Numerical Methods

By

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Introduction

- Bisection Method = a numerical method in Mathematics to find a root of a given *function*

Introduction (cont.)

- *Root* of a function:

  - Root of a function $f(x)$ = a value a such that:

    - $f(a) = 0$

Introduction (cont.)

- Example:

Function: $f(x) = x^2 - 4$

Roots: $x = -2, \; x = 2$

Because:

$$f(-2) = (-2)^2 - 4 = 4 - 4 = 0$$

$$f(2) = (2)^2 - 4 = 4 - 4 = 0$$

A Mathematical Property

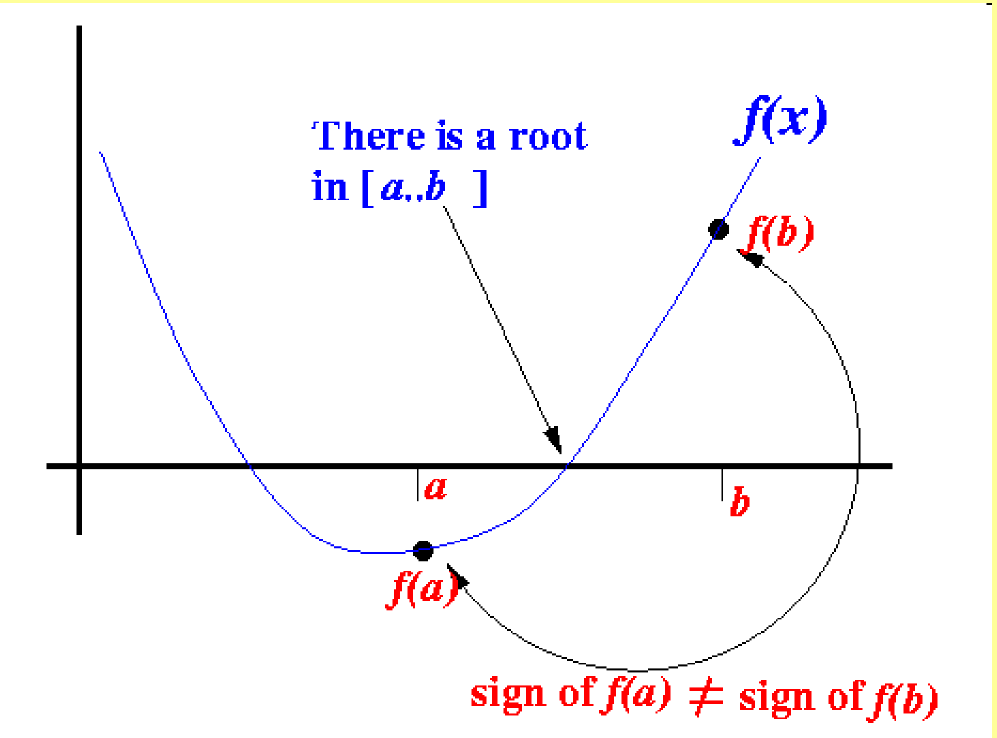
- Well-known Mathematical Property:

• If a function $f(x)$ is continuous on the interval $[a..b]$ and sign of $f(a) \neq$ sign of $f(b)$, then

• There is a value $c \in [a..b]$ such that: $f(c) = 0$ I.e., there is a root c in the interval $[a..b]$

A Mathematical Property (cont.)

- Example:



The *Bisection* Method

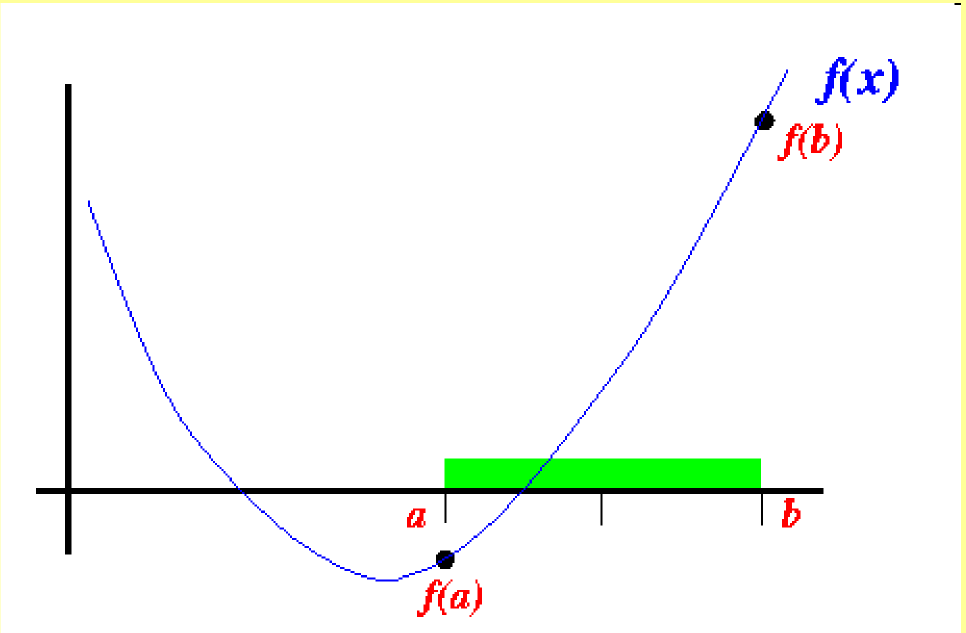
- The Bisection Method is a *successive* approximation method that narrows down an interval that contains a root of the function *f(x)*
- The Bisection Method is *given* an initial interval [*a*..*b*] that contains a root (We can use the property sign of *f(a)* ≠ sign of *f(b)* to find such an initial interval)
- The Bisection Method will *cut the interval* into 2 halves and check which half interval contains a root of the function
- The Bisection Method will keep *cut the interval* in halves until the resulting interval is extremely small

  The root is then *approximately equal* to *any value* in the final (very small) interval.

The *Bisection* Method (cont.)

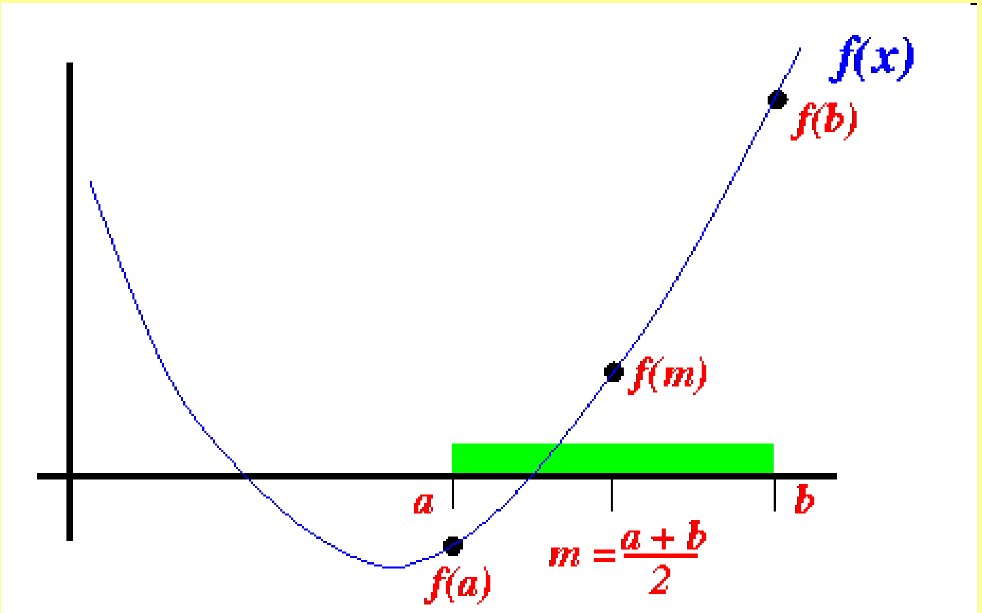
- Example:

- Suppose the interval [*a*..*b*] is as follows:



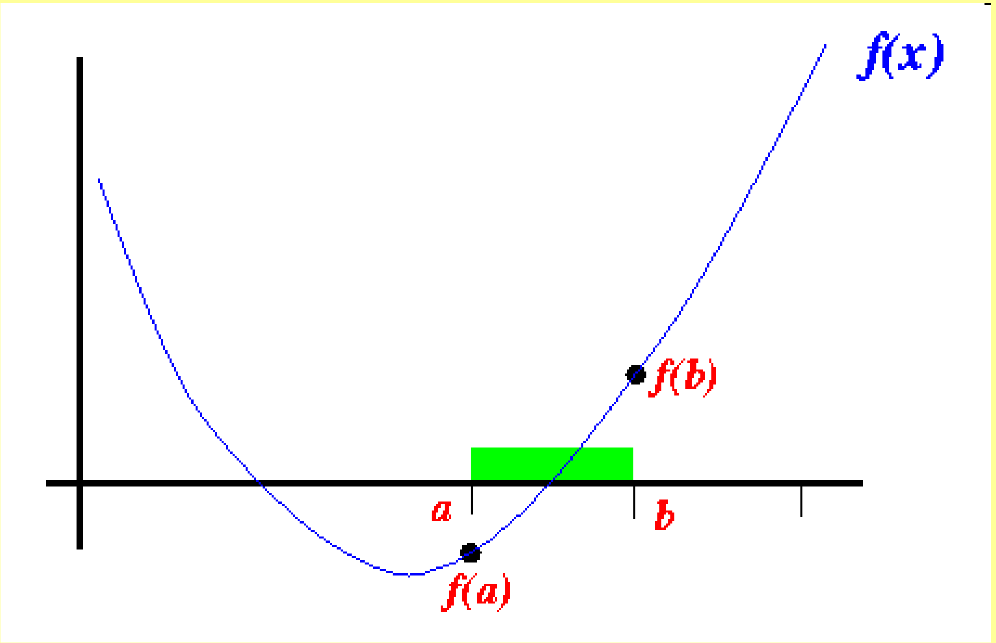
The *Bisection* Method (cont.)

- We cut the interval [*a*..*b*] in the middle: $m = (a+b)/2$



The *Bisection* Method (cont.)

- Because sign of *f(m)* ≠ sign of *f(a)* , we *proceed* with the search in the *new* interval [*a*..*b*]:



The *Bisection* Method (cont.)

We can use this statement to change to the new interval:

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b = m;
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- In the above example, we have changed the end point *b* to obtain a smaller interval that still contains a root
- In other cases, we may need to changed the end point *b* to obtain a smaller interval that still contains a root

The *Bisection* Method

This method is based on Theorem 1.1 which states that if a function $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one root between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the root lies between a and b and let its approximate value be given by $x_0 = (a+b)/2$. If $f(x_0) = 0$, we conclude that x_0 is a root of the equation $f(x) = 0$. Otherwise, the root lies either between x_0 and b, or between x_0 and a depending on whether $f(x_0)$ is negative or positive. We designate this new interval as $[a_1, b_1]$ whose length is $|b-a|/2$. As before, this is bisected at x_1 and the new interval will be exactly half the length of the previous one. We repeat this process until the latest interval (which contains the root) is as small as desired, say ε. It is clear that the interval width is reduced by a factor of one-half at each step and at the end of the nth step, the new interval will be $[a_n, b_n]$ of length $|b-a|/2^n$. We then have

- Follow the below procedure to get the solution for the continuous function:

1. Choose two real numbers a and b such that $f(a)\,f(b) < 0$.
2. Set $x_r = (a+b)/2$.
3. (a) If $f(a)\,f(x_r) < 0$, the root lies in the interval (a, x_r). Then, set $b = x_r$ and go to step 2 above.

   (b) If $f(a)\,f(x_r) > 0$, the root lies in the interval (x_r, b). Then, set $a = x_r$ and go to step 2.

   (c) If $f(a)\,f(x_r) = 0$, it means that x_r is a root of the equation $f(x) = 0$ and the computation may be terminated.

Example

Find a root of an equation $f(x)=x^3-x-1$ using Bisection method

Solution:

Here $x^3-x-1=0$

Let $f(x)= x^3-x-1$

Here

x	0	1	2
$f(x)=$	-1	-1	5

Example

n	a	$f(a)$	b	$f(b)$	$c = \frac{a+b}{2}$	$f(c)$	**Update**
1	1	-1	2	5	1.5	0.875	$b = c$
2	1	-1	1.5	0.875	1.25	-0.29688	$a = c$
3	1.25	-0.29688	1.5	0.875	1.375	0.22461	$b = c$
4	1.25	-0.29688	1.375	0.22461	1.3125	-0.05151	$a = c$
5	1.3125	-0.05151	1.375	0.22461	1.34375	0.08261	$b = c$
6	1.3125	-0.05151	1.34375	0.08261	1.32812	0.01458	$b = c$
7	1.3125	-0.05151	1.32812	0.01458	1.32031	-0.01871	$a = c$
8	1.32031	-0.01871	1.32812	0.01458	1.32422	-0.00213	$a = c$
9	1.32422	-0.00213	1.32812	0.01458	1.32617	0.00621	$b = c$
10	1.32422	-0.00213	1.32617	0.00621	1.3252	0.00204	$b = c$
11	1.32422	-0.00213	1.3252	0.00204	1.32471	-0.00005	$a = c$

Example

1st iteration :

Here $f(1)=-1<0$ and $f(2)=5>0$

∴ Now, Root lies between 1 and 2

$c=(1+2)/2=1.5$

$f(c)=f(1.5)=0.875$

So, $f(a)*f(c)=f(1)*f(1.5)=-1*0.875<0$

∴ From now, the root will be lie in the interval (1,**1.5**)

Example

2*nd* iteration :

Here $f(1)=-1<0$ and $f(1.5)=0.875>0$

c=1+1.52=1.25

$f(c)=f(1.25)= - 0.29688 <0$

So, $f(1)* f(-0.29688)= -1* - 0.29688 > 0$

∴ From now, the root will be lie in the interval (**1.25**,1.5)

Approximate root of the equation $f(x)=x^3-x-1$ using Bisection method is **1.32471**