Numerical Methods

Adapted By Manik Hosen

- What are Numerical Method?
- Ans: In numerical analysis, numerical methods are mathematical tools designed to solve numerical problems.
- What are the reasons to study Numerical Method?
- Ans: There are many things that you want to compute that can not be computed exactly. The roots of high degree polynomials. The solution of simple differential equations on irregular domains. Any calculation with real numbers that you do with a computer.
  Numerical analysis is the branch of mathematics that is about approximate computing. It tells you how quickly you can get how close to the true solution. If you study any sort of engineering science you need inevitably learn some corner of numerical analysis.

Numerical Methods

- Define accuracy and precision.
- Ans:
- Accuracy: Accuracy refers to the closeness of a measured value to a standard or known value.
- Precision: The quality of being exact and accurate is called precision.

Numerical Errors

Bisection Method

- Describe the bisection method for finding root of equation $f(x)=0$.
- Ans: If the function $f(x)=0$ is continuous between a and b, and f(a) and f(b) are off opposite signs, then there exists at least one root between a and b. Let f(a) is negative and f(b) is positive for definiteness. Then the root lies between a and b and let its approximate value be given by $x_0=(a+b)/2$. If $f(x_0)=0$, we conclude that x_0 is a root of the equation f(x). Otherwise, the root lies between either x_0 and b or between x_0 and a depending on whether $f(x_0)$ is negative or positive. Now we designate new interval $[a_1,b_1]$ whose length is $|a-b|/2$. As before this is bisected at x_1 and the new interval will be exactly half of the length of the previous one. We repeat that process until the latest interval is as small as desired. At the end of the process which is the bisected value is the root of the equation $f(x)=0$.

- What are the merits and demerits of bisection method?
- Ans:
- Merits:
  - Simple and easy to implement.
  - One function evaluation per iteration.
  - The size of the interval is reduced after each iteration.
  - The function does not have to be differentiable.
- Demerits:
  - Slow convergence rate.
  - It is unable to detect multiple roots.
  - It takes so many iterations.
  - It requires a fixed accuracy level called by tolerance.

Bisection Method

- Find out Newton's Forward Difference Interpolation Formula.
- Ans:

Given the set of (n+1) values are, $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$, of x and y. It is required to find $y_n(x)$, a polynomial of n^{th} degree such that y and $y_n(x)$ agree at the tabulated points. Let the value of x be equidistant, i.e. let,

$$x_i = x_0 + ih, \qquad i = 0, 1, 2, 3, \ldots\ldots, n.$$

Since $y_n(x)$ is a polynomial of n^{th} degree, it may be written as,

$$y_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \ldots\ldots\ldots\ldots + a_n(x-x_0)(x-x_1)(x-x_2)\ldots\ldots(x-x_{n-1}) \text{ \_\_\_\_\_\_\_\_(i)}$$

Imposing now the condition that y and $y_n(x)$ should agree at the set of tabulated points, we obtain,

$$A_0 = y_0;\ a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h};\ a_2 = \frac{\Delta^2 y_0}{h^2 2!};\ a_3 = \frac{\Delta^3 y_0}{h^3 3!};\ \ldots\ldots\ldots\ldots;\ a_n = \frac{\Delta^n y_0}{h^n n!};$$

Setting $x = x_0 + ph$ and substituting $a_0, a_1, a_2, \ldots\ldots$, an from (i) we get,

$$y_n(x) = y_0 + p\Delta y_0 + = \frac{p(p-1)\Delta^2 y_0}{2!} + \frac{p(p-1)(p-2)\Delta^3 y_0}{3!} + \ldots\ldots\ldots\ldots\ldots + \frac{p(p-1)(p-2)\ldots\ldots(p-n+1)\Delta^n y_0}{n!}$$

This is Newton's Forward Difference Interpolation Formula.

Newton's Interpolation

Curve Fitting For Polinomial

- Find out the equation of n degree for curve fitting by polynomial.
- Ans:

Let the polynomial of the n^{th} degree,

$$Y=a_0+a_1x+a_2x^2+a_3x^3+ \ldots\ldots +a_nx^n$$

Be fitted to the data points (xi, yi), i=1,2,3, ..., m. Then we have,

$$S=[y_1-(a_0+a_1x_1+a_2x_1{}^2+ \ldots\ldots +a_nx_1{}^n]^2+[y_2-(a_0+a_1x_2+a_2x_2{}^2+ \ldots\ldots +a_nx_2{}^n]^2+ \ldots\ldots\ldots +[y_m-(a_0+a_1x_m+a_2x_m{}^2+ \ldots\ldots +a_nx_m{}^n]^2$$

Equating to zero the first partial derivatives and simplifying, we obtain the normal equation:

$$ma_0+a_1\Sigma x_i+a_2\Sigma x_i{}^2+ \ldots\ldots +a_n\Sigma x_i{}^n=\Sigma y_i$$

$$a_0\Sigma x_i+a_1\Sigma x_i{}^2+a_2\Sigma x_i{}^3+ \ldots\ldots +a_n\Sigma x_i{}^n=\Sigma x_iy_i$$

$$\vdots$$

$$a_0\Sigma x_i{}^n+a_1\Sigma x_i{}^{n+1}+ \ldots\ldots +a_n\Sigma x_i{}^{2n}=\Sigma x_i{}^ny_i$$

Where the summations are performed from i=1 to i=m.

The system (ii) constitutes (n+1) equations with (n+1) unknown, and hencecan be solved for a0, a1, ..., an. Then equation (i) gives the required polynomial of n degree.

Gauss Sheidel Method

- Explain Gauss Sheidel Method for solution of linear system.
- Ans: We wish to solve Laplace's equation

$$u_{xx}+u_{yy}=0$$

in a bounded region R with boundary C. Let R be a square region so that it can be divided into network of small squares of side h. Let the values of u(x,y) on the boundary C be given by c_i. The aaproximate function values at the interior mesh points can now be computed according to the scheme, we first use the diagonal five-point formula and compute u_5, u_7, u_9, u_1 and u_3 in this order. Thus we obtain,

$$u_5=1/4(c_1+c_5+c_9+c_{13}) \qquad u_7=1/4(c_{15}+u_5+c_{11}+c_{13})$$
$$u_9=1/4(u_5+c_7+c_9+c_{11}) \qquad u_1=1/4(c_1+c_3+u_5+u_{15}) \qquad u_3=1/4(c_3+c_5+c_7+u_5)$$

we then compute u_8, u_4, u_6 and u_2 by the standard five-point formula. Thus we have,

$$u_8=1/4(u_5+u_9+c_{11}+u_7) \qquad u_4=1/4(u_1+u_5+u_7+c_{15}) \qquad u_6=1/4(u_3+c_7+u_9+u_5)$$
$$u_2=1/4(c_3+u_3+u_5+u_1)$$

Now let $u_{i,j}^{(n)}$ denotes the n^{th} iterative value of $u_{i,j}$. Then the iterative formula by Gauss sheidel Method is,

$$u_{i,j}^{(n+1)}=\frac{1}{4}[u_{i-1,j}^{(n+1)}+u_{i+1,j}^{(n)}+u_{i,j-1}^{(n+1)}+u_{i,j+1}^{(n)}]$$

This method is also reffered to as Liebmann's method.

- Solve the following system using Gaussian Elimination Method:

$$2x+y+z=10 \qquad 3x+2y+3z=18 \qquad x+4y+9z=16$$

- Ans:

We first eliminate x from 2nd and 3rd equation. For this we multiply 2nd and 3rd equation by (-2/3) and (-2) respectively and add to 1st equation to get 4th and 5th equation,

$$-1/3y-z=-2 \quad \text{and} \quad -7y-17z=-22$$

Now we elimininate y from 5th equation. For this we multiply 5th equation by (-1/21) and add to 4th equation to get,

$$-4/21z=-20/21 \quad \text{or, } 4z=20 \quad \text{or, } z=5$$

The upper triangular form is therefore given by,

$$2x+y+z=10$$

$$-1/3y-z=-2$$

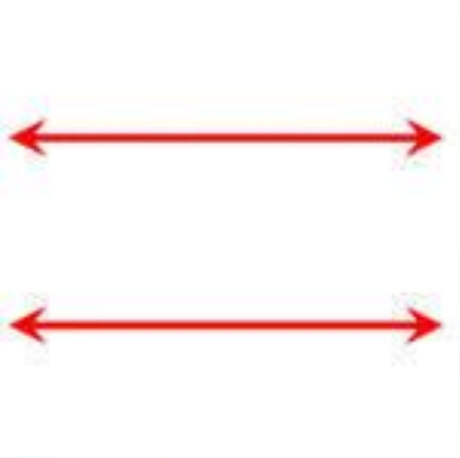
$$z=5$$

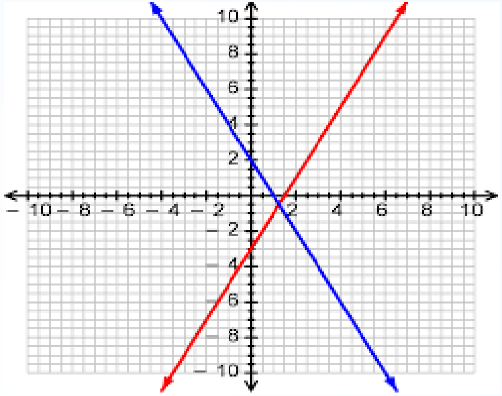
It follows the required solution, x=7, y=-9, z=5.

Gaussian Elimination

Linear Systems

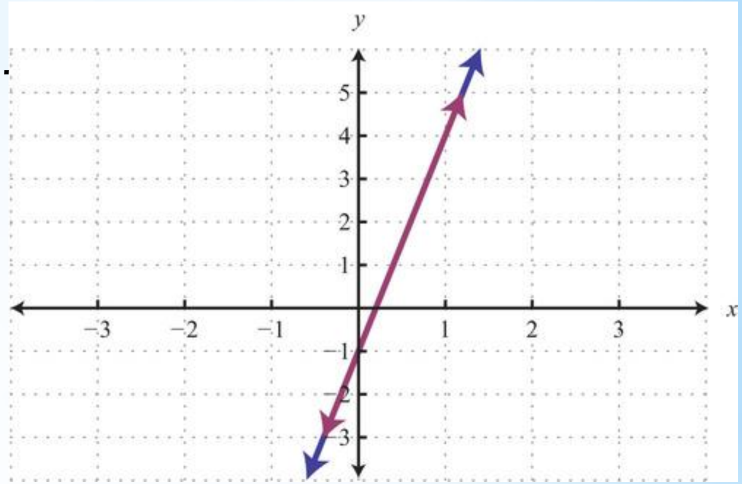
- Clasify system of linear equations and explain them based on graphical representation.
- Ans:
  - Inconsistant: A system of equation that has no solution.

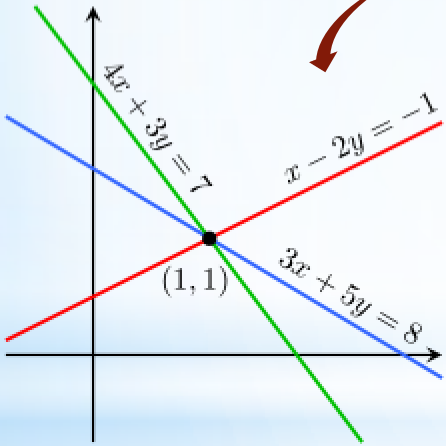




- Consistant: A system of equations that has one or more solutions.

- Dependent: A system of equation that has infinitly many solution.





- Independent: A system of equation that has only one solution.

Linear Systems

Gaussian Elimination

- Solve the following set of simultaneous equations using Gauss Elimination.

  $2x-4y+6z=5$ $\quad$ $x+3y-7z=2$ $\quad$ $7x+5y+9z=4$

- Ans: First multiply 2nd and 3rd equation by (-2) and (-2/7) respectively and add to 1st equation to get 4th and 5th equation,

  $-10y+20z=1$ $\quad$ $-38/7y+24/7z=27/7$ $\quad$ or $-38y+24z=27$

  Then we multiply 5th equation by (-5/19) and add to 4th equation to get,

  $260/19z=-116/19$ $\quad$ or, $260z=116$ $\quad$ or, $z=29/65$

  The upper triangular form therefore,

  $2x-4y+6z=5$

  $-10y+20z=1$

  $z=29/65$

  Solving these equation we get required solutions, $x=293/130$, $y=103/130$, $z=29/65$

- Describe the geometrical meaning of Trapezoidal Rule.
- Ans: The geometrical significance of trapezoidal rule is that the curve $y=f(x)$ is replaced by n straight lines joining the points (x_0, y_0) and (x_1, y_1), (x_1, y_1) and (x_2, y_2), ……, (x_{n-1}, y_{n-1}) and (x_n, y_n). The area bounded by the curve $y=f(x)$, the ordinates $x=x_0$ and $x=x_n$, and the x-axis is then approximately equivalent to the sum of the areas of the n trapeziums obtained.

Trapezoidal Rule

Gaussian Elimination

- Explain Gaussian Elimination method to solve linear system of equation.
- Ans: Let the linear system of equations in n unknowns be given by,

$$a_{11}x_1+a_{12}x_2+\ldots\ldots+a_{1n}x_n=b_1$$
$$a_{21}x_1+a_{22}x_2+\ldots\ldots+a_{2n}x_n=b_2$$
$$\vdots$$
$$a_{n1}x_1+a_{n2}x_2+\ldots\ldots+a_{nn}x_n=b_n$$

There are two steps in gaussian elimination.

Step 1: Eliminate the unknowns to obtain upper triangular system. To eliminate x_1 from2nd equation multiply it by $(-a_{11}/a_{21})$ and add to 1st equation to obtain,

$$(-a_{11}/a_{21})\, a_{22}x_2+\ldots\ldots+ (-a_{11}/a_{21})\, a_{2n}x_n= (-a_{11}/a_{21})\, b_2$$

Let write it,

$$a'_{22}x_2+\ldots\ldots+a'_{2n}x_n=b'_2$$

Similrly eliminate x_1 from all equation except 1st equation.

And by this way eliminate other variable from below equations and get upper triangle.

Now the upper triangle form,

$$a_{11}x_1+a_{12}x_2+\dots\dots\dots\dots+a_{1n}x_n=b_1$$

$$a'_{22}x_2+\dots\dots\dots\dots+a'_{2n}x_n=b'_2$$

$$\vdots$$

$$a'_{nn}x_n=b'_n$$

Step 2: Now solve the equation to get required solutions. From the last equation of the system we obtain,

$$x_n=\frac{b_n^{(n-1)}}{a_n^{(n-1)}}$$

Similarly, we can solve for all unknown.

Gaussian Elimination

Gauss Sheidel Method

- Solve the following equations using Gauss Sheidel Method.

  $10x+2y+z=9$ $\quad$ $2x+20y-2z=-44$ $\quad$ $-2x+3y+10z=22$

- Ans:

  We get from the equations,

  $x=9/10-1/5y-1/10z$ ....(i)

  $y=-11/5-1/10x+1/10z=11/5-1/10(9/10-1/5y-1/10z)+1/10z=211/100+1/50y+11/100z$

  or, $49/50y=211/100+11/100z$

  or, $y=211/98+11/98z$ ....(ii)

  $z=22/10+1/5x-3/10y=22/10+1/5(9/10-1/5y-1/10z)-3/10y=119/50-17/50y-1/50z$

  or, $49/50z=119/50-17/50y$

  or, $49z=119-17y=119-17(211/98+11/98z)=8075/98+187/98z$

  or, $34015/98z=8075/98$

  $z=8075/34015=0.265$

  From (i) and (ii) we get,

  $x=0.473,\ y=2.183,\ z=0.265$

- Derive Lagrange's Interpolation Formula for unequal distance.
- Ans:

Let y(x) be continuous and differentiable (n+1) times in the interval (a, b). Given (n+1) unequally distanced points $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_n, y_n)$. We wish to find a polynomial of degree n, say $L_n(x)$. Such that,

$$L_n(x_i)=y(x_i)=y_i; \qquad i=0,1,2,\ldots\ldots,n$$

Then the polynomial is,

$$L_n(x)=\sum_{i=0}^{n} l_i(x) y_i$$

Where,

$$l_i(x) = \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{i-1})(x-x_{i+1})\ldots\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots\ldots(x_i-x_n)}$$

which obviously satisfies the condition,

$$l_i(x_j) = \begin{cases} 1, if\ i = j \\ 0, if\ i \neq j \end{cases}$$

Lagrange Interpolation

If we set,

$$\prod_{n+1}(x) = (x - x_0)(x - x_1) \ldots \ldots (x - x_{i-1})(x - x_{i+1}) \ldots \ldots (x - x_n)$$

$$\prod'_{n+1}(x_i) = \frac{d}{dx}\left[\prod_{n+1}(x)\right]_{x=x_i} =$$
$$(x_i - x_0)(x_i - x_1) \ldots \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots \ldots (x_i - x_n)$$

So,

$$l_i(x) = \frac{\prod_{n+1}(x)}{(x-x_i)\prod'_{n+1}(x_i)}$$

Hence,

$$L_n(x) = \sum_{i=0}^{n} \frac{\prod_{n+1}(x)}{(x-x_i)\prod'_{n+1}(x_i)} y_i$$

This is lagrange interpolation formula.

Lagrange Interpolation

- Find the value of tan(0.05) from the following data: (0.10, 0.1003), (0.15, 0.1511), (0.20, 0.2027), (0.25, 0.2553), (0.30, 0.3039).
- Ans:

The table of difference is in right:

To find tan(0.05), we have,

0.05=0.10+p(0.05), which gives, p=-1

Hence, according to Newton's forward difference interpolation formula,

$$\tan(0.05) = 0.1003 + (-1)0.0508 + \frac{(-1)(-1-1)}{2}(0.0008) + \frac{(-1)(-1-1)(-1-2)}{6}(0.0002) + \frac{(-1)(-1-1)(-1-2)(-1-3)}{24}(0.0002)$$

$$=0.0503$$

Hence, tan(0.05)=0.0503.

x	y	Δ	Δ^2	Δ^3	Δ^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3039				

Newton's Interpolation

- Define curve fitting. Explain the purpose of it.
- Ans:

  Curve Fitting: Curve fitting is the process of constructing a curve, or mathematical function, that has the best fit to a series of data points, possibly subject to constraints.

  Purpose: Curve fitting, also known as regression analysis, is used to find the "best fit" line or curve for a series of data points. Most of the time, the curve fit will produce an equation that can be used to find points anywhere along the curve.

Curve Fitting

- Describe the least square curve fitting procedure for a straight line.
- Ans:

Let $Y=a_0+a_1x$ be a straight line to be fitted to the given data (x_i, y_i). Then we have,

$$S=[y_1-(a_0+a_1x_1)]^2+[y_2-(a_0+a_1x_2)]^2+\ldots\ldots\ldots+[y_m-(a_0+a_1x_m)]^2$$

For S to be minimum,

$$\frac{dS}{da_0}=0=-2[y_1-(a_0+a_1x_1)]-2[y_2-(a_0+a_1x_2)]-\ldots\ldots\ldots-2[y_m-(a_0+a_1x_m)]$$

and $$\frac{dS}{da_1}=0=-2x_1[y_1-(a_0+a_1x_1)]-2x_1[y_2-(a_0+a_1x_2)]-\ldots\ldots\ldots-2x_1[y_m-(a_0+a_1x_m)]$$

The above equation simplify to,

$$ma_0+a_1(x_1+x_2+\ldots\ldots+x_m)=y_1+y_2+\ldots\ldots.+y_m$$

and $$a_0(x_1+x_2+\ldots\ldots+x_m)+a_1(x_1^2+x_2^2+\ldots\ldots+x_m^2)= x_1y_1+x_2y_2+\ldots\ldots.+x_my_m$$

or more compactly to,

$$ma_0+a_1\sum_{i=1}^{m}x_i=\sum_{i=1}^{m}y_i$$ and $$a_0\sum_{i=1}^{m}x_i+a_1\sum_{i=1}^{m}{x_i}^2=\sum_{i=1}^{m}x_iy_i$$

Now we can easily solve for a_0 and a_1,

$$A^1=\frac{m\sum_{i=1}^{m}x_iy_i-\sum_{i=1}^{m}x_i\sum_{i=1}^{m}y_i}{m\sum_{i=1}^{m}x_i^2-\left(\sum_{i=1}^{m}x_i\right)^2}$$ and $$A_0=\bar{Y}-A_1\bar{X}$$

Curve Fitting

- The exponential function $y=ae^{bx}$ is fitted to the data: (1.0, 40.170), (1.2, 73.196), (1.4, 133.372), (1.6,243.02). Find the value of a and b.
- Ans:

We have,

$y=ae^{bx}$

Therefore,

$\ln y=\ln a+bx \quad \Rightarrow Y=A_0+A_1X$

Where, $Y=\ln y$, $A_0=\ln a$, $A_1=b$ and $X=x$.

The table of values is given right:

We obtain, $m=\bar{X} = 1.3$, $\bar{Y} = 4.593$

Then, $A^1=\frac{m\sum_{i=1}^{m} x_i y_i-\sum_{i=1}^{m} x_i \sum_{i=1}^{m} y_i}{m\sum_{i=1}^{m} x_i{}^2-\left(\sum_{i=1}^{m} x_i\right)^2}$

$=\frac{1.13(24.484)-5.2(18.372)}{1.3(6.96)-27.04} = 3.772$

$A_0=\bar{Y} - A_1\bar{X}$=4.593-4.904=-0.311

$a = e^{A_0} = 0.733 \quad b = 3.772$

X	Y=lny	X²	XY
1.0	3.693	1.0	3.693
1.2	4.293	1.44	5.152
1.4	4.893	1.96	6.850
1.6	5.493	2.56	8.789
5.2	18.372	6.96	24.484

Curve Fitting

- The exponential function $y=ae^{bx}$ is fitted to the data: (0, 0.10), (0.5, 0.45), (1.0,2,15), (1.5, 9,15), (2.0, 40.35), (2.5, 180.75). Find the value of a and b.
- Ans:

We have,

$y=ae^{bx}$

Therefore,

$\ln y=\ln a+bx \quad \Rightarrow Y=A_0+A_1X$

Where, $Y=\ln y$, $A_0=\ln a$, $A_1=b$ and $X=x$.

The table of values is given right:

We obtain, $m=\bar{X} = 1.25$, $\bar{Y} = 1.445$

Then, $A^1=\frac{m\sum_{i=1}^{m} x_i y_i-\sum_{i=1}^{m} x_i \sum_{i=1}^{m} y_i}{m\sum_{i=1}^{m} x_i^2-\left(\sum_{i=1}^{m} x_i\right)^2}$

$=\frac{1.25(24.025)-7.5(8.672)}{1.25(13.75)-56.25} = 0.896$

$A_0=\bar{Y} - A_1\bar{X}$=1.445-1.12=0.325

$a = e^{A_0} = 1.384 \quad b = 0.896$

X	Y=lny	X^2	XY
0	-2.303	0	0
0.5	-0.898	0.25	-0.449
1.0	o.765	1.0	0.765
1.5	2.214	2.25	3.321
2.0	3.698	4.0	7.396
2.5	5.197	6.25	12.992
7.5	8.672	13.75	24.025

Curve Fitting