Fixed Point Iteration Method

Fixed point : A point, say, **s** is called a fixed point if it satisfies the equation **x = g(x)**.

Fixed point Iteration :

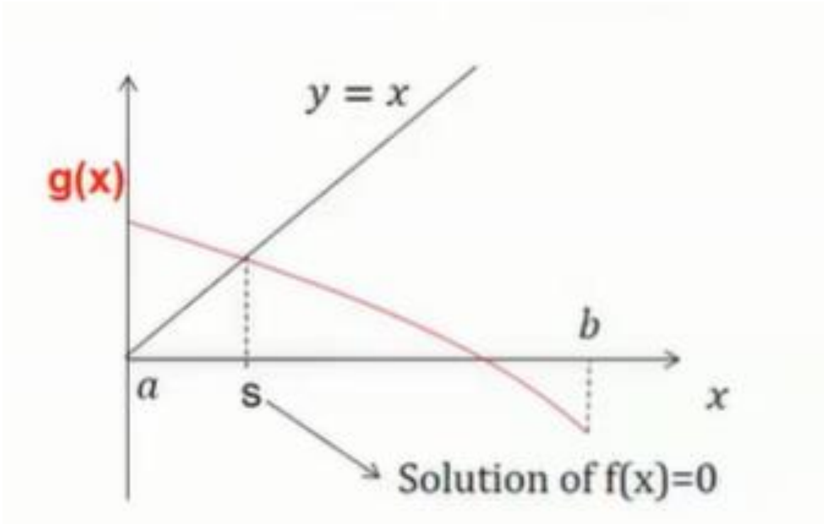
Fixed point iteration method is open and simple method for finding real root of non-linear equation by successive approximation.

It requires only one initial guess to start. Since it is open method its convergence is not guaranteed.

This method is also known as **Iterative Method.**

Fixed point Iteration method

It consist of converting the problem f(x) = 0 into x = g(x)



Example :

If f (x) = x – e^{-x}

A possible choice for g(x) is :

g(x)= e^{-x}

The fixed point iteration algorithm :

1) Choose an initial point x_0

2) Compute $x_{n+1}=g(x_n)$

3) Repeat till : $|x_{n+1} - x_n| \leq \varepsilon$

Algorithm:

Input: initial guess P0; tolerance epsilon;

No max no. of iteration.

Output: approximation solution P or failure

Step-1: i=1;

Step-2: while i<= N_0 do steps 3-6

Step-3: Set p= $q(p_0)$;

Step-4: If $|p-p_0|$< epsilon then output

Step 5: set i=i+1;

Step-6: set P_0 = P

Step-7: Output failure message

Algorithm - Fixed Point Iteration method Steps (Rule)

Step-1:	First write the equation $x=f(x)$
Step-2:	Find points a and b such that $a < b$ and $f(a)\cdot f(b) < 0$.
Step-3:	If $f(a)$ is more closer to 0 then $f(b)$ then $x_0=a$ else $x_0=b$
Step-4:	$x_1=f(x_0)$ $x_2=f(x_1)$ $x_3=f(x_2)$ ... ... Repeat until $\lvert f(x_i)-f(x_i-1)\rvert \approx 0$

Fixed Point Iteration

- Given $f(x) = 0$ write x in terms of $x = \;$ ...
- Label left side as x_{n+1} and right side with x_n
- Pick x_1 and plug into equation
- Repeat until converges

- Example: find where $x^2 - x - 1 = 0$

$$x^2 - x - 1 = 0$$

$$x^2 = x + 1$$

$$x = 1 + \frac{1}{x}$$

$$x_{n+1} = 1 + \frac{1}{x_n}$$

$$x^2 - x = 1$$

$$x(x - 1) = 1$$

$$x = \frac{1}{x - 1}$$

$$x_{n+1} = \frac{1}{x_n - 1}$$

$$x^2 = x + 1$$

$$x = \pm\sqrt{x + 1}$$

$$x_{n+1} = \pm\sqrt{x_n + 1}$$

$$x^2 - x - 1 = 0$$

$$x^2 = x + 1$$

$$x = 1 + \frac{1}{x}$$

$$x_{n+1} = 1 + \frac{1}{x_n}$$

Pick $x_1 = 2$

$$x_2 = 1 + \frac{1}{2} = 1.5$$

$$x_3 = 1 + \frac{1}{1.5} = 1.6666$$

$$x_4 = 1 + \frac{1}{1.6666} = 1.6$$

$$x_5 = 1 + \frac{1}{1.6} = 1.625$$

$$x_6 = 1 + \frac{1}{1.625} = 1.612538462$$

Converging to 1.618 ...

$$x^2 - x = 1$$

$$x(x - 1) = 1$$

$$x = \frac{1}{x - 1}$$

$$x_{n+1} = \frac{1}{x_n - 1}$$

Pick $x_1 = 1.6$

$$x_2 = \frac{1}{1.6 - 1} = 1.6666$$

$$x_3 = \frac{1}{1.6666 - 1} = 1.5$$

$$x_4 = \frac{1}{1.5 - 1} = 2$$

$$x_5 = \frac{1}{2 - 1} = 1$$

Not converging

When does it converge?

$$x_{n+1} = \ \ldots$$
$$x_{n+1} = g(x_n)$$
$$f(root) = 0$$
$$x_{n+1} - root = g(x_n) - g(root)$$

Expand $g(x_n)$ using Taylor Series

$$error_{n+1} = \ [g(root) + g'(\varphi)(x_n - root)] \ - g(root)$$
$$error_{\mathrm{n+1}} = g'(\varphi) \ \times error_n$$
$$|error_{\mathrm{n+1}}| \leq |g'(\varphi)||error_n|$$

If $|g'(root)| < 1 \rightarrow$ Converges

Convergence of the examples

$$x_{n+1} = 1 + \frac{1}{x_n}$$

$$g(x) = 1 + \frac{1}{x}$$

$$g'(x) = -\frac{1}{x^2}$$

$$g'\left(\frac{1+\sqrt{5}}{2}\right) = -\frac{1}{\left(\frac{1+\sqrt{5}}{2}\right)^2}$$

$$= -0.3819660112501$$

$$|-0.3819660112501| < 1$$

Converges

$$x_{n+1} = \frac{1}{x_n - 1}$$

$$g(x) = \frac{1}{x-1}$$

$$g'(x) = -\frac{1}{(x-1)^2}$$

$$g'\left(\frac{1+\sqrt{5}}{2}\right)$$

$$= -\frac{1}{\left(\left(\frac{1+\sqrt{5}}{2}\right) - 1\right)^2}$$

$$= -2.6180339887499$$

$$|-2.6180339887499| \geq 1$$

Does not converge

About the Order

- The order of fixed point iteration depends on $f(x)$
- Remember that $x_{n+1} = g(x_n)$
- If $|g'(r)| < 1 \rightarrow$ Converges
- If $g'(r) = 0 \rightarrow$ Converges Quadraticly (at least)
- And if $g''(r) = 0 \rightarrow$ Converges Order 3 (at least)
- And so on.

- Newton's Method is special case of Fixed Point Iteration

- Rearrange the function so that x is on the left side of the equation:

  f (x) = 0 => g(x) = x

  $x_i+1 = g(x_i)$
- Bracketing methods are "convergent".

- Fixed-point methods may sometime "diverge", depending on the stating point (initial guess) and how the function behaves.

Examples:

1. $f(x) = x^2 - x - 2 \qquad x > 0$

$$g(x) = x^2 - 2$$

or

$$g(x) = \sqrt{x+2}$$

or

$$g(x) = 1 + \frac{2}{x}$$

2. $f(x) = x^2 - 2x + 3 \Rightarrow x = g(x) = (x^2+3)/2$
3. $f(x) = \sin x \Rightarrow x = g(x) = \sin x + x$
4. $f(x) = e^{-x} - x \Rightarrow x = g(x) = e^{-x}$

Example-1

Find a root of an equation $f(x)=x^3-x-1$ using Fixed Point Iteration method

Solution:

Method-1

Let $f(x)=x^3-x-1$

$x^3-x-1=0$

$\therefore x^3=x+1$

$\therefore x=\sqrt[3]{x+1}$

$\therefore \phi(x)=\sqrt[3]{x+1}$

x	0	1	2
$f(x)$	-1	-1	5

Here $f(1) = -1 < 0$ and $f(2) = 5 > 0$

$\therefore$ Root lies between 1 and 2

$x_0 = \frac{1+2}{2} = 1.5$

$X1 = \phi(x_0) = \phi(1.5) = 1.35721$

$x2 = \phi(x_1) = \phi(1.35721) = 1.33086$

$x3 = \phi(x_2) = \phi(1.33086) = 1.32588$

$x4 = \phi(x_3) = \phi(1.32588) = 1.32494$

$x5 = \phi(x_4) = \phi(1.32494) = 1.32476$

Approximate root of the equation x^3-x-1=0 using Iteration method is 1.32476

n	X_0	$x_1=\varphi(x_0)$	Update	Difference x_1-x_0
2	1.5	1.35721	$X_0 = x_1$	0.14279
3	1.35721	1.33086	$X_0 = x_1$	0.02635
4	1.33086	1.32588	$X_0 = x_1$	0.00498
5	1.32588	1.32494	$X_0 = x_1$	0.00094
6	1.32494	1.32476	$X_0 = x_1$	0.00018