$x_1 = 1$	1	-2.1780	0.315	0.519
2	$x_1 = 1$	$x_2 = 0.315$	-2.1780	0.519	0.447	0.203
3	$x_2 = 0.315$	$x_3 = 0.447$	0.519	0.203	0.532	-0.043
4	$x_3 = 0.447$	$x_4 = 0.532$	0.203	-0.043	0.517	0.002
5	$x_4 = 0.532$	$x_5 = 0.517$	-0.043	0.002	0.518	-0.001
6	$x_5 = 0.517$	$x_6 = 0.518$	0.002	-0.001	0.518	

Here, x_5 and x_6 are same up to three decimal.
So, root of equation is $x = 0.518$.

Example

Find the positive solution of $\boldsymbol{f(x) = x - 2sin\,x = 0}$ by the secant method, starting from $\boldsymbol{x_0 = 2, x_1 = 1.9}$.

Solution:

We have $f(x) = x - 2\sin x$ and also starting points $x_0 = 2$ & $x_1 = 1.9$ are given.

By Secant method,

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n); n \geq 1.$$

Let, $n = 1$

1st Iteration:

$$\therefore x_2 = x_1 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) . f(x_1)$$

$$\therefore x_2 = 1.9 - \left(\frac{1.9-2}{f(1.9)-f(2)}\right).f(1.9)$$

$$\therefore x_2 = 1.9 - \left(\frac{-0.1000}{0.0074-0.1814}\right)(0.0074)$$

$$= 1.9 - (0.5747)(0.0074)$$

$$\therefore x_2 = 1.8957$$

So, $f(x_2) = 0.0003$

n	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	$x_0 = 2$	$x_1 = 1.9$	0.1814	0.0074	1.8957	0.0003
2	$x_1 = 1.9$	$x_2 = 1.8957$	0.0074	0.0003	1.8955	0.0000
3	$x_2 = 1.8957$	$x_3 = 1.8955$	0.0003	0.0000		

$$\left(\because x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n); n \geq 1\right)$$

Here, $f(x_3) = f(1.8955) = 0$

$\therefore root\ is\ x = 1.8955$

Find the positive solution of $f(x) = x - 2sin\,x = 0$ by the secant method, starting from $x_0 = 2, x_1 = 1.9$.

We have, $f(x) = x - 2\sin x$.

Starting points $x_0 = 2$ & $x_1 = 1.9$ are given.

By Secant method ,

$$x_2 = x_1 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) \cdot f(x_1)$$

$$x_2 = 1.9 - \left(\frac{1.9 - 2}{f(1.9) - f(2)}\right) . f(1.9)$$

$$\Longrightarrow x_2 = 1.9 - (0.5747)(0.0074)$$

$$\Longrightarrow x_2 = 1.8957$$

So, $f(x_2) = 0.0003$

n	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}
1	2	0.1814	1.9	0.0074	1.8957
2	1.9	0.0074	1.8957	0.0003	1.8955
3	1.8957	0.0003	1.8955	0.0000	

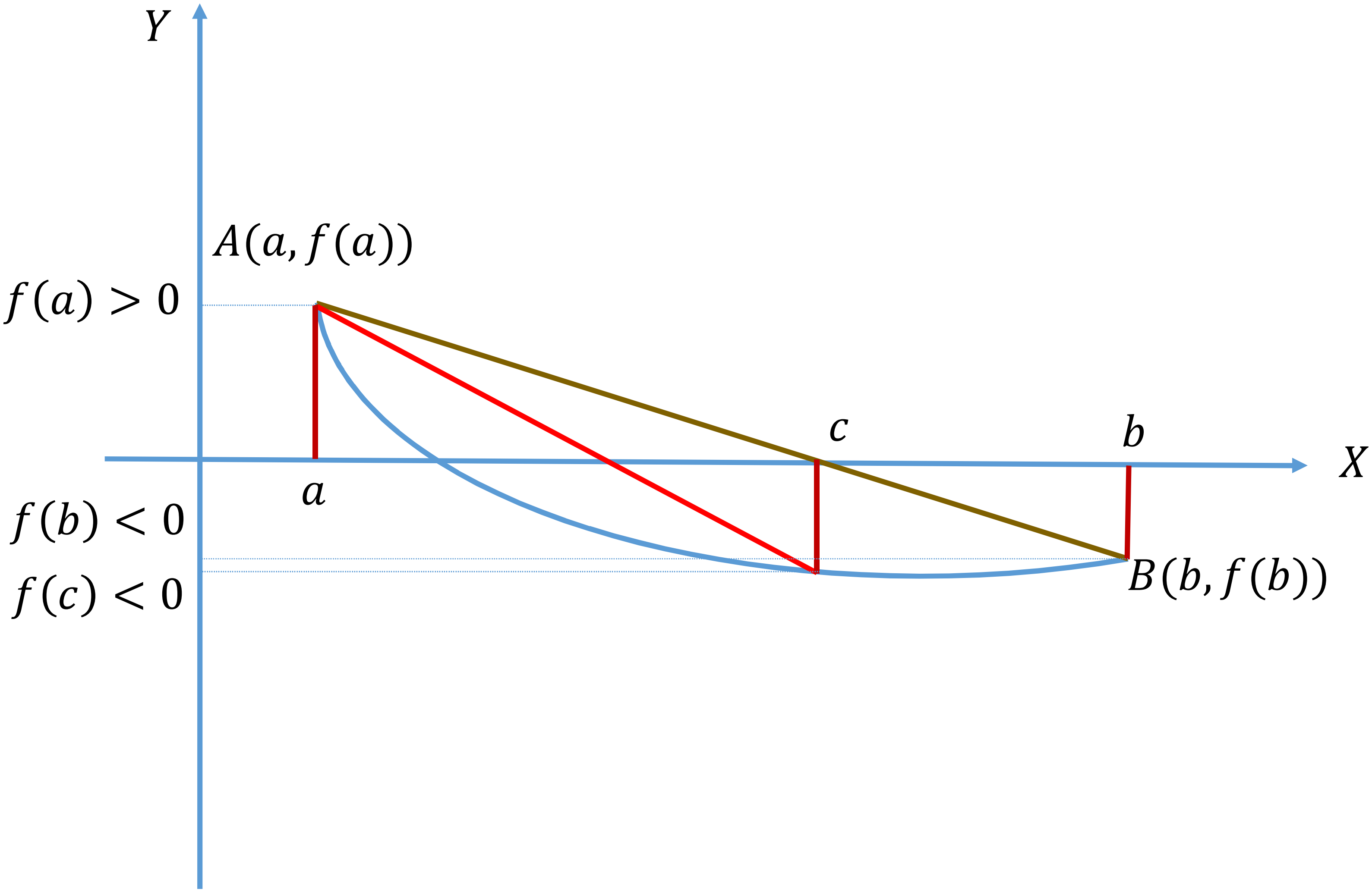
Here, $f(x_3) = f(1.8955) = 0$

So, Root is 1.8955

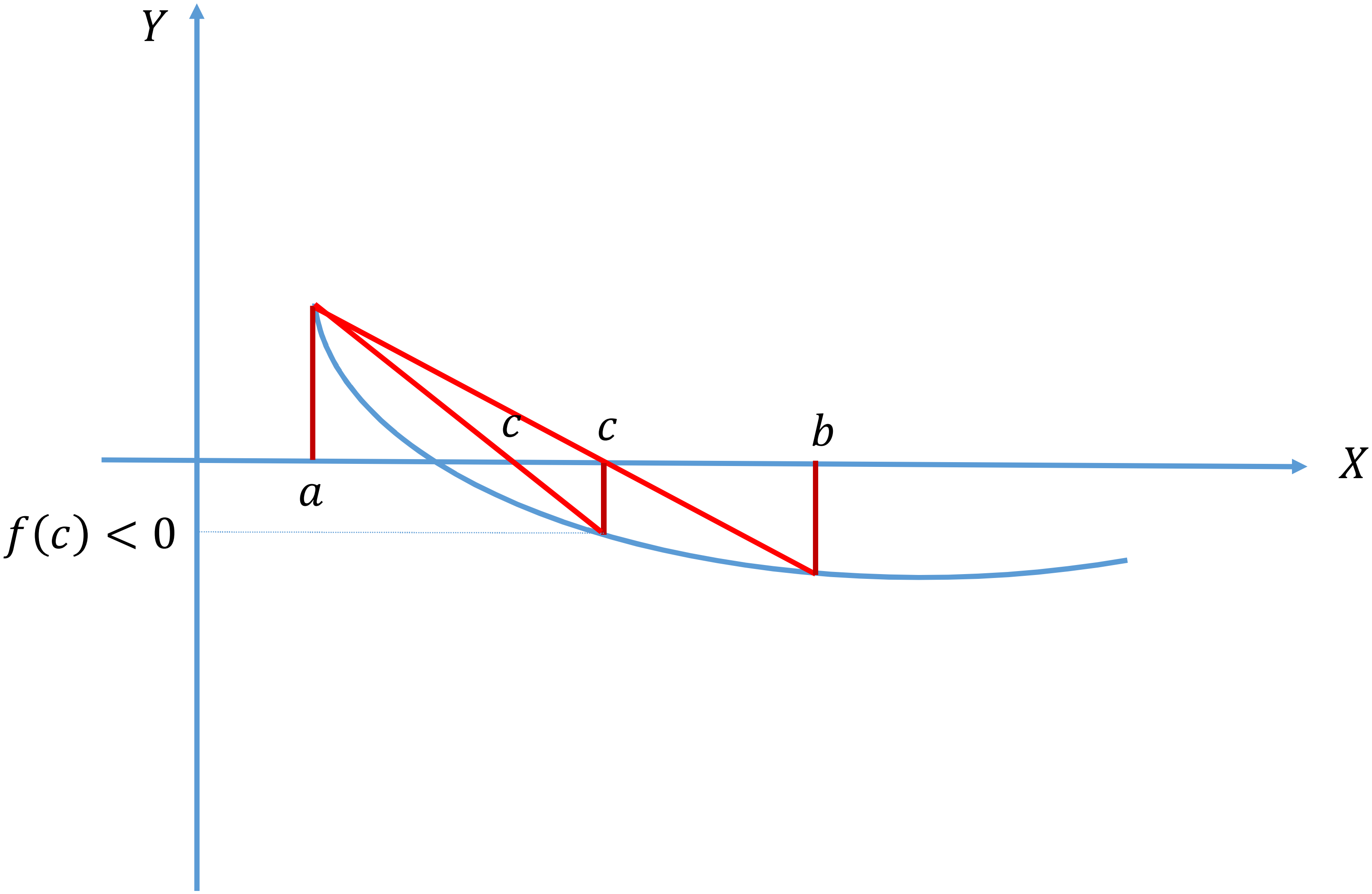
Regula Falsi method

- Find a and b such that $f(a) < 0$ & $f(b) > 0$.
- Root lies in interval $[a, b]$.
- Find c such that $c = a - \frac{b-a}{f(b)-f(a)} f(a)$. Check $f(c)$.
- If $f(c) < 0$, then replace a. New Interval will be $[c, b]$.
- If $f(c) < 0$, then replace b. New Interval will be $[a, c]$.
- Repeat same steps till $f(c)$ is equal or nearer to zero.

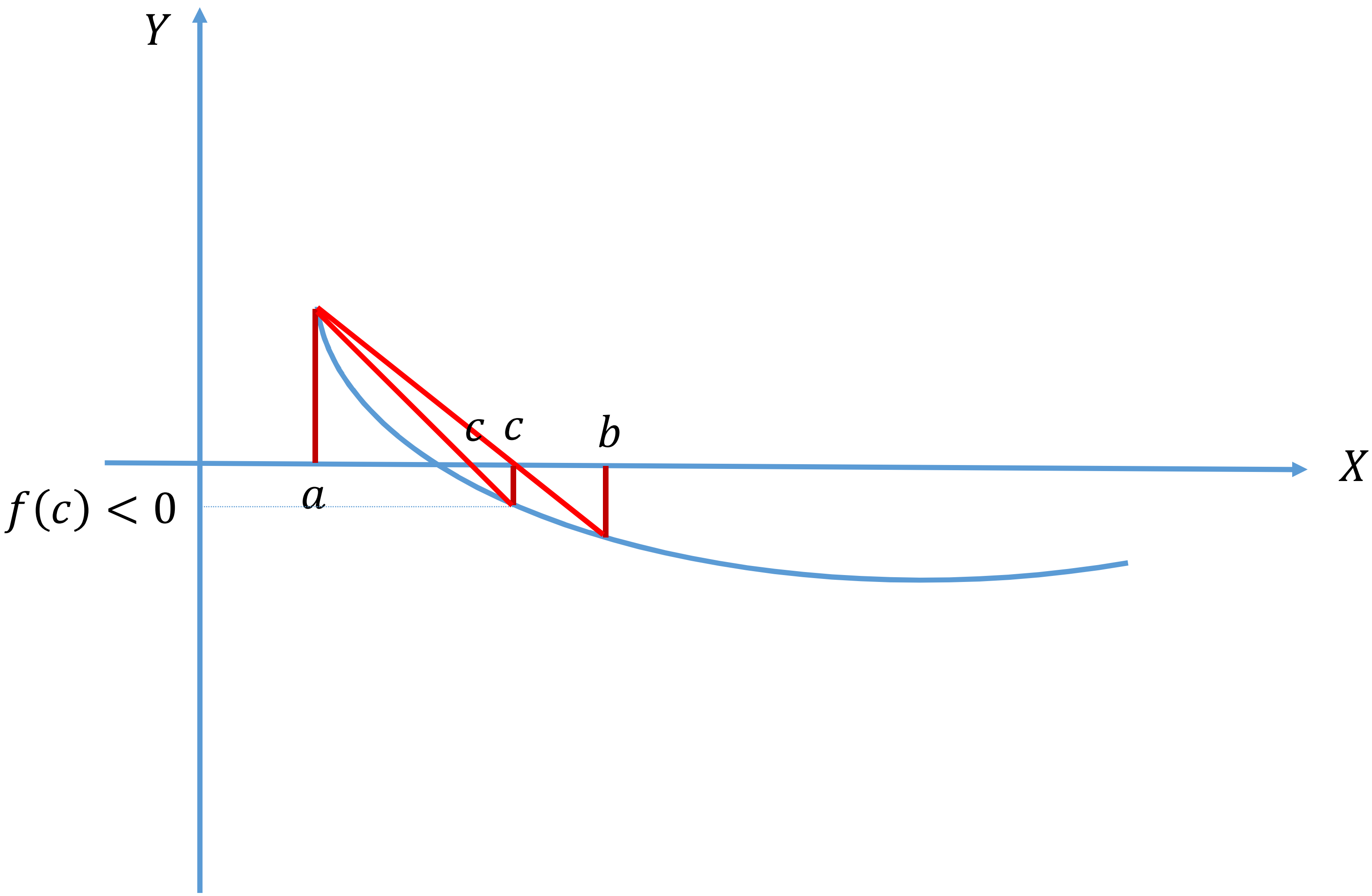
Regula Falsi method(Geometrically)



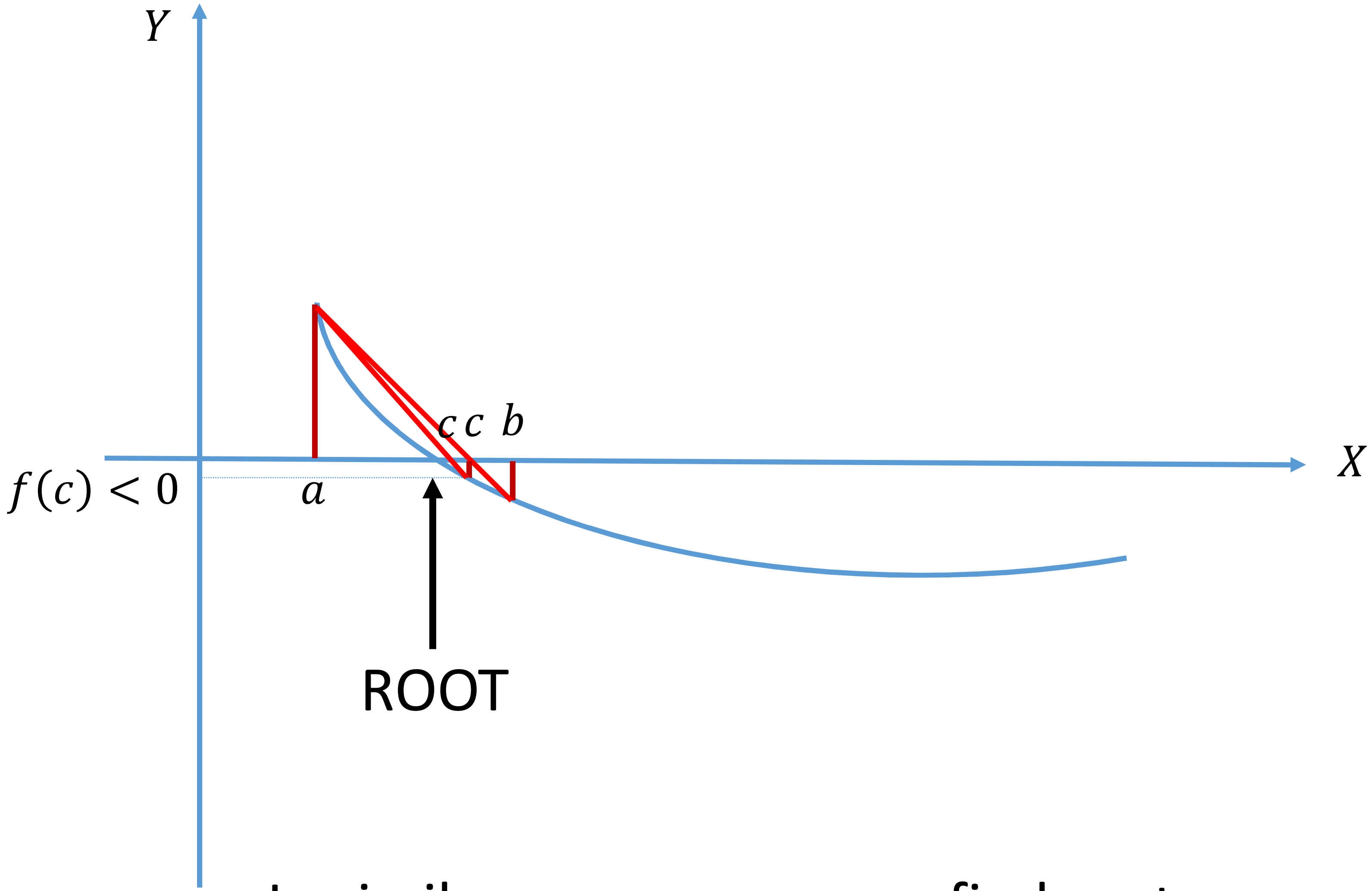
Regula Falsi method(Geometrically)



Regula Falsi method(Geometrically)



Regula Falsi method(Geometrically)

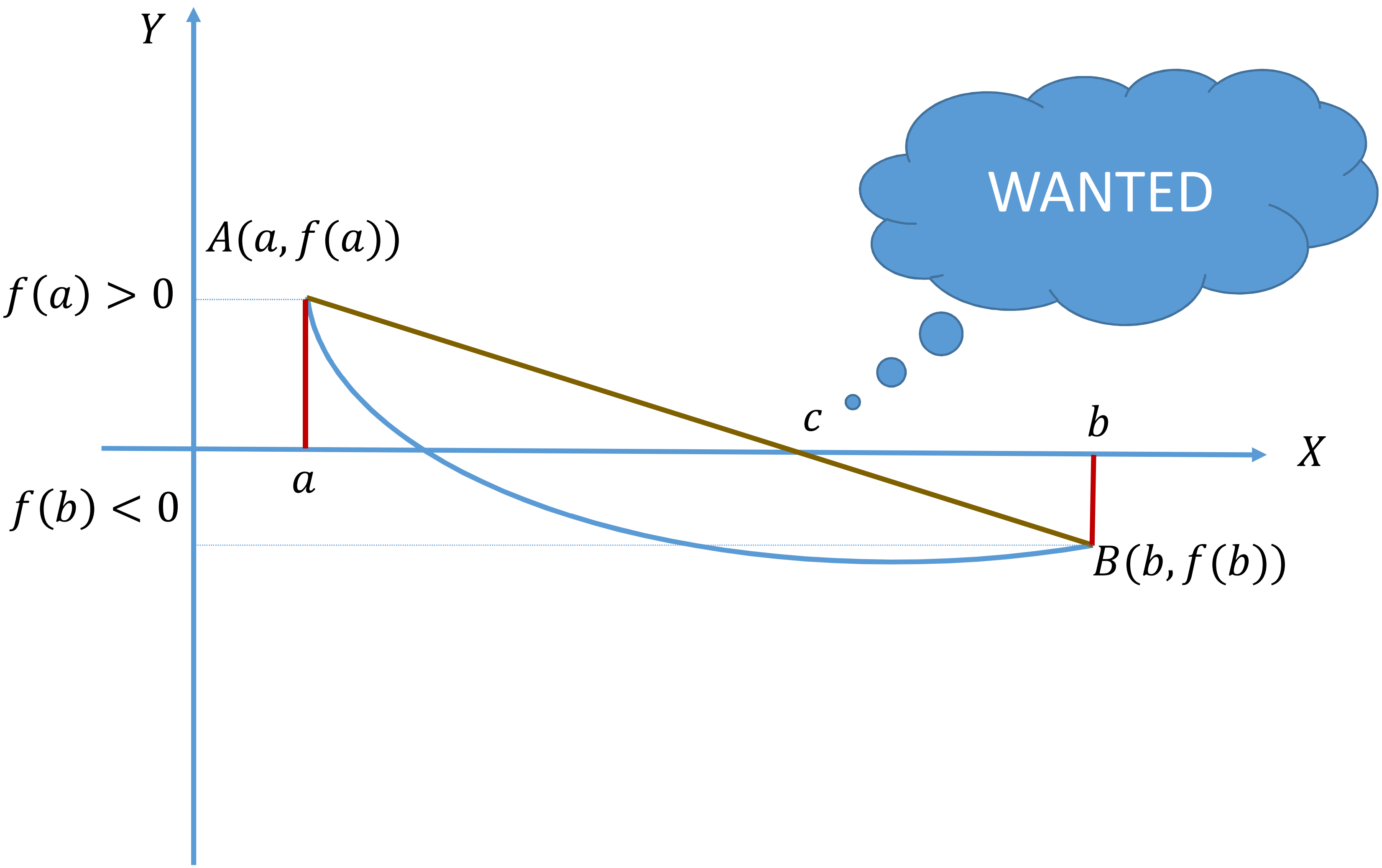


In similar manner, we can find root.

Derivation of Formula

- Let two points $\left(a, f(a)\right)$ & $\left(b, f(b)\right)$ be on curve such that $f(a) > 0$ & $f(b) < 0$.

Regula Falsi method



Derivation of formula

- Let two points $\left(a, f(a)\right)$ & $\left(b, f(b)\right)$ on curve such that $f(a) > 0$ & $f(b) < 0$.
- An equation of line Passing through (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$.
- Slope of line passing through $\left(a, f(a)\right)$ & $\left(b, f(b)\right)$ is,

$$m = \frac{f(b) - f(a)}{b - a}$$

Derivation of formula

- An equation of line Passing through $\left(a, f(a)\right)$ with slope m is,

$$y - f(a) \; = \frac{f(b) - f(a)}{b - a}(x - a)$$

- Note that, above equation represents all points (x, y) of line. We can find co ordinates of point " c ".
- Since, point " c " is on X-axis. So, y co-ordinate is zero.

Derivation of formula

- An equation of line Passing through $\left(a, f(a)\right)$ with slope m is,

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

- Since, $(c, 0)$ is on line.

$$0 - f(a) = \frac{f(b) - f(a)}{b - a}(c - a)$$

Derivation of formula

$$\Longrightarrow -f(a) = \boxed{\frac{f(b) - f(a)}{b - a}}(c - a)$$

$$\Longrightarrow -f(a)\frac{b - a}{f(b) - f(a)} = c - a$$

$$\Longrightarrow \qquad -f(a)\frac{b - a}{f(b) - f(a)} = c - a$$

Derivation of formula

$$\Longrightarrow a - f(a)\frac{b-a}{f(b)-f(a)} = c$$

$$\Longrightarrow c = a - \frac{b-a}{f(b)-f(a)}f(a)$$

So, we can find value of "c" by below equation

$$c = a - \frac{b-a}{f(b)-f(a)}f(a)$$

Find +ve root of $xe^x - 2 = 0$ by the method of False Position.

$f(x) = xe^x - 2$

- $f(0) = -2 < 0$
- $f(1) = 0.7183 > 0$
- $f(0.9) = 0.2136 > 0$
- $f(0.8) = -0.2136 < 0$

Let, $a = 0.8$ & $b = 0.9$

Since, $f(0.8) < 0$ & $f(0.9) > 0$

The root lies between 0.8 & 0.9 .

$$c = a - \frac{(a-b)}{\big(f(a) - f(b)\big)}\big(f(a)\big)$$

$$c = 0.8000 - \frac{(0.8000 - 0.9000)}{(-0.2136 - 0.2136)}(-0.2136)$$

Basic information and shortcuts

- For Value of " c ", Insert below expression

$$\boxed{0.8000-\big((0.8000-0.9000)(-0.2136)\big)\div(-0.2136-0.2136)}$$

- Press " = ", Note the value of " c ".
- For Value of function Insert Below expression

$$\boxed{\mathrm{Ans}\ \mathrm{eAns}-2}$$

- Press " = ", and Note Value of " $f(c)$ ".

$$c = a - \frac{(a-b)}{\big(f(a) - f(b)\big)}\big(f(a)\big)$$

$$c = 0.8000 - \frac{(0.8000 - 0.9000)}{(-0.2136 - 0.2136)}(-0.2136)$$

$$c = 0.8500$$

$$f(0.8500) = -0.0113 < 0$$

Since, $f(0.8500\ ) < 0$ & $f(1) > 0$

The roots lies between 0.8500 & 1 .

$$f(x) = xe^x - 2$$

$$c = a - \frac{(a-b)}{\big(f(a) - f(b)\big)}\big(f(a)\big)$$

n	a	$f(a) < 0$	b	$f(b) > 0$	c	$f(c)$
1	0.8	-0.2196	0.9	0.2136	$\boxed{0.8507}$	$-0.0083 < 0$
2	$\boxed{0.8507}$	-0.0083	0.9	0.2136	0.8525	$-0.0003 < 0$
3	0.8525	-0.0003	0.9	0.2136	0.8526	$-0.0002 < 0$
4	0.8526	-0.0002	0.9	0.2136	0.8526	

Since, 3rd and 4th iteration are same up to four decimal places.

$\therefore$ Required root is 0.8526.

Apply false position method to find the negative root of the equation $\boldsymbol{x^3 - 2x + 5 = 0}$ correct to four decimal places.

Solution : Here $f(x) = x^3 - 2x + 5$

- $f(0) = 5 > 0$
- $f(-1) = -1 + 2 + 5 = 6 > 0$
- $f(-2) = -8 + 4 + 5 = \boxed{1 > 0}$
- $f(-3) = -27 + 6 + 5 = -16 < 0$
- $f(-2.1) = \boxed{-0.061 < 0}$

Since $f(-2.1) < 0$ and $f(-2) > 0.$

So, Root lies between -2.1 and $-2.$

$$f(x) = x^3 - 2x + 5$$

$$c = a - \frac{(a-b)}{\big(f(a) - f(b)\big)}\big(f(a)\big)$$

n	a	$f(a) < 0$	b	$f(b) > 0$	c	$f(c)$
1	-2.1	-0.061	-2	1	-2.0943	$0.0034 > 0$
2	-2.1	-0.061	-2.0943	0.0034	-2.0946	$-0.0006 < 0$
3	-2.0946	-0.0006	-2.0943	0.0034	-2.0946	

Since, 2nd and 3rd iteration are same up to four decimal places.

$\therefore$ Required root is -2.0946.

Newton Raphson Method (NR Method)

Analytic Derivation

- ✓ Let $f(x) = 0$ be the given Non-Linear equation, and x_0 be the initial approximation, and h the **correction** to x_0 so that

$$f(x_0 + h) = 0$$

Analytic Derivation

- ✓ Expanding $f(x_0+h)$ by Taylor's series, We get

$$f(x_0+h)=f(x_0)+\frac{h^1}{1!}f'(x_0)+\frac{h^2}{2\,!}f''(x_0)+\cdots+\frac{h^n}{n!}f^n(x_0)$$

Where, $f^n(x_0)$ is n^{th} ordered derivative at x_0.

Hence,

$$f(x_0)+\frac{h^1}{1!}f'(x_0)+\frac{h^2}{2\,!}f''(x_0)+\cdots+\frac{h^n}{n!}f^n(x_0)=0$$

Analytic Derivation

- ✓ Since the value of h is too small such that,

$$\lim_{n\to\infty} \frac{h^n}{n!} f^n(x_0) = 0$$

- ✓ So, We neglect the higher order term in h, i.e. $h^2, h^3, \ldots$

- ✓ We get,

$$f(x_0) + \frac{h^1}{1!} f^1(x_0) + \frac{h^2}{2\,!} f''(x_0) + \cdots + \frac{h^n}{n!} f^n(x_0) = 0$$

Analytic Derivation

$$\Longrightarrow \quad f(x_0) \quad + \frac{h^1}{1!} f'(x_0) = 0$$

$$\Longrightarrow h\, f'(x_0) = -f(x_0)$$

$$\Longrightarrow h \;=-\frac{f(x_0)}{f'(x_0)}$$

- ✓ Therefore if x_0 is the initial approximation, by adding to this the value of h, we get the next approximation to the root.

Analytic Derivation

i.e. $x_1 = x_0 + h$

$$\Longrightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, taking x_1 as the starting value, we get

$x_2 = x_1 + h$

$$\Longrightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Analytic Derivation

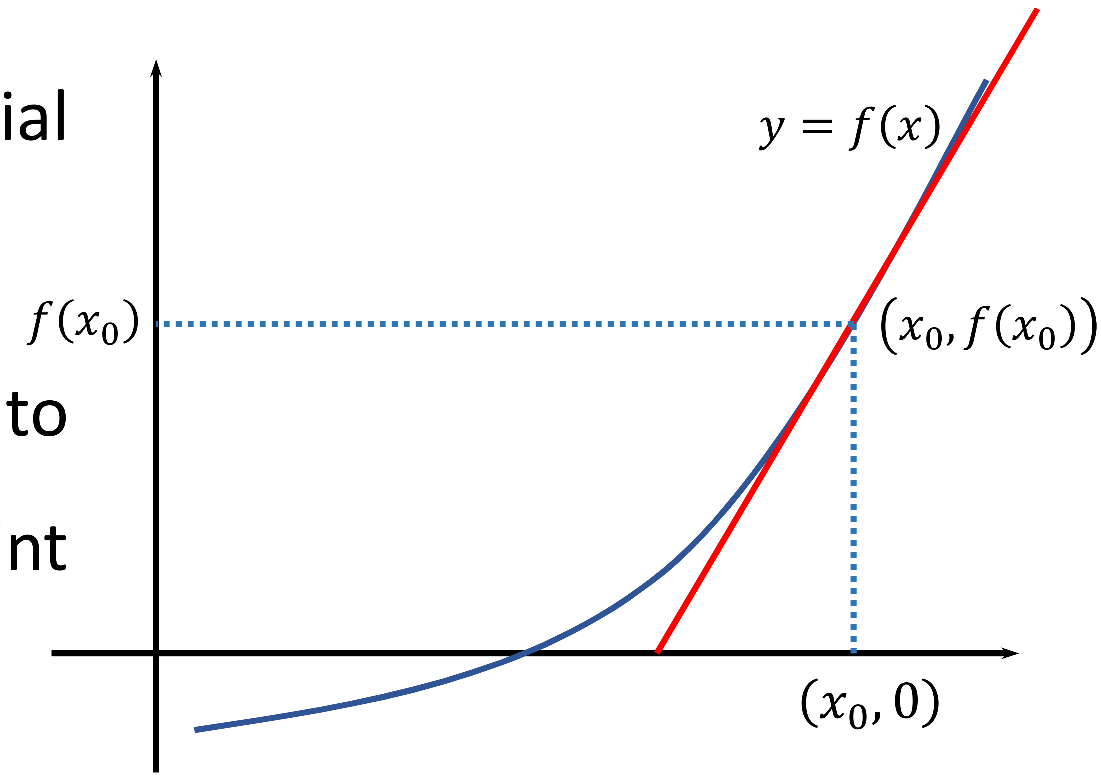
- ✓ In general, from the i^{th} approximation, we get the $(i+1)^{th}$ observation as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- ✓ The iterative procedure terminates when the difference between two successive iteration is about to zero OR up to prescribed tolerance.

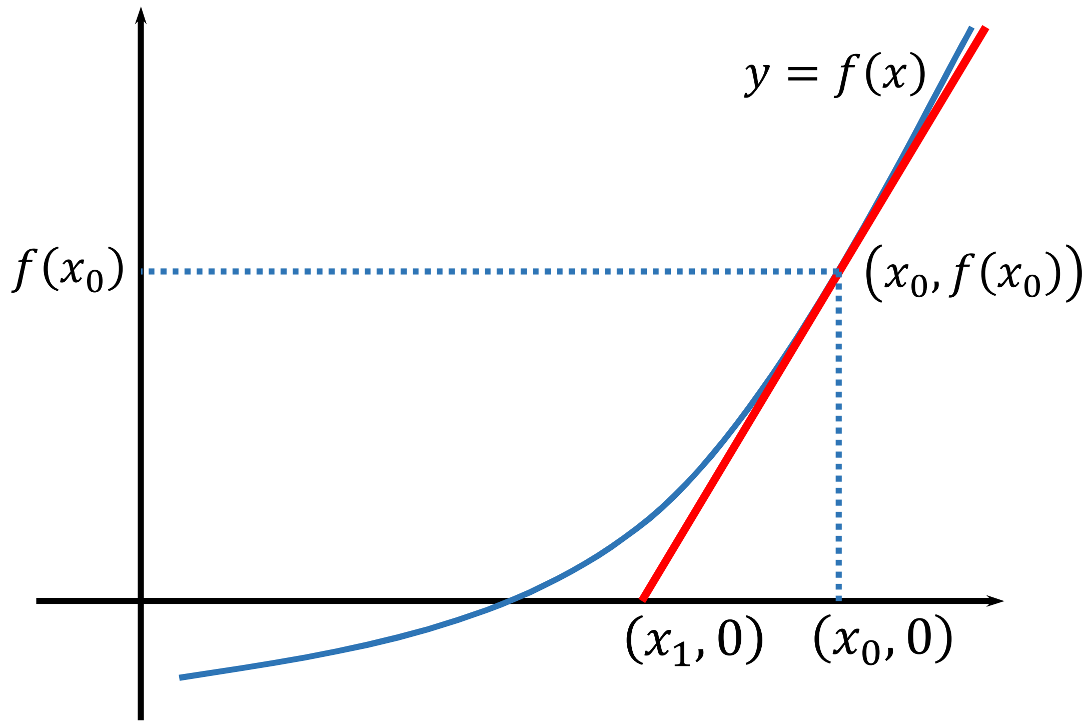
Geometric Derivation

- ✓ Let $f(x) = 0$.
- ✓ Consider that the initial approximation is x_0.
- ✓ Now draw a tangent to the curve at the point $\left(x_0, f(x_0)\right)$.



Geometric Derivation

- ✓ The point $(x_1, 0)$, where the tangent intersects the axis gives us next approximation.

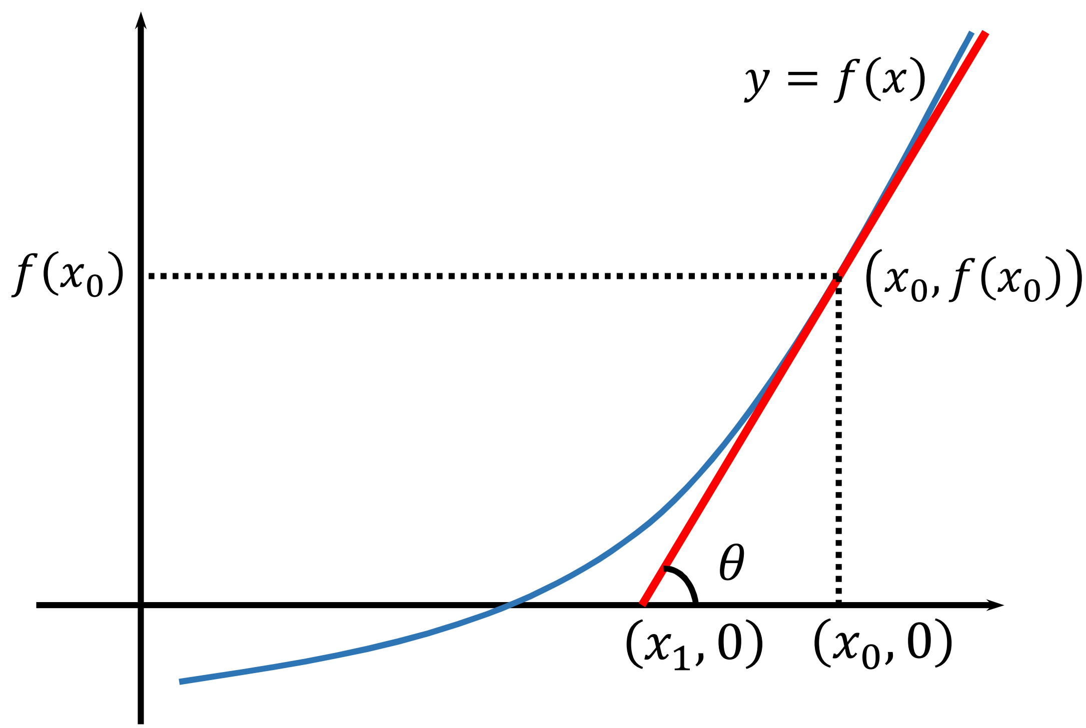


Geometric Derivation

✓ The slope of the curve at point $(x_0, f(x_0))$ is given by

$$tan\theta = f^1(x_0)$$

$$\Longrightarrow f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1}$$



Geometric Derivation

$$\Longrightarrow f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1}$$

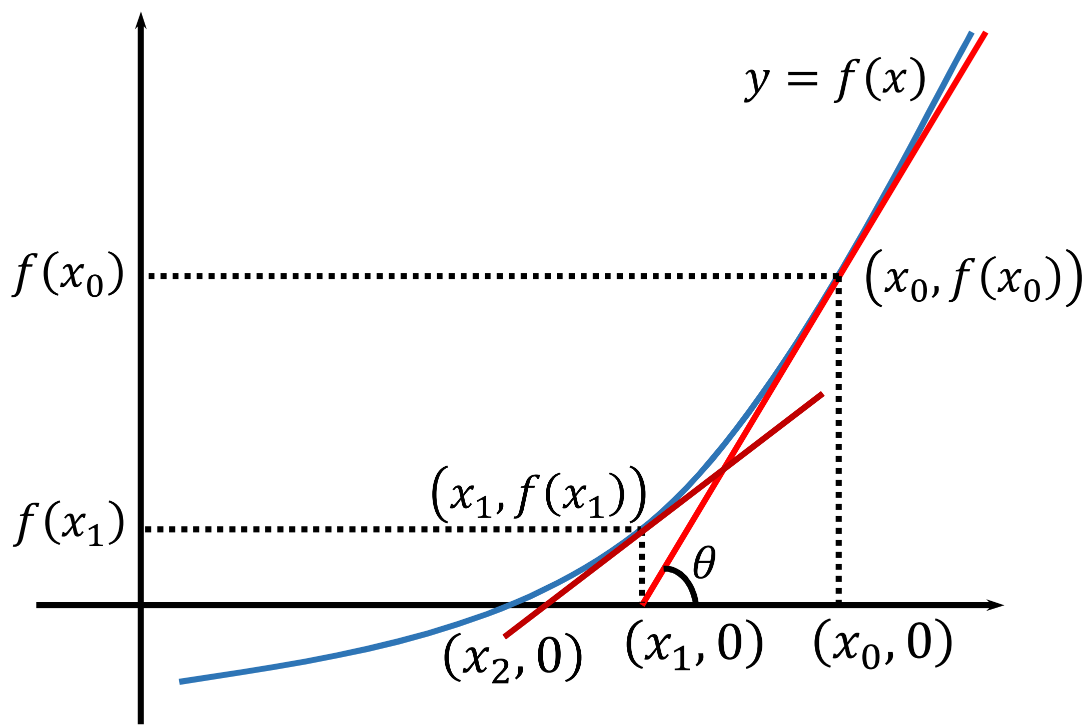
$$\Longrightarrow (x_0 - x_1)f'(x_0) = f(x_0)$$

$$\Longrightarrow x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$\Longrightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Geometric Derivation

- ✓ Now, repeating the same process from new approximation $(x_1, 0)$.
- ✓ Now draw a tangent to the curve at the point $(x_1, f(x_1))$.



Geometric Derivation

✓ The slope of the curve at point $(x_1, f(x_1))$ is given by

$$tan\theta = f'(x_1)$$

The slope of the line passing through $(x_2, 0)$ & $(x_1, f(x_1))$ is

$$\Rightarrow f'(x_1) = \frac{f(x_1) - 0}{x_1 - x_2} \qquad \left(\because tan\theta = \frac{f(x_1) - 0}{x_1 - x_2}\right)$$

$$\Rightarrow (x_1 - x_2)f'(x_1) = f(x_1)$$

$$\Rightarrow x_1 - x_2 = \frac{f(x_1)}{f'(x_1)}$$

Geometric Derivation

$$\Longrightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

- ✓ In general, the $(i+1)^{th}$ approximation is obtained from the i^{th} approximation as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- ✓ Here, we have assumed that $f'(x_i) \neq 0$.

Solved Examples

Find the positive root of $x = \cos x$ correct up to 3 decimal points.

Solution:

Here, $f(x) = x - \cos x$

$\Longrightarrow f'(x) = 1 + \sin x$

Let, $a = 0 \Longrightarrow \ f(0) = 0 - \cos 0 = -1 < 0$

Let, $b = 1 \Longrightarrow \ f(1) = 1 - \cos 1 = 0.460 > 0$

Hence, root lies between 0 and 1.

Find the positive root of $x = \cos x$ correct up to 3 decimal points.

Since, $f(0.6) = 0.6 - \cos 0.6 = -0.225$

$f(0.7) = 0.7 - \cos 0.7 = -0.065$

$f(0.8) = 0.8 - \cos 0.8 = 0.103$

Let, initial solution be $x_0 = 0.7$

Here, $f(x) = x - \cos x$

$\Longrightarrow f(0.7) = 0.7 - \cos 0.7$

$\Longrightarrow f(0.7) = -0.065$

$f'(x) = 1 + \sin x$

$\Longrightarrow f'(0.7) = 1 + \sin 0.7$

$\Longrightarrow f'(0.7) = 1.644$

Find the positive root of $x = \cos x$ correct up to 3 decimal points.

By Newton Raphson's Formula,

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$$x_1 = 0.7 - \frac{-0.065}{1.644}$$

$$x_1 = 0.740.$$

We have,

- $x_0 = 0.7$
- $f(x_0) = f(0.7) = -0.065$
- $f'(x_0) = f'(0.7) = 1.644$

Find the positive root of $x = \cos x$ correct up to 3 decimal points.

For, 1st iteration $x_1 = 0.740$

Here, $f(x_1) = x_1 - \cos x_1$

$\Longrightarrow f(0.740) = 0.740 - \cos 0.740$

$\Longrightarrow f(0.740) = 0.002$

- $f'(x_1) = 1 + \sin x_1$

$\Longrightarrow f'(0.740) = 1 + \sin 0.740$

$\Longrightarrow f'(0.740) = 1.674$

Find the positive root of $x = \cos x$ correct up to 3 decimal points.

By Newton Raphson's Formula,

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$$

$$x_2 = 0.740 - \frac{0.002}{1.674}$$

$$x_2 = 0.739.$$

We have,

- $x_1 = 0.740$
- $f(x_1) = f(0.740) = 0.002$
- $f'(x_1) = f'(0.740) = 1.674$

Find the positive root of $x = \cos x$ correct up to 3 decimal points.

For, 2nd iteration $x_2 = 0.739$

Here, $f(x_2) = x_2 - \cos x_2$

$\Longrightarrow f(0.739) = 0.739 - \cos 0.739$

$\Longrightarrow f(0.739) = 0$

No further calculation is needed,

$Solution = root = 0.739.$

Use of Calculator $fx - 82MS$

Basic information and shortcuts

- ✓ To clear previous data, [Shift + CLR + 3 + ==]
- ✓ Insert initial solution and press “ = ”

[Meaning : value is stored in memory as variable “ ans ”. If we type 5 and press “ = ”. It means whenever we use “ ans ” it takes value 5.]

- ✓ Type NR formula and press “ ans ” in place of x_n .

Basic information and shortcuts

- ✓ Now, press “ = ”, it gives you 1^{st} iteration value.
- ✓ Now, again press “ = ”, it gives you 2^{nd} iteration value.
- ✓ By continuing same process (by pressing “ = ”), we have next iteration.
- ✓ At some place, we will get same value. That repeated value is solution (root) of function.

Find the approximation solution of $x^3 + x - 1 = 0$ correct up to 3 decimal points.

Solution:

Here, $f(x) = x^3 + x - 1$

$\Longrightarrow f'(x) = 3x^2 + 1$

Let, $a = 0 \Longrightarrow f(0) = 0^3 + 0 - 1 = -1 < 0$

Let, $b = 1 \Longrightarrow f(1) = 1^3 + 1 - 1 = 1 > 0$

Hence, root lies between 0 and 1.

Find the approximation solution of $x^3 + x - 1 = 0$ correct up to 3 decimal points.

Let, initial solution be $x_0 = 0.6$

Here, $f(x) = x^3 + x - 1$

$\Longrightarrow f(0.6) = (0.6)^3 + 0.6 - 1$

$\Longrightarrow f(0.6) = -0.184$

- $f'(x) = 3x^2 + 1$

$\Longrightarrow f'(0.6) = 3(0.6)^2 + 1$

$\Longrightarrow f'(0.6) = 2.080$

Find the approximation solution of $x^3 + x - 1 = 0$ correct up to 3 decimal points.

By Newton Raphson's Formula,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.6 - \frac{-0.184}{2.080}$$

$$x_1 = 0.688.$$

We have,

- $x_0 = 0.6$
- $f(x_0) = f(0.6) = -0.184$
- $f'(x_0) = f'(0.6) = 2.080$

Find the approximation solution of $x^3 + x - 1 = 0$ correct up to 3 decimal points.

For, 1st iteration $x_1 = 0.688$

Here, $f(x_1) = x_1^3 + x_1 - 1$

$\Longrightarrow f(0.688) = (0.688)^3 + 0.688 - 1$

$\Longrightarrow f(0.688) = 0.014$

- $f'(x_1) = 3x_1^2 + 1$

$\Longrightarrow f'(0.688) = 3(0.688)^2 + 1$

$\Longrightarrow f'(0.688) = 2.420$

Find the approximation solution of $x^3 + x - 1 = 0$ correct up to 3 decimal points.

By Newton Raphson's Formula,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We have,

- $x_1 = 0.688$
- $f(x_1) = f(0.688) = 0.014$
- $f'(x_1) = f'(0.688) = 2.420$

$$x_2 = 0.688 - \frac{0.014}{2.420}$$

$$x_2 = 0.682.$$

Find the approximation solution of $x^3 + x - 1 = 0$ correct up to 3 decimal points.

For, 2nd iteration $x_2 = 0.682$

Here, $f(x_2) = x_2^3 + x_2 - 1$

$\Longrightarrow f(0.682) = (0.682)^3 + 0.682 - 1$

$\Longrightarrow f(0.682) = -0.001$

- $f'(x_2) = 3x_2^2 + 1$

$\Longrightarrow f'(0.682) = 3(0.682)^2 + 1$

$\Longrightarrow f'(0.682) = 2.395$

Find the approximation solution of $x^3 + x - 1 = 0$ correct up to 3 decimal points.

By Newton Raphson's Formula,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

We have,

- $x_2 = 0.682$
- $f(x_2) = f(0.682) = -0.001$
- $f'(x_2) = f'(0.682) = 2.395$

$$x_3 = 0.682 - \frac{-0.001}{2.395}$$

$$x_3 = 0.682.$$

Since, Solution $= x_3 = x_2 = 0.682$

Shortcuts

- ✓ For last example go through with following steps.
  1. Insert initial solution. i.e. $x_0 = 0.6$.
  2. Insert

$$\boxed{\boldsymbol{ans - \left(\left(ans^3 + ans - 1\right) \div \left(3ans^2 + 1\right)\right)}}$$

  3. On first press, we have 0.688.
  4. After 2nd press, we have 0.682 and on each press it will remain same.

Find the root of $x^4 - x^3 + 10x + 7 = 0$ correct up to 3 decimal points and between -2 and -1.

Solution:

Here, $f(x) = x^4 - x^3 + 10x + 7$

$\Longrightarrow f'(x) = 4x^3 - 3x^2 + 10$

Since, $f(-2) = 11 > 0$

$f(-1) = -1 < 0$

Let, initial solution be $x_0 = -2$.

So, $f(-2) = 11$ and $f'(-2) = -34$.

Find the root of $x^4 - x^3 + 10x + 7 = 0$ correct up to 3 decimal points and between -2 and -1.

By Newton Raphson's Formula,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We have,

- $x_0 = -2$
- $f(x_0) = f(-2) = 11$
- $f'(x_0) = f'(-2) = -34$

$$x_1 = -2 - \frac{11}{-34}$$

$$x_1 = -1.676.$$

So, $f(-1.676) = 2.838$ and $f'(-1.676) = -17.258$.

Find the root of $x^4 - x^3 + 10x + 7 = 0$ correct up to 3 decimal points and between -2 and -1.

By Newton Raphson's Formula,

We have,

- $x_1 = -1.676$
- $f(-1.676) = 2.838$
- $f'(-1.676) = -17.258$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = -1.676 - \frac{2.838}{-17.258}$$

$$x_2 = -1.512.$$

Now, $f(-1.512) = 0.563$ and $f'(-1.512) = -10.685$.

Find the root of $x^4 - x^3 + 10x + 7 = 0$ correct up to 3 decimal points and between -2 and -1.

By Newton Raphson's Formula,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

We have,

- $x_2 = -1.512$
- $f(-1.512) = 0.563$
- $f'(-1.512) = -10.685$

$$x_3 = -1.512 - \frac{0.563}{-10.685}$$

$$x_3 = -1.459.$$

So, $f(-1.459) = 0.047$ and $f'(-1.459) = -8.809$.

Find the root of $x^4 - x^3 + 10x + 7 = 0$ correct up to 3 decimal points and between -2 and -1.

By Newton Raphson's Formula,

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

We have,

- $x_3 = -1.459$
- $f(-1.459) = 0.047$
- $f'(-1.459) = -8.809$

$$x_4 = -1.459 - \frac{0.047}{-8.809}$$

$$x_4 = -1.454.$$

So, $f(-1.454) = 0.003$ and $f'(-1.454) = -8.638$.

Find the root of $x^4 - x^3 + 10x + 7 = 0$ correct up to 3 decimal points and between -2 and -1.

By Newton Raphson's Formula,

We have,

- $x_4 = -1.454$
- $f(-1.454) = 0.003$
- $f'(-1.454) = -8.638$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$

$$x_5 = -1.454 - \frac{0.003}{-8.638}$$

$$x_5 = -1.454.$$

Since, $x_4 = x_5 = -1.454$. So, $root = -1.454$.

Application of NR Method

1. q^{th} root of a number
2. Reciprocal of a number

Formula for q^{th} root of a number N.
where $q = 1,2,3, \ldots ; N \in \mathbb{N}$

Solution:

Let , $x = N^{\frac{1}{q}}$; where $q = 1,2,3, \ldots ; N \in \mathbb{N}$

$$\implies x^q = N$$

$$\implies x^q - N = 0$$

So, $f(x) = x^q - N$

$$\implies f'(x) = q \cdot x^{q-1}$$

Formula for q^{th} root of a number N.
where $q = 1,2,3, \ldots ; N \in \mathbb{N}$

By Newton Raphson's Formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

$$\Longrightarrow x_{n+1} = x_n - \frac{{x_n}^q - N}{q \cdot {x_n}^{q-1}}$$

$$\Longrightarrow x_{n+1} = x_n - \frac{{x_n}^q}{q \cdot {x_n}^{q-1}} + \frac{N}{q \cdot {x_n}^{q-1}}$$

Formula for q^{th} root of a number N.
where $q = 1,2,3, \dots ; N \in \mathbb{N}$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n^{\,q}}{q \cdot x_n^{\,q} \cdot x_n^{\,-1}} + \frac{N}{q \cdot x_n^{\,q-1}}$$

$$\Rightarrow \boldsymbol{x_{n+1} = x_n} - \frac{x_n^{\,1}\,\cancel{x_n^{\,q}}}{q\,\cancel{x_n^{\,q}}} + \frac{N}{q \cdot x_n^{\,q-1}}$$

$$\Rightarrow x_{n+1} = \frac{q \cdot x_n - x_n}{q} + \frac{N}{q \cdot x_n^{\,q-1}}$$

Formula for q^{th} root of a number N.
where $q = 1,2,3, \dots ; N \in \mathbb{N}$

$$\Rightarrow x_{n+1} = \frac{1}{q}\left[(q-1)x_n + \frac{N}{{x_n}^{q-1}}\right]$$

Where, $n = 0,1,2, \dots$

For $q = 2$,

$$\Rightarrow x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$

A general formula for finding square root of any number.

Find value of $\sqrt{28}$.

Solution:

Here, $q = 2$ & $N = 28$.

Let initial solution be $x_0 = \sqrt{25} = 5$. (Choose nearest perfect number for initial solution.)

Find value of $\sqrt{28}$.

We have,

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{28}{x_n}\right]$$

Iteration (n)	x_n
0	5
1	5.3000
2	5.2915
3	5.2915

Find value of $\sqrt[3]{58}$.

Solution:

Here, $q = 3$ & $N = 58$.

Let initial solution be $x_0 = \sqrt{64} = 4$.

We have , $x_{n+1} = \frac{1}{q}\left[(q-1)x_n + \frac{N}{{x_n}^{q-1}}\right]$

$$x_{n+1} = \frac{1}{3}\left[(3-1)x_n + \frac{N}{{x_n}^{3-1}}\right]$$

Find value of $\sqrt[3]{58}$.

We have,

$$x_{n+1} = \frac{1}{3}\left[2\,x_n + \frac{N}{x_n^2}\right]$$

$$x_{n+1} = \frac{1}{3}\left[2\,x_n + \frac{58}{x_n^2}\right]$$

Iteration (n)	x_n
0	4
1	3.8750
2	3.8709
3	3.8709

Formula for reciprocal of a number N.

Solution:

Let , $x = \frac{1}{N}$

$$\Rightarrow \frac{1}{x} = N$$

$$\Rightarrow \frac{1}{x} - N = 0$$

So, $f(x) = \frac{1}{x} - N$

$$\Rightarrow f'(x) = -\frac{1}{x^2}$$

Formula for reciprocal of a number N.

$$f(x) = \frac{1}{x} - N$$

$$\Rightarrow f(x) = \frac{1 - x\,N}{x}$$

$$\Rightarrow f(x_n) = \frac{1 - x_n\,N}{x_n}$$

$$f'(x) = -\frac{1}{x^2}$$

$$\Rightarrow f'(x_n) = -\frac{1}{x_n^2}$$

$$\Rightarrow \frac{1}{f'(x_n)} = -x_n^2$$

Formula for reciprocal of a number N.

By Newton Raphson's Formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Longrightarrow x_{n+1} = x_n - \frac{1}{f'(x_n)} \cdot f(x_n)$$

$$\Longrightarrow x_{n+1} = x_n - (-x_n^2) \cdot \frac{1 - x_n N}{x_n}$$

Formula for reciprocal of a number N.

$$\Rightarrow x_{n+1} = x_n - (-\cancel{x_n^2}) \cdot \frac{1 - x_n N}{\cancel{x_n}}$$

$$\Rightarrow x_{n+1} = x_n + x_n(1 - x_n N)$$

$$\Rightarrow x_{n+1} = x_n + x_n - x_n^2 N$$

$$\Rightarrow x_{n+1} = 2x_n - x_n^2 N$$

Where, $n = 0,1,2,\ldots$

Find value of $\frac{1}{3}$.

Solution:

Here, $N = 3$.

Let initial solution be $x_0 = \frac{1}{4}$.

We have, $x_{n+1} = 2\,x_n - x_n^2\,N$

For $n = 0$,

$$\Longrightarrow x_1 = 2\,x_0 - x_0^2\,N = 2\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)^2 (3) = 0.3125$$

Find value of $\frac{1}{3}$.

For $n = 1$,

$$x_{n+1} = 2\,x_n - x_n^2\,N$$

$$\Longrightarrow x_2 = 2\,x_1 - x_1^2\,N = 2\,(0.3125) - (0.3125)^2(3)$$

$$= 0.3320$$

We have,

Iteration (n)	x_n
0	0.25
1	0.3125
2	0.3320
3	0.3333
4	0.3333

Hence, Solution is 0.3333.

Rate of convergence of the NR Method.

- ✓ Let α be exact solution of curve $y = f(x)$.

  i.e. $f(\alpha) = 0$

- ✓ Let x_n and x_{n+1} be two successive approximations to the actual roots.

- ✓ If $\in_n$ and $\in_{n+1}$ are the corresponding errors then

  $x_n = \alpha + \in_n$ and $x_{n+1} = \alpha + \in_{n+1}$

Rate of convergence of the NR Method.

By NR formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$\Longrightarrow \alpha + \epsilon_{n+1} = \alpha + \epsilon_n - \frac{f(\alpha + \epsilon_n)}{f'(\alpha + \epsilon_n)}$$

$$\Longrightarrow \epsilon_{n+1} = \epsilon_n - \frac{f(\alpha) + \epsilon_n f'(\alpha) + \frac{\epsilon_n^2}{2!} f''(\alpha) + \cdots}{f'(\alpha) + \epsilon_n f''(\alpha) + \frac{\epsilon_n^2}{2!} f'''(\alpha) + \cdots}$$

Rate of convergence of the NR Method.

Since, $f(\alpha) = 0$.

$$\Longrightarrow \epsilon_{n+1} = \epsilon_n - \frac{f(\alpha) + \epsilon_n f'(\alpha) + \frac{\epsilon_n^2}{2!} f''(\alpha) + \cdots}{f'(\alpha) + \epsilon_n f''(\alpha) + \frac{\epsilon_n^2}{2!} f'''(\alpha) + \cdots}$$

Neglecting the derivative of order higher then two,

$$\Longrightarrow \epsilon_{n+1} = \epsilon_n - \frac{\epsilon_n f'(\alpha) + \frac{\epsilon_n^2}{2!} f''(\alpha) + \cdots}{f'(\alpha) + \epsilon_n f''(\alpha) + \frac{\epsilon_n^2}{2!} f'''(\alpha) + \cdots}$$

Rate of convergence of the NR Method.

So, We have

$$\Longrightarrow \in_{n+1} = \in_n - \frac{\in_n f'(\alpha) + \frac{\in_n^2}{2!} f''(\alpha)}{f'(\alpha) + \in_n f''(\alpha)}$$

$$\Longrightarrow \in_{n+1} = \frac{\in_n [f'(\alpha) + \in_n f''(\alpha)] - \in_n f'(\alpha) - \frac{\in_n^2}{2!} f''(\alpha)}{f'(\alpha) + \in_n f''(\alpha)}$$

$$\Longrightarrow \in_{n+1} = \frac{\cancel{\in_n f'(\alpha)} + \in_n^2 f''(\alpha) - \cancel{\in_n f'(\alpha)} - \frac{\in_n^2}{2!} f''(\alpha)}{f'(\alpha) + \in_n f''(\alpha)}$$

Rate of convergence of the NR Method.

So, We have

$$\Longrightarrow \epsilon_{n+1} = \frac{\frac{\epsilon_n^2}{2} f''(\alpha)}{f'(\alpha) + \epsilon_n f''(\alpha)}$$

Dividing by $f'(\alpha)$ in numerator and denominator,

$$\Longrightarrow \epsilon_{n+1} = \frac{\epsilon_n^2}{2} \frac{\frac{f''(\alpha)}{f'(\alpha)}}{1 + \epsilon_n \frac{f''(\alpha)}{f'(\alpha)}}$$

Rate of convergence of the NR Method.

Here, $1+\epsilon_n \frac{f''(\alpha)}{f'(\alpha)} \approx 1$

So, We have $\Longrightarrow \epsilon_{n+1} = \frac{\epsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)}$

- ✓ The last equation shows that the error at stage is proportional to the square of the error in the previous stage.
- ✓ Hence, the NR Method has a quadratic convergence and the convergence is of order 2.

Bairstow's Method for Quadratic factors

Example 1

Using the approximation factor $\mathrm{x}^2+\mathrm{x}+1$ of $\mathrm{x}^3-\mathrm{x}-1$, find the factor performing two iterations.

Take, $\mathrm{p}_0=\mathrm{q}_0=-1$.

Solution:

Here $p=p_0=-1$, $\mathrm{q}=q_0=-1$ and
$n = maximum\ power\ of\ polynomial = 3$

Now by synthetic division rule for two variables, we get

$p_0 = -1$ $q_0 = -1$	$a_0 = 1$ - -	$a_1 = 0$ $p_0 b_0 = -1$ -	$a_2 = -1$ $p_0 b_1 = 1$ $q_0 b_0 = -1$	$a_3 = -1$ $p_0 b_2 = 1$ $q_0 b_1 = 1$
	$b_0 = 1$	$b_1 = -1$	$b_2 = -1$	$b_3 = 1$
$p_0 = -1$ $q_0 = -1$	- -	$p_0 c_0 = -1$ -	$p_0 c_1 = 2$ $q_0 c_0 = -1$	
	$c_0 = 1$	$c_1 = -2$	$c_2 = 0$	

Now equations are

$b_n + c_{n-1}h + c_{n-2}k = 0$

$b_{n-1} + c_{n-2}h + c_{n-3}k = 0$

Putting obtained value of $b_n, b_{n-1}, c_{n-1}, c_{n-2}, c_{n-3}$

from the above table.

So, we have equations are ,

$1 + 0h - 2k = 0 \; and \; -1 - 2h + k = 0.$

By solving above equations we get,

$k = 0.5$ and $h = -0.25.$

Now improved values of p and q are:

$p_1 = p_0 + h = -1 - 0.25 = -1.25$

$q_1 = q_0 + k = -1 + 0.5 = -0.5$

By taking $p_1 = -1.25$ & $q_1 = -0.5$ we repeat the synthetic division rule as we have discussed above for more accurate values of p & q.

$p_1 = -1.25$ $q_1 = -0.5$	$a_0 = 1$ - -	$a_1 = 0$ $p_1 b_0 = -1.25$ -	$a_2 = -1$ $p_1 b_1 = 1.5625$ $q_1 b_0 = -0.5$	$a_3 = -1$ $p_1 b_2 = -0.0781$ $q_1 b_1 = 0.625$
$p_1 = -1.25$ $q_1 = -0.5$	$b_0 = 1$	$b_1 = -1.25$	$b_2 = 0.0625$	$b_3 = -0.4531$
	- -	$p_1 c_0 = -1.25$ -	$p_1 c_1 = 3.125$ $q_1 c_0 = -0.5$	
	$c_0 = 1$	$c_1 = -2.5$	$c_2 = 2.6875$	

Now equations are

$b_n + c_{n-1}h + c_{n-2}k = 0$

$b_{n-1} + c_{n-2}h + c_{n-3}k = 0$

Putting obtained value of $b_n, b_{n-1}, c_{n-1}, c_{n-2}, c_{n-3}$

from the above table.

So ,we have equations are ,

$-0.4531 + 2.6875h - 2.5k = 0$ and $0.0625 - 2.5h + k = 0.$

By solving above equations we get,

$k = -0.2708$ and $h = -0.0833$

Now improved values of p and q are:

$p_2 = p_1 + h = -1.25 - 0.0833 = -1.3333$

$q_2 = q_1 + k = -0.5 - 0.2708 = -0.7708$

The required quadratic factor is,

$x^2 - px - q = 0$

$\therefore x^2 + 1.3333x + 0.7708 = 0$

By comparing with a$x^2 + bx + c = 0$, we have

$a = 1, b = 1.3333 \; and \; c = 0.7708$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-1.3333 \pm \sqrt{(1.3333)^2 - 4(1)(0.7708)}}{2(1)}$$

$\therefore x = -0.6667 \pm 0.5713i$

So required complex roots are: $-0.6667 \pm 0.5713i$

Example 5

$x^3 - 5x^2 - 2x + 24 = 0$, find the factor performing two iterations. Take, $p_0 = q_0 = 0$.

Solution:

Here $p = p_0 = 0$, $q = q_0 = 0$ and
$n = maximum\ power\ of\ polynomial = 3$

Now by synthetic division rule for two variables, we get

$p_0 = 0$ $q_0 = 0$	$a_0 = 1$ - -	$a_1 = -5$ $p_0 b_0 = 0$ -	$a_2 = -2$ $p_0 b_1 = 0$ $q_0 b_0 = 0$	$a_3 = 24$ $p_0 b_2 = 0$ $q_0 b_1 = 0$
	$b_0 = 1$	$b_1 = -5$	$b_2 = -2$	$b_3 = 24$
$p_0 = 0$ $q_0 = 0$	- -	$p_0 c_0 = 0$ -	$p_0 c_1 = 0$ $q_0 c_0 = 0$	$p_0 c_2 = 0$ $q_0 c_1 = 0$
	$c_0 = 1$	$c_1 = -5$	$c_2 = -2$	$c_3 = 24$

Now equations are
$b_n + c_{n-1}h + c_{n-2}k = 0$
$b_{n-1} + c_{n-2}h + c_{n-3}k = 0$

Putting obtained value of $b_n, b_{n-1}, c_{n-1}, c_{n-2}, c_{n-3}$ from the above table.

So ,we have equations are ,

$24 - 2h - 5k = 0$ *and* $-2 - 5h + k = 0$.

By solving above equations we get,

$k = 4.5926$ and $h = 0.5185$.

Now improved values of p and q are:
$p_1 = p_0 + h = 0 + 0.5185 = 0.5185$
$q_1 = q_0 + k = 0 + 4.5926 = 4.5926$

$p_1 = 0.5185$ $q_1 = -0.5$	$a_0 = 1$ - -	$a_1 = -5$ $p_1 b_0 = 0.5185$ -	$a_2 = -2$ $p_1 b_1 = -2.3237$ $q_1 b_0 = 4.5926$	$a_3 = 24$ $p_1 b_2 = 0.1394$ $q_1 b_1 = -20.5817$
$p_1 = -1.25$ $q_1 = -0.5$	$b_0 = 1$	$b_1 = -4.4815$	$b_2 = 0.2689$	$b_3 = 3.5577$
	- -	$p_1 c_0 = 0.5185$ -	$p_1 c_1 = -2.0548$ $q_1 c_0 = 4.5926$	$p_1 c_2 = 1.4553$ $q_1 c_1 = -18.2005$
	$c_0 = 1$	$c_1 = -3.9630$	$c_2 = 2.8067$	$c_3 = -13.1875$

Now equations are

$b_n + c_{n-1}h + c_{n-2}k = 0$

$b_{n-1} + c_{n-2}h + c_{n-3}k = 0$

Putting obtained value of $b_n, b_{n-1}, c_{n-1}, c_{n-2}, c_{n-3}$ from the above table.

So ,we have equations are ,

$3.5577 + 2.8067h - 3.9630k = 0$ *and*

$0.2689 - 3.9630h + 1k = 0.$

By solving above equations we get,

$k = 1.1512$ and $h = 0.3584$

Now improved values of p and q are:

$p_2 = p_1 + h = 0.5185 + 0.3584 = 0.8769$

$q_2 = q_1 + k = 4.4926 + 1.1512 = 5.6438$

The required quadratic factor is,

$x^2 - px - q = 0$

$\therefore x^2 - 0.8769x - 5.6438 = 0$

By comparing with a$x^2 + bx + c = 0$, we have

$a = 1, b = -0.8769 \ and \ c = -5.6438$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{0.8769 \pm \sqrt{(-0.8769)^2 - 4(1)(-5.6438)}}{2(1)}$$

$\therefore x = -1.9773$ or $x = 2.8542$

So required roots are: -1.9773 or 2.8542