What is the secant method and why would I want to use it instead of the Newton-Raphson method?

- The Newton-Raphson method of solving a nonlinear equation *f(x)=0* is given by the iterative formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (1)$$

- One of the drawbacks of the Newton-Raphson method is that you have to evaluate the derivative of the function.
- It can be a laborious process, and even intractable if the function is derived as part of a numerical scheme.
- To overcome these drawbacks, the derivative of the function, is approximated as

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \qquad (2)$$

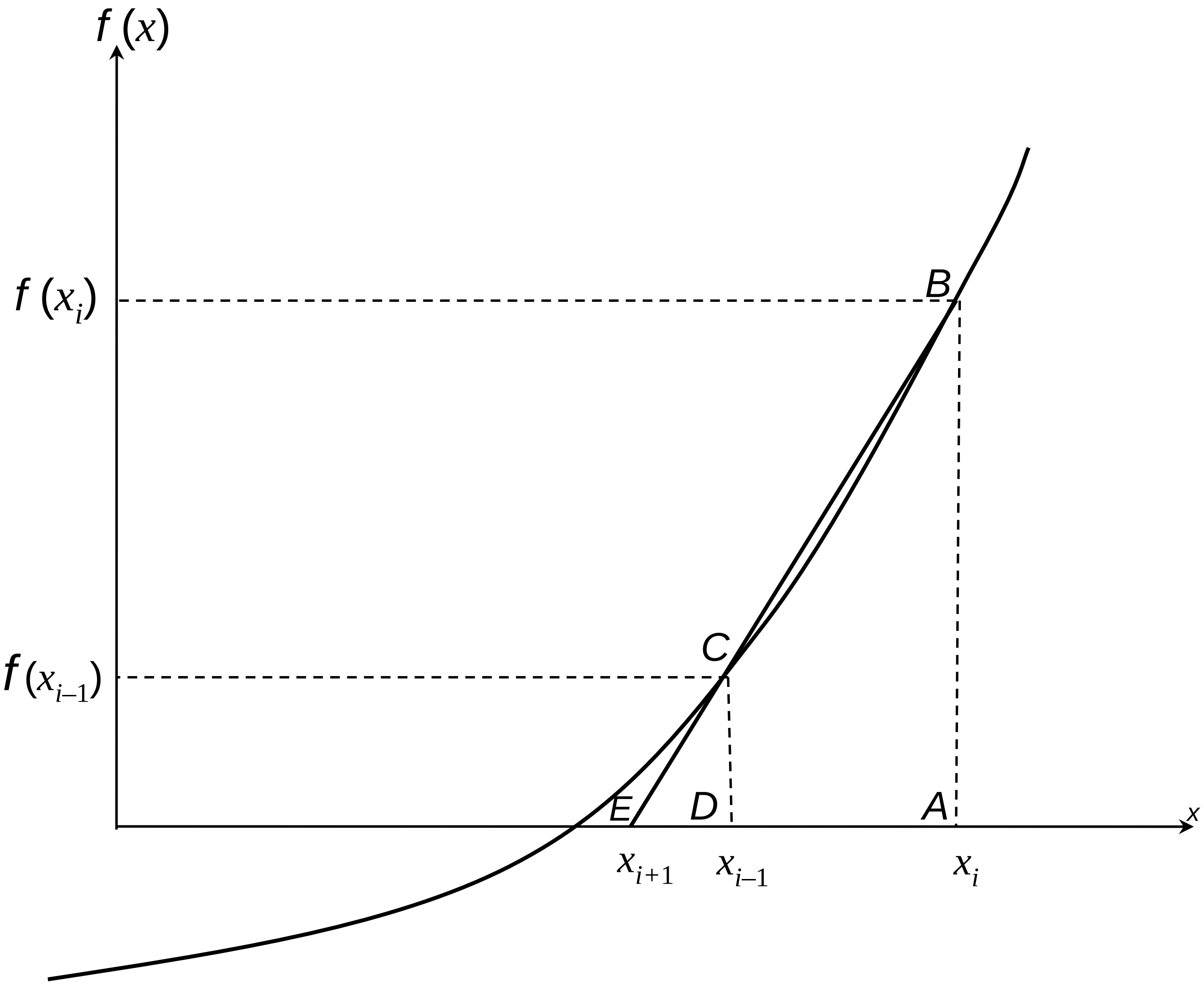
Derivation of Secant Method

- Substituting Equation (2) in Equation (1) gives

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \qquad (3)$$

- The above equation is called the secant method.
- This method now requires two initial guesses, but unlike the bisection method, the two initial guesses do not need to bracket the root of the equation.
- The secant method is an open method and may or may not converge.
- However, when secant method converges, it will typically converge faster than the bisection method.
- However, since the derivative is approximated as given by Equation (2), it typically converges slower than the Newton-Raphson method.

Figure 1 Geometrical representation of the secant method



Derivation of Secant Method (continued)

- The secant method can also be derived from geometry, as shown in Figure 1. Taking two initial guesses, and , one draws a straight line between and passing through the *x*-axis at x_i. *ABE* and *DCE* are similar triangles.
- Hence

$$\frac{AB}{AE} = \frac{DC}{DE}$$

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

- On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Apply Secant Method in the floating ball problem

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$

- Let us assume the initial guesses of the root of f(x)=0 as x_{-1}=0.02 and x_0=0.05

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$$= x_0 - \frac{\left(x_0^3 - 0.165x_0^2 + 3.993\times10^{-4}\right)\times\left(x_0 - x_{-1}\right)}{\left(x_0^3 - 0.165x_0^2 + 3.993\times10^{-4}\right) - \left(x_{-1}^3 - 0.165x_{-1}^2 + 3.993\times10^{-4}\right)}$$

$$= 0.05 - \frac{\left[0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right]\times\left[0.05 - 0.02\right]}{\left[0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right] - \left[0.02^3 - 0.165(0.02)^2 + 3.993\times10^{-4}\right]}$$

$$= 0.06461$$

Secant Method: Floating ball problem (continued)

- The absolute relative approximate error $|\in_a|$ at the end of Iteration 1 is

$$|\in_a| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100 = 22.62\%$$

- As you need an absolute relative approximate error of 5% or less so you need more iteration to carry on.

Iteration 2 $$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 0.06241$$

- The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is 3.525%

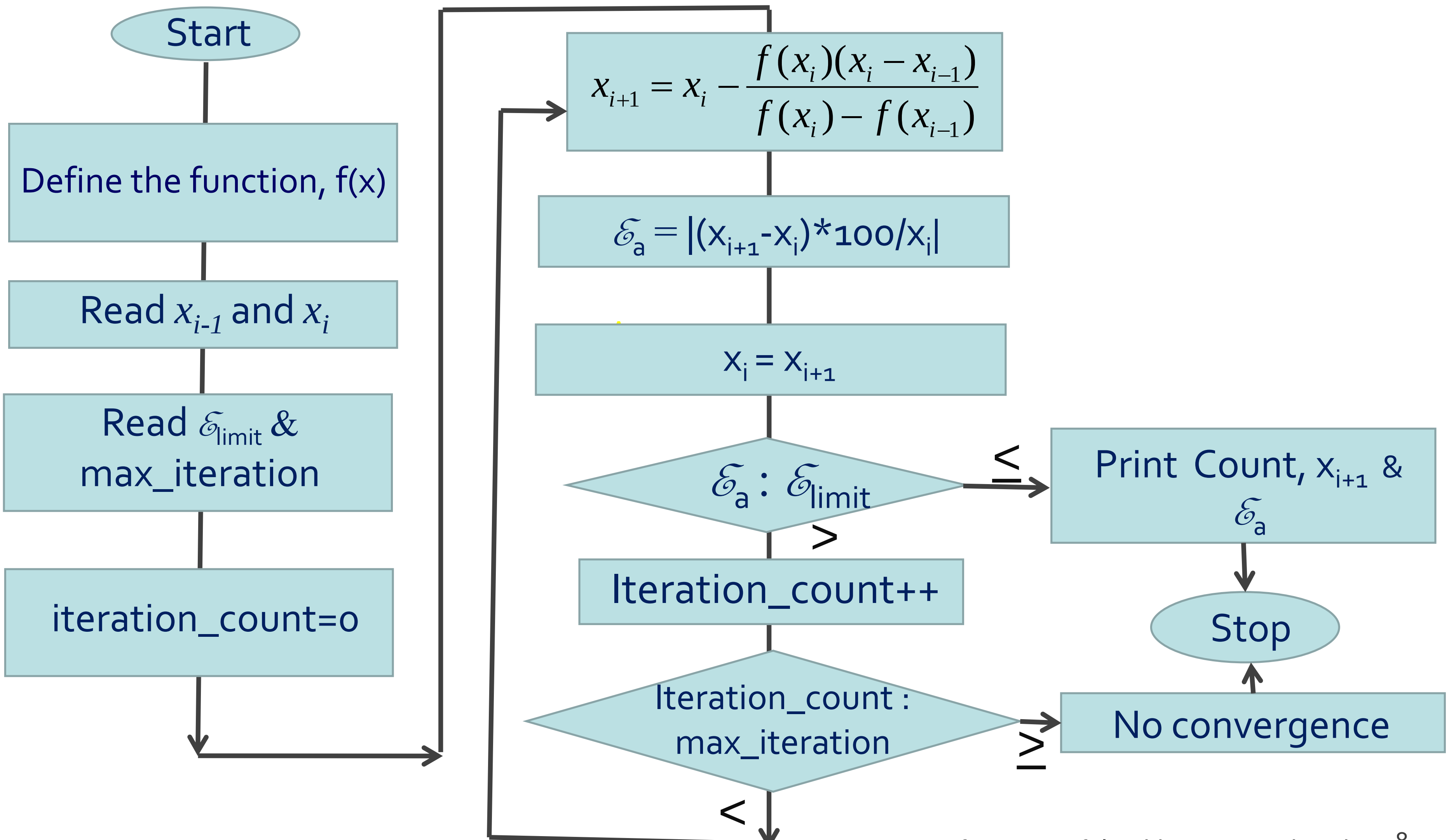
Iteration 3 for the floating ball problem (Secant Method)

- $X_3 = 0.06238$
- The absolute relative approximate error at the end of Iteration 3 is 0.0595%
- Table 1 shows the secant method calculations for the results from the above problem.

Table 1: Secent Method Result as Function of Iteration

Iteration Number, I	x_{i-1}	x_i	x_{i+1}	$\lvert\in_a\rvert\%$	$f(x_{i+1})$
1	0.02	0.05	0.06461		$-1.9812X10^{-5}$
2	0.05	0.06461	0.06241	22.62	$-3.2852X10^{-7}$
3	0.06461	0.06241	0.06238	3.525	$2.0252X10^{-9}$
4	0.06241	0.06238	0.06238	0.0595	$-1.8576X10^{-13}$

Flow Chart of Secant's Method



Advantages of Secant Method

1. It converges at faster than a linear rate, so that it is more rapidly convergent than the bisection method.
2. It does not require use of the derivative of the function, something that is not available in a number of applications.
3. It requires only one function evaluation per iteration, as compared with Newton's method which requires two.

Disadvantages of Secant Method

1. It may not converge.
2. There is no guaranteed error bound for the computed iterates.
3. It is likely to have difficulty if $f'(\alpha) = 0$. This means the x-axis is tangent to the graph of $y = f(x)$ at $x = \alpha$.
4. Newton's method generalizes more easily to new methods for solving simultaneous systems of nonlinear equations.

Systems of non-linear equations

- Usually to solve a system of non-linear equations, we will use an extension of open methods.
- An example of a system of non-linear equations is:

$$f_1(x_1, x_2) = x_1^2 + x_2^2 - 8x_1 - 4x_2 + 11 = 0$$
$$f_2(x_1, x_2) = x_1^2 + x_2^2 - 20x_1 + 75 = 0$$

Fixed point iteration for systems of non-linear equations

- One of the most important drawbacks of the fixed iteration method is that the convergence of the method is dependent on how the equations are formulated.
- It can be shown that sufficient convergence criteria for two equations are:

$$\left|\frac{\partial f_1}{\partial x_1}\right| + \left|\frac{\partial f_1}{\partial x_2}\right| < 1$$

$$and$$

$$\left|\frac{\partial f_2}{\partial x_1}\right| + \left|\frac{\partial f_2}{\partial x_2}\right| < 1$$

This represents a very restrictive criteria and that’s why fixed point iteration method is not used to solve systems of non-linear equations.

Newton-Raphson for systems of non-linear equations

- The Newton-Raphson formula is the following:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

considering a Taylor series that account for the presence of both variables

$$f_{1(i+1)} = f_{1(i)} + \left(x_{1(i+1)} - x_{1(i)}\right)\frac{\partial f_{1(i)}}{\partial x_1} + \left(x_{2(i+1)} - x_{2(i)}\right)\frac{\partial f_{1(i)}}{\partial x_2} + \ldots$$

$$and$$

$$f_{2(i+1)} = f_{2(i)} + \left(x_{1(i+1)} - x_{1(i)}\right)\frac{\partial f_{2(i)}}{\partial x_1} + \left(x_{2(i+1)} - x_{2(i)}\right)\frac{\partial f_{2(i)}}{\partial x_2} + \ldots$$

:

- For the root estimate and must be equal zero. Therefore:

$$\frac{\partial f_{1(i)}}{\partial x_1} x_{1(i+1)} + \frac{\partial f_{1(i)}}{\partial x_2} x_{2(i+1)} = -f_{1(i)} + x_{1(i)} \frac{\partial f_{1(i)}}{\partial x_1} + x_{2(i)} \frac{\partial f_{1(i)}}{\partial x_2}$$

$$and$$

$$\frac{\partial f_{2(i)}}{\partial x_1} x_{1(i+1)} + \frac{\partial f_{2(i)}}{\partial x_2} x_{2(i+1)} = -f_{2(i)} + x_{1(i)} \frac{\partial f_{2(i)}}{\partial x_1} + x_{2(i)} \frac{\partial f_{2(i)}}{\partial x_2}$$

- Finally;

$$x_{1(i+1)} = x_{1(i)} - \frac{f_{1(i)} \dfrac{\partial f_{2(i)}}{\partial x_2} - f_{2(i)} \dfrac{\partial f_{1(i)}}{\partial x_2}}{\dfrac{\partial f_{1(i)}}{\partial x_1} \dfrac{\partial f_{2(i)}}{\partial x_2} - \dfrac{\partial f_{1(i)}}{\partial x_2} \dfrac{\partial f_{2(i)}}{\partial x_1}}$$

$$x_{2(i+1)} = x_{2(i)} - \frac{f_{2(i)} \dfrac{\partial f_{2(i)}}{\partial x_1} - f_{1(i)} \dfrac{\partial f_{1(i)}}{\partial x_1}}{\dfrac{\partial f_{1(i)}}{\partial x_1} \dfrac{\partial f_{2(i)}}{\partial x_2} - \dfrac{\partial f_{1(i)}}{\partial x_2} \dfrac{\partial f_{2(i)}}{\partial x_1}}$$

The denominator is called the Jacobian.
These two equations are Newton-Raphson method for systems of non-linear equations.

Systems of Nonlinear Equations

- Roots of a set of simultaneous equations:

$$f_1(x_1,x_2,\dots\dots,x_n)=0$$

$$f_2(x_1,x_2,\dots\dots,x_n)=0$$

$$f_n(x_1,x_2,\dots\dots,x_n)=0$$

- The solution is a set of x values that simultaneously get the equations to zero.

Systems of Nonlinear Equations

Example: $x^2 + xy = 10$ & $y + 3xy^2 = 57$

$$u(x,y) = x^2 + xy - 10 = 0$$

$$v(x,y) = y + 3xy^2 - 57 = 0$$

- The solution will be the value of x and y which makes $u(x,y)=0$ and $v(x,y)=0$
- These are $x=2$ and $y=3$
- Numerical methods used are extension of the open methods for solving single equation; Fixed point iteration and Newton-Raphson. (we will only discuss the Newton Raphson)

Systems of Nonlinear Equations:
2. Newton Raphson Method

- Recall the standard Newton Raphson formula:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- which can be written as the following formula

$$x_{i+1} = x_i + \Delta x_i$$

$$where \quad \Delta x_i = -\frac{f(x_i)}{f'(x_i)}$$

$$f'(x_i) \cdot \Delta x_i = -f(x_i)$$

Systems of Nonlinear Equations:
2. Newton Raphson Method

- By multi-equation version (in this section we deal only with two equation) the formula can be derived in an identical fashion:

- *u(x,y)=0 and v(x,y)=0*

$$\begin{bmatrix} \dfrac{\partial u_i}{\partial x} & \dfrac{\partial u_i}{\partial y} \\ \dfrac{\partial v_i}{\partial x} & \dfrac{\partial v_i}{\partial y} \end{bmatrix} \begin{Bmatrix} \Delta x_i \\ \Delta y_i \end{Bmatrix} = -\begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta x_i \\ \Delta y_i \end{Bmatrix} = -\begin{bmatrix} \dfrac{\partial u_i}{\partial x} & \dfrac{\partial u_i}{\partial y} \\ \dfrac{\partial v_i}{\partial x} & \dfrac{\partial v_i}{\partial y} \end{bmatrix}^{-1} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

Systems of Nonlinear Equations:
2. Newton Raphson Method

$$\begin{bmatrix} \dfrac{\partial u_i}{\partial x} & \dfrac{\partial u_i}{\partial y} \\ \dfrac{\partial v_i}{\partial x} & \dfrac{\partial v_i}{\partial y} \end{bmatrix}^{-1} = \frac{1}{\dfrac{\partial u_i}{\partial x}\cdot\dfrac{\partial v_i}{\partial y} - \dfrac{\partial v_i}{\partial x}\dfrac{\partial u_i}{\partial y}} \begin{bmatrix} \dfrac{\partial v_i}{\partial y} & -\dfrac{\partial u_i}{\partial y} \\ -\dfrac{\partial v_i}{\partial x} & \dfrac{\partial u_i}{\partial x} \end{bmatrix}$$

- And thus

$$\boxed{x_{i+1} = x_i - \frac{u_i \cdot \dfrac{\partial v_i}{\partial y} - v_i \dfrac{\partial u_i}{\partial y}}{\dfrac{\partial u_i}{\partial x}\cdot\dfrac{\partial v_i}{\partial y} - \dfrac{\partial v_i}{\partial x}\dfrac{\partial u_i}{\partial y}}}$$

$$\boxed{y_{i+1} = y_i + \frac{u_i \cdot \dfrac{\partial v_i}{\partial x} - v_i \dfrac{\partial u_i}{\partial x}}{\dfrac{\partial u_i}{\partial x}\cdot\dfrac{\partial v_i}{\partial y} - \dfrac{\partial v_i}{\partial x}\dfrac{\partial u_i}{\partial y}}}$$

Systems of Nonlinear Equations:

2. Newton Raphson Method

- $x^2 + xy = 10$ *and* $y + 3xy^2 = 57$

are two nonlinear simultaneous equations with two unknown x and y they can be expressed in the form: use the point (1.5,3.5) as initial guess.

$$\frac{\partial u}{\partial x} = 2x + y, \quad \frac{\partial u}{\partial y} = x$$

$$\frac{\partial v}{\partial x} = 3y^2, \quad \frac{\partial v}{\partial y} = 1 + 6xy$$

i	x_i	y_i	U_i	V_i	$u_{i,x}$	$u_{i,y}$	$v_{i,x}$	$v_{i,y}$	$\varepsilon_{a,x}$	$\varepsilon_{a,y}$
0	1.5	3.5	-2.5	1.625	6.5	1.5	36.75	32.5		
1	2.03603	2.84388	-.06435	-4.7560	6.91594	2.03603	24.26296	35.74135	26.3	23.1
2	1.9987	3.00229							1.87	5.27