MATH 2241

Subject : 2018.

Date :

1.(a) Define vector space and it's subspace with example. Let $V = \mathbb{R}^3$ and let $W = \{(x,y,z) : x+y+z = 0\}$. Prove that W is subspace of V.

Vector space: ~~Let V be a non-empty se~~ The following defines the notion of a vector space V where K is the field of scalars. Let V be a non-empty set with two operations.

(i) Vector Addition: This assigns to any $u, v \in V$ a sum $u+v$ in V.

(ii) Scalar multiplication: This assigns to any $u \in V$, $k \in K$ a product $ku \in V$.

Then V is called vector space over the field K.

Subspace: Let V be a vector space over a field K and let W be a subset of V. Then W is a subspace of V if W is itself a vector space over K with respect to the operation of vector addition and scalar multiplication on V.

$$-b^2 \; \frac{(-b) \pm \sqrt{b^2-4ac}}{2a} = 0$$

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2 (b)(c) Show that $(1,1,1)$, $(0,1,1)$ and $(0,1,-1)$ generate $\mathbb{R}^3$.

Ans: In order to show that $(1,1,1)$, $(0,1,1)$ and $(0,1,-1)$ generates $\mathbb{R}^3$, we have to show that any vector $\alpha \in \mathbb{R}^3$ is a linear combination of other three vectors (given).

Let $\alpha \in \mathbb{R}^3$

$\alpha = (a,b,c)$

$$\alpha = a_1(1,1,1) + a_2(0,1,1) + a_3(0,1,-1)$$

$a_1, a_2, a_3 \in \mathbb{R}^3$

A U
AV - echelon form

$a = a_1$ ——— (i)

$b = a_1 + a_2 + a_3$ — (ii)

$c = a_1 + a_2 - a_3$ — (iii)

(i) $\Rightarrow a_1 = a$

(ii) - (iii) $\Rightarrow b - c = \cancel{a_1} + \cancel{a_2} + a_3 - \cancel{a_1} - \cancel{a_2} + a_3$

$\Rightarrow b - c = 2a_3$

$\Rightarrow a_3 = \frac{b-c}{2}$

(ii) + (iii) $\Rightarrow b + c = a_1 + a_2 + \cancel{a_3} + a_1 + a_2 - \cancel{a_3} = 2a + 2a_2$

$b + c - 2a = 2a_2$

$\therefore a_2 = \frac{b+c-2a}{2}$

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Thus, the system (i), (ii) & (iii) is consistent and has a solution. Hence $(1,1,1)$, $(0,1,1)$ and $(0,1,-1)$ generates $\mathbb{R}^3$

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Given, $V = \mathbb{R}^3$ and let $W = \{(x,y,z) : x+y+z=0\}$
Prove that W is subspace of V.

Clearly $(0,0,0) \in W$. $\therefore$ W is a non-empty subspace of $\mathbb{R}^3$

Let $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2) \in W$

Then $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$

$\therefore \alpha + \beta = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in \mathbb{R}^3$

Now, $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = 0$

$\therefore \alpha + \beta \in W$

Thus, $\alpha, \beta \in W$, $\alpha + \beta \in W$.

Again, let $c \in \mathbb{R}$, $c\alpha = (cx_1, cy_1, cz_1) \in \mathbb{R}^3$

Now, $cx_1 + cy_1 + cz_1 = 0$.

$\therefore c\alpha \in W$.

Thus $c \in W, \alpha \in W$, $c\alpha \in W$

Thus $\alpha, \beta \in W$, $\alpha + \beta \in W$

and $c \in \mathbb{R}$, $\alpha \in W$, $c\alpha \in W$

Thus W is a subspace of Real vector space $\mathbb{R}^3$.

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Q1(b) Prove that vector space V is the direct sum of it's subspace U and W, if and only if
(i) $V = U + W$ and (ii) $U \cap W = \{0\}$

~~Proof~~: Direct Sum: A vector space V is said to be direct sum of it's two subspaces if U and W if every $v \in V$ can be uniquely expressed as
$v = u + w$. where $u \in U$ & $w \in W$

Proof: The conditions are necessary:
Let $V = U \oplus W$ [direct sum]
$\therefore V = U + W$ [linear sum]

If possible, if $0 \neq v \in U \cap W$
so, $v \in U$ & $v \in W$
also $v \in V$
$v = 0 + v$ where $0 \in U$, & $v \in W$
$v = v + 0$ when $v \in U$ & $0 \in W$
So, v can be expressed in two different ways. (এখানে contradiction হচ্ছে ~~কম~~ এসেছে। direct sum এ uniquely express করতে হবে, but এখানে ২টা form ~~এর~~ এ আছে).
Hence, the only vector which is common to both U & W is 0. Hence $U \cap W = \{0\}$

The conditios are sufficient:
Let, $V = U + W$ —(1) & $U \cap W = \{0\}$ —(ii)

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Required to prove $V = U \oplus W$ (uniquely expre… প্রকাশ হবে)

Now (1) implies that every element of V can be expressed as sum of an element of U, & an element of W.

We shall prove that this expression is unique if possible. Let $v = u + w$, $u \in U$ & $w \in W$

& $v = u' + w'$; $u' \in U$ & $w' \in W$

$\Rightarrow u + w = u' + w'$

$\Rightarrow u - u' = w' - w$

Since U is a subspace

$\therefore u, u' \in W \Rightarrow u - u' \in W$

Similarly, $w, w' \in W \Rightarrow w - w' \in W$.

$\Rightarrow u - u' = w' - w \in U \cap W$

$\Rightarrow u - u' = 0$ & $w' - w = 0$

$\Rightarrow u = u'$ & $w' = w$

This shows that expression for each $v \in V$ is unique. Hence V is direct sum of U & W.

$V = U \oplus W$

[Proved]

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2017 1(a) If U and W be subspace of vector space V. Show that $U \cap W$ is a subspace of V.

X is subspace of V if
(1) $X \neq \phi$
(2) $\vec{x}, \vec{y} \in X \Rightarrow \vec{x}+\vec{y} \in X$
(3) $\vec{x} \in X$, $c \in F \Rightarrow c\vec{x} \in X$

Proof: Suppose, U, W are subspace of V. Since $\vec{0} \in U$ and $\vec{0} \in W$

So, $\vec{0} \in U \cap W$, hence $U \cap W = \Phi$

Take any $\vec{x}, \vec{y} \in U \cap W$. This means $\vec{x}, \vec{y} \in U$ and $\vec{x}, \vec{y} \in W$

Then $\vec{x}+\vec{y} \in U$ and $\vec{x}+\vec{y} \in W$.

Hence, $\vec{x}+\vec{y} \in U \cap W$. Take any $\vec{x} \in U \cap W$ and $c \in F$. Then $\vec{x} \in U$, $c \in F$ and $\vec{x} \in W$, $c \in F$ then $c\vec{x} \in U$ and $c\vec{x} \in W$

So, $c\vec{x} \in U \cap W$.

Thus, $U \cap W$ is a a subspace.

1(b) Let U and W be the subspace of $V = \mathbb{R}^3$ defined by $U = \{(a,b,c) \mid a=b=c\}$ and $W = \{(0,b,c)\}$. Show that $V = U \oplus W$

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1 (b) Define Subspace of a vector space with example. Show that the intersection of any two subspace is also a subspace.

Subspace: Let W be a non-empty subset of vector space V over the ~~fie~~ field F, We call W a subspace of V if and only if W is a vector space over the field F under the laws of vector addition and scalar multiplication defined on V, or equivalently, W is a subspace of V whenever $w_1, w_2 \in W$, $\alpha, \beta \in F$ implies that

$\alpha w_1 + \beta w_2 \in W$.

Example: Consider vector space $\mathbb{R}^2 = \{(a,b) | a, b \in \mathbb{R}\}$
Then $W_1 = \{(a,0) | a \in \mathbb{R}\}$; $W_2 = \{(0,b) | b \in \mathbb{R}\}$
and $W_3 = \{(a,b) | a = b$ and $a, b \in \mathbb{R}$ are subspace of $\mathbb{R}^2$.

W_1 and W_2 represent the sets of all points on x-axis and y-axis respectively.
Also W_3 represents the set of all points on the line $y = x$.

Let S and T be two subspaces of a vector space V. We have to prove the intersection of S & T is also a subspace of V.

Since S and T are subspace of V, they are non-empty and Clearly $0 \in S$ and $0 \in T$. Therefo $0 \in S, T$ and hence $S \cap T \neq \phi$

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Now let $u, v \in S \cap T$ the $u, v \in S$ and and $u, v \in T$.
Since S and T are subspace of V, $u, v \in S$ implies $\alpha u + \alpha v \in S$ where $\alpha, \beta \in F$. Similarly $u, v \in T$ that implies $\alpha u + \beta v \in T$ where $\alpha, \beta \in F$.

Hence $u, v \in S \cap T$ implies $\alpha u + \alpha \beta \in S \cap T$ for $\alpha, \beta \in F$
Therefore $S \cap T$ is subspace of vector space V.

(c) Determine whether or not W is a subspace of $\mathbb{R}^3$ if
(i) $W = \{(a, b, 0) : a, b \in \mathbb{R}\}$ (ii) $\{(a, b, c) : a^2 + b^2 + c^2 \leq 1\} = W$.

?

(i) For $0 \in \mathbb{R}^3$, $0 = (0,0,0) \in S$
Since the third component of 0 is 0; and so
W is a non-empty set.
For any vectors $u = (a, b, 0)$ and $v = (a', b', 0)$ in W and any scalars (real numbers) α, β we have

$$\alpha u + \beta v = \alpha(a, b, 0) + \beta(a', b', 0)$$
$$= (\alpha a, \alpha b, 0) + (\beta a', \beta b', 0)$$
$$= (\alpha a + \beta a', \alpha b + \beta b' + 0)$$

Since the 3rd component is zero.
Thus $\alpha u + \beta v \in W$ and so W is a subspace of $\mathbb{R}^3$.

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(iii) ~~Let~~ $\omega = \{(a,b,c) : a^2+b^2+c^2 \le 1\}$ ~~= W~~.

Let, $u = (0,1,0)$, $v = (0,0,1)$

Then $u \in W$ since $0^2+1^2+0^2 = 1 \le 1$

and $v \in W$ since $0^2+0^2+1^2 = 1 \le 1$

Now for any two scalars $\alpha = \beta = 1 \in \mathbb{R} = F$

$$\alpha u + \beta v = 1(0,1,0) + 1(0,0,1)$$
$$= (0,1,0) + (0,0,1)$$
$$= (0,1,1) \notin W$$

Since $0^2 + 1^2 + 1^2 = 2 \nleq 1$

Hence w is not a subspace of V,

2019
2(a) Define pivot position and pivot column in a matrix.

Pivot position: A position of a leading entry in an echelon form of the matrix..

A pivot position in a matrix, A, is a position in the matrix that corresponds to a row-leading 1 in the reduced row echelon form of A.

Pivot Column: If a matrix is in row-echelon form, then the first non-zero entry of each row is called a ~~piv~~ pivot, and the column in which pivot appears are called pivot column.

$$\text{P. P.} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

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2(c) (3) Let $u=(1,-2,3)$, $v=(5,-13,-3)$ $b=(-3,8,1)$. Span $\{u,v\}$ is a plane through the origin in $\mathbb{R}^3$ (0,0,0). Is b in the plane.

[Span of a set of vectors is the collection of all possible linear combination of those vectors.]

Sol We reduce the vectors u, v and b by using the argument matrix $[u \ v \ b]$:

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \quad \begin{matrix} R_3' = R_3 - 3R_1 \\ R_2' = R_2 + 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix} \quad R_3' = R_3 - 6R_2$$

The third equation is $0=-2$, which shows that the system has no solution. The vector equation $x_1u + x_2v = b$ has no solution, and so b is not in Span $\{u, v\}$.

Trivial Soln: very simple & straightforward solution Single soln.

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2.(a) Define linearly independent set. Let V be the vector space of 2×2 matrices over $\mathbb{R}$. Are the matrices $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}$, $B=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $C=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ linearly dependent? (3)

LIS: An indexed set of vectors $\{v_1,\dots v_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation $x_1v_1+x_2v_2+\dots+x_pv_p=0$ has only the trivial solution.

LDS: The set $\{v_1,v_2,\dots,v_p\}$ is said to be linearly dependent if there exist weights $c_1,\dots c_p$, not all zero, such that,
$c_1v_1+c_2v_2+\dots+c_pv_p=0$.

$$span(W)=\begin{pmatrix}1&1\\1&1\end{pmatrix}x_1+\begin{pmatrix}1&0\\0&1\end{pmatrix}x_2+\begin{pmatrix}1&1\\0&1\end{pmatrix}x_3$$

$$=\begin{pmatrix}x_1&x_1\\x_1&x_1\end{pmatrix}+\begin{pmatrix}x_2&0\\0&x_2\end{pmatrix}+\begin{pmatrix}x_3&x_3\\0&x_3\end{pmatrix}$$

$$=\begin{pmatrix}x_1+x_2+x_3&x_1+x_3\\x_1&x_1+x_2+x_3\end{pmatrix}$$

$$=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$x_1+x_2+x_3=0$
$x_1+0+x_3=0$
$x_1+0+0=0$
$x_1+x_2+x_3=0$

$$\left[\begin{array}{ccc|c}1&1&1&0\\1&0&1&0\\1&0&0&0\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&1&1&0\\0&-1&0&0\\0&-1&-1&0\end{array}\right] \quad R_2'=R_2+(-1)R_1,\ R_3'=R_3+(-1)R_1$$

$$\sim\left[\begin{array}{ccc|c}1&1&1&0\\0&1&0&0\\0&-1&-1&0\end{array}\right] \quad R_2'=(-1)R_2$$

$$\sim\left[\begin{array}{ccc|c}1&1&1&0\\0&1&0&0\\0&0&-1&0\end{array}\right] \quad R_3'=R_3+R_2$$

$$\sim\left[\begin{array}{ccc|c}1&1&1&0\\0&1&0&0\\0&0&1&0\end{array}\right] \quad R_3'=(-1)R_3$$

$$\sim\left[\begin{array}{ccc|c}1&0&1&0\\0&1&0&0\\0&0&1&0\end{array}\right] \quad R_1'=R_1+(-1)R_2$$

$$\sim\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right] \quad R_1'=R_1+(-1)R_3$$

Therefore set W is linearly independent since we get indentity matrix I_3 & so each component has unique value $x_1=0, x_2=0$, $x_3=0$

[এখানে যদি ~~has~~ infinite soln হয় then এটা dependent হত]

$$\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&0&0\end{array}\right] \rightarrow$$ last line causes infinite soln

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2(b) Define basis and dimension of vector space. Determine whether or not (1,1,2), (1,2,5) and (5,3,4) form a basis for the vector space $\mathbb{R}^3$ (3)

Basis: A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that span's H. OR, A set S = $\{u_1, u_2, \dots u_n\}$ of vectors is a basis of V if it has the following two properties: (1) S is linearly independent. (2) S spans V.

The elements ~~in Ro~~ forming a matrix whose rows are the given vectors.

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix}$$

$R_2 = R_2 - R_1$
$R_3 = R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \begin{matrix} R_2' = R_2 + R_1(-1) \\ R_3' = R_3 + (-5)R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} R_3' = R_3 + 2R_2$$

~~The last row~~ The echelon matrix has a zero row, hence the three vectors are linearly dependent and so they do not form a basis of $\mathbb{R}^3$.

Dimension: The dimension of a nonzero subspace H, denoted by dim H, is the number of vectors in any basis for H. The dimension of the zero subspace {0} is defined to be zero. The space $\mathbb{R}^n$ has dimension n. Every basis for $\mathbb{R}^n$ consist of n vectors. A plane through 0 in $\mathbb{R}^3$ is two-dimensional, and a line through 0 is one-dimentional.

Let V be a vector space such that one basis has m elements and another basis has n elements. Then m = n. A vector space V is said to be finite dimensional n or n-dimensional, written dim V = n if V has a basis with n elements.

2 ⓒ Let U and W be the following subspace of $\mathbb{R}^4$: $U = \{(a,b,c,d) : b+c+d=0\}$, $W = \{(a,b,c,d) : a+b=0,\ c=2d\}$. Find dimension and basis of (i) U and (ii) W. (2.75)

(i) Given,

$U = \{(a,b,c,d) : b+c+d=0\}$

$= \{(a,-c-d,c,d)\}$ $[b=-c-d]$

$= \{(a,0,0,0)\} + \{(0,-c,c,0)\} + \{(0,-d,0,d)\}$

$= a(1,0,0,0) + c(0,-1,1,0) + d(0,-1,0,1)$

$\therefore$ The dimension of U. $\dim(U) = 3$.

The Basis of U $\{(1,0,0,0),(0,-1,1,0),(0,-1,0,1)\}$

(ii) ~~W~~ Given,

$W = \{(a,b,c,d) : a+b=0, c=2d\}$

$= \{(a,b,2d,d) \mid a+b=0\}$

$= \{(-b,b,2d,d)\}$

$= \{(-b,b,0,0)\} + \{(0,0,2d,d)\}$

$= b(-1,1,0,0) + d(0,0,2,1)$

$\therefore$ The dimension of W: $\dim(W) = 2$

The Basis of W $\{(-1,1,0,0),(0,0,2,1)\}$

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(b) Let U and W be subspace of $\mathbb{R}^4$ generated by $\{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$ and $\{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$ respectively. (3)

Find dim (U+W)

Ans: (b) U+W is subspace spanned (or generated) by all given six vectors. Hence form the matrix whose rows are given six vectors and then reduce this matrix to row echelon form by elementary row operation.

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \quad R_2' = R_2 + (-1)R_1$$

~~This matrix is in~~

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix} \begin{matrix} R_2' = R_2 - R_1 \\ R_4' = R_4 - R_1 \\ R_6' = R_6 - R_1 \\ R_3' = R_3 - 2R_1 \\ R_5' = R_5 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix} \begin{matrix} R_3' = R_3 - R_2 \\ R_4' = R_4 - R_2 \\ R_5' = R_5 - R_2 \\ R_6' = R_6 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_5' = R_5 + (-1)R_4 \\ R_6' = R_6 + (-2)R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form having three non-zero $(1,1,0,-1)$, $(0,1,3,1)$ and $(0,0,-1,-2)$ which will form a basis of U+W & thus dim(U+W) = 3.

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2(a) Let $V=\{at^2+bt+c \mid a,b,c \in \mathbb{R}\}$ be vector space. The polynomials $e_1=1$, $e_2=t-1$ and $e_3=(t-1)^2$ form a basis of V. Let $v=2t^2-5t+6$, Find $[v]_e$.

$$\mathcal{E}=\{e_1, e_2, e_3\} \quad V=\{at^2+bt+c\}$$
$$v=\{2t^2-5t+6\}$$

$$v = x e_1 + y e_2 + z e_3$$
$$2t^2-5t+6 = x(1) + y(t-1) + z(t^2-2t+1)$$
$$= x + yt - y + zt^2 - 2zt + z$$
$$= zt^2 + (y-2z)t + (x-y+z).$$

$\therefore$ $z = 2$

$y - 2z = -5$

$x - y + z = 6.$

~~or~~

$y = -5 + 2z = -5 + 4 = -1$

$x = 6 + y - z = 6 - 1 - 2 = 3$

$\therefore x = 3, y = -1, z = 2$

Thus, $v = 3e_1 - e_2 + 2e_3$

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2(c) Let $W=\{(a,b,c,d) \mid a=d, b=2c\}$ be subspace of $\mathbb{R}$ 2·7

Find a basis and dimension of W.

$\{a,b,c,d\}$

$=\{a,2c,c,a\}$

$=\{(a,0,0,a)+(0,2c,c,0)\}$

$=\{a(1,0,0,1)+c(0,2,1,0)\}$

$\therefore$ Dimension $(W)=2$

Basis $=\{(1,0,0,1),(0,2,1,0)\}$

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Subject : Subspace definition Date :

A subspace of a vector space V is a subset H of V ~~that~~ has three properties:

(a) The zero vector of V is in H^2.

(b) H is closed under vector addition. That is for each u and v in H, the sum $u+v$ is in H.

(c) H is closed under multiplication by scalars. That is, for each u in H and each scalar c, the vector cu is in H.

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3. a) Given the system $x_1+2x_2-x_3=4$ (2)

$-5x_2+3x_3=1$

Write the system as a matrix times a vector form i.e. $Ax=b$ form.

Solⁿ Given, $1x_1+2x_2-x_3=4$ — (1)

$0x_1-5x_2+3x_3=1$ — (2)

The two equations has 3 variables x_1, x_2 & x_3.

$$\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ -5x_2 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 3x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad — (3)$$

The equation (3) has the form $Ax=b$. This equation is called a matrix equation.

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3(c) Let $u_1=(1,2,3)$; $u_2=(4,5,6)$ and $u_3=(2,1,0)$ Determine if the set $\{u_1,u_2,u_3\}$ is linearly independent. (2)

Soln The matrix form of the given sets:

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\2 & 1 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\\0 & -3 & -6\end{bmatrix} \quad R_2'=R_2+(-4)R_1,\quad R_3'=R_3+(-2)R_1$$

$$\sim\begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\\0 & 0 & 0\end{bmatrix} \quad R_3'=R_3+(-1)R_2$$

$$\sim\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix} \quad R_2'=\left(\frac{1}{-3}\right)R_2.$$

$$\begin{matrix}1 & 4 & 2\\2 & 5 & 1\\3 & 6 & 0\end{matrix}$$

$$\sim\begin{bmatrix}1 & 4 & 2\\0 & -3 & -3\\0 & -6 & -6\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 4 & 2\\0 & 1 & 1\\0 & -6 & -6\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 4 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

∴ There is one row of 0 values. So, it has infinite solution. So, the ~~matrix~~ set is linearly dependent.

3b) Define linear mapping with example. Let $F: V \to U$ be a linear mapping. Show that the kernel of F is a subspace of V. (3)

Sol^n Let U and V be two vector spaces over the same field F. ~~doo~~. Then the linear mapping from U to V is a function that preserves vector addition and scalar multiplication. ~~The~~ linear mapping is also known as linear transformation or vector space homomorphism. The linear mapping ~~T~~ of U into V, written as $T: U \to V$ is a transformation of U into V such ~~tat~~ that

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$ for all $u_1, u_2 \in U$

(ii) $T(\alpha u) = \alpha T(u)$ for all $u \in U$ and all $\alpha \in F$

Example: The zero map $0: V \to W$ mapping every element $v \in V$ to $0 \in W$ is linear.

The identity map $I: V \to V$ defined as $Iv = v$ is linear.

Let, $F: V \to U$ be a linear mapping. To show that ker(F) is a ~~bed~~ subspace of V, we must show that it is closed under vector addition and scalar multiplication.

Since $F(0) = 0$, $0 \in \ker(F)$

Let $x, y \in \ker(F)$ and let α be any scalar.

Then $F(x) = 0$, $F(y) = 0$.

Thus $F(x+y) = F(x) + F(y) = 0 + 0 = 0$.

So that $x+y \in \ker(F)$

Also $F(\alpha x) = \alpha F(x) = \alpha . 0 = 0$ so that $\alpha x \in \ker(F)$

Hence, the kernel of F is a subspace of V.

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3(c) Let $F: \mathbb{R}^3 \to \mathbb{R}$ be defined by $(x, y, z) = 8x - 3y + 5z$. Is the map F linear. 2.5

Soln: Let, $u = (x, y, z)$ and $v = (x', y', z')$ be any two elements of spac $\mathbb{R}^3$ and α be scalar belonging to $\mathbb{R}$, then we see that,

$$F(u+v) = F((x+x', y+y', z+z')) = 8(x+x') - 3(y+y') + 5(z+z')$$

$$= (8x - 3y + 5z) + (8x' - 3y' - 5z') = F(u) + F(v) \text{ and}.$$

$$F(\alpha u) = F((\alpha x, \alpha y, \alpha z)) = 8(\alpha x) - 3(\alpha y) + 5(\alpha z)$$

$$= \alpha(8x - 3y + 5z) = \alpha . F(u)$$

$\therefore$ F is linear.

2017 3(a) Define linear mapping and kernel. Let f and g be defined by $F(x) = 2x+1$ and $g(x) = x^2 - 2$. Find $(gof)(4$ and $(fog)(4$ Also find formula for fog and gof.

Linear Mapping: A linear map is a mapping $V \to W$ betw- two vector spaces that preserves the operations of vec addition and scalar multiplication.

Kernel: Let T be a linear transformation of U int V. Then the kernel (or null space) of T is the subset of all $x \in U$ for which $T(x) = 0$ wher of U consisting $0 \in V$. The kernel of T is generally denoted by ker(T The range of T is the subset of V consisting of all $y \in$ such that $T(x) = y$ for all $x \in U$. It is generally deno by $R(T)$.

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$(g\circ f)(4) = g(f(4))$. $f(x) = 2x+1$ $g(x) = x^2-2$

$f(4) = 2.4+1 = 9$ $g(f(4)) = 9^2-2 = 79$

$(f\circ g)(4) = f(g(4))$

$g(4) = 4^2-2 = 14$ $f(g(4)) = 2.14+1 = 29$

$g(x) = x^2-2$ $f(g(x)) = 2.(x^2-2)+1 = 2x^2-4+1 = 2x^2-3$

$f(x) = 2x+1$ $g(f(x)) = (2x+1)^2-2 = 4x^2+4x+1-2$
$= 4x^2+4x-1$

2017
3(b) Define image of a linear mapping. Let $F: V\to U$ be a linear mapping. Show that image of F is a subspace of U. (3)

Image of linear mapping: Let V and W be vector spaces and let $T: V\to W$ be a linear transformation. Then the image of T denoted as $im(T)$ is defined to be set $\{T(\vec{v}) : \vec{v}\in V\}$. In ~~words~~ words, it consists of all vectors in W which equal $T(\vec{v})$ for some $\vec{v}\in V$. the kernel $ker(T)$, consists of all $\vec{v}\in V$ such that $T(\vec{v}) = \vec{0}$. That is, $ker(T) = \{\vec{v}\in V : T(\vec{v}) = \vec{0}\}$.
Both $im(T)$ and $ker(T)$ are subspace of W and V respectively.

Let $F: V\to U$ be linear mapping. $u_1, u_2 \in R(F)$; then we have to prove that $u_1+u_2 \in R(F)$ and $\alpha u_1 \in R(F)$ for any scalar α; that is, we must find vectors $v, v' \in V$ such that $F(v) = u_1+u_2$ and $F(v') = \alpha u_1$. Since $u_1, u_2 \in R(F)$, there exist vectors v_1, v_2 in V such that $F(v_1) = u_1$ and $F(v_2) = u_2$.
Let $v = v_1+v_2$ and $v' = \alpha v_1$. Then $F(v) = F(v_1+v_2) =$
$F(v_1)+F(v_2) = u_1+u_2$.
$F(v') = F(\alpha v_1) = \alpha F(v_1) = \alpha u_1$ which completes the proof.

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Suppose $\{v_1, v_2, \dots, v_n\}$ be a basis for a vector space V and $F: V \to U$ be a linear transformation. If we happen to know the image of the basis vectors that is $F(v_1), F(v_2) \dots F(v_n)$ then we can obtain the image $F(v)$ of any vector v by first expressing v in terms of the basis say $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ where α_i are scalars in F. Now if $v_1, v_2, \dots v_n$ are vectors in V and $\alpha_1, \alpha_2, \dots \alpha_n$ are scalar then we can write.

$$F(v) = F(\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n)$$
$$= \alpha_1 F(v_1) + \alpha_2 F(v_2) + \dots + \alpha_n F(v_n)$$

In short, a linear transformation is completely determined by its values at a basis.

3(c) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by $T(x, y, z) = (x + 2y - z,\ y + z,\ x + y - 2z)$. Find a basis and the dimension of (i) Image of T and (ii) Kernel of T

2.5

Given,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] R_3 = R_3 + (-1)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3 = R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 = R_1 + (-2)R_2$$

$x - 3z = 0$
$y + z = 0$
$z = 0$

$x = 3s$
$y = -s$
$z = s$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \therefore \text{Ker}(T) \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \text{basis}$$

~~basis~~ $\dim(T) = 1$.

$$\text{im}(T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \cancel{\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}} \right\}$$

Leading pivot in x & y.

$$\therefore \text{im}(T) \text{ basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\dim(\text{im}(T)) = 2$$

Ans

2016 Subject : .... Date :

~~3(a) Define Image of linear mapping~~

3(b) Find $T(a,b,c)$ where $T : \mathbb{R}^3 \to \mathbb{R}$ is defined by (3)
$T(1,1,1) = 3$, $T(0,1,-2) = 1$, and $T(0,0,1) = -2$

Sol^n Given, $T(1,1,1)$, $T(0,1,-2)$, $T(0,0,1) \in \mathbb{R}^3$

To find $(T[a,b,c])$ for any $[a,b,c] \in \mathbb{R}^3$, we now reduce,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 1 & 1 & 0 & b \\ 1 & -2 & 1 & c \end{array}\right] \qquad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b-a \\ 0 & -2 & 1 & c-a \end{array}\right] \begin{array}{l} R_2' \Rightarrow R_2 + (-1)R_1 \\ R_3' \Rightarrow R_3 + (-1)R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b-a \\ 0 & 0 & 1 & c+2b-3a \end{array}\right] R_3' \Rightarrow R_3 + 2R_2$$

$c-a+2b-2a$
$c+2b-3a$

Thus, $[a,b,c] = a v_1 + (b-a) v_2 + (c+2b-3a) v_3$

Subject :.. Date :

16 (3c) Show that mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x,y) = (x+y, x)$ is linear.

2-75

Let, $u = (a,b)$, $v = (a',b')$ & α be any scalar component.

$$F(u+v) = F(a+a', b+b') = (a+a'+b+b', a+a')$$
$$= (a+b, a) + (a'+b', a')$$
$$= F(a,b) + F(a',b')$$
$$= F(u) + F(v)$$

$$F(\alpha \vec{u}) = F(\alpha\{a,b\})$$
$$= F(\alpha a, \alpha b)$$
$$= (\alpha a + \alpha b, \alpha a)$$
$$= \alpha(a+b, a)$$
$$= \alpha F(u)$$

$\therefore F: \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

2015 3(a) Suppose $F: V \to U$ and $G: V \to U$ are linear mapping over a field K. The mappings $F+G: V \to U$ and $KF: V \to U$ [illegible] defined by $(F+G)(u) = F(V) + G(V)$ and $(kF)(v)$ [illegible] $kF(v)$, $k \in K$. Show that $(F+G)$ and (KF) are linear.

Digital Products

Subject: ~~2019~~ 2018 Date:

7(a) ~~Define inner product~~. Verify inner product $\mathbb{R}^2$

$\langle u,v\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ $u=(x_1,x_2)$

$v=(y_1,y_2)$

Given,

$\langle u,v\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$

(i) We take,

$\langle v,v\rangle = y_1y_1 - y_2y_2 - y_2y_1 + 3y_2y_2 \ge 0$

$= y_1y_1 - y_2y_1 + 2y_2y_2 = 0$ if & only

if $y_1=0$ & $y_2=0$.

~~(i) $\langle u,v\rangle = 3$~~. that is, if and only if

$v = (y_1, y_2) = (0,0) = 0$.

(ii) $\langle u,v\rangle = x_1y_1 - x_2y_2 - x_2y_1 + 3x_2y_2$

$\langle v,u\rangle = y_1x_1 - y_2x_2 - y_2x_1 + 3y_2x_2$

$= x_1y_1 - x_2y_2 - x_1y_2 + 3x_2y_2$

$\ne \langle u,v\rangle$

$\therefore$ $\langle u,v\rangle \ne \langle v,u\rangle$

$\therefore$ The equation is not symmetric.

Hence, not satisfying axioms of inner product Ans.

$\langle u,u\rangle$

$\langle u,v\rangle = \langle v,u\rangle$

Subject : 2018 Date :

8 (a) Definition: Inner Product.

Let $\mathbb{R}_{n,1}$ be the set of all matrices having n rows and 1 column. If $x, y \in \mathbb{R}_{n,1}$ then

$$\langle x, y\rangle = x^T y = [x_1, x_2, \ldots x_n]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

and $x^T y$ is called an inner product of x and y. It is to be noted that $x^T y$ is just a real number.

8 (c) Define orthonormal set, Find an orthonormal basis of subspace W of $\mathbb{C}^3$ spanned by $v_1 = (1, i, 0)$ and $v_2 = (1, 2, 1-i)$

Soln: The set of vectors $\{v_i\}$ in V is an orthonormal set if,

(i) each v_i is of length 1 i.e, $\langle v_i, v_i\rangle = 1$.

(ii) for $i \neq j$, $\langle v_i, v_j\rangle = 0$.

Orthogonal : $\langle u_i, u_j\rangle = 0$ for $i \neq j$

Orthonormal : $\langle u_i, u_j\rangle = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$