Differential and Integral Calculus
math (1211)

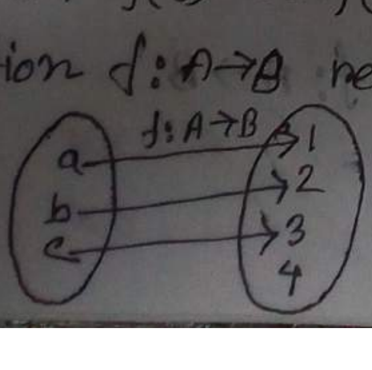
Calculus: is the mathematical study of continuous changes.

Function: Let A and B be two nonempty site. If each elements of the set A is assigned by some menner or other, to a unique element of the site B. Then these assigments is called a function. If we let 'f' denote these assignments, the we write $f: A \to B$, which reads as "f is a function of A into B".

The set A is called the domain of 'f' and the set B is called the co-domain of 'f'. Again if $a \in A$, then the elements in B which is assign to 'a' is called the image of 'a' and is denoted by $f(a)$. The set of all image is called the range of 'f'.

Example: Let $A = \{a, b, c\}$, $B \{1, 2, 3, 4\}$

Define the function of A into B by the correspondence $f(a) = 1$, $f(b) = 2$, $f(c) = 3$. Using this definition, the function $f: A \to B$ represented by following diagram.




Here domain $A = \{a, b, c\}$

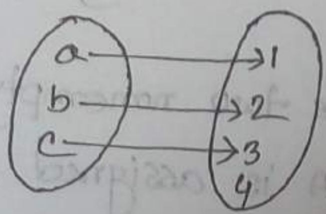
Codomain $B = \{1, 2, 3, 4\}$

Range $= \{1, 2, 3\}$
$\{f(a), f(b), f(c)\}$

## One-One function:

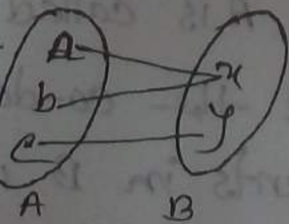
Let $f: A \to B$ be a function. Then $f$ is called one-one, if each elements of $A$ has district images.

Example:



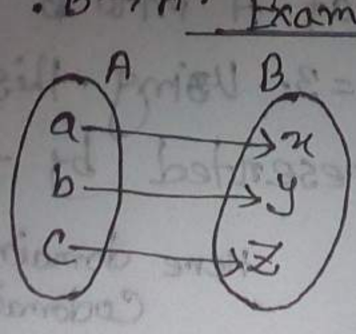
## Onto function:

Let $f: A \to B$ be a function. The $f$ is called onto if every element of $B$ appears as the image of at least one elements of $A$. In this case, $f(A) = B$. Example:



## Inverse function:

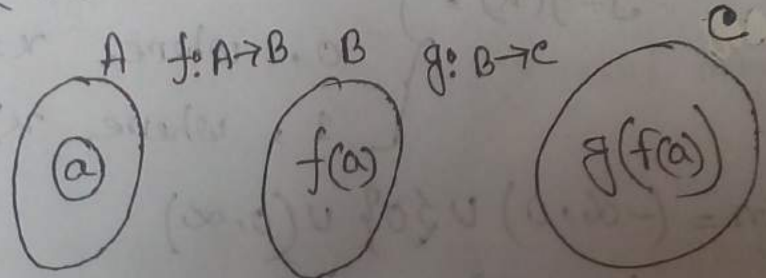
Let $f: A \to B$ be a function, if $f$ is one-one and onto, then, there exists a function $f^{-1}$ from $B$ to $A$, which is called the inverse function. In this case, we write $f^{-1}: B \to A$. Example:



$f: A \to B$

## Composition function / Product function:

Let $f: A \to B$ be a function $A$ into $B$ and $g$ be a function of $B$ into $C$. Then we define a function $(g\circ f): A \to C$ be $(g\circ f)(a) = g(f(a))$, where $a \in A$ implies $g(f(a)) \in C$. We consider the following diagram-




$(g\circ f): A \to C$

Example: $f(x) = x^2$ $\quad g(x) = x+1$

$$\therefore f\circ g = f(g(x))$$
$$= f(x+1) = (x+1)^2$$

## Step function / Box function / Greatest Integer function:

A function $[x]$ is said to be greatest intege function, where $[x]$ is the greatest. less then or eq to $x$ that is not exceeding $x$.

Mathematically, $f(x) = [x] = n$, whenever, $n \leq x < n+1$, n

Example: $x = 3$, $\quad [x] = 3$

$x = 2.3$, $\quad [2.3] = 2$

$x = -1.5$, $\quad [-1.5] = -2$

Domain Range: $y = \frac{x+1}{x-1}$

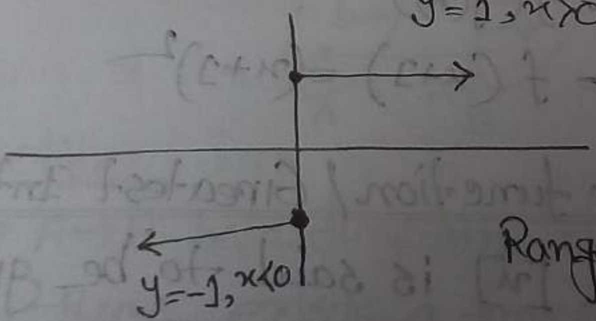
Graph of function:

~~A function~~ Find the domain and range to the given function and draw the graph.

$$y = f(x) = \begin{cases} -1, & \text{where } x<0 \\ 0, & \text{where } x=0 \\ 1, & \text{where } x>0 \end{cases}$$

Domain $= (-\infty, 0) \cup \{0\} \cup (0, \infty)$
$= (-\infty, \infty)$
$= \mathbb{R}$

The graph of the given function as follow:-

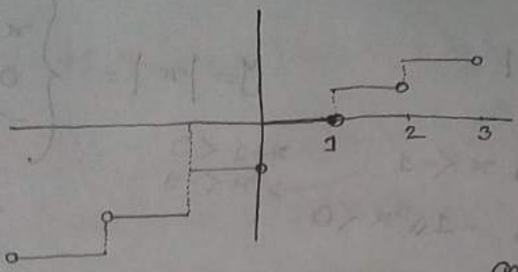



Range: $\{-1, 0, 1\}$.

Draw the graph $y=[x]$, where $[x]$ is the greatest integer not exceeding $x$.

$$y = [x] = \begin{cases} -2, & -2 \leq x < 1 \\ -1, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & 2 \leq x < 3 \end{cases}$$

Domain all real numbers $\mathbb{R}$.



This is clear that:
Domain = $\mathbb{R}$
and Range = $\{\cdots -2, -1, 0, 1, 2 \cdots\}$
= Set of integers.

Sol$^n$:
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8. (vi) $y = x - [x]$, where, $[x]$ is the greatest integer, not exceeding $x$.

For $x = 0$, $y = 0 - [0] = 0$ $\boxed{y = x}$

For $0 \leq x < 1$, $y = x - 0 = x$

For $1 \leq x < 2$, $y = x - 1$

For $2 \leq x < 3$, $y = x - 2$

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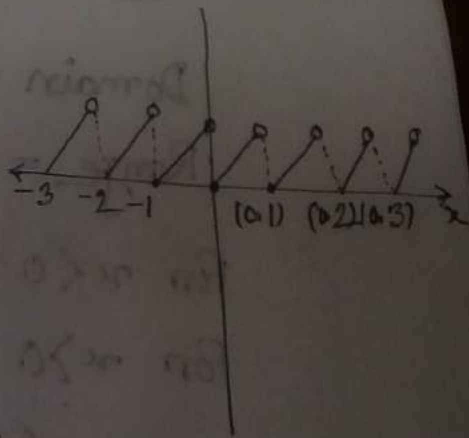
For $-1 \leq x < 0$, $y = x + 1$

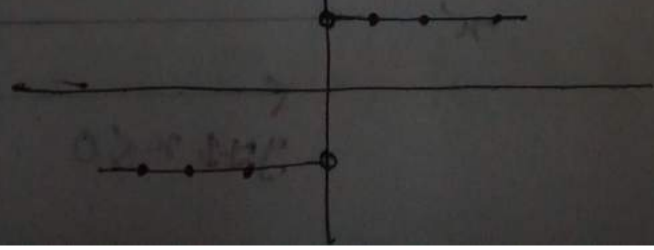
For $-2 \leq x < -1$, $y = x + 2$




Domain = $\mathbb{R}$

It is clear that Range = $[0, 1)$

Page: 34. $f(x) = \dfrac{|x|}{x}$



Solⁿ:

$y = |x+1| + |x|$

$$y = |x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

$$= \begin{cases} -(x+1)-x, & x < -1 \\ (x+1)-x, & -1 \leq x < 0 \\ (x+1)+x, & x \geq 0 \end{cases}$$

$x+1 < 0 \rightarrow x < -1$

$x+1 = 0,\ x = 1$

$x = 0$

$$= \begin{cases} -2x-1, & x < -1 \\ 1, & -1 \leq x < 0 \\ 2x+1, & x \geq 0 \end{cases}$$

$y = |x+1| + |x+2|$

$x < -2$

$-2 \leq x < -1$

$x \geq -1$

Range $= [1, \infty)$

Solⁿ:

Page: 44 (viii) $f(x) = \dfrac{|x|}{x}$

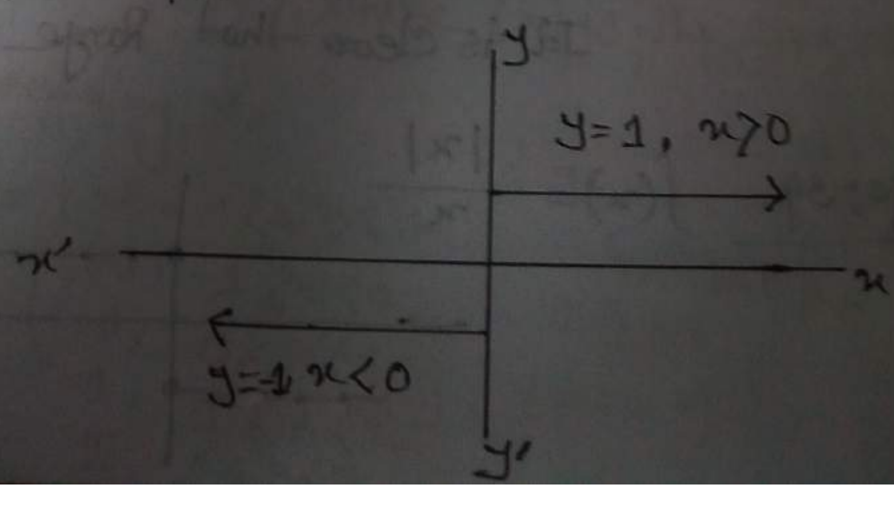
Domain $= \mathbb{R} - 0$

Range $= \mathbb{R} - 0$

For $x < 0$, $y = -1$

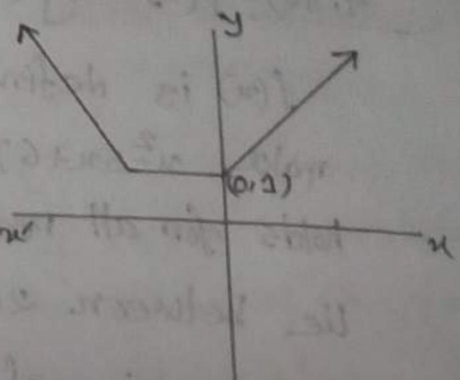
For $x > 0$, $y = 1$

The graph of the function:



$\boxed{\text{ii}}$ $y = |x+1| + |x|$

$$= \begin{cases} -(x+1) - x, & x < -1 \\ (x+1) - x, & -1 \leq x < 0 \\ (x+1) + x, & x \geq 0 \end{cases}$$




$$= \begin{cases} -2x-1, & x < -1 \\ 1, & -1 \leq x < 0 \\ 2x+1, & x \geq 0 \end{cases}$$

Domain $= \mathbb{R}$

Range $= [1, \infty)$

Page: 31 $\boxed{4(1)}$ $f(x) = \sqrt{x-1} + \sqrt{5-x}$

Solⁿ: Since, $f(x)$ is real, the values of $x$ ~~$f(x)$~~ if

$x+1 \geq 0$ and $5-x \geq 0$

$x \geq 1$ and $x \leq 5$

if $x \in (\infty, 1]$ and $(-\infty, 5]$

Now, $(\infty, 1] \cap (-\infty, 5] = [1, 5]$

Hence, the domain of $f(x)$ is $[1, 5]$

Soln.

4. iv) $f(x) = \log(x^2 - 5x + 6)$

$f(x)$ is defined for all real values of $x$ that make $x^2 - 5x + 6 > 0$ or $(x-2)(x-3) > 0$. The inequality holds for all real values of $x$, except those that lie between 2 and 3 including $x = 2$ and $x = 3$.

So, domain of definition of $f(x)$ is all real value of $x$, ~~ex~~ except $2 \leq x \leq 3$.

Soln.

Page: 40 $\boxed{\text{Ex: 16}}$ $f(x) = \sqrt{4+x} + \sqrt{9-x}$

$f(x)$ has real value, if $4+x \geq 0$ and $9-x \geq 0$.

i.e. $x \geq -4$ and $x \leq 9$

i.e, if $x \in [-4, \infty)$ and $(-\infty, 9]$

Now, $[-4, \infty) \cap (-\infty, 9] = [-4, 9]$

Hence, the domain of $f(x)$ is $[-4, 9]$.

## Limits

**Limit of a function:** If the variable $x$ taking the values greater or less than 'a' in such a way that $x$ is very close to 'a' and hence the values of $f(x)$ approaches to a definite number 'L' (say). Then 'L' is called the limit of the function $f(x)$ which we denote by $\lim_{x \to a} f(x) = $ 'L'.

**Example:** let, $\lim_{x \to 2} \frac{x^2-4}{x-2}$

When, $x \to 1.9$, $f(x) = 3.9$

$x \to 1.99$, $f(x) = 3.99$

Thus, $x \to 2^-$, $f(x) = 4^-$

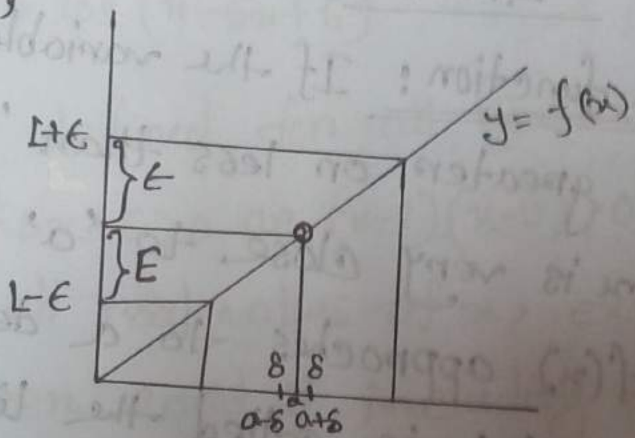
Again when, $x \to 2.1$, $f(x) = 4.1$

$x \to 2.01$, $f(x) = 4.01$

Thus, $x \to 2^+$, $f(x) \to 4^+$

In this case, we can write $\lim_{x \to 2} f(x) = 4$

Graphically,




$L-\epsilon < f(x) < L+\epsilon$

$\Rightarrow -\epsilon < f(x) - L < \epsilon$

$\Rightarrow |f(x) - 2| < \epsilon \ , \ \epsilon > 0$

$a-\delta < x < a+\delta$

$\Rightarrow |x-a| < \delta \ , \ \delta > 0.$

<u>Cauchy's definition:</u> A function $f(x)$ is said to have a limit 'L' at $x \to a$ if for every $\epsilon > 0$, there exists ~~&~~ $\delta > 0$ ($\delta$ depends on $\epsilon$) such that, $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$. In this case we write,

$$\lim_{x \to a} f(x) = L$$

<u>Example:</u> $\lim_{x \to 1} (3x+4) = ?$

Let, $f(x) = 3x+4$

Now, we see that,

for, $x = 0.9$, $f(x) = 6.7$

$x = 0.99$, $f(x) = 6.97$

again when, $x = 1.1$, $f(x) = 7.3$

$x = 1.01$, $f(x) = 7.03$

We observe that, when $x$ comes closer and closer to 1, then $f(x) = 3x+4$ comes closer and closer to 7.

In this case, if we take,

$$|f(x) - 7| < \epsilon, \ \epsilon > 0$$

$$\Rightarrow |3x+4-7| < \epsilon$$

$$\Rightarrow |3x-3| < \epsilon$$

$$\Rightarrow 3|x-1| < \epsilon$$

$$\Rightarrow |x-1| < \epsilon/3 = \delta \text{ (say)}$$

Hence, we see that,

when $|f(x)-7|<\epsilon$ whenever $|x-1|<\delta$

$\therefore \lim_{x\to 1} f(x)=7$

** L.H.L (Left Hand Limit):

Note: $x\to a$ $[x\to \bar{a}$ or $x\to a^+]$

$$\lim_{x\to a} f(x) \rightarrow \lim_{x\to a^+} f(x) = \lim_{x\to \bar{a}} f(x)$$

** L.H.L (Left Hand limit):

Let $f(x)$ be any function and $x\to a$. Then

$\lim_{x\to \bar{a}} f(x)$ is called the L.H.L of $f(x)$ and

$\lim_{x\to a^+} f(x)$ .... R.H.L of $f(x)$.

Existance of a limit:

A function $f(x)$ have a limit $L$ at $x\to a$ if

$$L.H.L = R.H.L = L$$

That is $\lim_{x\to \bar{a}} f(x) = \lim_{x\to a^+} f(x) = L$

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* Distinction between $\lim_{x\to a} f(x)$ and $f(x)$:

~~The~~

3rd-Analytical

# Distinction between $\lim_{x \to a} f(x)$ and $f(a)$:

The statement $\lim_{x \to a} f(x)$ is a statement about the value of $f(x)$ when $x$ has any value arbitrarily near to a, except a. In this case, we do not care about to know what happens when $x$ is put equal to a. But $f(x)$ stands for the value of $f(x)$ when $x$ is exactly equal to a, obtained ~~eq~~ either by the definition of the function at a, or else by substitution of a for $x$ in the expression $f(x)$, when it exists.

Problem: If $f(x) = \begin{cases} 4, & \text{when } x > 5 \\ 0, & \text{when } x = 5 \\ -4, & \text{when } x < 5 \end{cases}$

Does the ~~function~~ limit of a function exists at=5?

Now, L.H.L, $\lim_{x \to 5^-} f(x) = -4$

R.H.L, $\lim_{x \to 5^+} f(x) = 4$

Since, L.H.L $\neq$ R.H.L. Thus $\lim_{x \to 5} f(x)$ does not exist.

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Ex: 81 $\lim_{x \to 4} (2x-2) = 6$

Solution: Let us choose $\epsilon = 0.01$.

Then, $|(2x-2)-6| < 0.01$ if $|2x-8| < 0.01$

if $|x-4| < 0.005$, i.e.

$\delta = 0.005$. Similarly, if $\epsilon = 0.001$, $\delta = 0.0005$, and so on.

Thus, $\delta$ depends upon $\epsilon$, i.e. the nearer $(2x-2)$ is to 6, the nearer $x$ is to 4. We have,

$$|(2x-2)-6| < 0.01 \quad \text{if } 0 < |x-4| < 0.005,$$

$$|(2x-2)-6| < 0.01 \quad \text{if } 0 < |x-4| < 0.0005,$$

and generally, $|(2x-2)-6| < \epsilon$ if $0 < |x-4| < \frac{1}{2}\epsilon$

Hence, 6 is the limit of $2x-2$ as $x \to 4$.

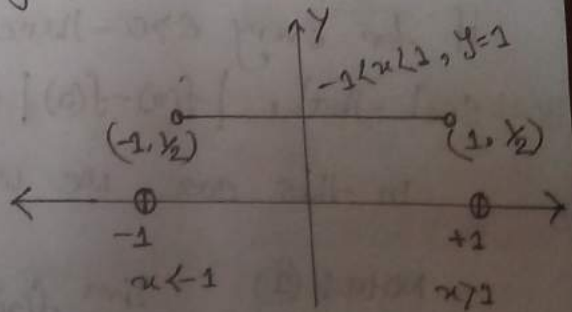
Soln:

Page-85 (19): $f(x) = \lim_{n \to \infty} \frac{1}{1+x^{2n}}$

$f(x) = 1, \frac{1}{2}, 0$

$|x| < x = 1$ or $> 1$

In this case draw the graph;




for, $|x| < 1 = -1 < x < 1$

$\therefore f(x) = \lim_{n \to \infty} \frac{1}{1+x^{2n}}$

for $|x| = 1$, $x = \pm 1$;

$$\therefore f(x) = \lim_{n \to \infty} \frac{1}{1+x^{2n}} = \frac{1}{1+(\pm 1)^{2n}}$$

$$= \frac{1}{2}$$

for $|x| > x$

$-1 > x > 1$

$x > 1$, $x < -1$

$$f(x) = \lim_{n \to \infty} \frac{1}{1+x^{2n}} = 0$$

$-1 < x < 1$, $y = 1$

$x = \pm 1$, $y = \frac{1}{2}$

$x < -1$, $x > 1$. $y = 0$.

## Continuity of function:

A function $f(x)$ is said to be continuous at $x=a$ if $\lim_{x \to a} f(x) = f(a)$ in finit.

## Cauchy's definition ($\delta$-$\epsilon$) of continuity:

A function $f(x)$ is said to be continuous at $x=a$ if for every $\epsilon>0$ there exists $\delta>0$. ($\delta$ depends on $\epsilon$) Such that, $|f(x)-f(a)|<\epsilon$ where ever $|x-a|<\delta$.

In this case, we write ①. $\lim_{x\to a} f(x) = f(a) =$ finit

Note: ① $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} =$ fraction value $f(a)$.

$\Rightarrow$ L.H.L = R.H.L = FV

② $\lim_{x\to a} f(x) = f(a) = f(x)$ is discontinuous at $x=a$.

## Classification of discontinuity:

① Ordinary discontinuity: A function $f(x)$ is said to be have an ordinary discontinuity at $x=a$. if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

Example: $\lim_{x\to 0} (2+e^{1/x})$ has an ordinary discontinuity at $x=0$.

② **Removable discontinuity:** A function $f(x)$ is said to have an removable discontinuity at $x=0$, if

$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(a)$, or $f(a)$ can't be define.

Example: if $f(x) = \frac{x^2-a^2}{x-a}$, this $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \neq f$

this $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \neq f(a)$.

③ **Infinite discontinuity:** A function $f(x)$ is said to have an infinite discontinuity at $x=a$, if one or both of $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ tend to $+\infty$ to $-\infty$.

Here, $f(a)$ may not exists.

Example: If $f(x) = \frac{x^2}{x-3}$, then $\lim_{x\to 3^-} f(x) \to -\infty$

$\lim_{x\to 3^+} f(x) \to +\infty$

but $f(3)$ can not be exists / define.

④ **Oscillatory discontinuity:** A function $f(x)$ is said to have oscillatory discontinuity if one or both of $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ does not tend to $+\infty$ to $-\infty$.

**Example:** $f(x) = \sin\frac{1}{x}$, Here, $\sin\frac{1}{x}$ oscillatory between $-1$ to $+1$ and at $x=0$, $f(x)$ is discontinuity.

## Differentiability

**Derivatives:** Let $f(x)$ be a function defined on $[a,b]$ and $c \in (a,b)$. Then $f(x)$ is said to be derivative at $x=c$. If $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists and finite.

This limit is known as derivative of $f(x)$, at $x=c$ and which are denote it by $f'(c)$. That is $f'(c)$

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \quad \text{——— (1)}$$

* If we put $x = c+h$, then $h \to 0$ (but +ve) as $x \to c$ from (1) becomes,

$$f'(c) = \lim_{h \to 0} \frac{f(c+h)-f'(c)}{h}$$

* Here, $\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$ or $\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}$

That is called the RHD of $f(x)$ at $x=c$. $[x = c+h, h>0]$ and which we define by,

$$Rf'(c) = \lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$$

and $\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c}$ or $\lim_{h \to 0^+} \frac{f(c-h)-f(c)}{-h}$ $[x=c-h, h \to 0$

That is called the LHD of $f(x)$ at $x=c$ and which we define by,

$$Lf'(c) = \lim_{h \to 0^+} \frac{f(c+h)-f(c)}{-h} \text{ or } \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c}$$

* Now if $Lf'(c) = Rf'(c)$ finite, then we say that $f(x)$ is differentiable at $x=c$.

Theorem: prove that, every differentiable function

$x=a$. $\quad f'(a) = \lim_{x \to 0} \frac{f(a+h)-f(a)}{h}$

Now, we write, $f(a+h)-f(a) = \frac{f(a+h)-f(a)}{h} \cdot h$

$$\lim_{h \to 0} \{f(a+h)-f(a)\} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \lim_{h \to 0} h$$

$$\cancel{\lim_{h \to 0} \{\ldots\}}$$

$$= \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \cdot \lim_{h \to 0} h$$

$$\Rightarrow f'(a) \times 0 = 0$$

$$\Rightarrow \lim_{h \to 0} f(a+h) - f(a) = 0$$

$$\Rightarrow \lim_{h \to 0} f(a+h) = f(a)$$

$\Rightarrow \lim_{x \to a} f(x) = f(a)$

$\Rightarrow f(x)$ is continuous at $x = a$.

$a + h = x$
$h \to 0$
$x \to a$

Example: For the converse case, we consider $f(x) = |x|$, which is obviously continuous at $x = 0$.

we find, $Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0^+} \frac{h - 0}{h} = 1$

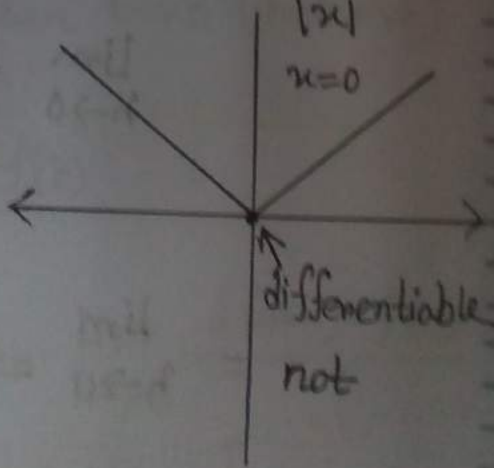
$|x|$
$\Rightarrow x, \ x > 0$
$0, \ x = 0$
$-x, \ x < 0$

$Lf'(0) = \lim_{h \to 0^-} \frac{f(0-h) - f(0)}{-h}$

$= \lim_{h \to 0^-} \frac{-(-h) - 0}{-h} = -1$

Since, $Rf'(0) \neq Lf'(0)$

$\therefore f(x)$ is not continuous.



Page - 188 (17): V.V.I

if $f(x) = \begin{cases} 1+x, & x<0 \\ 1, & 0 \leq x < 1 \\ 2x^2+4x+5, & \text{for } x>1 \end{cases}$

Finds $f'(x)$ for all values of $x$ for which it exists

$\lim_{x \to 0} f'(x)$ exists ?

Sol$^n$: For $x<0$, $f(x) = 1+x$, $f'(x) = 1$

For $0 \leq x < 1$, $f'(x) = 0$

For $x>1$, $f'(x) = 4x+4$

Now, we test the differentiate $x=1$ and $x=0$.

For $x=1$,

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{1-1}{-h}$$

$$= 0$$

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h \to 0} \frac{2(1+h)^2+4(1+h)+5-1}{h}$$

$$= \infty$$

$\therefore$ $f'(x)$ does not exists at $x=1$.

Again, for $x=0$;

$$Lf'(0) = 1$$
$$Rf'(0) = 0$$

$\Rightarrow f'(x)$ does not exists at $x=0$

Hence, $f'(x)$ exists excluding $x=0, 1$.

Last proof:

Not important

For $x>0$, $f'(x)=1$

For $x=0$, $f'(x)$ does not exists

For $x<0$, $f'(x) = 1+x$

$$\Rightarrow \lim_{x\to 0} f'(x) = \lim_{h\to 0} f(a+h)$$

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Example – VII (A)

7(i)| $f(x) = 3+2x$ for $-\frac{3}{2} < x \leq 0$,
$= 3-2x$ for $0 < x < \frac{3}{2}$.

Show that $f(x)$ is continuous at $x=0$ but does not exists?

Soln: Given $f(x) = 3+2x$ when $-\frac{3}{2} < x \leq 0$
$= 3-2x$ when $0 < x < \frac{3}{2}$

Left Hand Limit,

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} (3-2h) = 3$$

And Right Hand Limit,

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} (3-2h) = 3$$

also, $f(0) = 3$

Thus, $f(x)$ is continuous at $x=0$.

Now, $$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{3+2(-h)-3}{-h} = 2$$

and $$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{3-2(h)-3}{h} = -2$$

Since, $Lf'(0) \neq Rf'(0)$

Hence $f'(0)$ does not exist.

81 $f(x) = 1$ for $x<0$,

$= 1+\sin x$ for $0 \le x < \frac{1}{2}\pi$,

$= 2+\left(x-\frac{1}{2}\pi\right)^2$ for $\frac{1}{2}\pi \le x$;

Show that $f'(x)$ exists at $x=\frac{1}{2}\pi$ but does not exists at $x=0$?

Soln: Given that, $f(x) = 1$ when $x<0$

$= 1+\sin x$ when $0 \le x < \frac{\pi}{2}$

$= 2+\left(x-\frac{\pi}{2}\right)^2$ when $\frac{\pi}{2} \le x$

$$Rf'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}+h\right) - f\left(\frac{\pi}{2}\right)}{+h}$$

$$= \lim_{h\to 0} \frac{2+\left(\frac{\pi}{2}+h-\frac{\pi}{2}\right)^2 - \{2+0^2\}}{+h}$$

$$= \lim_{h\to 0} \frac{2+h^2-2}{+h} = 0$$

$$Lf'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}-h\right) - f\left(\pi/2\right)}{-h}$$

$$= \lim_{h\to 0} \frac{1+\sin\left(\frac{\pi}{2}-h\right)-2}{-h}$$

$$= \lim_{h\to 0} \frac{1+\cos h}{-h}$$

$$= \lim_{h\to 0} \frac{-1+1-\frac{h^2}{2}!+\dots}{-h}$$

$$= 0.$$

Again $Lf'(0) = \lim\limits_{h \to 0} \dfrac{f(0-h) - f(0)}{-h}$

$= \lim\limits_{h \to 0} \dfrac{1 - (1 + \sin)}{-h} = 0$

$Rf'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$= \lim\limits_{h \to 0} \dfrac{1 + \sin h - 1}{h} = 1$

Hence $f'(0)$ does not exist.

8(c) $f(x) = x$ for $0<x<1$

$= 2-x$ for $1 \le x \le 2$

$= x - \frac{1}{2}x^2$ for

Is $f(x)$ continuous at $x=1$ and 2? Does $f'(x)$ exist for these values?

Soln: Given that $f(x) = x$ when $0<x<1$

$= 2-x$ when $1 \le x \le 2$

$= x - \frac{1}{2}x^2$ when $x>2$

Consider the point $x=1$ for continuity.

Left hand limit,

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} (1-h) = 1$$

Right hand limit,

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 2-(1+h) = 1$$

Also $f(1) = 1$

so $f(x)$ is continuous at $x=1$.

Consider $x=2$ for continuity.

Left hand limit,

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} (2-h) = 0$$

Right hand limit,

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2+h) - \frac{1}{2}(2+h)^2 = 0$$

So, $f(x)$ is continuous at $x = 2$.

For derivative at $x = 1$.

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{(1-h) - 1}{-h}$$

$$= 1$$

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{2-(1+h)-1}{h}$$

$$= -1$$

So, $f'(1)$ does not exist.

Now, derivative at $x = 2$.

$$Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \frac{2-(2-h)-0}{h}$$

$$= -1$$

$$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h) - \frac{1}{2}(2+h)^2 - 0}{h}$$

$$= -1$$

Hence, $f'(2)$ exist and its value $-1$.

18(ii) If $f(x) = [x]$ where $[x]$ denotes the greatest integer not exceeding $x$, find $f'(x)$ and draw its graph.

Soln: Given that, $f(x) = [x]$

when $1 \leq x < 2$, $f(x) = 1$ $\therefore f'(x) = 0$

when $2 \leq x < 3$, $f(x) = 2$ $\therefore f'(x) = 0$

when $3 \leq x < 4$, $f(x) = 3$ $\therefore f'(x) = 0$.

and so on.

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{1-1}{-h} = 0$$

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim \frac{2-1}{h} = \infty$$

$\therefore f'(1)$ does not exist.

Similarly it can shown that $f'(0), f'(2), f'(3)$, etc does not exist. From above we conclude that exists $= 0$ for all values of $x$; except when $x$ is zero or any integer when $f'(x)$ does not exists. Thus the graph of $f'(x)$ consists of broken segments of line on the $x$-axis. The points $(0,0), (1,0), (2,0), (3,0)$ etc. do not lie on the graph.

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Ex: 41 If $f(x) = 2|x| + |x-2|$ find $f'(1)$.

Soln: we have $|x| = x$ for $x > 0$

$= 0$ for $x = 0$ ..... ①

$= -x$ for $x < 0$

and $|x-2| = x-2$ when $x > 2$

$= 0$ when $x = 2$ ..... ②

$= -(x-2)$ when $x < 2$

To find $f'(1)$, we are concerned with values of $x$ in the neighbourhood of $x = 1$.

From ① and ②, $f(x) = 2x + (2-x)$
$= x + 2$ | for $0 < x < 2$

Now, $f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x-1}$

$= \lim_{x \to 1} \frac{(x+2) - 3}{x-1}$ $[x - 1 \neq 0]$

$= \lim_{x \to 1} \frac{x-1}{x-1}$

$= 1$

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Ex: 12 (ii) / Show that the function $f(x) = x|x|$, is differentiable at $x=0$.

Soln: Here, $f(x) = x^2, \quad x>0$

$= 0, \quad x=0$

$= -x^2, \quad x<0$

$$Lf'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{-x^2}{x}$$

$$= 0$$

$$Rf'(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^2}{x}$$

$$= 0$$

$\therefore Lf'(0) = Rf'(0)$; $f(x)$ is differentiable at $x=0$, and $f'(0) = 0$.

Page: 105 / Ex: 7 / $f(x) = [x] + [-x]$.

Soln:

Let $x = k$ be any integer.

$\therefore [k] = k$ and $[-k] = k$

Now, $\lim_{x \to k^+} f(x) = \lim_{h \to 0} f(k+h)$

$= \lim_{h \to 0} [k-h] + \lim_{h \to 0} [-k-h]$

$= -1 \quad k - (k+1) = -1$

and. $\lim_{x \to k^-} f(x) = \lim_{h \to 0} f(k-h)$

$= \lim_{h \to 0} [k-h] + \lim_{h \to 0} [-k+h]$

$= (k-1) - (k) = -1$

$\therefore \lim_{x \to k} f(x) = -1$, but $f(k) = 0$.

So, ) f has a discontinuity at $x = k$) where $k$ is any integer. If however, we define $f(k) = -1$, then the function becomes continuous at $x = k$.

Hence, the function has a removable discontinuity for integeral values of $x$.

# Indeterminate Forms

The following forms.

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty^0, 0^0, 1^\infty, \infty\pm\infty.$$

are called Indeterminate forms.

1. Hospital Theorem:

Statement: If $\varphi(x)$, $\psi(x)$ as also their derivation $\varphi'(x)$, $\psi'(x)$ are continuous at $x=a$ and if $\varphi(a)=\psi(a)=0$. Then,

$$\lim_{x\to a}\frac{\varphi'(x)}{\psi'(x)} = \frac{\varphi'(a)}{\psi'(a)} \qquad [\psi'(a)\neq 0]$$

Resolving $\frac{\infty}{\infty}$:

Suppose, $\lim_{x\to a}\varphi(x)=\infty$, $\lim_{x\to a}\psi(x)=\infty$

Then, $$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)} = \lim_{x\to a}\frac{1/\psi(x)}{1/\varphi(x)},$$

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Evaluate: 6 (***)

$$\lim_{x \to 0} (\cos x)^{\cot^2 x}$$

let $y = (\cos x)^{\cot^2 x}$

$\Rightarrow \log y = \cot^2 x \log \cos x$

$\Rightarrow \lim_{x \to 0} \log y = \lim_{x \to 0} \cot^2 x \log(\cos x)$

$\Rightarrow \lim_{x \to 0} \log y = \lim_{x \to 0} \cot^2 x \cdot \log(\cos x) \cdot$ [$\infty \times 0$ form]

$\Rightarrow \lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log(\cos x)}{\tan^2 x}$ [$\frac{0}{0}$ form]

$\Rightarrow \lim_{x \to 0} \log y = \lim_{x \to 0} \frac{-\frac{\sin x}{\cos x}}{2 \tan x \sec^2 x}$

$\Rightarrow \lim_{x \to 0} \log y = \lim_{x \to 0} -\frac{1}{2} \cos^2 x$

$\Rightarrow \lim_{x \to 0} \log y = -\frac{1}{2}$

$\Rightarrow$ ~~log y~~ $\lim_{x \to 0} y = e^{-\frac{1}{2}}$

$\therefore \lim_{x \to 0} (\cos x)^{\cot^2 x} = e^{-\frac{1}{2}}$

Ex: 41 $\lim_{x \to \frac{1}{2}\pi} (1 - \sin x) \tan x$

$= \lim_{x \to \frac{1}{2}\pi} \frac{1 - \sin x}{\cot x}$ $\left[\text{form } \frac{0}{0}\right]$

$= \lim_{x \to \frac{1}{2}\pi} \frac{-\cos x}{-\operatorname{cosec}^2 x} = 0$

Since, $\cos x = 0$ and $\operatorname{cosec} x = 1$ as $x \to \frac{1}{2}\pi$.

Ex: 2/ If $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3}$ be finite, find the value of 'a' and the limit.

Solⁿ: The given limit, being of the form 0/0.

$= \lim_{x \to 0} \frac{2\cos 2x + a \cos x}{3x^2}$

When, $x \to 0$, the denominator $3x^2 = 0$; hence, in order that the limiting value of the expression may be finite, the numerator $(2\cos 2x + a \cos x)$ should be zero, as $x \to 0$.

$\therefore 2 + a = 0$

$a = -2$

Ex: 41 $\lim_{x \to \frac{1}{2}\pi} (1-\sin x)\tan x$

$$= \lim_{x \to \frac{1}{2}\pi} \frac{1-\sin x}{\cot x} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to \frac{1}{2}\pi} \frac{-\cos x}{-\operatorname{cosec}^2 x} = 0$$

Since, $\cos x = 0$ and $\operatorname{cosec} x = 1$ as $x \to \frac{1}{2}\pi$.

Ex: 9/ If $\lim_{x \to 0} \frac{\sin 2x + a\sin x}{x^3}$ be finit, find the value of 'a' and the limit.

Soln: The givin limit, being of the form 0/0.

$$= \lim_{x \to 0} \frac{2\cos 2x + a\cos x}{3x^2}$$

When, $x \to 0$, the denominator $3x^2 = 0$; hence, in order that the limiting value of the expression may be finit, the numerator $(2\cos 2x + a\cos x)$ should be zero, as $x \to 0$.

$\therefore 2 + a = 0$

when $a=-2$, the given limit becomes

$$= \lim_{x\to 0} \frac{\sin 2x - 2\sin x}{x^3}$$

$$= \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2}$$

$$= \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x}$$

$$= \lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6}$$

$$= -\frac{6}{6} = -1.$$

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Ex: 4-(1) $\lim_{x\to 0} (\cos x)^{1/x^2}$

Solⁿ: Let $y = (\cos x)^{1/x^2}$

$$\therefore \log y = \frac{1}{x^2} \log \cos x$$

$$= \frac{\log \cos x}{x^2}$$

$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \left\{ \frac{\log \cos x}{x^2} \right\}$$

$$= \lim_{x\to 0} \frac{-\tan x}{2x}$$

$$= \lim_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}$$

or, $\log \left\{ \lim_{x\to 0} y \right\} = -\frac{1}{2}$ $\therefore \lim_{x\to 0} = e^{-1/2}$.

* (ii) $\lim_{x \to 1} \left( x^{\frac{1}{1-x}} \right)$

Soln: Let $y = x^{\left(\frac{1}{1-x}\right)}$

$\Rightarrow \log y = \frac{1}{1-x} \log x = \frac{\log x}{1-x}$

$\Rightarrow \lim_{x \to 1} \{\log y\} = \lim_{x \to 1} \frac{\log x}{1-x}$

$\Rightarrow \quad = \lim_{x \to 1} \left(-\frac{1}{x}\right) = -1$

$\Rightarrow \log \left\{ \lim_{x \to 1} y \right\} = -1$

$\Rightarrow \cancel{\log} \lim_{x \to 1} y = e^{-1}$

$\therefore \lim_{x \to 1} x^{\frac{1}{1-x}} = e^{-1}$

* (iii) $\lim_{x \to \alpha} (1+x)^{1/x}$

Soln: Let, $z = 1/x$ then $z \to 0$ as $x \to \alpha$

Let $y = (1+x)^{1/x}$

then, $\log y = z \cdot \log \left(1 + 1/z\right)$

$\therefore \lim_{z \to 0} \{\log y\} = \lim_{z \to 0} \left\{ z \cdot \log \left(1 + 1/z\right) \right\}$

$= \lim_{z \to 0} \frac{\log(1 + 1/z)}{1/z}$

$$= \lim_{z \to 0} \frac{\left(\frac{1}{1+\frac{1}{z}} \times \left(-\frac{1}{z^2}\right)\right)}{\left(-\frac{1}{z^2}\right)}$$

$$= \lim_{z \to 0} \frac{z}{1+z} = 0$$

$$\therefore \log \left\{ \lim_{x \to \infty} y \right\} = 0$$

$$\Rightarrow \lim_{x \to \infty} y = e^0 = 1.$$

$$\therefore \lim_{x \to \infty} (1+x)^{1/x} = 1.$$

* (iv) $\lim_{x \to 0} \left(x^{2\sin x}\right)$

Solⁿ: Let, $y = x^{2\sin x}$

$$\Rightarrow \log y = 2\sin x \cdot \log x$$

$$\Rightarrow \lim_{x \to 0} (\log y) = \lim_{x \to 0} \frac{2\log x}{\operatorname{cosec} x}$$

$$= \lim_{x \to 0} \frac{2/x}{-\operatorname{cosec} x \cot x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{-x\cos x}$$

$$= -2 \lim_{x \to 0} \frac{2\sin x \cos x}{\cos x - x\sin x}$$

$$= -2 \frac{\lim_{x \to 0} \sin 2x}{\lim_{x \to 0} (\cos x - x\sin x)} = -2 \times 0/1$$

$$= 0.$$

* (ii) $\lim_{x\to 1}\left(x^{\frac{1}{1-x}}\right)$

Soln: let $y = x^{\left(\frac{1}{1-x}\right)}$

$$\Rightarrow \log y = \frac{1}{1-x}\log x = \frac{\log x}{1-x}$$

$$\Rightarrow \lim_{x\to 1}\{\log y\} = \lim_{x\to 1}\frac{\log x}{1-x}$$

$$\Rightarrow \qquad = \lim_{x\to 1}\left(-\frac{1}{x}\right) = -1$$

$$\Rightarrow \log\left\{\lim_{x\to 1} y\right\} = -1$$

$$\Rightarrow \cancel{\log}\lim_{x\to 1} y = e^{-1}$$

$$\therefore \lim_{x\to 1} x^{\frac{1}{1-x}} = e^{-1}$$

* (iii) $\lim_{x\to \alpha}(1+x)^{1/x}$

Soln: let, $z = 1/x$ then $z \to 0$ as $x \to \alpha$

let $y = (1+x)^{1/x}$

then, $\log y = z.\log\left(1+\frac{1}{z}\right)$

$$\therefore \lim_{z\to 0}\{\log y\} = \lim_{z\to 0}\left\{z.\log\left(1+\frac{1}{z}\right)\right\}$$

$$= \lim_{z\to 0}\frac{\log\left(1+\frac{1}{z}\right)}{1/z}$$

$$= \lim_{z\to 0} \frac{\left(\frac{1}{1+\frac{1}{z}} \times \left(-\frac{1}{z^2}\right)\right)}{\left(-\frac{1}{z^2}\right)}$$

$$= \lim_{z\to 0} \frac{z}{1+z} = 0$$

$$\therefore \log\left\{\lim_{x\to\infty} y\right\} = 0$$

$$\Rightarrow \lim_{x\to\infty} y = e^0 = 1.$$

$$\therefore \lim_{x\to\infty} (1+x)^{1/x} = 1.$$

* (iv) $\lim_{x\to 0} \left(x^{2\sin x}\right)$

Soln: Let, $y = x^{2\sin x}$

$$\Rightarrow \log y = 2\sin x \cdot \log x$$

$$\Rightarrow \lim_{x\to 0} (\log y) = \lim_{x\to 0} \frac{2\log x}{\operatorname{cosec} x}$$

$$= \lim_{x\to 0} \frac{2/x}{-\operatorname{cosec} x \cot x}$$

$$= \lim_{x\to 0} \frac{2\sin^2 x}{-x\cos x}$$

$$= -2 \lim_{x\to 0} \frac{2\sin x\cos x}{\cos x - x\sin x}$$

$$= -2\, \frac{\lim_{x\to 0} \sin 2x}{\lim_{x\to 0} (\cos x - x\sin x)} = -2\times \frac{0}{1} = 0.$$

$\Rightarrow \log\left\{\lim_{x\to 0} y\right\} = 0$

$\Rightarrow \lim_{x\to 0} y = e^0 = 1$

$\therefore \lim_{x\to 0} x^{2\sin x} = 1.$

Page-364 | Ex: 5 (1): $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$

Soln: | let $y = \left(\frac{\tan x}{x}\right)^{1/x}$

$\therefore \lim_{x\to 0}\{\log y\} = \lim_{x\to 0}\left\{\frac{1}{x}\log\left(\frac{\tan x}{x}\right)\right\}$

$= \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x}$

[Form $\frac{0}{0}$, $x\to 0$ since as $x\to 0$ $\frac{\tan x}{x}\to 1$, ie. $\log\left(\frac{\tan x}{x}\right)\to 0$]

$= \lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{1}$

$= \lim_{x\to 0}\left(\frac{1}{\sin x\cos x} - \frac{1}{x}\right)$

$= \lim_{x\to 0}\frac{2x - \sin 2x}{x\sin 2x}$

$= 2.\lim_{x\to 0}\frac{1-\cos 2x}{\sin 2x + 2x\cos 2x}$

$$= 2 \lim_{x \to 0} \frac{2 \sin 2x}{2\cos 2x + 2\cos 2x - 4x \sin 2x}$$

$$= 4 \frac{\lim_{x \to 0} \sin 2x}{\lim_{x \to 0} (4\cos 2x - 4x \sin 2x)}$$

$$= 4 \times \frac{0}{4} = 0$$

$$\Rightarrow \log \left\{ \lim_{x \to 0} y \right\} = 0$$

$$\Rightarrow \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x} = e^0 = 1.$$

(11)

Soln: $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

Let $y = \left(\frac{\sin x}{x}\right)^{1/x^2}$

$$\Rightarrow \log y = \frac{1}{x^2} \log \frac{\sin x}{x} = \frac{\frac{\sin x}{x}}{x^2}$$

$$\Rightarrow \lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1, \quad \lim_{x \to 0} \log\left(\frac{\sin x}{x}\right) = 0$$

$$\therefore \lim_{x \to 0} (\log y) = \lim_{x \to 0} \frac{\log \frac{\sin x}{x}}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{x}{\sin x} \times \frac{x\cos x - \sin x}{x^2}}{2x}$$

$$= \lim_{x \to 0} \frac{x\cos x - \sin x}{2x^2 \sin x}$$

)→0

$$= \lim_{x \to 0} \frac{\cos x - x\sin x - \cos x}{4x\sin x + 2x^2\cos x}$$

$$= \lim_{x \to 0} \frac{-\sin x}{2x\cos x + 4\sin x}$$

$$= \lim_{x \to 0} \frac{-\cos x}{2\cos x - 2x\sin x + 4\cos x}$$

$$= -1/6$$

$$\Rightarrow \log\left\{\lim_{x \to 0} y\right\} = -1/6$$

$$\Rightarrow \lim_{x \to 0} y = e^{-1/6}$$

$$\therefore \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}.$$

Page-367 / Ex-7(a):1 Find a, b such that,

$$\lim_{x\to 0} \frac{x(1+a\cos x)-b\sin x}{x^3} = 1$$

Soln: Here, $\lim_{x\to 0} \frac{x(1+a\cos x)-b\sin x}{x^3}$

$$= \lim_{x\to 0} \frac{1(1+a\cos x)-ax\sin x-b\cos x}{3x^2}$$

$$= \lim_{x\to 0} \frac{1+(a-b)\cos x-ax\sin x}{3x^2} \text{ ——— (1)}$$

For (1) to be form $\frac{0}{0}$, $1+(a-b)=0$

$\therefore b = 1+a$ ——— (2)

So, the given expression $= \lim_{x\to 0} \frac{1-\cos x-ax\sin x}{3x^2}$

$$= \lim_{x\to 0} \frac{\sin x-a\sin x-ax\cos x}{6x}$$

$$= \lim_{x\to 0} \frac{\cos x-a\cos x-a\cos x+ax\sin x}{6}$$

$$= \lim_{x\to 0} \frac{(1-2a)\cos x+ax\sin x}{6}$$

$$= \frac{1-2a}{6}$$

$$= 1.$$

$\therefore 1-2a=6$

$\therefore a=-5/2$

From (2) $b=1-5/2$

$=-3/2$

$\therefore a=-5/2$ and $b=-3/2$.

Page-370/ Ex: 5(iii) $\lim_{x\to 0}(\sin x)^{2\tan x}$

Soln:

If $y=(\sin x)^{2\tan x}$

$\Rightarrow \lim_{x\to 0} y = \lim_{x\to 0}(\sin x)^{2\tan x}$

$\Rightarrow \log\left(\lim_{x\to 0} y\right) = \log \lim_{x\to 0}(\sin x)^{2\tan x}$

$= \lim_{x\to 0}\left[2\tan x\cdot\log(\sin x)\right]$

$= \lim_{x\to 0}\dfrac{2\log\sin x}{\cot x}$

$= \lim_{x\to 0}\dfrac{2\cot x}{-\operatorname{cosec}^2 x}$

$= \lim_{x\to 0}(-\sin 2x)=0$

$\therefore \lim_{x\to 0} y = e^0 = 1.$

(v) $\lim_{x\to \pi/2} (\sin x)^{\tan x}$

If $y = (\sin x)^{\tan x}$

$\therefore \log\left(\lim_{x\to 0} y\right) = \lim_{x\to \pi/2} \tan x . \log(\sin x)$

$= \lim_{x\to \pi/2} \frac{\log \sin x}{\cot x} = \lim_{x\to \pi/2} \frac{\cot x}{-\operatorname{cosec}^2 x} = 0$

$\therefore \lim_{x\to 0} y = e^0 = 1.$

(vii) $\lim_{x\to 0} (\cot^2 x)^{\sin x}$

if, $\log y = \log \lim_{x\to 0} (\cot^2 x)^{\sin x}$

$= \lim_{x\to 0} \sin x \log \cot^2 x$

$= \lim_{x\to 0} \frac{2\log \cot x}{\operatorname{cosec} x} \left[\frac{\infty}{\infty}\right]$

$= \lim_{x\to 0} \frac{2\frac{2}{\cot x}(-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x \cot x}$

$= \lim_{x\to 0} \frac{2\sin x}{\cos^2 x} = 0$

$\therefore y = e^0 = 1.$

(xii) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x}$

If, $\log y = \lim_{x\to 0} \frac{1}{x}\log\left(\frac{\sin x}{x}\right)$

$$= \lim_{x\to 0} \frac{\log\frac{\sin x}{x}}{x}$$

$$= \lim_{x\to 0} \frac{\frac{\sin x}{x}\left(\frac{x\cos x - \sin x}{x^2}\right)}{1}$$

$$= \lim_{x\to 0}\left(\frac{x\cos x - \sin x}{x\sin x}\right)$$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{x\cos x + \sin x}$$

$$= \lim_{x\to 0} \frac{-x\cos x - \sin x}{\cos x - x\sin x + \cos x}$$

$$= 0$$

$\therefore y = e^0 = 1$

(XIII) $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{1/x^2}$

if, $\log y = \lim\limits_{x\to 0}\dfrac{1}{x^2}\log\left(\dfrac{\tan x}{x}\right)$

$$= \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2}$$

$$= \lim_{x\to 0}\frac{\frac{x}{\tan x}\left(\frac{x\sec^2 x-\tan x}{\tan x^2}\right)}{2x}$$

$$= \lim_{x\to 0}\frac{x\sec^2 x-\tan x}{2x^2\tan x}$$

$$= \lim_{x\to 0}\frac{x-\frac{1}{2}\sin 2x}{x^2\sin 2x}$$

$$= \lim_{x\to 0}\frac{1-\cos 2x}{x^2 2\cos 2x+2x\sin 2x}$$

$$= \lim_{x\to 0}\frac{2\sin 2x}{-4x^2\sin 2x+4x\cos 2x+4\cos 2x+2\sin 2x}$$

$$= \lim_{x\to 0}\frac{4\cos 2x}{-8x^2\cos 2x-8x\sin 2x-16x\sin 2x+8\cos 2x+4\cos 2x}$$

$$= \frac{4}{0-0-0+8+4} = \frac{1}{3}$$

$\therefore y = e^{1/3}.$

Page-581 / Ex: 20/. $\lim_{x\to 0} \dfrac{a\sin x - \sin 2x}{\tan^3 x}$

$$= \lim_{x\to 0} \frac{a\cos x - 2\cos 2x}{3\tan^2 x \sec^2 x}$$

$$= \lim_{x\to 0} \frac{a(\cos x - 2\cos 2x)\cos^4 x}{3\sin^2 x}$$

when $x\to 0$, the denominator $3\sin^2 x = 0$ hence, in order that the limiting value of the expression may be finite, the numerator $\dfrac{(a\cos x - 2\cos 2x)\cos^4 x}{3\sin^2 x}$ should be zero, as $x\to 0$.

$\therefore a - 2 = 0 \Rightarrow a = 2$.

Putting $a=2$, the above limit becomes,

$$= \lim_{x\to 0} \frac{(2\cos x - 2\cos 2x)\cos^4 x}{3\sin^2 x}$$

$$= \lim_{x\to 0} \frac{-10\cos^4 x\sin x + 4\sin 2x\cos^4 x - 8\cos 2x\cos^3 x\sin x}{6\sin x\cos x}$$

$$= \lim_{x\to 0} \frac{-5\cos^3 x\sin 2x + 4\sin 2x\cos^4 x + 2\cos^2 x\sin 4x}{3\sin 2x}$$

$$= \lim_{x\to 0} \frac{(15\cos^2 x\sin 2x - 10\cos^3 x\cos 2x) + (8\cos 2x\cos^4 x - 16\cos^2 x - 16\cos^2 x\sin x\sin 2x - 4\cos x\sin x\sin 4x + 8\cos^2 x\cos 4x)}{6\cos 2x}$$

$$= \frac{-10+8+8}{6}$$

$$= 1$$

## Differentiation Chapter-7(b)

Page-200/ Ex: 10/ If $\sin y = x\sin(a+y)$ prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

Solⁿ:/

From the given relation, we have,

$$x = \frac{\sin y}{\sin(a+y)}$$

$$\therefore \frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a-y)}{\sin^2(a+y)}$$

$$= \frac{\sin\{(a+y)-y\}}{\sin^2(a+y)}$$

$$= \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

Page - 205 / Ex - 8 (xiii) : / $(\sin x)^{\cos x} + (\cos x)^{\sin x}$

Sol^n: Suppose $u = (\sin x)^{\cos x}$, $v = (\cos x)^{\sin x}$ then $y = u + v$.

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{--- (1)}$$

Now, $u = (\sin x)^{\cos x}$

$$\log u = \cos x \log \sin x$$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = -\sin x \cdot \log(\sin x) + \cos x \cdot \frac{1}{\sin x} \cdot \cos x$$

$$\therefore \frac{du}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \log \sin x\right] \quad \text{--- (2)}$$

Again, $v = (\cos x)^{\sin x}$

$$\log v = \sin x \cdot \log(\cos x)$$

$$\frac{1}{v} \cdot \frac{dv}{dx} = -\cos x \log \cos x - \sin x \cdot \frac{1}{\cos x} \sin x .$$

$$\therefore \frac{dv}{dx} = (\cos x)^{\sin x}\left[\cos \cdot \log(\cos x) - \frac{\sin^2 x}{\cos x}\right] \quad \text{--- (3)}$$

From (2) and (3) putting the value in (1).

$$\frac{dy}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \log(\sin x)\right] + (\cos x)^{\sin x}\left[\cos x \cdot \log(\cos x) - \frac{\sin^2 x}{\cos x}\right]$$

(xiv) $(\tan x)^{\cot x} + (\cot x)^{\tan x}$

If $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$

Suppose, $u = (\tan x)^{\cot x}$, $v = (\cot x)^{\tan x}$. then, $y = u + v$.

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{du}{dx} \text{ ——— } (1)$$

Now, $u = \cancel{\tan x \cot})\ (\tan x)^{\cot x}$

$$\log u = \cot x \cdot \log \tan x$$

$$\frac{1}{u} \cdot \frac{du}{dx} = -\operatorname{cosec}^2 x \cdot \log \tan x + \cot x \cdot \frac{1}{\tan x} \cdot \sec^2 x$$

$$\therefore \frac{du}{dx} = (\tan x)^{\cot x} \left[ \frac{\sec^2 x}{\tan^2 x} - \operatorname{cosec}^2 x \log \tan x \right]$$

$$= (\tan x)^{\cot x} \left[ \operatorname{cosec}^2 x - \operatorname{cosec}^2 x \log \tan x \right]$$

$$\therefore \frac{du}{dx} = (\tan x)^{\cot x} \left[ \operatorname{cosec}^2 x \left( 1 - \log \tan x \right) \right] \text{ —— } (2)$$

Again, $v = (\cot x)^{\tan x}$

$$\log u = \tan x \log \cot x$$

$$\frac{1}{u} \cdot \frac{du}{dx} = \sec^2 x \cdot \log \cot x + \tan x \cdot \frac{1}{\cot x} \operatorname{cosec}^2 x$$

$$\frac{du}{dx} = (\cot x)^{\tan x} \left[ \sec^2 x \cdot \log \cot x + \frac{\operatorname{cosec}^2 x}{\cot^2 x} \right]$$

$$= (\cot x)^{\tan x} \left[ \sec^2 x \left( \log \cot x + 1 \right) \right] \text{ —— } (3)$$

(2) ও (3) নং সমীকৰণ (1) নং বহাই। (Ans).

Page-207/ Ex:12/ If $y = e^{\sin^{-1}x}$ and $z = e^{-\cos^{-1}x}$,
then show that $\frac{dy}{dz}$ is independent of $x$.

Sol$^n$: Given that, $y = e^{\sin^{-1}x}$, $z = e^{-\cos^{-1}x}$

$$\frac{dy}{dx} = e^{\sin^{-1}x} \cdot \frac{1}{\sqrt{(1-x^2)}}, \quad \frac{dz}{dx} = e^{-\cos^{-1}x} \cdot \frac{1}{\sqrt{(1-x^2)}}$$

$$\therefore \frac{dy}{dz} = \frac{dy/dx}{dz/dx}$$

$$= \frac{e^{\sin^{-1}x}}{e^{-\cos^{-1}x}}$$

$$= e^{\sin^{-1}x + \cos^{-1}x}$$

$$= e^{\pi/2} = \text{Const}.$$

Page-2101 Ex: 24/ If $f(x)=\left(\frac{a+x}{b+x}\right)^{a+b+2x}$, show that

$$f'(0)=\left(2\log\frac{a}{b}+\frac{b^2-a^2}{ab}\right)\left(\frac{a}{b}\right)^{a+b}.$$

Soln:/ Given that, $f(x)=\left(\frac{a+x}{b+x}\right)^{a+b+2x}$.

$$\log f(x)=(a+b+2x)\left[\log(a+x)-\log(b+x)\right]$$

Differentiating,

$$\frac{1}{f(x)}f'(x)=2\left[\log(a+x)-\log(b+x)\right]+(a+b+2x)\left[\frac{1}{a+x}-\frac{1}{b+x}\right]$$

$$\therefore f'(x)=f(x)\left[2\left\{\log\frac{(a+x)}{(b+x)}+(a+b+2x)\left\{\frac{1}{a+x}-\frac{1}{b+x}\right\}\right]$$

putting, $x=0$

$$f'(0)=\left(\frac{a}{b}\right)^{a+b}\left[2\log\frac{a}{b}+\frac{b^2-a^2}{ab}\right]$$

*

Page-285/ Ex: 9(i) if $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, show that

$$\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}.$$

Soln: let us assume, $x=\sin\theta$, and $y=\sin\phi$.

$$\therefore \frac{dx}{d\phi}=\cos\theta,\ \frac{dy}{d\phi}=\cos\phi$$

$$\therefore \sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y).$$

$\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

$\Rightarrow 2\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi) = 2a\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)$

$\Rightarrow \cot\frac{1}{2}(\theta-\phi) = a, \quad \because \cos\frac{1}{2}(\theta+\phi) \neq 0$

$\Rightarrow \theta - \phi = 2\cos^{-1}a = \text{Constant}$

$\Rightarrow \frac{d}{d\theta}(\theta-\phi) = 0$

$\Rightarrow 1 - \frac{d\phi}{d\theta} = 0 \quad \therefore \frac{d\phi}{d\theta} = 1$

Now, $\frac{dy}{dx} = \frac{dy}{d\phi}\cdot\frac{d\phi}{d\theta}\cdot\frac{d\theta}{dx} = \frac{\cos\phi}{\cos\theta} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$.

V.V.I

Page-237) Ex: 11(i) if $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x \cdots \text{to } \infty}}}$, show that, $\frac{dy}{dx} = \frac{\cos x}{2y-1}$.

Soln:
$y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x \cdots\cdots \text{to } \infty}}}$

$= \sqrt{\sin x + y}$

$\Rightarrow y^2 = \sin x + y$

Differentiating both the side w.r.t. $x$,

$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{\cos x}{2y}$.

(11) Show that, $\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \dots$ to $\infty$

$$= \frac{1}{1-x} \quad (0<x<1)$$

Soln: We have,

$$(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\dots(1+x^n)$$

$$= (1-x^2)(1+x^2)(1+x^4)(1+x^8)\dots(1+x^n)$$

$$= (1-x^4)(1+x^4)(1+x^8)\dots(1+x^n)$$

$$= (1-x^8)(1-x^8)\dots(1+x^n)$$

$$= (1-x^n)(1+x^n) = 1-x^{2n}$$

$\therefore$ $0<x<1$, $\lim_{x\to\infty} x^{2n} = 0$

$\therefore \lim_{x\to\alpha}\{(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\dots(1+x^n)\} = \lim_{x\to\alpha}(1-x^{2n}) = 1$

$\therefore \log(1-x) + \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8) + \dots$ to $\infty = \log 1$
$= 0$

Differentiating both the side w.r.t. we get,

$$\frac{-1}{1-x} + \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \dots \text{ to } \infty = 0$$

$$\Rightarrow \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \dots \text{ to } \infty = \frac{1}{1-x}.$$

(iii) If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \text{to}\infty}}}$, show that

$$\frac{dy}{dx} = \cfrac{1}{2 - \cfrac{x}{x + \cfrac{1}{x + \cfrac{1}{x + \text{to}\infty}}}}$$

Soln: Here, $y = x + \frac{1}{y}$

$\Rightarrow y^2 = xy + 1$

Differentiating both sides w.r.t. $x$,

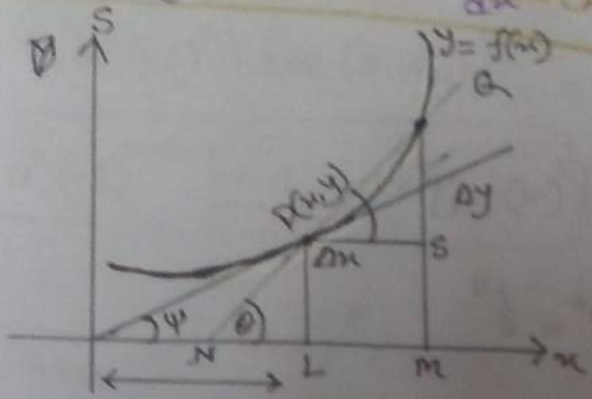
$$2y\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\therefore 2y\frac{dy}{dx} - x\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx}(2y - x) = y$$

$$\therefore \frac{dy}{dx} = \frac{y}{2y-x} = \frac{1}{2 - \frac{x}{y}} = \cfrac{1}{2 - \cfrac{x}{x + \cfrac{1}{x + \cfrac{1}{x + \text{to}\infty}}}}$$

## Geometrical significance of $\frac{dy}{dx}$ ?




P and Q be two points whose co-ordinates are (x,y), (x+Δx, y+Δy) respectively.

tan QPS

$\Rightarrow \tan\theta = \frac{QS}{PS}$

$\Rightarrow \tan\theta = \frac{\Delta y}{\Delta x}$

$Q \to P \Rightarrow \Delta x \to 0,\ \Delta y \to 0$, also $\Rightarrow$ QN = TR $[\theta = \psi]$

$Q \to P \lim_{\Delta x \to 0} \tan\theta = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta x} \Rightarrow \tan\psi = \frac{dy}{dx}$

The slop of the tangent drawn at p(x,y) is equal to the derivative of the function $y = f(x)$.

# Successive Differentiation

$y = f(x)$

$y_1 = \frac{dy}{dx} = f'(x) = y'$

$\frac{dy_1}{dx} = y_2 = y'' = f''$

(# 1) $y = x^n$

$y_1 = nx^{n-1}$

$y_2 = n(n-1)x^{n-2}$

$y_n = n(n-1)(n-2)\cdots\cdots (n-n+1)x^{n-n}$

$= 1\cdot2\cdot3\cdots(n-1)^n\cdot x^0$

$= \lfloor n$

$y_{n+1} = 0$

② $y = (ax+b)$

$y_n = \frac{\lfloor m}{\lfloor m+n} a^n (ax+b)^{(m-n)}$ $(m>n)$

③ $y = e^{ax+b}$

$y_n = a^x \cdot e^{ax+b}$

④ $y = \ln(ax+b)$

$y_1 = a\cdot\frac{1}{ax+b} = a(ax+b)^{-1}$

$y_2 = a^2(-1)^{2-1}(ax+b)^{-2}$

$$y_3 = a^3(-1)(-2)(ax+b)^{-3}$$

$$= a^3(-1)^{3-1}\ 1\cdot 2\ (ax+b)^{-3}$$

$$\boxed{y_n = a^n(-1)^{n-1}\ \lfloor n-1\ (ax+b)^{-n}}$$

⑤ $y = \dfrac{1}{x+a} = (x+a)^{-1}$

$$y_n = \frac{(-1)^n \lfloor n}{(x+a)^{n+1}}$$

$$\frac{dy}{dx} = Dy\left(\frac{1}{x+a}\right)$$

⑥ $y = \sin(ax+b)$

$$y_1 = a\cos(ax+b)$$

$$= a\sin\left\{1\cdot \pi/2 + (ax+b)\right\}$$

$$y_2 = a^2\cos\left\{\pi/2 + (ax+b)\right\}$$

$$= a^2\sin\left\{\pi/2 + \pi/2 + (ax+b)\right\}$$

$$y_3 = a^3\sin\left(3\,\pi/2 + ax+b\right)$$

$$y_n = a^n\sin\left(n\cdot\pi/2 + ax+b\right)$$

Page – 253 (4) :/ If $y = x^{2n}$, where $n$ is a positive integer, show that, $y_n = 2^n \{1.3.5.\ldots.(2n-1)\} x^n$.

Solⁿ: 

$$y = x^{2n}$$

$$y_1 = 2n x^{2n-1}$$

$$y_2 = 2n(2n-1)x^{2n-2}$$

$$y_3 = 2n(2n-1)(2n-2)x^{2n-3}$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$y_n = 2n(2n-1)(2n-2)\ldots(2n-n+1)x^{2n-n}$$

$$= (n+1)\ldots\ldots(2n-1)\,2n\,x^n$$

$$= \frac{1.2.3\ldots(2n-1)\,2n\,x^n}{\lfloor n}$$

$$y_n = \frac{\{2.4.6\ldots(2n-1)2n\}\{1.3.5\ldots(2n-1)\}x^n}{\lfloor n}$$

$$= \frac{\{1.2.3\ldots n\}\{1.3.5\ldots(2n-1)\}.x^n}{\lfloor n}$$

$$= 2^n\{1.3.5\ldots(2n-1)\}x^n$$

Page-259(4):/ Find the value of $y_n$ for $x=0$, when $y=e^{a\sin^{-1}x}$. From the value of $y$, when $x=0$, $y=1$.

Soln: Here $y_1 = e^{a\sin^{-1}x} \cdot a\frac{1}{\sqrt{1-x^2}}$ —— ①

$$= ay\frac{1}{\sqrt{1-x^2}}$$

$$\therefore y_1^2(1-x^2) = a^2y^2$$

Differentiating, $2y_1y_2(1-x^2)+y_1^2(-2x) = 2a^2yy_1$,

$$\Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0 \quad \text{—— ②}$$

Differentiating this $n$ times by Leibnitz's Theorem, as

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$$

Putting, $x=0$, $(y_{n+2})_0 = (n^2+a^2)(y_n)_0$ —— ③

Replacing $n$ by $n-2$, we get, similarly

$$(y_n)_0 = \{(n-2)^2+a^2\}(y_{n-2})_0$$

$$= \{(n-2)^2+a^2\}\{(n-4)^2+a^2\}(y_{n-4})_0$$

From ① and ② $(y_1)_0 = a$, $(y_2)_0 = a^2$

Thus, $(y_n)_0 = \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\} \cdots \{4^2+a^2\}\{$
$(4^2+a^2)(2^2+a^2)a^2$, if $n$ is even

and $= \{(n-2)^2+a^2\}\{(n-4)^2+a^2\} \cdots\cdots$
$(3^2+a^2)(1^2+a^2)a^2$, if $n$ is odd.

NOTE: The value of $y_n$ for $x=0$ is shortly denoted by $(y_n)_0$.

V.V.I.

## Leibnitz's Theorem:

If $u$ and $v$ are two function of $x$ is possessing derivatives upto $n$th order, then the $n$th derivatives of their product that is,

$$(uv)_n = u_n v + {}^nC_1 u_n v_1 + {}^nC_2 u_{n-2} v_2 + \cdots\cdots + {}^nC_r u_{n-r} v_r + \cdots + uv_n$$

where

Proof: let $y = uv$

$y_1 = (uv)_1$

$= u_1 v + uv_1$

$y_2 = (u_1 v + uv_1)_1$

$= u_2 v + u_1 v_1 + u_1 v_1 + uv_2$

$y_2 = (u_2 v) + 2u_1 v_1 + uv_2 = u_2 v + 2C_1 u_{2-1} v_1 + uv_2$

$y_3 = u_3 v + 3u_2 v_1 + 3u_1 v_1 + uv_3$

$= u_3 v + 3C_1 u_{3-1} v_1 + {}^3C_2 u_{3-2} v_2 + uv_3$, Proceeding in this way, upto n times, we can write.

Page-261(2): If $y^{1/m}+y^{-1/m}=2x$, prove that $(x^2-1)y_2+xy_1-m^2y=0$

where, $y_1=\frac{dy}{dx}$, $y_2=\frac{d^2y}{dx^2}$.

Soln: $\because y^{1/m}+y^{-1/m}=2x$, $a^2-2ax+1=0$, where $a=y^{1/m}$

$$\therefore a=\frac{2x\pm\sqrt{4x^2-4}}{2}=x\pm\sqrt{x^2-1}$$

$$\Rightarrow y^{1/m}=x\pm\sqrt{x^2-1}$$

$$\Rightarrow \frac{1}{m}\log y=\log\left(x\pm\sqrt{x^2-1}\right)$$

Differentiating both sides, w.r.t. $x$,

$$\frac{1}{m}\cdot\frac{1}{y}\cdot y_1=\frac{1}{\left(x\pm\sqrt{x^2-1}\right)}\times\left(1\pm\frac{x}{\sqrt{x^2-1}}\right)$$ where $y_1=\frac{dy}{dx}$

$$\Rightarrow \frac{y_1}{my}=\pm\frac{1}{\sqrt{x^2-1}}$$

$$\Rightarrow (x^2-1)y_1^2=m^2y^2$$

Differentiating again both sides w.r.t. $x$,

$(x^2-1)2y_1y_2+2xy_1^2=m^2\cdot 2y\cdot y_1$, where $y_2=\frac{d^2y}{dx^2}$

$\Rightarrow (x^2-1)y_2+xy_1-m^2y=0$ $\qquad 2y_1\neq 0$.

Page-263 4(III):/ If $y = 2\cos x(\sin x - \cos x)$

Show that $(y_{10})_0 = 2^{10}$.

$y = 2\cos x(\sin x - \cos x)$

$= 2\sin x\cos x - 2\cos^2 x$

$= \cancel{\sin 2x - 1 - \sin^2 x}$

$= \sin 2x - (1 + \cos 2x)$

$= \sin 2x - \cos 2x - 1$

$y_{10} = 2^{10}\sin\left(10\cdot\frac{\pi}{2} + 2x\right) - 2^{10}\cos\left(10\cdot\frac{\pi}{2} + 2x\right)$

If $y = \sin ax$, $y_n = a^n \cdot \sin\left(n\frac{\pi}{2} + x\right)$, and

if $y = \cos ax$, $y_n = a^n \cdot \cos\left(n\frac{\pi}{2} + x\right)$

$\therefore (y_{10})_0 = 2^{10}\cdot\sin 5\pi - 2^{10}\cos 5\pi$

$= 2^{10}\cdot 0 - 2^{10}\times(-1)$

$= 2^{10}$.

Page-269(5): If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ ; $|x| < 1$, show that

i) $(1-x^2)y_2 - 3xy_1 - y = 0$

ii) $(1-x^2)\,y_{n+2} - (2n+3)\,xy_{n+1} - (n+1)^2 y_n = 0$

Sol$^n$:

$$y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$(1-x^2)\,y^2 = (\sin^{-1}x)^2$$

Differentiating w.r.t. $x$,

$$(1-x^2)\,2yy_1 - 2xy^2 = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} = 2y$$

or, $(1-x^2)\,y_1 - xy = 1$

Differentiating again w.r.t. $x$,

$$(1-x)^2 y_2 - 2xy_1 - y - xy = 0$$

$$\Rightarrow (1-x^2)y_2 - 3xy_1 - y = 0 \quad \text{—— (1)}$$

(Ans).

Differentiating (1) n times by leibnitz's theorem,

$$y_{n+2}(1-x^2) + n\cdot y_{n+1}(-2x) + \frac{n(n-1)}{1\cdot 2}\cdot y_n(-2) - 3\{xy_{n+1} + n\cdot y_n\} - y_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - (2n+3)\,xy_{n+1} - (n+1)^2 y_n = 0$$

(Ans)

Page-263 4(iii):/ If $y = 2\cos x(\sin x - \cos x)$

Show that $(y_{10})_0 = 2^{10}$.

$$y = 2\cos x(\sin x - \cos x)$$

$$= 2\sin x\cos x - 2\cos^2 x$$

$$= \cancel{\sin 2x - 1 - \sin^2 x}$$

$$= \sin 2x - (1 + \cos 2x)$$

$$= \sin 2x - \cos 2x - 1$$

$$y_{10} = 2^{10}\sin\left(10\cdot\tfrac{\pi}{2} + 2x\right) - 2^{10}\cos\left(10\cdot\tfrac{\pi}{2} + 2x\right)$$

If $y = \sin ax$, $y_n = a^n\cdot\sin\left(n\tfrac{\pi}{2} + x\right)$, and

if $y = \cos ax$, $y_n = a^n\cdot\cos\left(n\tfrac{\pi}{2} + x\right)$

$$\therefore (y_{10})_0 = 2^{10}\cdot\sin 5\pi - 2^{10}\cos 5\pi$$

$$= 2^{10}\cdot 0 - 2^{10}\times(-1)$$

$$= 2^{10}.$$

Page-269(5): If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, $|x| < 1$, show that

i) $(1-x^2)y_2 - 3xy_1 - y = 0$

ii) $(1-x^2)\,y_{n+2} - (2n+3)\,xy_{n+1} - (n+1)^2 y_n = 0$

Solⁿ:

$$y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$(1-x^2)\,y^2 = (\sin^{-1}x)^2$$

Differentiating w.r.t. $x$,

$$(1-x^2)\,2yy_1 - 2xy^2 = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} = 2y$$

Or, $(1-x^2)\,y_1 - xy = 1$

Differentiating again w.r.t. $x$,

$$(1-x^2)y_2 - 2xy_1 - y - xy_1 = 0$$

$$\Rightarrow (1-x^2)y_2 - 3xy_1 - y = 0 \text{ (Ans).} \quad (1)$$

Differentiating (1) n-times by Leibnitz's theorem,

$$y_{n+2}(1-x^2) + n\cdot y_{n+1}(-2x) + \frac{n(n-1)}{1\cdot 2}\cdot y_n(-2) - 3\{y_{n+1}\cdot x + n\cdot y_n\} - y_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - (2n+3)\,xy_{n+1} - (n+1)^2 y_n = 0. \text{ (Ans)}$$

Page-249(b): If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, $|x| < 1$, show that

i) $(1-x^2)y_2 - 3xy_1 - y = 0$

ii) $(1-x^2)\, y_{n+2} - (2n+3)\, xy_{n+1} - (n+1)^2 y_n = 0$

Solⁿ:

$$y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$(1-x^2)\, y^2 = (\sin^{-1}x)^2$$

Differentiating w.r.t. $x$,

$$(1-x^2)\, 2yy_1 - 2xy^2 = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} = 2y$$

or, $(1-x^2)\, y_1 - xy = 1$

Differentiating again w.r.t. $x$,

$$(1-x^2)\, y_2 - 2xy_1 - y - xy_1 = 0$$

$$\Rightarrow (1-x^2)\, y_2 - 3xy_1 - y = 0 \quad \text{(Ans)} \quad \text{——— (1)}$$

Differentiating (1) n times by Leibnitz's theorem,

$$y_{n+2}(1-x^2) + n\, y_{n+1}(-2x) + \frac{n(n-1)}{1\cdot 2} \cdot y_n(-2) - 3\{y_{n+1}x + n y_n\} - y_n = 0$$

$$\Rightarrow (1-x^2)\, y_{n+2} - (2n+3)\, xy_{n+1} - (n+1)^2 y_n = 0 \quad \text{(Ans)}$$

Page- 265 (6):/ If $y = \cos(10\cos^{-1}x)$ show that.

$$(1-x^2)y_{12} = 21xy_{11}.$$

Soln: $y = \cos(10\cos^{-1}x)$ ——— (1)

$$y_1 = -\sin(10\cos^{-1}x)\frac{(-10)}{\sqrt{1-x^2}}$$

$$= \frac{10\sin(10\cos^{-1}x)}{\sqrt{1-x^2}}$$

$$(1-x^2)y_1^2 = 100\sin^2(10\cos^{-1}x)\}$$

$$= 100\{1-\cos^2(10\cos^{-1}x)\}$$

$$= 100(1-y^2) \quad [\text{from (1)}]$$

Differentiating again,

$$(1-x^2)2y_1y_2 - 2x\cdot y_1^2 = -2\cdot 100y\cdot y_1$$

$$\Rightarrow (1-x^2)y_2 = xy_1 - 100y \quad (\because 2y_1 \neq 0) \text{ ——— (2)}$$

Differentiating (2). 10 times with the help of Leibnitz's theorem,

$$(1-x^2)y_{12} + 10\cdot y_{11}(-2x) + \frac{10\cdot 9}{1\cdot 2}\cdot y_{10}(-2) = xy_{11} + 10\cdot y_{10}(1) - 100y_{10}$$

$$\Rightarrow (1-x^2)y_{12} = 21\cdot x\cdot y_{11}.$$

Page-[illegible] If $y=\cos(m\sin^{-1}x)$, show that

i) $(1-x^2)y_2 - xy_1 + m^2y = 0$

ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y = 0$

Also, find the value of $y_n$ when $x=0$.

Solⁿ: Given, $y=\cos(m\sin^{-1}x)$ ——— (1)

Differentiating $x$,

$$\frac{dy}{dx} = y_1 = -m\sin(m\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)y_1^2 = m^2\sin^2(m\sin^{-1}x) = m^2\{1-\cos^2(m\sin^{-1}x)\}$$

$$\Rightarrow (1-x^2)y_1^2 = m^2(1-y^2) \text{ ——— (2) [from (1)]}$$

Differentiating again,

$$(1-x^2)2y_1\cdot y_2 + y_1^2(-2x) = -m^2\cdot 2y\cdot y_1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0 \quad (\because 2y_1 \neq 0) \text{ ——— (3)}$$

Differentiating (3) n times by Leibnitz's theorem, we ge

$$(1-x^2)y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - y_{n+1}\cdot x - {}^nC_1 y_n(1) + m^2y$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0 \text{ ——— (4)}$$

Last Part: From (1), (2), (3) we have $y=1,\ y_1=0,\ y_2=-m^2$, when $x=0$.

Putting $n=1,2,3$ successively in (4), we get

$$y_3=-m^2y_1=-m^2\times 0=0$$

$$y_4=(2^2-m^2)y_2=-m^2(2^2-m^2)$$

$$y_5=(3^2-m^2)y_3=0$$

$$y_6=(4^2-m^2)y_4=-m^2(2^2-m^2)(4^2-m^2)$$

Thus, $y_n=0$, when $n$ is odd and

$$y_n=-m^2(2^2-m^2)(4^2-m^2)\cdots\{(n-2)^2-m^2\}$$, where $n$ is even.

Page-268(9): If $y = e^{\cos^{-1}x}$, show that an equation connecting $y_n$, $y_{n+1}$ and $y_{n+2}$ is given by $(1-x^2) y_{n+2}$

$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+1) y_n = 0$.

Sol$^n$: $y = e^{\cos^{-1}x}$ ——— (1)

$\therefore y_1 = e^{\cos^{-1}x} \times \frac{-1}{\sqrt{1-x^2}} = -\frac{y}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2) y_1^2 - y^2 = 0$ ——— (2)

Differentiating (2) again,

$(1-x^2)\, 2y_1 y_2 + y_1^2(-2x) - 2y \cdot y_1 = 0$

$\Rightarrow (1-x^2) y_2 - x y_1 - y = 0$ ——— (3)

Differentiating (3) n times with the help of Leibnitz's theorem, we have,

$(1-x^2) y_{n+2} + {}^nc_1\, y_{n+1}(-2x) + {}^nc_2\, y_n(-2) - x y_{n+1}$

$- {}^nc_1\, y_n(1) - y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+1) y_n = 0$

Page-269 (8①) If $y = \sin^{-1}x$, then show that,

i) $(1-x^2)y_2 - xy_1 = 0$

ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$

Find also the value of $(y_n)_0$.

Solⁿ: i) Given that, $y = \sin^{-1}x$

$$y_1 = \frac{1}{\sqrt{1-x^2}} \quad \text{———} \quad (1)$$

$\Rightarrow y_1^2(1-x^2) = 1$

$\Rightarrow 2y_1y_2(1-x^2) - 2xy_1^2 = 0$

$\Rightarrow y_2(1-x^2) - xy_1 = 0$ ——— (2)

ii) Differentiating the above equation n times by Leibnitz's theorem,

$$y_{n+2}(1-x^2) - {}^nC_1 y_{n+1} 2x - {}^nC_2 y_n 2 - \left[y_{n+1}x + {}^nC_1 y_n\right] = 0$$

$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - xy_{n+1} - n^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$

Putting $x = 0$,

$$(y_{n+2})_0 = n^2(y_n)_0 \quad \text{———} \quad (3)$$

from (1), $(y_1)_0 = 1$

from (2), $(y_2)_0 = 0$

Putting $n = 1, 3, 5$ in (3),

$$(y_3)_0 = 1^2 (y_1)_0 = 1^2 \cdot 1$$

$$(y_5)_0 = 3^2 (y_3)_0 = 3^2 \cdot 1^2 \cdot 1$$

$$(y_7)_0 = 5^2 (y_5)_0 = 5^2 \cdot 3^2 \cdot 1^2 \cdot 1$$

Hence when $n$ is odd,

$$(y_n)_0 = (n-2)^2 (n-4)^2 (n-6)^2 \cdots\cdots 3^2 \cdot 1^2 \cdot 1$$

Again putting $n = 2, 4, 6$ in equation (3),

$$(y_4)_0 = 2^2 (y_2)_0 = 0$$

$$(y_6)_0 = 4^2 (y_4)_0 = 0$$

Thus when $n$ is even,

$$(y_n)_0 = 0.$$

Hence the proof is completed.

10/ If $\log y = \tan^{-1}x$, then prove that,

i) $(1+x^2)y_2 + (2x-1)y = 0$.

ii) $(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$.

Solⁿ: i) Given that, $\log y = \tan^{-1}x$

$$y = e^{\tan^{-1}x} \quad \text{——— (1)}$$

$$y_1 = e^{\tan^{-1}x}\frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)y_1 = e^{\tan^{-1}x} \quad (\text{From } 1)$$

$$\Rightarrow (1+x^2)y_1 = y$$

Differentiating again with respect to $x$,

$$y_2(1+x^2) + 2xy_1 - y_1 = 0 \quad \text{——— (2)}$$

ii) Differentiating the last equation $n$ times by Leibnitz's theorem,

$$y_{n+2}(1+x^2) + 2x.ny_{n+1} + \frac{2.n(n-1)}{2!}y_n + 2[y_{n+1}x + ny_n] - y_{n+1} = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + y_{n+1}(2nx + 2x - 1) + (n^2+n)y_n = 0.$$

Page-270 (11) / If $y = a\cos(\log x) + b\sin(\log x)$, then prove that

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n = 0.$$

Solⁿ: Given that, $y = a\cos(\log x) + b\sin(\log x)$ — ①

$$y_1 = \frac{-a\sin(\log x)}{x} + \frac{b}{x}\cos(\log x)$$

$$\Rightarrow x y_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating it again,

$$y_2 x + y_1 = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$\Rightarrow x^2 y_2 + x y_1 = -y \quad \text{(From 1)}.$$

Differentiating this equation $n$ times by Leibnitz's theor

$$y_{n+2}x^2 + n y_{n+1} 2x + \frac{2n(n-1)}{2} y_n + y_{n+1} x + n y_n + y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n = 0$$

13/ If $y = e^{a\sin^{-1}x}$, then prove that,

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$$

Given that, $y = e^{a\sin^{-1}x}$ ——— (1)

$$y_1 = ae^{a\sin^{-1}x}\frac{1}{\sqrt{1-x^2}}$$

$\Rightarrow (1-x^2)y_1 = a^2y^2$

Differentiating it again,

$$2y_1y_2(1-x^2) - 2xy_1^2 - 2ayy_1 = 0$$

$\Rightarrow (1-x^2) - xy_1 - a^2y = 0.$

Applying Leibnitz's theorem:

$$(1-x^2)y_{n+2} - 2xy_{n+1}.n - y_n n(n-1) - y_{n+1}x - ny_n - a^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$$

14) If $y = \sin(m\sin^{-1}x)$, then show that

i) $(1-x^2)y_2 - xy_1 + m^2 y = 0$

ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$

Solⁿ: Given that $y = \sin(m\sin^{-1}x)$ ——— ①

$$y_1 = m\cos(m\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1^2(1-x^2) = m^2\cos^2(m\sin^{-1}x)$$

$$= m^2\{1-\sin^2(m\sin^{-1}x)\}$$

$$\Rightarrow y_1^2(1-x^2) = m^2 - m^2y^2.$$

Differentiating again,

$$2y_1y_2(1-x^2) - 2xy_1^2 + m^2 y y_2 = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2 y = 0$$

ii) Applying Leibnitz's theorem, we get,

$$(1-x^2)y_{n+2} - 2xny_{n+1} - \frac{2n(n-1)}{2}y_n - xy_{n+1} - ny_n + m^2y_n$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0.$$

# Expration of Function

1. Rolle's Theorem
2. Mean value Theorem
3. Taylor's Theorem

## Statement of Rolle's Theorem:

If $f(x)$ be a function such that

I) $f(x)$ is continuous in $[a, b]$

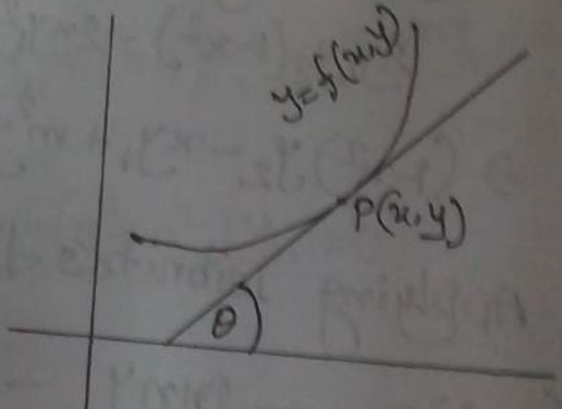
II) $f'(x)$ exists in $(a, b)$

III) $f(a) = f(b)$

Then there exists at least one value of $x$ (say $c$) in $(a, b)$ for which $f'(c) = 0$

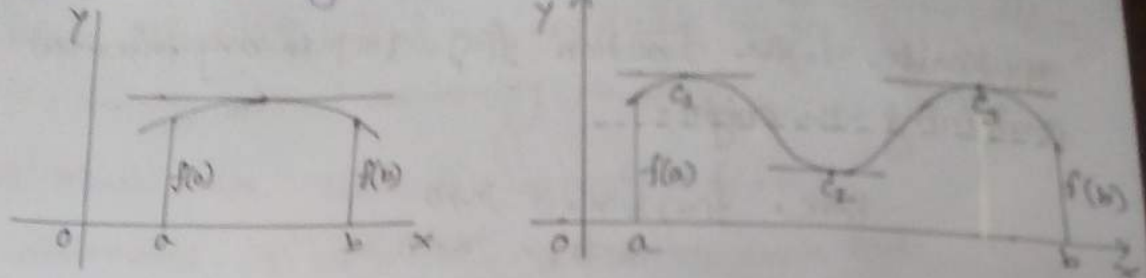
$y = f(x)$

$0 = \left.\frac{dy}{dx}\right|_{x=c} = \tan\theta$




$\frac{d f}{dx}$

## Geometrical significance of Rolle's Theorem:




Let the curve $y=f(x)$ is continuous in $[a,b]$ and $f'(x)$ exists in $(a,b)$ then, the theorem states that there exists at least one point (say $c$) in $(a,b)$ at which the ~~tangent~~ tangent to the curve $y=f(x)$ is parallel to the x-axis.

## Verify Rolle's Theorem:

$f(x)=x^2-6x+8$ in $[2,4]$

1) Since $f(x)$ is a polynomial and here it is continuous in $[2,4]$

2) $f'(x)=2x-6$ which exists in $(2,4)$

3) $f(2)=4-12+8=0$

$f(4)=16-24+8=0$

$\Rightarrow f(2)=f(4)$

$\Rightarrow f(x)$ ~~st~~ satisfies all the position of Rollo's theorem.

Then there exists at least one point (say $c$) in $(2,4)$ for which

$f'(c)=0$. $\Rightarrow 2c-6=0$

$\therefore c=3\in(2,4)$

$\therefore$ The Rolle's Theorem is varified.

Page-302 (Ex-2)/ Explain whether Rolle's theorem is applicable to the function $f(x) = |x|$ in any interval containing the origin.

Here, $f(x) = x, \quad x>0$
$= 0, \quad x=0$
$= -x, \quad x<0$

Let us consider an interval $-a \leq x \leq a$, where $a>0$.
obviously, this interval contains the origin.

Here, $f(a) = f(-a) = a$

and since, $\lim_{h \to 0^+} f(h) = 0$, $\lim_{h \to 0^-} f(h) = 0$

and $f(0)=0$ $f(x)$ is continuous at $x=0$ and so at all points in $-a \leq x \leq a$.

Now, $Lf'(0) = \lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h \to 0^+} \frac{-h}{h} = -1$

and $Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h \to 0^+} \frac{h}{h} = 1$

$\therefore$ $Lf'(0) \neq Rf'(0)$, $f(x)$ is not derivable at $x=0$.
So, it is not derivable at all point $-a \leq x \leq a$.

Second condition of Rolle's Theorem is not satisfied.

Hence, Rolle's Theorem is not applicable to $f(x) = |x|$, in $[-a, a]$, i.e., in any interval containing the origin.

## Mean value Theorem (Lagrange's Theorem):

Statement: If $f(x)$ be a function such that

1. If $f(x)$ is continuous in $[a, b]$
2. $f'(x)$ exists in $(a, b)$

Then there exists at least one point (say $c$) in $(a, b)$ for which

$$f(b) - f(a) = (b-a) f'(c)$$

Proof: Let $\phi(x) = f(x) - Ax$ ——— ① where $a$ is a constant to be determind later. Such that,

$$\phi(b) = \phi(a)$$

$$\Rightarrow f(b) - Ab = f(A) - Aa$$

$$A = \frac{f(b) - f(A)}{b-a}$$

$$① \Rightarrow \phi(x) = f(x) - \frac{f(b) - f(a)}{b-a} x \text{ ——— } ②$$

Since, ① $f(x)$ is continuous and $x$ is continuous in $(a, b)$.

$\Rightarrow f(x)$ is continuous in $[a, b]$

(ii) $f'(x)$ exists in $(a, b)$ also the derivative of $x$ exis in $(a, b)$.

$\Rightarrow \phi'(x)$ exists.

(iii) Also, $\phi(a) = \phi(b)$.

Thus, $\phi(x)$ satisfies all the condition of Rolle's theorem and hence there exists at least one point (say c) in $(a,b)$ for which,

$$\phi'(c)=0$$

$$f'(c) = \frac{f(b)-f(a)}{b-a} = 0$$

$$= f(b)-f(a) = (b-a)f'(c).$$

NOTE: 1

**Deducation of Rolle's Theorem of mean value theorem:**

We know, $f(b)-f(a) = (b-a)f'(c)$, $c \in (a,b)$ — ① $\boxed{a<c<b}$

Now, if $f(a) = f(b)$

$$① \Rightarrow f'(c) = 0, \quad c \in (a,b)$$

This is the Rolle's Theorem.

NOTE: 2

**Another form of mean value Theorem:**

Let us prove, $b-a = h$ ie. $b = a+h$

$$① \Rightarrow f(a+h)-f(a) = hf'(c)$$

$$\Rightarrow f(a+h)-f(a) = hf'(a+\theta h), \quad c = a+\theta h, \ 0<\theta<1$$

$$\Rightarrow f(a+h) = f(a) + hf'(a+\theta h)$$

$a < c < \theta$

$0 < c-a < b-a$

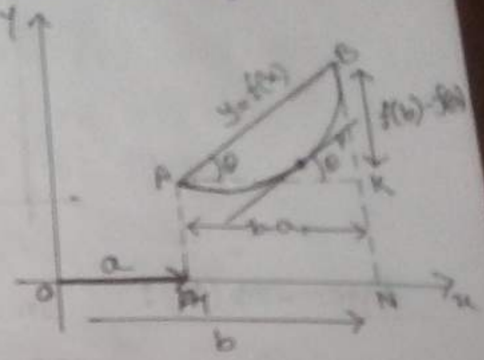
$0 < \frac{c-a}{b-a} < 1$

$0 < \frac{c-a}{h} < 1$

$\frac{c-a}{h} = \theta$

## ** Geometrical interpitation of mean value Theorem:

Let, A and B be two points on the curve $y = f(x)$.




$OM = a$ $\quad$ $AM = f(a)$

$ON = b$ $\quad$ $BN = f(b)$

Thus the co-ordinates of A is $(a, f(a))$ and B is $(b, f(b))$. In the adjoining figure, $AM \perp OX$, $BN \perp OX$, $AK \perp BN \Rightarrow AK \parallel OX$.

Let $\angle BAK = \theta$

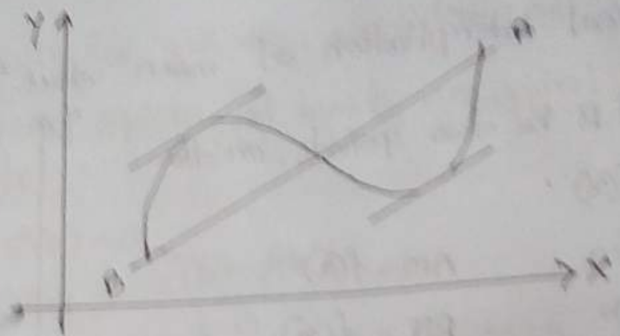
$$\therefore \tan\theta = \frac{BK}{AK}$$

$$\Rightarrow \tan\theta = \frac{f(b) - f(a)}{b-a} \quad \text{——— (1)}$$

Suppose, $f(x)$ satisfies all the condition of mean value theorem. Then there exists at least one point (say c) in $(a,b)$. Now, (1) will be true for $x = c$, if $\tan\theta = f'(c)$.

$$\text{Thus, (1)} \Rightarrow \frac{f(b) - f(a)}{b-a} = f'(c)$$

Hence, we conclude that, "If a curve has a tangent at each of it's point, then there exists at least one point (say P) on the curve, such that, the tangent at this point (P) parallel to the chord joining its extremities."




Page-285 (3): In the Mean Value Theorem

$$f(a+h) = f(a) + hf'(a+\theta h).$$

if $a=1$, $h=3$ and $f(x)=\sqrt{x}$, find $\theta$.

Soln: Given that, $f(x) = \sqrt{x}$, $a=1$, $h=3$

$\therefore f'(x) = \dfrac{1}{2\sqrt{x}}$

$f(a+h) = \sqrt{1+3} = 2$

$f'(a+\theta h) = \dfrac{1}{2\sqrt{(1+3\theta)}}$

$f(a+h) = \sqrt{x+h}$

$f(a+h) = f(a) + hf'(a+\theta h)$

$\Rightarrow 2 = 1 + 3\dfrac{1}{2\sqrt{(1+3\theta)}}$

$\Rightarrow 2\sqrt{(1+3\theta)} = 3$

$\Rightarrow 1+3\theta = 9/4$

$\therefore \theta = 5/12$

Q.) If $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h)$, $0<\theta<1$. find $\theta$, when $h=7$ and $f(x) = \frac{1}{1+x}$.

Given that, $f(x) = \frac{1}{1+x}$

$\therefore f'(x) = -\frac{1}{(1+x)^2}$ and $f''(x) = \frac{2}{(1+x)^3}$

Given relation is $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h)$

ie. $\frac{1}{1+h} = 1 - h + \frac{h^2}{2!} \frac{2}{(1+\theta h)^2}$

when $h=7$, it becomes,

$\frac{1}{8} = 1 - 7 + \frac{49}{2} \frac{2}{(1+7\theta)^2}$

$\therefore \theta = 1/7$.

Page-286 12(i): In the mean value theorem

$f(a+h) - f(a) = hf'(a+\theta h)$, $0<\theta<1$,

if $f(x) = 1/3 x^3 - 3/2 x^2 + 2x$, and $a=0$, $h=3$,

Show that $\theta$ has got two values and find them

Soln: Given that, $f(x) = \frac{x^3}{3} - 3/2 x^2 + 2x$; $a=0$, $h=3$.

$f'(x) = x^2 - 3x + 2$

so, $f(a+h) = f(3) = 9 - \frac{27}{2} + 6$

$= 3/2$

$f(a) = f(0) = 0$

Given relation is $f(a+h) - f(a) = hf'(a+\theta h)$.

It becomes

$f(3) - f(0) = 3(9\theta^2 - 9\theta + 2)$

$\Rightarrow \frac{3}{2} = 3(9\theta^2 - 9\theta + 2)$

$\Rightarrow 9\theta^2 - 9\theta + \frac{3}{2} = 0$

$\Rightarrow \theta^2 - \theta + \frac{1}{6} = 0$

$\Rightarrow \theta = \frac{1}{6}(3 \pm \sqrt{3})$.

Thus $\theta$ has got two values.

Page- 804 5(1): Discuss the applicability of the mean value theorem $f(b) - f(a) = (b-a) f'(c)$, $a < c < b$.

Find $c$, if the theorem is applicable.

(1) $f(x) = x(x-1)(x-3)$ $0 \leq x \leq 4$.

Solⁿ:

$f(x) = x(x-1)(x-3)$

$= x^3 - 4x^2 + 3x$

$f'(x) = 3x^2 - 8x + 3$

$f(x)$ being a polynomial function of $x$ is continuous in ~~0≤4~~ $0 \leq x \leq 4$.

$f'(x)$ also being a polynomial function of $x$, exists in $0 < x < 4$.

so, $f(x)$ satisfies the condition of ~~x exists~~ Lagrange's Mean value theorem in $0 \leq x \leq 4$.

There exists at least one point $\epsilon$ in $0<x<4$, such that $f(b)-f(a)=(b-a)f'\epsilon$

i.e., $f(4)-f(0)=(4-0)f'\epsilon$

$\Rightarrow 12-0=4(\epsilon^2-8\epsilon+3)$

$\Rightarrow 3\epsilon^2-8\epsilon=0 \quad \therefore \epsilon=0, 8/3$

$\because 0<\epsilon<4$, $\epsilon=8/3$ (The value $\epsilon=0$ is rejected).

(iii) $f(x)=|x|$, $-1\leq x<1$.

Soln: Here, $f(x)=|x|$ is continuous in $-1\leq x<1$ but $f'(x)$ does not exists at $x=0$, in the interval. so mean value theorem is not applicable for the function $f(x)=|x|$ in the interval $-1\leq x\leq 1$.

$\boxed{\text{See Rolle's Theorem applicable function}}$

(iv) $f(x)=x(x-1)(x-2)$, $a=0$, $b=1/2$

Soln: Here, $f(x)=x^3-3x^2+2x$

$f'(x)=3x^2-6x+2$

$f(x)$ being a polynomial function is continuous is $0 \leq x \leq \frac{1}{2}$ and $f'(x)$ also being a polynomial function exists in $0 < x < \frac{1}{2}$.

$$f(\tfrac{1}{2}) = \tfrac{3}{8} \quad \therefore f(0) = 0$$

$$f(\tfrac{1}{2}) - f(0) = (\tfrac{1}{2} - 0) f'\epsilon \text{ gives.}$$

$$\tfrac{3}{8} = \tfrac{1}{2}(3\epsilon^2 - 6\epsilon + 2)$$

$$\Rightarrow 12\epsilon^2 - 24\epsilon + 5 = 0$$

$$\therefore \epsilon = \frac{6 \pm \sqrt{21}}{6}$$

$$\because 0 < \epsilon < \tfrac{1}{2} \quad , \therefore \epsilon = \frac{6 - \sqrt{21}}{6}.$$

Page-281 Ex2/ If $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$, find $\theta$. when $h = 1$, and $f(x) = (1-x)^{5/2}$.

~~Ex:~~ Solⁿ: we have $f(h) = (1-h)^{5/2}$. Since $f(x) = (1-x)^{5/2}$

$$\therefore f'(h) = -\tfrac{5}{2}(1-h)^{3/2} \;;$$

$$f''(h) = \tfrac{15}{4}(1-h)^{1/2}$$

$$\therefore f(0) = 1, \; f'(0) = -\tfrac{5}{2}$$

from the given relation,

$$(1-h)^{5/2} = 1 - \tfrac{5}{2}h + \frac{h^2}{2!} \cdot \frac{15}{4}(1 - \theta h)^{1/2}.$$

Putting $h = 1$, $\theta$

$$0 = 1 - 5/2 + \frac{15}{4}(1-\theta h)^{1/2}. \quad \text{whence } (1-\theta)^{1/2} = 4/5,$$

$$\therefore 1-\theta = \frac{16}{25}$$

$$\therefore \theta = 9/25$$

## Taylor's Theorem in finite form:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n,$$

$$R_n = \frac{h^n}{n!} f^n(a+\theta h) \quad \text{——— (1)}$$

if $n \to \alpha$, $R_n \to 0$

(1) $\Rightarrow f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots$ to $\alpha$ [in infinite form]

## Maxima and minema (extenme value):

A function $f(x)$ is said to have a maximum value at $x=a$, if there exists a (+ve) number $h$, such that, $a-h < x < a+h$ for which $f(a) > f(x)$.

Here, $f(a+h) - f(a) < 0$ also

$f(a-h) - f(a) < 0$

minimum:
$f'(e-h) = -ve$
$f'(e+h) = +ve$

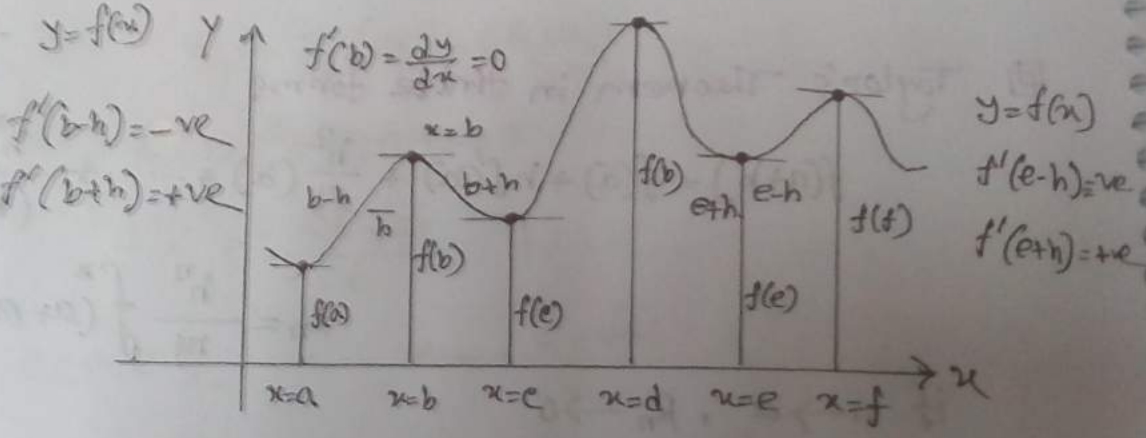
Absolute value:

A function $f(x)$ is said to have an absolute maximum at $x=a$ if $f(a) >$ all other values of $x$.

Maxima and minima (For a function):

$f'(x)=0$

$f'(e)=0$




$f(a)$ = absolute minimum

$f(b)$ = absolute maximum

$f(b)$ = Local maximum

$f(e)$ = Local minimum

→ definition

Local maximum

Local minimum

Global / absolute maximum, minimum.

$\boxed{\text{222 — " NOTE 1,2,3."}}$

# Necessary condition for maxima or minima: $y = f(x)$

__Statement:__ If a function $f(x)$ is extremum at $x=a$ and if $f'(a)$ exists then $f'(a)=0$.

__Proof:__ Suppose $f(x)$ is maxima at $x=a$.

Then, $f(a+h)-f(a)<0$ and $f(a-h)-f(a)<0$ where, h is very small+num.

$$\Rightarrow \frac{f(a+h)-f(a)}{h}<0 \text{ and } \frac{f(a+h)-f(a)}{-h}>0$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}<0 \text{ and } \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}>0$$

$$\Rightarrow Rf'(a)<0 \qquad \text{and} \qquad Lf'(a)>0$$

Since, $f'(a)$ exists, so $Rf'(a) = Lf'(a) = 0$

$$\Rightarrow f'(a)=0$$

Similarly, $f(x)$ is minimum at $x=0$ then also $f'(a)=$

*** Determination of maximam or minima:

Statement: If a function $f(x)$ is continuous at $x=a$ and $f'(a)=0$, $f''(a)\neq 0$ then,

i) $f(x)$ will have maximum values at $x=a$ if $f''(a)<0$.

ii) $f(x)$ will have minimum values at $x=a$ if $f''(a)>0$.

Proof: If $f(x)$ is maxima at $x=a$ then,

$f(a+h)-f(a)<0$, $h$ is very small ⊕ quantity

By Taylor's Theorem,

Now, $f(a+h)=f(a)+\frac{hf'(a)}{1!}+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)\cdots$

$\Rightarrow f(a+h)-f(a)<0$ whenever $f''(a)<0$.

Similarly, $f(x)$ will have minimum values at $x=a$ if $f''(a)>0$.

*** $f(a+h)=f(a)+\underset{0}{\underline{hf'(a)}}+\frac{h^2}{2!}\underset{0}{\underline{f''(a)}}+\frac{h^3}{3!}f'''(a)$

$\Rightarrow f(a+h)-f(a)=\frac{h^3}{6}\underset{0}{\underline{f'''(a)}}$

NOTE: ① If even order derivative is positive or negative, then maxima or minima exists.

ii) when all even order derivatives are zero add odd order derivatives are constant (not equat to zero) then, neighter maxima nor minima of the given function.

## Determination of absolute maxima / minima values:

Page-321 & 322

let $f(x)$ be a ~~condition~~ continuous function defined in $[a, b]$ and $\lambda_1, \lambda_2, \cdots \lambda_n \in [a, b]$, where $f'(\lambda_r) = 0$ for $r = 1, 2, \cdots n$.

Then, absolute maxima value $= \max\{f(a), f(\lambda_1), f(\lambda_2) \cdots f(\lambda_n), f(b)\}$

absolute minima value $= \min\{f(a), f(\lambda_1), f(\lambda_2) \cdots f(\lambda_n), f(b)\}$

Page-322: Ex(2) / Find the global minimum if the $f(x) = x^3 - 6x^2 + 9x + 1$ in $[0, 1]$.

Solⁿ:

$$f(x) = x^3 - 6x^2 + 9x + 1$$

$$f'(x) = 3x^2 - 12x + 9 = 0 \qquad [0, 1]$$

$$x = 1, 3 \qquad \text{But } 3 \notin [0, 1]$$

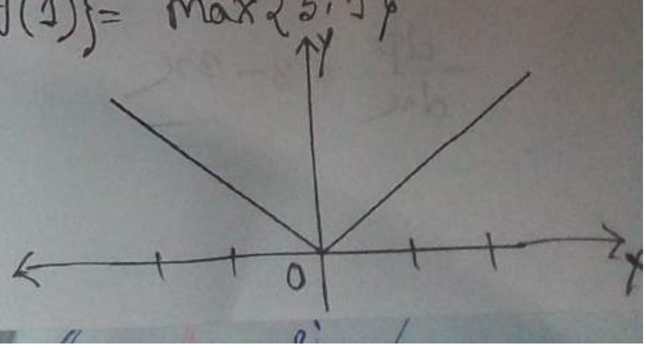
$$\therefore x = 1.$$

A/c to definition

Absolute max $= \text{Max}\{f(0), f(1)\} = \max\{5, 1\}$

$$f(x) = |x|, \quad -2 \le x \le 2$$

$$= \begin{cases} -x, & x < 0 \\ 0, & x = 0 \\ x, & x > 0 \end{cases}$$



$$Rf(2+h) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$l = f(2-h) = \frac{2+h-2}{h} = 1$$

$$= \lim_{h\to 0} \frac{f(2-h)-f(2)}{h}$$

$$\max = \{f(x), f(-2)\}$$
$$= \{2, 2\}$$

$\boxed{f(0)=0}$ → excluded.

Page-330 → Ex: 1, 2

Page-334 → Ex: 5 (ii)

Page → 343 → Ex: 17 (i) / Given that, $\frac{x}{2} + \frac{y}{3} = 1$. find the maximum value of $xy$ and minimum value of $x^2+y^2$.

Soln.
$$\frac{x}{2} + \frac{y}{3} = 1$$
$$\Rightarrow 3x + 2y = 6$$
$$\Rightarrow 2y = 6 - 3x$$
$$\therefore y = 3 - \frac{3}{2}x$$

Now, let, $P = xy$
$$= \cancel{x(} x\left(3 - \frac{3}{2}x\right)$$
$$\boxed{P = 3x - \frac{3}{2}x^2}$$

$$\frac{dP}{dx} = 3 - 3x$$

For maxima $\frac{dP}{dx} = 0$

$\Rightarrow 3 - 3x = 0$

$\therefore x = 1$

$\frac{d^2P}{dx^2} = -3 < 0 \Rightarrow P = xy$ is maximum at $x = 1$.

$\therefore$ Maximum value $= P$

$= 3x - \frac{3}{2}x^2$

$= 3.1 - \frac{3}{2}.1$

$= 3 - \frac{3}{2}$

$= \frac{3}{2} = \cancel{1.5}$

$= 1.5$.

Again, $v = x^2 + y^2 = x^2 + \frac{3}{2}(2-x)^2$

$\therefore \frac{dv}{dx} = 2x - \frac{9}{2}(2-x) = 0$ when, $x = \frac{18}{13}$

$\frac{d^2v}{dx^2} = 2 + \frac{3}{2}$ is positive.

$\therefore v$ is minimum at $x = \frac{18}{13}$

$\therefore$ Minimum value of $x^2 + y^2 = \left(\frac{18}{13}\right)^2 + \frac{9}{4}\left(2 - \frac{18}{13}\right)^2$

$= \frac{36}{13}$.

17(i) Show that the minimum value of $\frac{(2x-1)(x-8)}{(x-1)(x-4)}$ is greater than its maximum value.

---

17(ii) Given $xy=4$, find the maximum and minimum values of $4x+9y$.

Soln: Given $xy=4$ $\quad [\therefore y = 4/x]$

$\therefore y = 4/x$

Let $u = 4x+9y$

$= 4x + 36/x$

$\frac{du}{dx} = 4 - 36/x^2$

For maximum or minimum value $\frac{du}{dx} = 0$.

ie, $4 - 36/x^2 = 0$

$\therefore x = 3, -3$

now, $\frac{d^2u}{dx^2} = 0 + 72/x^3$ at $x=3$, $\frac{d^2u}{dx^2}$ is positive.

Hence, $u$ is minimum at $x=3$ and minimum value of $u = 4\times3 + \frac{36}{3}$

$= 24.$

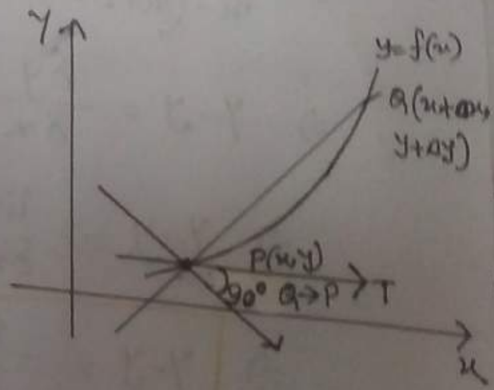
When, $x = -3$, $\frac{d^2u}{dx^2}$ is negative.

Hence $u$ is maximum at $x = -3$.

Hence, maximum value of $u$ at $x = -3$,

$$u = 4 \times (-3) + \frac{36}{3}$$
$$= -24.$$

## Tangent and Normal




Definition: Let P and Q be two neighbouring points on a curve $y = f(x)$ as shown in the adjoining figure.

Now, if $Q \to P$ then the straight line line $PQ \to PT$ which is on the tangent to the curve $y = f(x)$ at $P(x, y)$.

Normal in the straight line perpendicular to the tangent at the tangent point. In fig. PN is the normal at $P(x,y)$.

## Equation of tangent and Normal of the curve $Y = f(x)$

From the figure,

The equation of the line PQ

$Y = f(x) \rightarrow$ Explicit

$f(x,y) = 0 \rightarrow$ Implicit

$$\frac{X - x_1}{x_1 - x_2} = \frac{Y - y_1}{y_1 - y_2}$$

$$\Rightarrow \frac{X - x}{x - (x + \delta x)} = \frac{Y - y}{y - (y + \delta y)}$$

$$\Rightarrow Y - y = \frac{\delta y}{\delta x}(X - x)$$

$$\Rightarrow Y - y = \lim_{\delta x \to 0} \frac{\partial y}{\partial x}(X - x)$$

$$\Rightarrow \boxed{Y - y = \frac{\partial y}{\partial x}(X - x)}$$

$Q \rightarrow P$

$\partial x \rightarrow 0$

$\partial y \rightarrow 0$

$PQ \rightarrow PT$

$$Y - y = \frac{\partial y}{\partial x}(X - x) \longrightarrow \text{at } (x, y) \text{ PT}$$

The equation of tangent to the curve $Y = f(x)$ at $(x_1, y_1)$ then

$$Y - y_1 = \frac{dy}{dx}(X - x_1)$$

$x = x_1$

$y = y_1$

The equation of normal at $(x_1; y_1)$

$$y - y_1 = -\frac{1}{\frac{dy}{dx}}(x - x_1)$$

$$\Rightarrow (x - x_1) + (y - y_1)\frac{dy}{dx} = 0.$$

## [1] Equation of tangent or Normal for implicit function:

Let, $f(x,y)=0$

$\therefore df=0$

$\Rightarrow \frac{\partial f}{\partial x}\,dy = \frac{\partial f}{\partial y}\,dx=0$

$\Rightarrow f_x\,dx + f_y\,dy = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}$

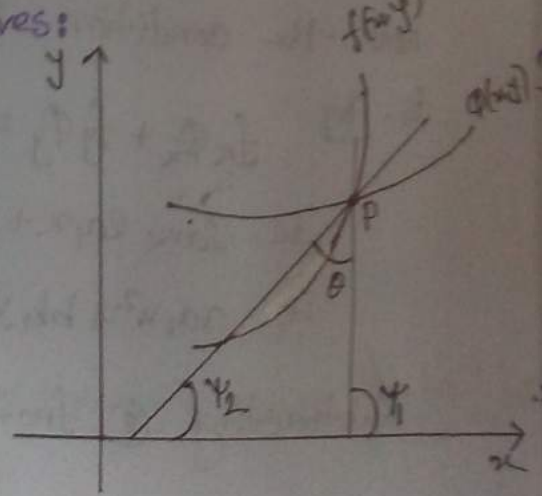
NOTE: 1) The equation tangent for the implicit function at $(x_1,y_1)$

$$(x-x_1)f_x + (y-y_1)f_y = 0$$

2) And the equation of Normal at $(x_1,y_1)$ is–

$$\frac{x-x_1}{f_x} = \frac{y-y_1}{f_y}.$$

## [1] Angle of intersection of two curves:

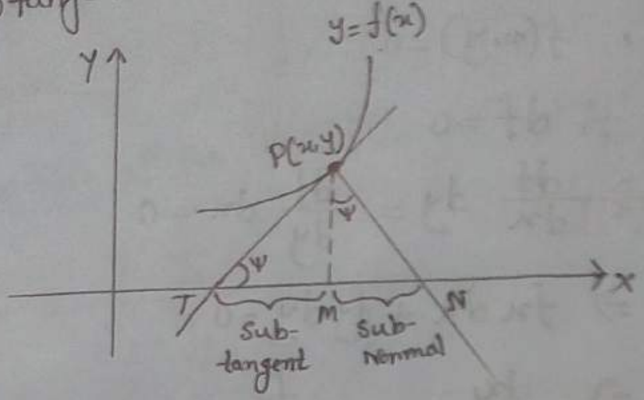



If $\theta$ be the angle between two tangents at $(x,y)$ then,

$$\tan\theta = \frac{\phi_x f_y - f_x \phi_y}{f_x\phi_x + f_y\phi_y}$$

## [1] Cartesian, Subtangent, subnormal.

Cartesian, Subtangent, SubNormal :




Page-443 Ex: 3/ Find the condition that the conics $ax^2+by^2=1$ and $a_1x^2+b_1y^2=1$.

Soln: shall cut orthogonally

The equation of the conics are

$$f(x,y) \equiv ax^2+by^2-1=0 \quad \text{———} \quad ①$$

$$\phi(x,y) \equiv ax_1^2+by_1^2-1=0 \quad \text{———} \quad ②$$

Now, the condition that they should cut orthogonally at $(x,y)$ is, by

$$f_x\phi_x + f_y\phi_y = 0$$

ie, $2ax \cdot 2a_1x + 2by \cdot 2b_1y = 0$,

ie, $aa_1x^2 + bb_1y^2 = 0$ ——— ③

Subtracting ② from ①,

$$(a-a_1)x^2+(b-b_1)y^2=0 \quad \text{———} \quad ④$$

Comparing ③ from ④, we get,

$$\frac{a-a_1}{aa_1} = \frac{b-b_1}{bb_1} \Rightarrow \frac{1}{a_1}-\frac{1}{a} = \frac{1}{b_1}-\frac{1}{b}.$$

Page-446 : 9(i)/ Show that the straight line $x/a + y/b = 2$ touches the curve $(x/a)^n + (y/b)^n = 2$ at the point $(a,b)$, whatever be the value of $n$.

Solⁿ: The given curve is $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 2$

Differentiating with respect to $x$

$$\frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x^{n-1}b^n}{a^n y^{n-1}}$$

$$\therefore \frac{dy}{dx}(a,b) = -\frac{b}{a}$$

Equation of tangent at $(a,b)$ is

$$y-b = -\frac{b}{a}(x-a)$$

$$\Rightarrow \frac{y}{b} - 1 = -\frac{x}{a} + 1$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 2.$$

9(ii)/ Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $\frac{x}{a} + \log\left(\frac{y}{b}\right) = 0$

Solⁿ: Given that, $\frac{x}{a} + \log\left(\frac{y}{b}\right) = 0$

Differentiating with respect to $x$,

$$\frac{1}{a}+\frac{b}{y}\cdot\frac{1}{b}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y}{a}.$$

Equation of the tangent is

$$Y-y=-\frac{y}{a}(X-x)$$

$$\Rightarrow \frac{Y}{y}=\frac{X}{a}+\frac{x}{a}$$

$$\Rightarrow \frac{X}{a}+\frac{Y}{y}=1+\frac{x}{a}$$

This becomes identical with $\frac{X}{a}+\frac{Y}{b}=1$ when $x=0$ and $y=b$ which point clearly satisfies the given curve.

Then $\frac{X}{a}+\frac{Y}{b}=1$ touches the given curve.

Page- 447: Ex-10 (i)] If $lx+my=1$ touches the curve $(ax)^n+(by)^n=1$. show that $al^3+2alm^2=m^2$.

Soln: Let the line $lx+my=1$ touches the given curve $(ax)^n+(by)^n=1$ at the point $(h,k)$,

then $(ah)^n+(bk)^n=1$ ——— (1)

Now from the curve on differentiating,

$$na^n x^{n-1}+nb^n y^{n-1}\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{a^n x^{n-1}}{b^n y^{n-1}}$$

value of $\frac{dy}{dx}$ at the point $(h,k)=-\frac{a^n h^{n-1}}{b^n k^{n-1}}$

Equation of the tangent to the curve on the point $(h,k)$ is

$$y-k=-\frac{a^n h^{n-1}}{b^n k^{n-1}}(x-h)$$

$$\Rightarrow yb^n k^{n-1}-b^n k^n=-xa^n h^{n-1}+a^n h^n$$

$$\Rightarrow xa^n h^{n-1}+yb^n k^{n-1}=(ah)^n+(bk)^n$$

$$\Rightarrow xa^n h^{n-1}+yb^n k^{n-1}=1 \quad [\text{from } (1)]$$

Now the equation must be identical with $lx+my=1$.

Hence on comparing, we have

$$\frac{a^n h^{n-1}}{l}=\frac{b^n k^{n-1}}{m}=1$$

$\Rightarrow a^n h^{n-1} = l$ and $b^n k^{n-1} = m$

$\Rightarrow a^{n-1} h^{n-1} = l/a$ $\Rightarrow b^{n-1} k^{n-1} = m/b$

$\therefore ah = (l/a)^{1/(n-1)}$ $\Rightarrow bk = (m/b)^{1/n-1}$

$\therefore (ah)^n = (l/a)^{n/n-1}$ $\Rightarrow (bk)^n = (m/b)^{n/n-1}$

But $(ah)^n + (bk)^n = 1$

Therefore, $(l/a)^{n/n-1} + (m/b)^{n/n-1} = 1$ which is the required condition.

10(ii) If $lx + my = 1$ is a normal to the parabola $y^2 = 4ax$, then show that $al^3 + 2alm^2 = m^2$.

Solⁿ: Here, $y^2 = 4ax$ —— (i)

Let, $f(x,y) = y^2 - 4ax = 0$

$f(x) = -4a$

$f_y = 2y$

Equation of normal at $(x,y)$ is,

$$\frac{X-x}{f_x} = \frac{Y-y}{f_y}$$

$$\Rightarrow \frac{X-x}{-4a} = \frac{Y-y}{2y}$$

$\Rightarrow 2Xy + 4aY = 2xy + 4ay$

$\Rightarrow Xy + 2aY = y(x+2a)$ —— (ii)

Also given that, $lx+my=1$ is the normal.

we rewrite, $lx+my=1$ ——— (3)

(2) and (3) must be identical. Thus,

$$\frac{y}{l}=\frac{2a}{m}=\frac{y(x+2a)}{1}$$

$$\Rightarrow y=\frac{2al}{m}, \quad y(x+2a)=\frac{2a}{m}$$

$$\Rightarrow x+2a=\frac{2a}{m}\cdot\frac{1}{y}$$

$$=\frac{2a}{m}\cdot\frac{m}{2al} \quad \boxed{y\text{'s value}}$$

$$=\frac{1}{l}$$

$$\therefore x=\frac{1}{l}-2a$$

$$\left(\frac{2al}{m}\right)^2=4a\left(\frac{1}{l}-2a\right)$$

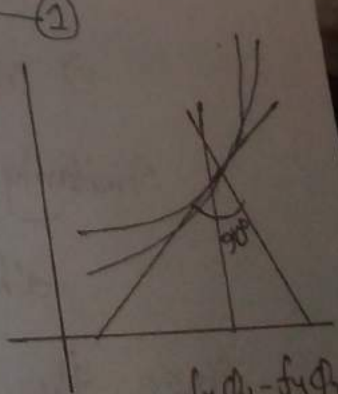
$$al^3+2alm^2=m^2.$$

**

Page-442 (Ex: 13(1)) / Prove that the curves $\frac{x^2}{a}+\frac{y^2}{b}=1$ and $\frac{x^2}{a'}+\frac{y^2}{b'}=1$ will cut orthogonally if $a-b=a'-b'$.

Solⁿ: Let $f(x,y)=\frac{x^2}{a}+\frac{y^2}{b}-1=0$ ——— (1)

$$\phi(x,y)=\frac{x^2}{a'}+\frac{y^2}{b'}-1=0 \text{ ——— (2)}$$




$$f_x=\frac{2x}{a}, \quad f_y=\frac{2y}{b}$$

$$\phi_x=\frac{2x}{a}, \quad \phi_y=\frac{2y}{b}$$

$$\tan\theta=\frac{f_x\phi_y-f_y\phi_x}{f_x\phi_x+f_y\phi_y}$$

If $\theta$ be the angle between (1) and (2)

then $\tan\theta=$ ___________

As the equation, $\theta = \pi/2$

$\therefore \tan \pi/2$ = undefined

$\Rightarrow f_x \phi_x + f_y \phi_y = 0$

$\Rightarrow a - b = a' - b'$.

** 18(b). Find the condition that the curves $ax^3 + by^3 = 1$ and $a'x^3 + b'y^3 = 1$ should cut orthogonally.

Soln: Suppose two curves are $ax^3 + by^3 = 1$ and $a'x^3 + b'y^3 = 1$ intersect at the point $(h, k)$,

Then, $ah^3 + bk^3 = 1$ and $a'h^3 + b'k^3 = 1$

Subtracting, $(a-a')h^3 = -k^3(b-b')$

$$\Rightarrow \frac{h}{k} = -\left(\frac{b-b'}{a-a'}\right)^{1/3} \quad \text{--- (1)}$$

Now equation of tangent at $(h,k)$ on the first curve is $y - k = -\frac{ah^2}{bk^2}(x-h)$

$\Rightarrow ah^2x + bk^2y = ah^3 + by^3$

$\Rightarrow ah^2x + bk^2y = 1$

Similarly tangent for the second curve is

~~$a'h^2x \cdot a'h^3 + bk^2 \cdot b'k^2 = 0$~~ $a'h^2x + b'k^2y = 2$ —— (2)

Since the curve intersect orthogonally, then tangents (1) and (2) must be perpendicular.

$$\therefore ah^2 \div a'h^2 + bk^2 \div b'k^2 = 0$$

$$\Rightarrow \frac{h^4}{k^4} = -\frac{bb'}{aa'}$$

Eliminating $h, k$ from ① ~~and ②~~,

$$-\frac{bb'}{aa'} = \frac{(b-b')^{4/3}}{(a-a')^{4/3}} \longleftarrow \Big| \because -\left(\frac{bb'}{aa'}\right)^4 = \left(\frac{bb'}{a-a'}\right)^{1/3}$$

$$\Rightarrow aa'(b-b')^{4/3} + bb'(a-a')^{4/3} = 0.$$

16/ If the tangent at $(x_1, y_1)$ to the curve $x^3+y^3=a^3$ meets the curve in $(x_2, y_2)$ show that $x_2/x_1 + y_2/y_1 = -1$.

The equation of the tangent at $(x_1, y_1)$ on $x^3+y^3=a^3$ is

$$y - y_1 = -\frac{x_1^2}{y_1^2}(x - x_1)$$

$$\Rightarrow xx_1^2 + yy_1^2 = x_1^3 + y_1^3$$

$$\Rightarrow xx_1^2 + yy_1^2 = a^3 \qquad \Big| \because (x_1, y_1) \text{ lies on } x^3+y^3=a^3$$

If this tangent passes through $(x_2, y_2)$ which lies on the given curve, then,

$$x_1^2 x_2 + y_1^2 y_2 = a^3 \text{ ——— } ①$$

$$x_1^3 + y_1^3 = a^3 \text{ ——— } ②$$

$$x_2^3 + y_2^3 = a^3 \text{ ——— } ③$$

from ① ② on subtraction

$$x_1^2(x_2-x_1)+y_2^2(y_2-y_1)=0 \quad \text{——— ④}$$

and from ①, ③

$$x_2(x_2^2-x_1^2)+y_2(y_2^2-y_1^2)=0 \quad \text{——— ⑤}$$

From ④ and ⑤ transposing and dividing (5) by (4)

$$\frac{x_2(x_2+x_1)}{x_1^2} = \frac{y_2(y_2+y_1)}{y_1^2}$$

$$\Rightarrow \left(\frac{x_2^2}{x_1^2}-\frac{y_2^2}{y_1^2}\right)+\frac{x_2}{x_1}+\frac{y_2}{y_1}=0$$

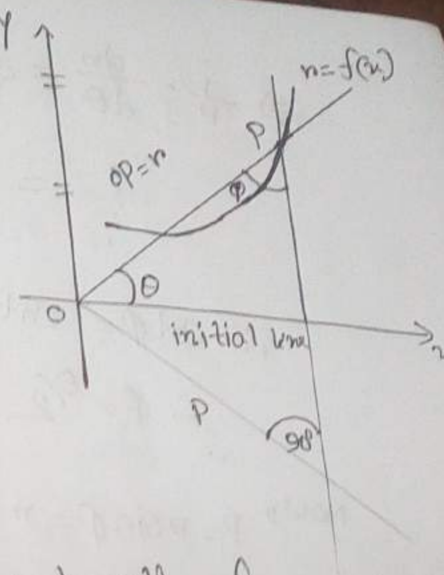
$$\Rightarrow \left(\frac{x_2}{x_1}-\frac{y_2}{y_1}\right)\left(\frac{x_2}{x_1}+\frac{y_2}{y_1}+1\right)=0$$

$$\therefore \frac{x_2}{x_1}+\frac{y_2}{y_1}+1=0 \qquad \left[\because \frac{x_2}{x_1}\neq\frac{y_2}{y_1}\right].$$

Note: i) Angle between Radius vector and tangent :—

If $\phi$ be the angle between the tangent and the radius vector at any point on the curve $r = f(\theta)$ then,

$$\tan\phi = r\frac{d\theta}{dr}$$




ii) If p be the perpendicular length from pole to the tangent, $\boxed{p = r\sin\phi.}$

Pedal equation: The relation between p and r is called pedal equation. P is the perpendicular length from pole to the tangent at any point of the given curve, and r is the radius vector.

Page-465 : 6(ii)] Find the pedal equation of the ~~elli~~ followi

Curves: $r = a(1-\cos\theta)$.

Soln: Given that, $r = a(1-\cos\theta)$

$$= 2a\sin^2\theta/2 \text{ ——— (1)}$$

$$\log r = \log 2a + 2\log\sin\theta/2$$

$$\Rightarrow \frac{1}{r}\cdot\frac{dr}{d\theta} = 2\cdot\frac{\frac{1}{2}\cos\theta/2}{\sin\theta/2}$$

$$= \cot\theta/2$$

$$\therefore \cot\phi = \cot\theta/2$$

$$\therefore \phi = \theta/2$$

Now, $p = r\sin\phi = r\sin\theta/2$

$$\Rightarrow p^2 = r^2\sin^2\theta/2 = r^2\cdot\frac{r}{2a} \quad \left[\text{From } ①\right]$$

$$\Rightarrow 2ap^2 = r^3.$$

Page-469: 18(i):1. Show that the pedal equation of the cardioide $r = a(1+\cos\theta)$ is $r^3 = 2ap^2$.

Solⁿ:

$$r = a(1+\cos\theta) \text{ ——— } ①$$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \frac{1}{r}\cdot\frac{dr}{d\theta} = -\frac{a\sin\theta}{r}$$

$$= -\frac{a\sin\theta}{a(1+\cos\theta)}$$

$$\therefore r\frac{d\theta}{dr} = -\frac{1+\cos\theta}{\sin\theta} = -\cot\theta/2$$

$$\Rightarrow \tan\phi = \tan\left(\pi/2 + \theta/2\right)$$

$$\phi = \pi/2 + \theta/2 \text{ ——— } ②$$

NOW, $P = r\sin\phi$

$= r\sin\left(\pi/2 + \theta/2\right)$

$= r\cos\theta/2$ ——— (3)

From (1) $\Rightarrow r = 2a\cos^2\theta/2$

from (3) $\Rightarrow p^2 = r^2\cos^2\theta/2$ ——— (4)

$= r^2 \cdot \dfrac{r^2}{2a}$

$\Rightarrow r^3 = 2ap.$ [by (4)]

18 (ii) / The parabola $r = 2a/(1-\cos\theta)$ is $p^2 = ar$.

Given that, $r = 2a/(1-\cos\theta)$

$\Rightarrow 1-\cos\theta = 2a/r$ ——— (1)

$\therefore \sin\theta\,\dfrac{d\theta}{dr} = -\dfrac{2a}{r^2}$

$\Rightarrow r\,\dfrac{d\theta}{dr} = -\dfrac{2a}{r\sin\theta}$

$= -\dfrac{1}{\sin\theta}(1-\cos\theta)$

$= -\tan\theta/2$

from (1)
$2\sin^2\theta/2 = \dfrac{2a}{r}$
$\therefore \sin^2\theta/2 = a/r$

$\tan\phi = \tan\left(\pi - \theta/2\right)$

$\phi = \pi - \theta/2$

Now. $P = r\sin\phi = r\sin\left(\pi - \theta/2\right)$

$\therefore \cancel{p^2 \sin}$ $\therefore p^2 = r^2 \sin\theta/_2$

$= r^2 \cdot \frac{a}{r}$

$= ar$

Hence, $p^2 = ar$.

18 (iv) / The lemniscate ~~spiral~~ $r^2 = a^2\cos 2\theta$ is $r^3 = a^2 p$.

Given that, $r^2 = a^2\cos 2\theta$

$\therefore r\frac{dr}{d\theta} = -a^2\sin 2\theta$

$\Rightarrow \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{-a^2\sin 2\theta}{a^2\cos 2\theta} = -\tan 2\theta$

$\therefore \tan\phi = -\cot 2\theta$ | $\cot 2\theta = \tan(\pi + \theta/_2)$

$\therefore \phi = \pi + \theta/2$

Now, $P = r\sin\phi = r\sin\left(\pi + \theta/_2\right)$

$\Rightarrow r\cos 2\theta = r\,\frac{r^2}{a^2}$

$\therefore Pa^2 = r^3$.

18(vii)/ The reciprocal spiral $r\theta = a$ is $p^2(a^2+r^2) = a^2r^2$.

Given that, $r\theta = a$

$$\Rightarrow \theta = a/r.$$

$$\therefore \frac{d\theta}{dr} = -a/r^2$$

$$\Rightarrow r\frac{d\theta}{dr} = -a/r$$

$$\therefore \tan\phi = -a/r$$

$$\therefore \sin\phi = \frac{a}{\sqrt{r^2+a^2}}$$

Now, $p = r\sin\phi = r\frac{a}{\sqrt{r^2+a^2}}$

$$\therefore p^2(r^2+a^2) = a^2r^2.$$

# Partial Differentiation

**Partial Derivation:** let $u = f(x, y, z)$ or $z = f(x, y)$.

Now, if we put like to differentiate $u$ with respect to $x$ we have to assume all other variables are constant. This type of differentiation is known as partial differentiation. And the result obtained by this process is known as partial derivatives / partial differential coefficient.

Mathmatically, if $u = f(x, y, z)$, then the derivative of $u$ w.r.t to $x$ is denoted and defined by

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

Similarly, $$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k, z) - f(x, y, z)}{k}$$

**Successive partial Derivatives:**

let, $v = f(x, y, t)$

$$\frac{\partial u}{\partial x} = u_x, \quad \frac{\partial u}{\partial y} = u_y \quad \text{etc}$$

Then, $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} = u_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x \partial y} = u_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y \partial x} = u_{yx}$$

② (a) Chain Rule (For ordinary derivatives)

If $F = f\,[u(x)]$

$$\frac{dF}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Exp. $F = P(x)$ , $x = t^2$

$\therefore \frac{dF}{dt} = ?$

$$\frac{dF}{dt} = \frac{dP}{dx} \cdot \frac{dx}{dt}$$

$$= 2t \frac{dP}{dt}$$

② (b) Let, $u = f(x, y)$ , $x = x(t)$ , $y = y(t)$

Then, $$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dy} \cdot \frac{dy}{dt}$$

② (c) If $r = f_1(x, y)$ , $s = f_2(x, y)$ and $u = f(r, s)$

Then $\frac{\partial u}{\partial x} = ?$ $\quad \frac{\partial u}{\partial y} = ?$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{\partial f_1}{\partial x} + \frac{\partial f}{\partial s} \cdot \frac{\partial f_2}{\partial x}$$

② (d) Chain Rule fo partial differentiation:

If $f = F(u_1, u_2, \cdots u_n)$ where, $u_1 = u_1(x_1, x_2, \cdots x_n)$
$u_2 = u_2(x_1, x_2, \cdots x_n)$

Then, $$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_2} + \frac{\partial F}{\partial u_3} \cdot \frac{\partial u_1}{\partial x_1} + \cdots + \frac{\partial F}{\partial u_n} \cdot \frac{\partial u_4}{\partial x_2}$$

## Total derivatives

let $u = f(x, y)$, $x = \varphi(t)$, $y = \psi(t)$.

## Derivative of implicit function:

let, $u = f(x, y) = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial f}{\partial x} \cdot \frac{\partial y}{\partial x}$$

$$\Rightarrow f_x + f_y \frac{\partial y}{\partial x} = 0$$

$$\Rightarrow \frac{\partial y}{\partial x} = -\frac{f_x}{f_y}.$$

**Differential:** let, $u = f(x, y)$; Then the differential of $u$ is denoted and defined by $du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

Exp: Let, $u = x^2 y^3$ Then $du$?

$$y^3 d(x^2) + x^2 d(y^3)$$

$$= y^3 2x\,dx + x^2 3y^2\,dy$$

**Exact or perfect different:** The expression $M(x, y)\,dx + N(x, y)\,dy$ is called an exact differential if there exists a function $u(x, y)$ for which $du = M dx + N dy$.

Example: $\underset{N}{x dy} + \underset{M}{y dy} = d(x, y)$

$$d(x, y) = y\,d(x) + x\,d(y)$$

$$= y\,dy + x\,dy$$

Or, $M = x$
$N = y$ $\qquad \frac{dy}{dx} = 1 \qquad \frac{\partial N}{dx} = 1 \qquad xdy + ydx$

$=$

Page-383 Ex:15/ If $u = \log r$ and $r^2 = x^2 + y^2 + z^2$, prove that

$$r^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1.$$

Soln: Given that, $u = \log(x^2+y^2+z^2)^{1/2} = \frac{1}{2}\log(x^2+y^2+z^2)$

Now,
$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2+z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

Similarly, $\frac{d^2 u}{\partial y^2} = \frac{z^2+x^2-y^2}{(x^2+y^2+z^2)^2}$

and $\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$

Therefore, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} = \frac{1}{r^2}$

$$\Rightarrow r^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1.$$

16/ If $Y = f(x+ct) + \varphi(x-ct)$, show that $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$.

Solⁿ: Given that, $y = f(x+ct) + \varphi(x-ct)$

Now, $\frac{\partial y}{\partial t} = f'(x+ct)\cdot c + \varphi'(x-ct)(-c)$

$\therefore \frac{\partial^2 y}{\partial t^2} = f''(x+ct)c^2 + c^2\varphi''(x-ct)$ —— (1)

Again, $\frac{\partial y}{\partial x} = f'(x+ct)\cdot 1 + \varphi'(x-ct)\cdot 1$

$\therefore \frac{\partial^2 y}{\partial x^2} = f''(x-ct) + \varphi''(x-ct)$ —— (2)

From (1) and (2), we see that,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

Page-416 : 5(i)/ If $u = x\varphi(y/x) + \psi(y/x)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\varphi(y/x).$$

Solⁿ: let $u = v + F$ where $v = x\varphi(y/x)$ and $F = \psi(y/x)$

Now, $x\varphi(y/x)$ is a homogeneous function of degree 1 and $\psi(y/x)$ is a homogeneous function of degree 0.

$$\therefore x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v$$

and $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 0$

Now, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = x\frac{\partial}{\partial x}(v+F)+y\frac{\partial}{\partial y}(v+F)$

$$= x\frac{\partial v}{\partial x}+x\frac{\partial F}{\partial x}+y\frac{\partial v}{\partial y}+y\frac{\partial F}{\partial y}$$

$$=\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial x}\right)+\left(x\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial F}\right)$$

$$= v+0 = x\phi(y/x)$$

i.e. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = x\phi(y/x)$

5(ii) / $x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}=0.$

Solⁿ: In part (i) we have proved,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = v \quad \text{———} (1)$$

Differentiating partially both sides with respect to $x$,

$$\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial v}{\partial x} \quad \text{———} (2)$$

Again Differentiating (1) with respect to $y$,

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial y^2} = \frac{\partial v}{\partial y} \quad \text{———} (3)$$

Multiplying (2) by $x$ and (3) by $y$ adding we get

$$x\frac{\partial u}{\partial x}+x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial u}{\partial x}+y^2\frac{\partial^2 u}{\partial y^2}$$

$$= x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}.$$

$$\Rightarrow x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} + xy\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial u}{\partial x} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$= x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$\Rightarrow \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = v$$

[By Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = v$]

$$\Rightarrow v + x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = v$$

[By relation (1). $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = v$]

Therefore, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

Page-417 (11) | If $x^2+y^2+z^2-2xyz = 1$, show that

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

Sol$^{n}$: Given that, $x^2+y^2-z^2-2xyz-1=0$ —— (1)

Taking differentials.

$$2x\,dx + 2y\,dy - 2z\,dz - 2yz\,dx - 2xz\,dy - 2xy\,dz = 0$$

$$\Rightarrow (x-yz)\,dx + (y-zx)\,dy + (z-xy)\,dz = 0 \text{ —— (2)}$$

from (1) $\Rightarrow x^2 - 2xyz + y^2z^2 = 1 - y^2 - z^2 + y^2z^2$

$$\Rightarrow (x-yz)^2 = (1-x^2)(1-y^2)(1-z^2) = k^2$$

$$\therefore x - yz = \frac{k}{\sqrt{(1-x^2)}}$$

Similarly, $y - zx = \frac{k}{\sqrt{(1-y^2)}}$

and, $z - xy = \frac{k}{\sqrt{(1-z^2)}}$

Putting this value in (2),

$$\frac{dxk}{\sqrt{(1-x^2)}} + \frac{dyk}{\sqrt{(1-y^2)}} + \frac{dzk}{\sqrt{(1-z^2)}} = 0$$

$$\Rightarrow \frac{dx}{\sqrt{(1-x^2)}} + \frac{dy}{\sqrt{(1-y^2)}} + \frac{dz}{\sqrt{(1-z^2)}} = 0.$$

Page-418 (16) // If $u = F(y-z, z-x, x-y)$, Prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Given that, $u = F(y-z, z-x, x-y)$.

Suppose, $x_1 = y - z$, $x_2 = z - x$, $x_3 = x - y$.

$$\therefore \frac{\partial x_1}{\partial x} = 0, \quad \frac{\partial x_2}{\partial x} = -1, \quad \frac{\partial x_3}{\partial x} = 1$$

$$\frac{\partial x_1}{\partial y} = 1, \quad \frac{\partial x_2}{\partial y} = 0, \quad \frac{\partial x_3}{\partial y} = -1$$

$$\frac{\partial x_1}{\partial z} = -1, \quad \frac{\partial x_2}{\partial z} = 1 \quad \frac{\partial x_3}{\partial z} = 0.$$

$\because$ The given function becomes $u = f(x_1, x_2, x_3)$.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x^2} \cdot \frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_2}{\partial x} + \frac{\partial u}{\partial x_3} \cdot \frac{\partial x_3}{\partial x}$$

$$= \frac{\partial u}{\partial x_1} \cdot 0 + \frac{\partial u}{\partial x_2} \cdot (-1) + \frac{\partial u}{\partial x_3} \cdot (1) \text{ ——— } (1)$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial y} + \frac{\partial u}{\partial x_3} \cdot \frac{\partial x_3}{\partial x}$$

$$= \frac{\partial u}{\partial x_1} (1) + \frac{\partial u}{\partial x_2} \cdot (0) + \frac{\partial u}{\partial x_3} (-1) \text{ — } (2)$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_1} \cdot (-1) + \frac{\partial u}{\partial x_2} \cdot (1) + \frac{\partial u}{\partial x_3} \cdot (0) \text{ ——— } (3)$$

Adding (1), (2), (3) we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \left(\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_1}\right) + \left(\frac{\partial u}{\partial x_2} - \frac{\partial u}{\partial x_2}\right) + \left(\frac{\partial u}{\partial x_3} - \frac{\partial u}{\partial x_3}\right) = 0.$$

Page — 418 (17, 18)

Page → 419 (21).

Page → 416 (3(ii), 4).

## Homogeneous Function:

A function $u=f(x,y)$ is said to be homogeneous to a degree $n$, if it can be expressed in the form $x^n\phi(y/x)$ or $y^n\phi(x/y)$.

Ex: $u=f(x,y)=x^2+3xy+y^2$

$$=x^2\left(1+\frac{3y}{x}+\frac{y^2}{x^2}\right)$$

$$=x^2\phi(x/y)$$

or, $u=y^2\left(1+3x/y+x^2/y^2\right)$

$$=y^2\phi(x/y)$$

## Euler's theorem on Homogeneous function:

Statement: If $u=f(x,y)$ be a homogeneous function of degree $n$, then the theorem states that,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Proof: Since $u=f(x,y)$ is a homogeneous function of degree $n$ then we may write,

$$u=x^n(y/x)$$

$$\therefore \frac{\partial u}{\partial x}\; nx^{n-1}\phi(y/x)+x^n\phi(y/x)(y/x^2)$$

$$x\frac{\partial u}{\partial x}=nx^n\phi(y/x)-yx^{n-1}\phi'(y/x) \quad \text{——} (1)$$

$$\frac{\partial u}{\partial y} = x^n \phi'(y/x) \cdot \frac{1}{x}$$

$$\therefore y \frac{\partial u}{\partial y} = x^{n-1}\, y\, \phi'(y/x) \quad \text{--- (2)}$$

Now, $(1)+(2) \Rightarrow x \frac{du}{dx} + y \frac{du}{dy} = n x^n \phi(y/x)$

$$= nu$$

Page-393 (Ex:1) / If $u = \tan^{-1} \frac{x^3+y^3}{x-y}$, show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u.$$

Soln: From the given relation, we get,

$$\tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\{1+(y/x)^3\}}{x\{1-(y/x)\}}$$

$$= x^2 \phi(y/x)$$

$\therefore \tan u$ is a homogeneous function of degree 2.

Let $v = \tan u$ ; $\therefore$ by Euler's Theorem,

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 2v$$

$$\therefore x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2\tan u$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$= 2 \sin u \cos u$$

$$= \sin 2u.$$

Page-399 : 3(i) / Show that $f(x, y) = \tan^{-1} y/x + \sin^{-1} x/y$ is a homogeneous function of $x, y$. Determine the degree of homogeneity. Hence, or otherwise, find the value of

$$x \frac{\delta f}{\delta x} + y \frac{\delta f}{\delta y}.$$

Soln: $f(x, y) = \tan^{-1} y/x + \sin^{-1} x/y$

$$= \tan^{-1} y/x + \operatorname{cosec}^{-1} y/x$$

$$= x^0 \{ \tan^{-1} (y/x) + \operatorname{cosec}^{-1} (y/x) \}$$

$$= x^0 \phi (y/x).$$

Hence, $f(x, y)$ is a homogeneous function of $x, y$ of degree 0.

By Euler's theorem on homogeneous functions.

$$x \frac{\delta f}{\delta x} + y \frac{\delta f}{\delta y}$$

$$= 0 \times f = 0.$$

[∵ hence $n = 0$.]

Page-399 Ex: 4(ii) / If $u = x \cdot \phi (y/x) + \psi(x)$, prove that

$$x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = x \phi (y/x).$$

Soln: Let $v = x \cdot \phi (y/x)$ and $w = \psi (y/x)$

Then, $v$ is the homogeneous function of $x, y$ of degree 1, ~~is the homogeneous function of $x, y$ of degree 0.~~

and $w$ is a homogeneous function

so, $x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y} = 1.v = v$

and $x\frac{\delta w}{\delta x} + y\frac{\delta w}{\delta y} = 0.w = 0$

$\because$ $u = v + w$ , $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta x} + \frac{\delta w}{\delta x}$

or, $x\frac{\delta u}{\delta x} = x\frac{\delta v}{\delta x} + x\frac{\delta w}{\delta x}$

similarly, $y\frac{\delta u}{\delta y} = y\frac{\delta v}{\delta y} + y\frac{\delta w}{\delta y}$

$$\therefore x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \left(x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y}\right) + \left(x\frac{\delta w}{\delta x} + y\frac{\delta w}{\delta y}\right)$$

$$= v + 0 = v$$

$$= x \cdot \phi(y/x).$$

Page-401 Ex: 5(1) / If $v = 2\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y} + \cot\frac{v}{2} = 0.$$

Sol$^n$: $v = 2\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$\Rightarrow \cos\frac{v}{2} = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\{1+(y/x)\}}{x^{1/2}\{1+\sqrt{y/x}\}}$$

$$= x^{1/2}\phi(y/x)$$

$\therefore \cos\left(\frac{v}{2}\right)$ is a homogeneous function of $x, y$ of degree $1/2$.

Hence by Euler's theorem,

$$x\frac{\delta}{\delta x}\left(\cos\frac{v}{2}\right)+y\frac{\delta}{\delta y}\left(\cos\frac{v}{2}\right)=\frac{1}{2}\cos\frac{v}{2}$$

$$\Rightarrow x\left(-\frac{1}{2}\sin\frac{v}{2}\right)\frac{\delta v}{\delta x}+y\left(-\frac{1}{2}\sin\frac{v}{2}\right)\frac{\delta v}{\delta y}=\frac{1}{2}\cos\frac{v}{2}$$

$$\Rightarrow x\frac{\delta v}{\delta x}+y\frac{\delta v}{\delta y}+\cot\frac{v}{2}=0.$$

Page- 416 : 3(ii) | If $u=\cos^{-1}\{(x+y)/(\sqrt{x}+\sqrt{y})\}$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0.$

Here, $u=\cos^{-1}\{(x+y)/(\sqrt{x}+\sqrt{y})\}$

$$\Rightarrow \cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}}=F(x,y) \quad (\text{say})$$

Here, f is a homogeneous function of degree $1-\frac{1}{2}=\frac{1}{2}$.

By Euler's theorem we can write

$$x\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial y}=\frac{1}{2}F$$

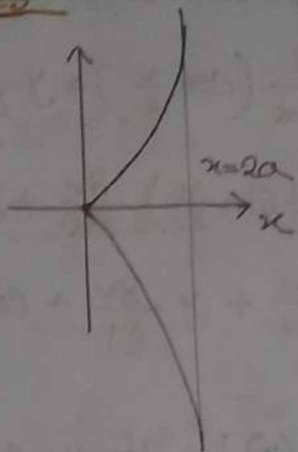
$$\Rightarrow x\frac{\partial \cos u}{\partial x}+y\frac{\partial \cos u}{\partial y}=\frac{1}{2}\cos u$$

$$\Rightarrow -x\sin u\frac{\partial u}{\partial x}-y\sin u\frac{\partial u}{\partial y}=\frac{1}{2}\cos u$$

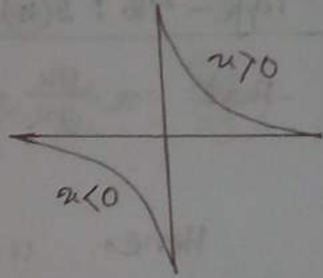
$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\cot u.$$

# Asymptotes

$$y^3 = \frac{x^3}{x-2a}$$

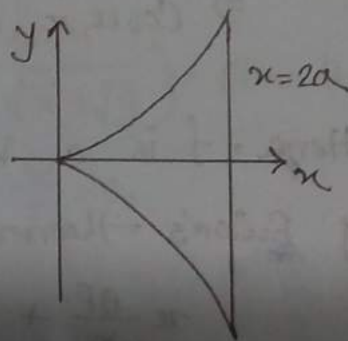



$$y = \frac{1}{x}$$




It seems that the line which touches the axis at infinite point but actually not.

A asymptotes is a stright line which touches at infinity but does not lie wholly infinity.




degree of equation = no. of asymptotes

$$y^2 = \frac{x^3}{x-2a}$$

## II Determination of Asymptotes which are all to x or y axis:

(1) $x^2y^2 - 4(x-y)^2 + 2y - 3 = 0$

We see that given equation is of degree 4 but $x^4$ and $y^4$ are ~~constant~~ absent.

In this case,

(i) The asymptotes parallel to $x$ axis are obtain by equation the coefficient of $x^2=0$.

$$\Rightarrow y^2-4=0$$
$$\Rightarrow y+2=0 \quad , y-2=0$$

(ii) The asymptote parallel to $y$ axis are obtain by equation the co-efficient of $y^2=0$.

$$x^2-4=0$$
$$\Rightarrow x+2=0 \quad , x-2=0$$

Page-512/ Special case: If the equation can be put is the form $F_n+F_{n-2}=0$, then the asymptotes are given by $F_n=0$

where, $F_n$ is a polynomial of degree $n$ of $n$ linear factors. None of which is repeated $F_{n-2}$ is at most of degree $n-2$.

Page-451 Ex:1(i)/ $x^2-4y=1$ ——— (1)

$$\Rightarrow (x+2y)(x-2y)-1=0$$

$$\Rightarrow F_2+F_0=0$$

$\Rightarrow$ the asymptotes of (1) are given by $F_2=0$

$$\Rightarrow (x+2y)=0,$$
$$(x-2y)=0.$$

Page- 509 Ex: 2(i) Find the asymptotes of :

$$x^3 + 2x^2y + xy^2 - x + 1 = 0 \quad \text{--- (1)}$$

$$\Rightarrow x(x+y+1)(x+y-1) + 1 = 0$$

$$\Rightarrow F_3 + F_0 = 0$$

where $F_3 = x(x+y+1)(x+y-1)$ is of degree 3 and it has three no-repeated linear factors and $F_0 = 1$ which is degree of 0.

The asymptotes of the curve (1) are given by.

$$F_3 = 0 \Rightarrow x(x+y+1)(x+y-1) = 0$$

$$\therefore \quad x = 0 \;,\; x+y+1 = 0 \;,\; x+y-1 = 0 .$$

~~(ii) $(x+y)^2 + 2x^2y$~~

(ii) $(x+y)^2 (x+2y+2) = x+9y+2$ --- (1)

$$\Rightarrow (x+y)^2 (x+2y+2) = (x+2y+2) + 7y$$

$$\Rightarrow (x+y)^2 (x+2y+2) - (x+2y+2) - 7y = 0$$

$$\Rightarrow (x+2y+2) \{(x+y)^2 - 1\} - 7y = 0$$

$$\Rightarrow (x+2y+2)(x+y+1)(x+y-1) - 7y = 0$$

$$\Rightarrow F_3 + F_1 = 0 .$$

The asymptotes of the curve (1) is given by,

$$F_3 = 0$$

$$\Rightarrow (x+2y+2)(x+y+1)(x+y-1) = 0$$

$$\therefore (x+2y+2) = 0, \quad x+y+1 = 0, \quad x+y-1 = 0.$$

(iii) $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$ —— 1

$$\Rightarrow 4x^3 - 4xy^2 + xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$$

$$\Rightarrow 4x(x+y)(x-y) + y^2(x-y) + 2x^2 - xy - y^2 - 1 = 0$$

$$\Rightarrow (x-y)(2x+y)^2 + 2x^2 - 2xy + xy - y^2 - 1 = 0$$

$$\Rightarrow (x-y)(2x+y)^2 + 2x(x-y) + y(x-y) - 1 = 0$$

$$\Rightarrow (x-y)(2x+y)(2x+y+1) - 1 = 0$$

$$\Rightarrow F_3 + F_0 = 0$$

$\therefore$ The asymptotes of the curve (1) is given by,

$$f_3 = 0$$

$$\Rightarrow (x-y)(2x+y)(2x+y-1) = 0$$

$$\therefore x-y = 0, \quad 2x+y = 0, \quad 2x+y-1 = 0.$$

# Integral Calculus

**Indefinite integral:** If $f(x)$ be a given function and if $F(x)$ be another function such that $\frac{d}{dx}F(x) = f(x)$; than $F(x)$ is defined as an idefinite integral of $f(x)$ w.r. to $x$. Symbodically, if $\frac{d}{dx}F(x) = f(x)$ then,

$$\int \underbrace{f(x)}_{\text{Integrand}} dx = \underbrace{F(x)}_{\text{Integral}}$$

Here, $\int f(x)\,dx$ is called indfinite integral of $f(x)$ w.r. to $x$.

In addition, if $\frac{d}{dx}\{F(x)+c\} = f(x)$, then

$$\int f(x)\,dx = F(x) + c$$

Here, $c$ is called the constant of integration.

Again, if $f(x)$ is ~~constant~~ continuous in $[a,b]$, then the definite integral of $f(x)$ w.r. to $x$ in $[a,b]$ is denoted and defined by $\int_a^b f(x)\,dx = \Big[F(x)+c\Big]_a^b$

$$= F(a) - F(b)$$

## Standard / Fundamental Integral:

1. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

2. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$ , $(|x| \neq |a|)$

3. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|$

4. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left|x+\sqrt{x^2 \pm a^2}\right|$

$\rightarrow \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| = \sinh^{-1}\frac{x}{a}$

$\rightarrow \int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| = \cosh^{-1}\frac{x}{a}$

5. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

## Method of substitution (IIB):

Rule-1: $\int \frac{dx}{ax^2+bx+c}$

Rule→2: $\int \frac{Px+q}{ax^2+bx+c}$

Rule-3: $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Rule-4: $\int \frac{Px+q}{\sqrt{ax^2+bx+c}}$

Rule-5: $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$

Rule-6: $\int \frac{dx}{(Px+q)\sqrt{ax^2+bx+c}}$

Page-34 (Ex:8) | $\int \frac{dx}{\sqrt{\{(x-\alpha)(\beta-x)\}}}$ $(\beta > \alpha)$

Put $x-\alpha = z^2$ $\therefore dx = 2z\,dz$

and $\beta - x = \beta - (z^2+\alpha)$ $\Big|\ \because x = z^2+\alpha$

$= \beta - z^2 - \alpha$

$$\therefore I = \int \frac{2z\,dz}{\{\sqrt{z^2(\beta-\alpha-z^2)}\}}$$

$$= \int \frac{2z\,dz}{z\sqrt{(k^2-z^2)}} \quad \Big|\ k^2 = \beta-\alpha$$

$$= 2\int \frac{dz}{\sqrt{(k^2-z^2)}}$$

$$= 2\sin^{-1}\frac{z}{k} = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}.$$

Page-39 (Ex:9) | $\int \frac{x\,dx}{x^4-x^2-2}$

$$= \frac{1}{2}\int \frac{dz}{z^2-z-2} \quad \Big|\ x^2 = z \Rightarrow 2x\,dx = dz \Rightarrow x\,dx = \frac{1}{2}dz$$

$$= \frac{1}{2}\int \frac{dz}{z^2 - 2\cdot\frac{1}{2}z + \frac{1}{4} - \frac{1}{4} - 2}$$

$$= \frac{1}{2}\int \frac{dz}{(z-\frac{1}{2})^2 - (3/2)^2}$$

$$= \frac{1}{2} \cdot \frac{1}{2 \cdot 3/2} \cdot \log \left| \frac{z - \frac{1}{2} - 3/2}{z - 1/2 + 3/2} \right| + c$$

$$= \frac{1}{6} \log \left| \frac{x^2 - 2}{x^2 + 1} \right| + c.$$

## Examples II(B)

Page-491

7(i): $\int \frac{dx}{1+x+x^2} = \int \frac{dx}{(x+1/2)^2 + 3/4}$

let, $z = x + \frac{1}{2}$

$\therefore dx = dz$

$$\therefore \text{Given integral} = \int \frac{dz}{z^2 + (\sqrt{3}/2)^2}$$

$$= \frac{1}{\sqrt{3}/2} \tan^{-1} \frac{2z}{\sqrt{3}} + c$$

$$= 2/\sqrt{3} \tan^{-1} \frac{2z}{\sqrt{3}} + c$$

$$= 2/\sqrt{3} \tan^{-1} \left( \frac{2x+1}{\sqrt{3}} \right) + c$$

(ii) $\int \frac{dx}{4x^2+4x+5} = \int \frac{dx}{(2x+1)^2+4}$

$$= \int \frac{dx}{(2x+1)^2 + 2^2} = \frac{1}{2} \cdot \frac{1}{2} \tan^{-1} \left( \frac{2x+1}{2} \right) + c$$

$$= \frac{1}{4} \tan^{-1} \left( \frac{2x+1}{2} \right) + c.$$

8(i) $\int \frac{dx}{1+x-x^2}$

$= \int \frac{dx}{\frac{5}{4} - \left(\frac{1}{4} - x + x^2\right)}$

$= \int \frac{dx}{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}$

$= \frac{1}{2 \cdot \frac{\sqrt{5}}{2}} \log \frac{\frac{\sqrt{5}}{2} + x - \frac{1}{2}}{\frac{\sqrt{5}}{2} - x + \frac{1}{2}} + c$

$= \frac{1}{\sqrt{5}} \log \frac{\sqrt{5} + x - 1}{\sqrt{5} - x + 1} + c$

(ii) $\int \frac{dx}{6x^2 + 7x + 2}$

$= \int \frac{dx}{\left(\sqrt{6}\,x + \frac{7}{2\sqrt{6}}\right)^2 - \frac{1}{24}}$

let, $\sqrt{6}\,x + \frac{7}{2\sqrt{6}} = z$ $\quad \therefore \sqrt{6}\,dx = dz$

Given integral $= \frac{1}{\sqrt{6}} \int \frac{dz}{z^2 - \left(\frac{1}{2\sqrt{6}}\right)^2}$

$= \log \frac{z - \frac{1}{2\sqrt{6}}}{z + \frac{1}{2\sqrt{6}}} + c$

$= \log \cancel{2z} \frac{2\sqrt{6}\,z - 1}{2\sqrt{6}\,z + 1} + c_1$

$$= \log \frac{12x+6}{12x+8} + c_1$$

$$= \log \frac{3(2x+1)}{4(3x+2)} + c_1$$

$$= \log \frac{2x+1}{3x+2} + c \quad , \quad \text{where } c_1 + \log 3/4 = c$$

* 9) $\displaystyle\int \frac{x\,dx}{x^4 + 2x^2 + 2}$

$$= \int \frac{x\,dx}{(x^2+1)^2 + 1}$$

$$= \frac{1}{2 \cdot 1} \tan^{-1}(x^2+1) + c$$

$$= \frac{1}{2} \tan^{-1}(x^2+1) + c$$

* 10) $\displaystyle\int \frac{\cos x\,dx}{\sin^2 x + 4\sin x + 3}$

$$= \int \frac{\cos x\,dx}{(\sin x + 2)^2 - 1}$$

$$= \frac{1}{2 \cdot 1} \log \frac{\sin x + 2 - 1}{\sin x + 2 + 1} + c$$

$$= \frac{1}{2} \log \frac{\sin x + 1}{\sin x + 3} + c$$

15 (i)

$$\int \frac{x\,dx}{x^2+2x+1}$$

$$= \frac{1}{2}\int \frac{(2x+2)-1}{x^2+2x+1}\,dx$$

$$= \int \frac{2(x+1)}{x^2+2x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+2x+1}\,dx$$

$$= \frac{1}{2}\log\left(x^2+2x+1\right) - \frac{1}{2}\int \frac{dx}{(x+1)^2}$$

$$= \frac{1}{2}\log\left(x^2+2x+1\right) + \frac{1}{2}\cdot\frac{1}{x+1} + c$$

$$= \frac{1}{2}\log\left(x+1\right)^2 + \frac{1}{2}\cdot\frac{1}{x+1} + c$$

(ii)

$$\int \frac{x+1}{3+2x-x^2}\,dx$$

$$= \int \frac{x+1}{(3-x)(1+x)}\,dx$$

$$= \int \frac{dx}{3-x} = -\int \frac{dx}{x-3}$$

$$= -\log(x-3)+c.$$

16 (i) $\int \frac{x+1}{x^2+4x+5}\,dx$

$= \int \frac{x+1}{(x+2)^2+1}\,dx$

Let, $x+2 = z$ $\therefore dx = dz$

Given integral $= \int \frac{z-1}{z^2+1}\,dz$ | $x+2-1 = z-1$

$= \frac{1}{2}\int \frac{2z\,dz}{z^2+1} - \int \frac{dz}{z^2+1}$

$= \frac{1}{2}\log(z^2+1) - \tan^{-1} z + c$

$= \frac{1}{2}\log(x^2+4x+5) - \tan^{-1}(x+2) + c$

(ii) $\int \frac{2x+3}{4x^2+1}\,dx$

Let $2x = z$ $\therefore 2dx = dz$

Given integral $= \int \frac{z+3}{2z^2+1}\,dz$

$= \frac{1}{4}\int \frac{2z\,dz}{z^2+1} + \frac{3}{2}\int \frac{dz}{z^2+1}$

$= \frac{1}{4}\log(z^2+1) + \frac{3}{2}\tan^{-1} z + c$

$= \frac{1}{4}\log(4x^2+1) + \frac{3}{2}\tan^{-1}(2x) + c$

17 (i) / $\int \frac{(4x+3)}{3x^2+3x+1}dx$

$= \frac{2}{3}\int \frac{(6x+3)+1}{3x^2+3x+1}dx$

$= \frac{2}{3}\int \frac{(6x+3)}{3x^2+3x+1}dx + \frac{1}{3}\int \frac{1}{3x^2+3x+1}dx$

$= \frac{2}{3}\log(3x^2+3x+1) + \frac{1}{3}\int \frac{1}{x^2+x+1/3}dx$

$= \frac{2}{3}\log(3x^2+3x+1) + \frac{1}{3}\int \frac{1}{(x+1/3)^2+\left(\frac{1}{2\sqrt{3}}\right)^2}dx$

$= \frac{2}{3}\log(3x^2+3x+1) + \frac{1}{3}\cdot 2\sqrt{3}\tan^{-1}\left\{(x+1/2)2\sqrt{3}\right\}+c$

Page-50/ * 23/ $\int \frac{dx}{\sqrt{2x^2+3x+4}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+3/2\, x+2}}$

$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+2\cdot x\cdot \frac{3}{4}+\frac{9}{16}-\frac{9}{16}+2}}$

$= \frac{1}{\sqrt{2}}\int \frac{}{\sqrt{(x+3/4)^2-\frac{23}{16}}}$

$= \frac{1}{\sqrt{2}}\log\left\{(x+3/4)+\sqrt{(x+3/4)^2-\frac{23}{26}}\right\}+c$

$= \frac{1}{\sqrt{2}}\log\left((x+3/4)+\sqrt{x^2+3/2\, x+2}\right)+c$

$*$

25) $\int \frac{dx}{\sqrt{6+11x-10x^2}}$

$= \int \frac{dx}{\sqrt{(3-2x)(2+5x)}} = \frac{1}{\sqrt{10}} \int \frac{dx}{\sqrt{(\frac{3}{2}-x)(\frac{2}{5}+x)}}$

Let, $\frac{2}{5}x = z^2$ $\therefore dx = 2z\,dz$

$\therefore$ Given integration $= \frac{1}{\sqrt{10}} \int \frac{2z\,dz}{z\sqrt{\frac{19}{10}-z^2}}$

$= \frac{2}{\sqrt{10}} \int \frac{dz}{\sqrt{\frac{19}{10}-z^2}}$

$= \frac{2}{\sqrt{10}} \sin^{-1}\left(z\sqrt{\frac{10}{19}}\right)+c$

$= \frac{2}{\sqrt{10}} \sin^{-1}\left(\sqrt{\frac{10}{19}} \times \sqrt{\frac{2+5x}{5}}\right)$

$= \frac{2}{\sqrt{10}} \sin^{-1}\sqrt{\frac{10}{19} \times \frac{2+5x}{5}}$

$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{10}} \sin^{-1}\sqrt{\frac{4+10x}{\cancel{5}}}+c$

$= \sqrt{\frac{2}{5}} \sin^{-1}\sqrt{\frac{10x+4}{19}}+c$

*26/ $\displaystyle\int \frac{\cos x\,dx}{\sqrt{5\sin^2 x - 12\sin x + 4}}$

$$= \frac{1}{\sqrt{5}} \int \frac{\cos x\,dx}{\sqrt{(2-\sin x)\left(\frac{2}{5} - \sin x\right)}}$$

$$\left| \begin{aligned} &(2-\sin x) - \tfrac{8}{5} \\ &= \frac{10 - 5\sin x - 8}{5} \\ &= \frac{2 - 5\sin x}{5} \\ &= \tfrac{2}{5} - \sin x \end{aligned} \right.$$

Let $2 - \sin x = z^2$ ; $\therefore -\cos x\,dx = 2z\,dz$

$\therefore$ Given integral $= -\frac{1}{\sqrt{5}} \displaystyle\int \frac{-2z\,dz}{z\sqrt{z^2 - 8/5}}$

$$= -\tfrac{2}{\sqrt{5}} \int \frac{dz}{\sqrt{z^2 - 8/5}}$$

$$= -\tfrac{2}{\sqrt{5}} \log\left(z + \sqrt{z^2 - 8/5}\right) + c$$

$$= -\tfrac{2}{\sqrt{5}} \log\left(\sqrt{2-\sin x} + \sqrt{\tfrac{2}{5} - \sin x}\right) + c$$

$$= -\tfrac{2}{\sqrt{5}} \log\left(\frac{\sqrt{5}\cdot\sqrt{2-\sin x} + \sqrt{2 - 5\sin x}}{\sqrt{5}}\right) + c_1$$

$$= -\tfrac{2}{\sqrt{5}} \log\left(\sqrt{5(2-\sin x)} + \sqrt{(2 - 5\sin x)}\right) + c$$

where, $c = c_1 + \tfrac{2}{\sqrt{5}} \log\sqrt{5}$.

*27/

**29.** $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

Let $x-\alpha = z^2$ ; $\therefore dx = 2z\,dz$

$\Rightarrow x = z^2+\alpha$

$\therefore$ Given integral $= \int \frac{2z\,dz}{z\sqrt{z^2+\alpha-\beta}}$

$= 2\int \frac{dz}{\sqrt{z^2+\alpha-\beta}}$

$= 2\log\left(z+\sqrt{z^2+\alpha-\beta}\right)+c$

$= 2\log\left(\sqrt{x-\alpha}+\sqrt{x-\alpha+\alpha-\beta}\right)+c$

$= 2\log\left(\sqrt{x-\alpha}+\sqrt{x-\beta}\right)+c.$

**30.** $\int \frac{x-2}{\sqrt{2x^2-8x+5}}\,dx$

Let $2x^2-8x+5 = z^2$

$\Rightarrow (4x-8)\,dx = 2z\,dz$

$\Rightarrow (x-2)\,dx = \frac{z\,dz}{2}$

Given integral $= \int \frac{z\,dz}{2z} = \frac{1}{2}z+c$

$= \frac{1}{2}\sqrt{2x^2-8x+5}+c$

31 (i)/ $\int \frac{(x+1)}{\sqrt{4+8x-5x^2}}$

$$= -\frac{1}{10}\int \frac{8-10x}{\sqrt{4+8x-5x^2}} + 9/5 \int \frac{dx}{\sqrt{4+8x-5x^2}}$$

$$= -\frac{1}{10}\cdot 2\sqrt{4+8x-5x^2} + 9/5 \cdot \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{4/5 + 8/5 x - x^2}}$$

$$= -\frac{1}{5}\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}} \int \frac{dx}{\sqrt{\frac{36}{25} - (x-4/5)^2}}$$

$$= -1/5 \sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}} \cancel{\sin^{-1}} \int \frac{5x\, dx}{\sqrt{(6/5)^2 - (x-4/5)^2}}$$

$$= -1/5 \sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}} \sin^{-1}\left(\frac{5x-4}{6}\right) + c$$

$$= \frac{9}{5\sqrt{5}} \sin^{-1}\left(\frac{5x-4}{6}\right) - 1/5\sqrt{(4+8x-5x^2)} + c.$$

(ii) $\int \frac{(2x+1)\,dx}{\sqrt{4x^2+4x+2}}\, dx$

$$= \frac{1}{\cancel{2}} \int \frac{8x \cancel{+2} -1}{}$$

$$= \frac{1}{4}\int \frac{8x+4}{\sqrt{4x^2+4x+2}}\,dx - \int \frac{dx}{\sqrt{4x^2+4x+2}}$$

$$= \frac{1}{4} \cdot 2\sqrt{4x^2+4x+2} - \int \frac{dx}{\sqrt{(2x+1)^2+1}}$$

$$= \frac{1}{2}\sqrt{4x^2+4x+2} - \log\left|(2x+1) + \sqrt{(2x+1)^2+1}\right| + c$$

$$= \frac{1}{2}\sqrt{4x^2+4x+2} - \log\left((2x+1) + \sqrt{4x^2+4x+2}\right) + c$$

32 (1) $\displaystyle\int \frac{dx}{(2x-1)\sqrt{(1+x)}}$

Let, $1+x = z^2$ $\therefore dx = 2z\,dz$

Given integral $= \displaystyle\int \frac{2z\,dz}{(z^2+1)\sqrt{z^2}}$

$$= \int \frac{2z\,dz}{(z^2+1)z}$$

$$= \int \frac{dz}{z^2+1}$$

$$= 2\tan^{-1} z + c$$

$$= 2\tan^{-1}\sqrt{1+x} + c.$$

$1+x = z^2$

$\cancel{2+2x}$

$\cancel{2x+2-1}$

$1+x = z^2$

$x = z^2-1$

$2z^2-$

37(ii) $\int \frac{dx}{x+\sqrt{x-1}}$

put, $x-1=z^2 \quad \therefore dx = 2z\,dz$

Given integral $= \int \frac{2z\,dz}{(z^2+1)+z}$

$= \int \frac{(2z+1)-1}{z^2+z+1}\,dz$

$= \int \frac{2z+1}{z^2+z+1} - \int \frac{dz}{z^2+2\cdot\frac{1}{2}z+\frac{1}{4}-\frac{1}{4}+1}$

$= \log(z^2+z+1) - \int \frac{dz}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)}$

$= \log(z^2+z+1) - \int \frac{2}{\sqrt{3}} \tan^{-1} \frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$

$= \log(x+\sqrt{x-1}) - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\sqrt{x-1}+1}{\sqrt{3}}\right)$

# Chapter-3

## Integration by parts (III):

$$\int uv\,dx = ?$$

1. $$\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|$$
$$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}.$$

2. $$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left|x-\sqrt{x^2-a^2}\right|$$
$$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}.$$

3. $$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Rule-1: $\int \sqrt{ax^2+bx+c}\,dx$

Rule-2: $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

Rule-3: $\int e^x\{f(x)+f'(x)\}\,dx$

Page-64 (Ex: 3) $\int \sqrt{4+8x-5x^2}\, dx$

$$= \int \sqrt{5\left(\frac{4}{5}+\frac{8}{5}x-x^2\right)}\, dx$$

$$= \sqrt{5}\int \sqrt{\frac{36}{25}-\left(\frac{16}{25}-\frac{8}{5}x+x^2\right)}\, dx$$

$$= \sqrt{5}\int \sqrt{\left(\frac{6}{5}\right)^2-\left(x-\frac{4}{5}\right)^2}\, dx$$

$$= \sqrt{5}\left[\frac{\left(x-\frac{4}{5}\right)\sqrt{4+8x-5x^2}}{2} + \frac{\left(\frac{6}{5}\right)^2}{2} \cancel{\log} \sin^{-1}\left(\frac{5x-4}{5\times \frac{6}{5}}\right)\right]$$

$$= \sqrt{5}\left[\frac{(5x-4)\sqrt{4+8x-5x^2}}{10} + \frac{18}{25}\sin^{-1}\left(\frac{5x-4}{6}\right)\right]$$

$$= \sqrt{5}\left[\frac{(5x-4)\sqrt{4+8x-5x^2}}{10\sqrt{5}} + \frac{18}{25}\sin^{-1}\left(\frac{5x-4}{6}\right)\right]$$

Page-65 Ex: 4 | $\int (3x-2)\sqrt{x^2-x+1}\, dx$

Since, $(3x-2) = \frac{3}{2}(2x-1) - \frac{1}{2}$

$$\therefore I = \frac{3}{2}\int (2x-1)\sqrt{x^2-x+1}\, dx - \frac{1}{2}\sqrt{x^2-x+1}\, dx \quad \text{--- (1)}$$

put $= x^2-x+1 = z \quad \therefore dz = (2x+1)\, dx$

$$1\text{st integral} = \int \sqrt{z}\,dz$$

$$= \frac{2}{3} z^{3/2} = \frac{2}{3}\left(x^2 - x + 1\right)^{3/2}$$

$$2\text{nd integral} = \int \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}\; dx$$

$$= \sqrt{z^2 + a^2}\; dz$$

$$x^2 - x + 1$$

$$x^2 - 2 \cdot \frac{1}{2} x + \frac{1}{4} - \frac{1}{4} + 1$$

$$\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

Put, $x - \frac{1}{2} = z$ and $\frac{3}{4} = a^2$

$$= \frac{z\sqrt{z^2 + a^2}}{2} + \frac{a^2}{2} \log\left(z + \sqrt{z^2 + a^2}\right)$$

$$= \frac{1}{4}(2x+1)\sqrt{x^2 - x + 1} + \frac{3}{8} \log\left(x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right).$$

From ① $\Rightarrow$

$$\therefore I = \frac{3}{2}\left\{\frac{2}{3}\left(x^2 - x + 1\right)^{3/2}\right\} - \frac{1}{2}\left\{\frac{1}{4}(2x+1)\sqrt{x^2 - x + 1} + \frac{3}{8}\left(x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right.\right.$$

$$= \left(x^2 - x + 1\right)^{3/2} - \frac{1}{8}(2x+1)\sqrt{x^2 - x + 1} - \frac{3}{16}\left(x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right.$$

Page-69 Ex:2/ $\int x\sqrt{\frac{a-x}{a+x}}\,dx$

$$= \int x\sqrt{\frac{(a-x)(a-x)}{(a+x)(a-x)}}\,dx$$

$$= \int \frac{x(a-x)}{\sqrt{a^2-x^2}}\,dx$$

$$= \int \frac{ax-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= \int \frac{ax}{\sqrt{a^2-x^2}} + \int \frac{-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= \int \frac{ax}{\sqrt{a^2-x^2}} + \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= -\frac{a}{2}\int \frac{(-2x)\,dx}{\sqrt{a^2-x^2}} + \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx - a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$= -\frac{1}{2}a\int \frac{2t}{t}\,dt + \int\sqrt{a^2-x^2}\,dx - a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$\boxed{\because a^2-x^2=t^2}$$

$$= -a\int dt + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - a^2\sin^{-1}x/a + c$$

$$= -at + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - a^2\sin^{-1}x/a + c$$

$$= \frac{1}{2}(x-2a)\sqrt{a^2-x^2} - a^2/2\,\sin^{-1}x/a + c$$

Ex:3/ $\int$

Ex:31/ $\int f'(x) = e^x(\sin x - \cos x)$

$f'(x) = \int f'(x)\,dx + c$

$= \int e^x(\sin x - \cos x)\,dx + c$

$= \int e^x \sin x\,dx - \int e^x \cos x\,dx + c$

$= e^x \int \sin x\,dx - \int\left[\frac{d}{dx}(e^x)\int \sin x\,dx\right] - \int e^x \cos x\,dx + c$

$= -e^x \cos x - \int e^x(-\cos x)\,dx - \int e^x \cos x\,dx + c$

$= -e^x \cos x + \int e^x \cos x\,dx - \int e^x \cos x\,dx + c$

$= -e^x \cos x + c$

$\therefore f(0) = 1, \quad \Rightarrow 1 = -1 + c$

$\therefore c = 2$

$\therefore f(x) = -e^x \cos x + 2$

$= 2 - e^x \cos x.$

Page-87 Ex:30/ $I = \int \frac{x^2\,dx}{(x\sin x + \cos x)^2}$

$= \int \frac{x\cos x \cdot x\sec x}{(x\sin x + \cos x)^2}\,dx$

$= x\sec x \int \frac{d(x\sin x + \cos x)}{(x\sin x + \cos x)^2} - \int\left[\frac{d}{dx}(x\sec x)\int \frac{d(x\sin x + \cos x)}{(x\sin x + \cos x)^2}\right]dx$

$$= -\frac{x\sec x}{x\sin x+\cos x} + \int \frac{\sec x + x\sec x \tan x}{x\sin x+\cos x}\,dx$$

$$= -\frac{x\sec x}{x\sin x+\cos x} + \int \frac{\sec^2 x\,(x\sec x+\cos x)}{x\sin x+\cos x}\,dx$$

$$= -\frac{x\sec x}{x\sin x+\cos x} + \int \sec^2 x\,dx$$

$$= -\frac{x\sec x}{x\sin x+\cos x} + \tan x + c$$

$$= \tan x - \frac{x\sec x}{x\sin x+\cos x} + c.$$

Page-80: 21(1) / $\int e^x \frac{x^2+1}{(x+1)^2}\,dx$

$$= \int e^x \frac{(x^2-1)+2}{(x+1)^2}\,dx$$

$$= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right] dx$$

$$= e^x \cdot \frac{x-1}{x+1} + c$$

(ii) $\int e^x \frac{(1-x)^2}{(1+x^2)^2} dx$

$$= \int e^x \frac{1-2x+x^2}{(1+x^2)^2} dx = \int e^x \frac{1+x^2-2x}{(1+x^2)^2} dx$$

$$= \int e^x \left[\frac{1+x^2}{(1+x^2)^2} - \frac{2x}{(1+x^2)^2}\right] dx$$

$$= \int e^x \left[\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}\right] dx$$

$$= e^x \cdot \frac{1}{1+x^2} + c$$

(iii) $\int e^x \frac{x-1}{(x+1)^3} dx$

$$= \int e^x \frac{x+1-2}{(x+1)^3} dx$$

$$= \int e^x \left[\frac{x+1}{(x+1)^3} - \frac{2}{(x+1)^3}\right] dx$$

$$= \int e^x \left[\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right] dx$$

$$= \frac{e^x}{(x+1)^2} + c$$

22.(i) $\int e^x \frac{1+\sin x}{1+\cos x}\,dx$

$= \int e^x \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$

$= \int e^x \frac{1}{2\cos^2\frac{x}{2}}\,dx + \int e^x \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$

$= \frac{1}{2}\int e^x \sec^2\frac{x}{2}\,dx + \int e^x \tan\frac{x}{2}\,dx$

$= \cancel{\frac{1}{2}}\, e^x \tan\frac{x}{2} - \int e^x \tan\frac{x}{2}\,dx + \int e^x \tan\frac{x}{2}\,dx + c$

$= e^x \tan\frac{x}{2} + c$

(ii) $\int e^x \frac{1-\sin x}{1-\cos 2x}\,dx$

$= \int e^x \cdot \frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\,dx$

$= \int e^x \cdot \frac{1}{2\sin^2\frac{x}{2}}\,dx - \int e^x \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$

$= \int e^x \operatorname{cosec}^2\frac{x}{2}\,dx - \int e^x \cot\frac{x}{2}\,dx$

$= -e^x \cot\frac{x}{2} + \int e^x \cot\frac{x}{2}\,dx - \int e^x \cot\frac{x}{2}\,dx + c$

$= -e^x \cot\frac{x}{2} + c.$

(11) $\int e^x \frac{2-\sin 2x}{1-\cos 2x}\,dx$

$= \int e^x \cdot \frac{2-2\sin x\cos x}{2\sin^2 x}\,dx$

$= \int e^x \operatorname{cosec}^2 x - \int e^x \cot x\,dx$

$= -e^x \cot x + \int e^x \cot x\,dx - \int e^x \cot x\,dx$

$= -e^x \cot x + c$

12) $\int e^x \frac{2+\sin 2x}{1+\cos 2x}\,dx$

$= \int e^x \frac{2+2\sin x\cos x}{2\cos^2 x}\,dx$

$= \int e^x \sec^2 x\,dx + \int e^x \tan x\,dx$

$= e^x \tan x - \int e^x \tan x\,dx + \int e^x \tan x\,dx$

$= e^x \tan x + c$

25 (i) / $\int \sqrt{18x-65-x^2}\, dx$

$= \sqrt{16-(x^2-2.9x+81)}\, dx$

$= \sqrt{16-(x-9)^2}\, dx$

$= \dfrac{(x-9)\sqrt{16-(x-9)^2}}{2} + \dfrac{16}{2}\sin^{-1}\left(\dfrac{x-9}{4}\right) + c$

$= \dfrac{(x-9)\sqrt{18x-65-x^2}}{2} + 8\sin^{-1}\left(\dfrac{x-9}{4}\right) + c$

(ii) $\sqrt{4-3x-2x^2}\, dx$

$= \sqrt{2}\int\sqrt{2-\frac{3}{2}x-x^2}\, dx$

$= \sqrt{2}\sqrt{\dfrac{41}{16}-\left(x+\frac{3}{4}\right)^2}\, dx$

$= \sqrt{2}\left[\dfrac{\left(x+\frac{3}{4}\right)\sqrt{\frac{41}{16}-\left(x+\frac{3}{4}\right)^2}}{2} + \dfrac{41}{2\cdot 16}\sin^{-1}\dfrac{x+3/4}{\sqrt{41}/4}\right] + c$

$= \sqrt{2}\left[\dfrac{(4x+3)\sqrt{(4-3x-2x^2)}}{8\sqrt{2}} + \dfrac{41}{32}\sin^{-1}\dfrac{4x+3}{\sqrt{41}}\right] + c$

$= \dfrac{(4x+3)\sqrt{4-3x-2x^2}}{8} + \dfrac{41}{32}\cdot\sqrt{2}\sin^{-1}\dfrac{4x+3}{\sqrt{41}} + c.$

Page-90 (29) ***

$$\int \sqrt{(x-\alpha)(\beta-x)}\, dx$$

let. $x = \alpha \cos^2\theta + \beta \sin^2\theta$

$\therefore dx = 2(\beta-\alpha) \sin\theta \cos\theta \, d\theta.$

$x-\alpha = \beta \sin^2\theta + \alpha \cos^2\theta - \alpha = (\beta-\alpha)\sin^2\theta,$

$\beta-\alpha = \beta - \alpha\cos^2\theta - \beta \sin^2\theta = (\beta-\alpha)\cos^2\theta.$

Given integral $= \int \sqrt{(\beta-\alpha)^2 \sin^2\theta \cos^2\theta \, (\beta-\alpha)} \cdot 2(\beta-\alpha) \sin\theta\cos\theta \, d\theta$

$= \int \sqrt{(\beta-\alpha)^2 \sin^2\theta \cos^2\theta} \cdot 2(\beta-\alpha) \sin\theta \cos\theta \, d\theta$

$= (\beta-\alpha)^2 \int 2\sin^2\theta \cos^2\theta \, d\theta$

$= \frac{1}{4}(\beta-\alpha)^2 \int 2\sin^2 2\theta \, d\theta$

$= \frac{1}{4}(\beta-\alpha)^2 \int (1-\cos 4\theta)\, d\theta$

$= \frac{1}{4}(\beta-\alpha)^2 \left(\theta - \frac{\cancel{\sin 4}\ \sin 4\theta}{4}\right) + c$

$= \frac{1}{4}(\beta-\alpha)^2 \left[\theta - \sin\theta\cos\theta(2\cos^2\theta - 1)\right] + c$

$= \frac{1}{4}(\beta-\alpha)^2 \left[\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} - \frac{\sqrt{(x-\alpha)(\beta-x)}}{\beta-\alpha}\left(\frac{2\beta-2x}{\beta-\alpha} - 1\right)\right] + c$

$= \frac{1}{4}(\beta-\alpha)^2 \left[\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + \frac{\sqrt{(x-\alpha)(\beta-x)}\,(2x-\alpha-\beta)}{(\beta-\alpha)^2}\right] + c$

$= \frac{1}{4}\left[(2x-\alpha-\beta)\sqrt{(x-\alpha)(\beta-x)} + (\beta-\alpha)^2 \sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}\right] + c$

30(i)/ $\int (x-1)\sqrt{x^2-1}\,dx$ (11)

$= \int x\sqrt{x^2-1}\,dx - \int \sqrt{x^2-1}\,dx$

for 1st integral let, $x^2-1 = z^2$ ; $\therefore 2x\,dx = 2z\,dz$

Given integral $= \int z^2 dz - \int \sqrt{x^2-1}\,dx$

$= \frac{1}{3}z^3 - \left\{ \frac{x\sqrt{x^2-1}}{2} - \frac{1}{2}\log\left(x+\sqrt{x^2-1}\right) \right\}$

$= \frac{1}{3}(x^2-1)^{3/2} - \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\log\left(x+\sqrt{x^2-1}\right) + c.$

31(i)/ $\int (x-1)\sqrt{x^2-x+1}\,dx$

$x^2-x+1 = z^2$

$\therefore x-1 = \frac{1}{2}(2x-1) - \frac{1}{2}$ $\qquad \therefore 2x\,dx = 2z\,dz$

$\therefore$ Given integral $= \frac{1}{2}\int (2x-1)\sqrt{x^2-x+1}\,dx - \frac{1}{2}\int\sqrt{x^2-x+1}\,dx$

$= \int z^2 dz - \frac{1}{2}\int\sqrt{(x-\frac{1}{2})^2 + \frac{3}{4}}\,dx$

$= \frac{1}{3}z^3 - \frac{1}{2}\frac{(x-\frac{1}{2})\sqrt{(x-\frac{1}{2})^2+\frac{3}{4}}}{2} + \frac{3}{8}\log\left\{x-\frac{1}{2}+\sqrt{(x-\frac{1}{2})^2+\frac{3}{4}}\right\} + c$

$= \frac{1}{3}(x^2-x+1)^{3/2} - \frac{1}{8}(2x-1)\sqrt{x^2-x+1} - \frac{3}{16}\log\left(x-\frac{1}{2}+\sqrt{x^2-x+1}\right) + c$

(11) $\int (x+2)\sqrt{2x^2+2x+1}\,dx$

$$x+2 = \frac{1}{4}(4x+2) + \frac{3}{2}$$

$$\text{Integrated} = \frac{1}{4}(4x+2)\sqrt{2x^2+2x+1} + \frac{3}{2}\sqrt{2}\sqrt{(x+\tfrac{1}{2})^2+(\tfrac{1}{2})^2}$$

To get $I_1$ = integral of 1st term,

put, $2x^2+2x+1 = z$

Then, $(4x+2)\,dx = dz$

$$\therefore I_1 = \frac{1}{4}\int z^{\frac{1}{2}}dz = \frac{1}{6} z^{3/2} = \frac{1}{6}(2x^2+2x+1)^{3/2}$$

Also, if $x+\frac{1}{2} = u$, $\frac{1}{2} = a$

Then, $dx = du$ and

and $I_2$ = 2nd term $= \frac{3}{2}\sqrt{2}\int\sqrt{u^2+a^2}\,du$

$$= \frac{3}{2}\sqrt{2}\left[\frac{1}{2}u\sqrt{u^2+a^2} + \frac{1}{2}a^2\log\left\{u+\sqrt{u^2+a^2}\right\}\right]+c$$

$$= \frac{3}{4}\sqrt{2}\,(x+\tfrac{1}{2})\sqrt{(x+\tfrac{1}{2})^2+(\tfrac{1}{2})^2} + \frac{3}{4}\sqrt{2}\cdot\frac{1}{4}\log\left\{x+\tfrac{1}{2}+\sqrt{(x+\tfrac{1}{2})^2+(\tfrac{1}{2})^2}\right\}+\log 2\Big]$$

$$= \frac{3}{8}\left\{(2x+2)\sqrt{2x^2+2x+1}\right\} + \frac{3\sqrt{2}}{16\times 2}\left(\log\left(2x+1+\sqrt{2x^2+2x+1}\right)+\log 2\right)$$

$$= \frac{3}{8}(2x+2)\sqrt{2x^2+2x+1} + \frac{3}{8\sqrt{2}}\log\left(2x+1+\sqrt{2x^2+2x+1}\right)+c$$

$$\left[c = \frac{3\sqrt{2}}{16}\log 2.\right]$$

$\therefore I = I_1 + I_2$

$= \frac{1}{6}\left(2x^2+2x+1\right)^{3/2} + \frac{3}{8}(2x+1)\sqrt{2x^2+2x+1} + \frac{3}{8\sqrt{2}} \log\left(2x+1+\sqrt{2\sqrt{2x^2+2x+1}}\right) + c$

* 24(i) / $\int \frac{a+x}{\sqrt{a^2-x^2}}\,dx$

$= \int \frac{a\,dx}{\sqrt{a^2-x^2}} + \int \frac{x\,dx}{\sqrt{a^2-x^2}}$

$= a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + c$

* (ii) $\int x\sqrt{\frac{a-x}{a+x}}\,dx$

put, $x = a\cos\theta$.

$\therefore$ Given integral $= \int a\cos\theta \cdot (-a\sin\theta)\,d\theta \frac{\sin\frac{\theta}{2}}{\cos\theta/2}$

$= -a^2\int \cos\theta \cdot 2\sin^2\theta/2\,d\theta$

$= -a^2\int \cos\theta(1-\cos\theta)\,d\theta$

$= \frac{a^2}{2}\int (1+\cos 2\theta - 2\cos\theta)\,d\theta$

$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta - 2\sin\theta\right) + c$

$= \frac{a^2}{2}\left(\theta + \sin\theta\cos\theta - 2\sin\theta\right) + c$

$= \frac{a^2}{2}\left(\cos^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{a^2} - \frac{2\sqrt{a^2-x^2}}{a}\right) + c.$

# Chapter-4

## Special Trigonometric Functions:

Integration of the type: $\int \frac{dx}{a+b\sin x}$ or $\int \frac{dx}{a+b\cos}$ or $\int \frac{dx}{a+b\sin x + \cos x}$

Page-107 Ex:2/ $\int \frac{dx}{a\sin x + b\cos x}$

put $a = r\cos\theta$, $b = r\sin\theta$, then $a\sin x + b\cos x = r\sin(x+\theta)$.

Here, $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1} b/a$

$$I = \int \frac{dx}{r\sin(x+\theta)}$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\theta)\,dx$$

$$= \frac{1}{r}\int \operatorname{cosec} z\,dz \qquad \Big| z = x+\theta$$

$$= \frac{1}{r}\log\left|\tan\frac{z}{2}\right|$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log \tan\left|\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right|$$

Page- 108 Ex 2) $\int \frac{dx}{5-13\sin x}$

$$= \int \frac{dx}{5\left(\sin^2\frac{1}{2}x+\cos^2\frac{1}{2}x\right)-13\cdot 2\sin\frac{1}{2}x\cos\frac{1}{2}x}$$

Multiplying ~~the number~~ numerator and denominator by $\sec^2\frac{1}{2}x$, this,

$$= \int \frac{\sec^2\frac{1}{2}dx}{5\left(\tan^2\frac{1}{2}x+1\right)-26\tan\frac{1}{2}x}$$

$$= \int \frac{2dz}{5z^2-26z+5} \quad \Big| \text{putting } \tan\frac{1}{2}x = z$$

$$= \frac{2}{5}\int \frac{dz}{\left(z-\frac{13}{5}\right)^2-\left(\frac{12}{5}\right)^2}$$

$$= \frac{2}{5}\int \frac{du}{u^2-a^2} \qquad u = z-\frac{13}{5} \text{ and } a=\frac{12}{5}$$

$$= \frac{2}{5}\,\frac{1}{2a}\log\frac{u-a}{u+a} = \frac{1}{\cancel{5}_1}\cdot\frac{\cancel{5}^1}{2\times 12}\log\frac{z-\frac{13}{5}-\frac{12}{5}}{z-\frac{13}{5}+\frac{12}{5}}$$

$$= \frac{1}{12}\log\frac{5z-25}{5z-1}$$

$$= \frac{1}{12}\log\left|\frac{5\tan\frac{1}{2}x-25}{5\tan\frac{1}{2}x-1}\right| + c$$

Page - 109 Ex: 3 /

$$\int \frac{dx}{13+3\cos x+4\sin x}$$

$$= \frac{dx}{13\left(\sin^2\frac{1}{2}x+\cos^2\frac{1}{2}x\right)+3\left(\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x\right)+4\cdot 2\sin\frac{1}{2}\cos\frac{1}{2}x}$$

Multiplying the numerator and denominator by $\sec^2\frac{1}{2}x$, this

$$= \int \frac{\sec^2\frac{1}{2}x\,dx}{10\tan^2\frac{1}{2}x+8\tan\frac{1}{2}+16}$$

$$= \int \frac{2dz}{10z^2+8z+16}, \quad \left[\text{putting } z=\tan\frac{1}{2}x\right]$$

$$= \frac{1}{5}\int \frac{dz}{\left(z+\frac{2}{5}\right)^2+\left(\frac{6}{5}\right)^2} = \frac{1}{5}\int \frac{du}{u^2+a^2} \qquad \begin{vmatrix} u = z+\frac{2}{5} \\ a = \frac{6}{5} \end{vmatrix}$$

$$= \frac{1}{5}\cdot\frac{1}{a}\tan^{-1}\frac{u}{a}$$

$$= \frac{1}{5}\cdot\frac{5}{6}\tan^{-1}\frac{5z+2}{6}$$

$$= \frac{1}{6}\tan^{-1}\frac{5\tan\frac{1}{2}x+2}{6}.$$

***

Page-111 Ex: 11

$$\int \frac{dx}{4+3\sin x}$$

$$= \int \frac{dx}{4+3\times\dfrac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x}}$$

$$= \int \frac{\sec^2\frac{1}{2}x\,dx}{4+4\tan^2\frac{1}{2}x+6\tan\frac{1}{2}x}$$

Let us put, $\tan\frac{x}{2} = z$

$$\Rightarrow \frac{1}{2}\sec^2\frac{1}{2}x\,dx = dz$$

$$\Rightarrow \sec^2\frac{1}{2}x\,dx = 2dz$$

$$\therefore I = \int \frac{2dz}{4z^2+6z+4}$$

$$= \frac{1}{2}\int \frac{dz}{z^2+2\cdot z\cdot\frac{3}{4}+\frac{9}{16}-\frac{9}{16}+1}$$

$$= \frac{1}{2}\int \frac{dz}{\left(z+3/4\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{4}{\sqrt{7}}\tan^{-1}\frac{z+3/4}{\sqrt{7}/4}+c$$

$$= 2/\sqrt{7}\cdot\tan^{-1}\frac{4\tan\frac{1}{2}x+3}{\sqrt{7}}+c.$$

Ex: 2/ $\int \frac{dx}{5+4\cos x}$

$$= \int \frac{dx}{5 + \frac{4(1-\tan^2\frac{1}{2}x)}{1+\tan^2\frac{1}{2}x}}$$

$$= \int \frac{\sec^2\frac{1}{2}x}{9+\tan^2\frac{1}{2}x}\,dx$$

$$= 2\int \frac{dz}{9+z^2} \quad \Big| \; z = \tan\frac{1}{2}x$$

$$= 2\cdot\frac{1}{3}\tan^{-1}\frac{z}{3} + c$$

$$= 2\cdot\frac{1}{3}\tan^{-1}\left(\frac{\tan\frac{1}{2}x}{3}\right) + c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{1}{2}x\right) + c.$$

Ex: 3/ $\int \frac{dx}{3+2\sin x + \cos x}$

$$= \int \frac{dx}{3+2\cdot\frac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x} + \frac{1-\tan^2\frac{1}{2}x}{1+\tan^2\frac{1}{2}x}}$$

$$= \int \frac{(1+\tan^2\frac{1}{2}x)\,dx}{3+3\tan^2\frac{1}{2}x + 4\tan\frac{1}{2}x + 1 - \tan^2\frac{1}{2}x}$$

$$= \int \frac{\sec^2 \frac{1}{2}x\, dx}{2\tan^2\frac{1}{2}x + 4\tan\frac{1}{2}x + 4}$$

$$= \int \frac{2dz}{2z^2 + 4z + 4} \quad \Bigg| \; z = \tan\frac{1}{2}x$$

$$= \int \frac{dz}{(z+1)^2 + 1}$$

$$= \tan^{-1}(z+1) + c$$

$$= \tan^{-1}\left(1 + \tan\frac{1}{2}x\right) + c.$$

Page-114 Ex:4) $\displaystyle\int \frac{dx}{2\sin x + 3\cos x + 4}$

$$= \int \frac{dx}{2\cdot\frac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x} + \frac{3\left(1-\tan^2\frac{1}{2}x\right)}{1+\tan^2\frac{1}{2}x} + 4}$$

$$= \int \frac{\sec^2\frac{1}{2}x\, dx}{4\tan\frac{1}{2}x + 3 - 3\tan^2\frac{1}{2}x + 4 + 4\tan^2\frac{1}{2}x}$$

$$= \int \frac{\sec^2\frac{1}{2}x\, dx}{\tan^2\frac{1}{2}x + 4\tan\frac{1}{2} + 7} \quad \Bigg| \; z = \tan\frac{1}{2}x$$

$$= 2\int \frac{dz}{z^2+4z+7}$$

$$= 2\int \frac{dz}{(z+2)^2+(\sqrt{3})^2}$$

$$= 2 \cdot \frac{1}{\sqrt{3}} \tan^{-1} \frac{z+2}{\sqrt{3}} + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left\{\frac{1}{\sqrt{3}}\left(2+\tan\frac{1}{2}x\right)\right\} + c.$$

Page-116 Ex: C/ #

$$\int \frac{dx}{1+2\tan x}$$

$$= \int \frac{\cos x \, dx}{\cos x + 2\sin x}$$

Let, $\cos x = l(\cos x + 2\sin x) + m(-\sin x + 2\cos x)$

$$= (l+m)\cos x + (2l-m)\sin x$$

Equating the coefficients of $\cos x$ and $\sin x$ of both the sides, we have, $l+2m=1$ —— (i)

$2l-m=0$ —— (ii)

$l = \frac{1}{5}$, $m = \frac{2}{5}$

$$\therefore I = \int \frac{l(\cos x + 2\sin x) + m(-\sin x + 2\cos x)}{\cos x + 2\sin x} dx$$

$$= l\int dx + m\int \frac{dz}{z}, \quad \text{where, } z = \cos x + 2\sin x$$

$$= \frac{1}{5}x + \frac{2}{5}\log|z| + C$$

$$= \frac{1}{5}x + \frac{2}{5}\log\left|\cos x + 2\sin x\right| + C.$$

Ex:7/ $\displaystyle\int \frac{dx}{\tan(1+\tan x)}$

$$= \int \frac{(1+\tan x) - \tan x}{\tan x(1+\tan x)} \cdot dx$$

$$= \int \frac{dx}{\tan x} - \int \frac{dx}{1+\tan x}$$

$$= \int \frac{\cos x \, dx}{\sin x} - \int \frac{\cos x}{\sin x + \cos x} dx$$

$$= \int \frac{dz}{z} - \frac{1}{2}\int \frac{(\cos x + \sin x) - (\sin x - \cos x)}{\sin x + \cos x} dx$$

where, $z = \sin x$, $dz = \cos x$

$$= \int \frac{dz}{z} - \frac{1}{2}\int dz - \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

$$= \int \frac{dz}{z} - \frac{1}{2}\int dx - \frac{1}{2}\int \frac{du}{u}, \quad \text{where, } u = \sin x + \cos x$$

$$= \log|z| - \frac{1}{2}x - \frac{1}{2}\log|u| + C$$

$= \log|\sin x| - \frac{1}{2}x - \frac{1}{2}\log|\sin x + \cos x| + c.$

Page- 126: 25(i) / $\int \frac{dx}{5+4\sin x}$

$$= \int \frac{\cancel{dx}\ 1}{5\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) + 4 \cdot 2\sin\frac{x}{2}\cos\frac{x}{2}}\, dx$$

$$= \cancel{\tfrac{1}{5}} \int \frac{\sec^2\frac{x}{2}}{5\tan^2\frac{x}{2} + 8\tan\frac{x}{2} + 5}$$

multiplying $\sec^2 x/2$ both sides

$$= \frac{1}{5}\int \frac{\sec^2\frac{x}{2}\, dx}{\tan^2\frac{x}{2} + 8/5\tan\frac{x}{2} + 1}$$

$$= \frac{1}{5}\int \frac{\sec^2\frac{x}{2}\, dx}{\left(\tan\frac{x}{2} + \frac{4}{5}\right)^2 + \left(3/5\right)^2}$$

$$= 2/5 \int \frac{dz}{\left(z + 4/5\right)^2 + \left(3/5\right)^2}$$

$$= 2/5 \cdot 5/3 \tan^{-1}\left(\frac{5z+4}{3}\right) + c$$

$$= 2/3 \tan^{-1}\left\{\frac{1}{3}\left(5\tan\frac{x}{2} + 4\right)\right\} + c.$$

(11) $\int \frac{dx}{4+5\sin x}$

$= \int \frac{1}{4\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+5\cdot 2\sin\frac{x}{2}\cos\frac{x}{2}}$

$= \int \frac{\sec^2\frac{x}{2}}{4\tan^2\frac{x}{2}+10\tan\frac{x}{2}+4}$ | Multiplying $\sec^2\frac{x}{2}$

$= \frac{1}{4}\int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+\frac{5}{2}\tan\frac{x}{2}+1}$

$= \frac{1}{4}\int \frac{\sec^2\frac{x}{2}\,dx}{\left(\tan\frac{x}{2}+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$

$= \frac{1}{4}\int \frac{dz}{\left(z+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$ where, $\tan\frac{x}{2}=z$
$\sec^2\frac{x}{2}=dz$

$= \frac{1}{4}\times\frac{4}{3}\log\frac{4z+5-3}{4z+5+3}$

$= \frac{1}{3}\log\frac{4z+2}{4z+8}$

$= \frac{1}{3}\log\frac{2z+1}{2z+4}$

$= \frac{1}{3}\log\frac{2\tan\frac{x}{2}+1}{2\tan\frac{x}{2}+4}+c.$

26(i) $\int \frac{dx}{5+4\cos x}$

Suppose, $5+4\cos x = a+b\cos x$

Now, $\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{1}{2}x\right)$

when $a>b$

Here, $a=5$, $b=4$.

$\therefore I = \frac{2}{\sqrt{5^2-4^2}} \tan^{-1}\left(\sqrt{\frac{5-4}{5+4}} \tan\frac{1}{2}x\right)+c$

$= \frac{2}{3} \tan^{-1}\left(\frac{1}{3}\tan\frac{1}{2}x\right)+c$

(ii) $\int \frac{dx}{3+5\cos x}$

Suppose, $3+5\cos x = a+b\cos x$

Now, $\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}} \log \frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}$ when $b>a$

Here, $b=5$, $a=3$ $\therefore b>a$

$\therefore I = \frac{1}{\sqrt{16}} \log \frac{\sqrt{8}+\sqrt{2}\tan\frac{x}{2}}{\sqrt{8}-\sqrt{2}\tan\frac{x}{2}}$

$= \frac{1}{4}\log \frac{\sqrt{2}\left(2+\tan\frac{1}{2}x\right)}{\sqrt{2}\left(2-\tan\frac{1}{2}x\right)} = \frac{1}{4}\log\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}+c$

२७(1)/ $\displaystyle\int \frac{dx}{\cos\alpha + \cos x}$

$$= \int \frac{dx}{\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right)\cos\alpha + \left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{(1+\cos\alpha)\cos^2\frac{x}{2} - 2\sin^2\frac{\alpha}{2}\sin^2\frac{x}{2}}$$

$$= \frac{1}{2\sin^2\frac{\alpha}{2}} \int \frac{\sec^2\frac{x}{2}}{\cot^2\frac{\alpha}{2} - \tan^2\frac{x}{2}}\,dx$$

$$= \frac{1}{2\sin^2\frac{\alpha}{2}} \int \frac{dz}{\cot^2\frac{\alpha}{2} - z^2}, \quad z = \tan\frac{x}{2}$$

$$= \frac{1}{\sin^2\frac{\alpha}{2}\, 2\cot\frac{\alpha}{2}} \log\left(\frac{\cot\frac{\alpha}{2} + z}{\cot\frac{\alpha}{2} - z}\right) + c$$

$$= \frac{1}{\sin\alpha} \log \frac{\cot\frac{\alpha}{2} + \tan\frac{x}{2}}{\cot\frac{\alpha}{2} - \tan\frac{x}{2}} + c$$

$$= \frac{1}{\sin\alpha} \log \frac{\cos\frac{1}{2}(x-\alpha)}{\cos\frac{1}{2}(x+\alpha)} + c.$$

32 (i)

$$\int \frac{dx}{1-\cos x+\sin x}$$

$$= \int \frac{dx}{2\sin^2\frac{1}{2}x+2\sin\frac{1}{2}x\cos\frac{1}{2}x}$$

$$= \int \frac{\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}x}{1+\cot\frac{1}{2}x}\,dx \quad \Big| \text{ multiplying } \frac{1}{2}\operatorname{cosec}^2\frac{1}{2}x$$

$$= -\int \frac{dz}{z} \quad \text{where, } 1+\cot\frac{x}{2}=z \text{ and, } \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}dx=dz.$$

$$= -\log z+c$$

$$= -\log\left(1+\cot\frac{1}{2}x\right)+c$$

(ii)

$$\int \frac{dx}{3+2\sin x+\cos x}$$

$$= \int \frac{dx}{(1+\cos x)+2(1+\sin x)}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{2\cos^2\frac{x}{2}+2\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}$$

$$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}\,dx}{1+\left(1+\tan\frac{x}{2}\right)^2} \quad \text{where, } 1+\tan\frac{x}{2}=z,\ \sec^2\frac{x}{2}dx=dz$$

$$= \int \frac{dz}{1+z^2} = \tan^{-1}(z)$$

$$= \tan^{-1}\left(1+\tan\frac{x}{2}\right)+c$$

33/ $\int \frac{dx}{3+2\sin x+\cos x}$

let, $6+3\sin x+14\cos x$

$= A(3+4\sin x+5\cos x)+B\frac{d}{dx}(3+4\sin x+5\cos x)+C$

$= A(3+4\sin x+5\cos x)+B(4\cos x-5\sin x)+C$

Then, $3A+C=6$

$4A-5B=3$

$5A+4B=14$

$\therefore A=2,\ B=1,\ C=0$

$\therefore$ Integrand $= 2+\frac{1+\frac{d}{dx}(3+4\sin x+5\cos x)}{3+4\sin x+5\cos x}$

~~Also~~

$\therefore I = \int \frac{2(3+4\sin x+5\cos x)+1(4\cos x-5\sin x)}{3+4\sin x+5\cos x}dx$

$= 2\int dx + \int \frac{4\cos x-5\sin x}{3+4\sin x+5\cos x}dx$

$= 2x+\log(3+4\sin x+5\cos x)\,dx$

# Chapter-5

## Relational Functions:

Page-139: Ex:2/ $\int \frac{e^{-x}dx}{e^{x}+2e^{-x}+3}$

$$= \int \frac{e^{x}dx}{e^{3x}+2e^{x}+3e^{2x}} \quad \left[\text{multiplying } e^{2x}\right]$$

If we substitude $e^{x}=z$; $e^{x}dx=dz$ then,

$$I=\int \frac{dz}{z^3+3z^2+2z}$$

$$=\int \frac{dz}{z(z+1)(z+2)}$$

let, $\frac{1}{z(z+1)(z+2)} = \frac{a}{z}+\frac{b}{z+1}+\frac{c}{z+2}$

Then, $1=a(z+1)(z+2)+bz(z+2)+cz(z+1)$ ——— (1)

putting, $z=0,-1,-2$ successively in (1), we get,

$$a=\frac{1}{2},\ b=-1,\ c=\frac{1}{2}$$

$$\therefore I=\frac{1}{2}\int\frac{dz}{z}-\int\frac{dz}{z+1}+\frac{1}{2}\int\frac{dz}{z+2}$$

$$=\frac{1}{2}\log|z|-\log|z+1|+\frac{1}{2}\log|z+2|+c$$

$$=\frac{1}{2}\log|e^{x}|-\log|e^{x}+1|+\frac{1}{2}\log|e^{x}+2|+c$$

$$=\frac{1}{2}x-\log(e^{x}+1)+\frac{1}{2}\log|e^{x}+2|+c.$$

Page-149 22(i) ) $\int \frac{x^2 dx}{x^4 - x^2 - 12}$ ~~dz~~

$= \int \frac{x^2 dx}{(x^2-4)(x^2+3)}$

Let, $\frac{x^2}{(x^2-4)(x^2+3)} = \frac{A}{x^2-4} + \frac{B}{x^2+3}$

$\therefore x^2 = A(x^2+3) + B(x^2-4)$

putting, $x^2 = 4, -3$, we get, $A = 4/7$, $B = 3/7$.

$\therefore I = \frac{3}{7}\int \frac{dx}{x^2+3} + \frac{4}{7}\int \frac{dx}{x^2-4}$

$= \frac{3}{7\sqrt{3}} \tan^{-1}\frac{x}{\sqrt{3}} + \frac{4}{7 \cdot 4} \log \frac{x-2}{x+2} + c$

$= \frac{\sqrt{3}}{7} \tan^{-1}\frac{x}{\sqrt{3}} + \frac{1}{7} \log \frac{x-2}{x+2} + c$

$= \frac{1}{7} \log \frac{x-2}{x+2} + \frac{\sqrt{3}}{7} \tan^{-1}\frac{x}{\sqrt{3}} + c$.

ii) $\int \frac{x dx}{x^4 - x^2 - 2}$ ~~dz~~

Let $x^2 = z$ ; $\therefore 2x dx = dz$

$\therefore I = \frac{1}{2}\int \frac{dz}{z^2 - z - 2} = \frac{1}{2}\int \frac{dz}{(z-2)(z+1)}$

$= \frac{1}{2} \cdot \frac{1}{3} \int \frac{z - z + 3}{(z-2)(z+1)} dz$

$$= \frac{1}{6}\int \frac{(z+1)-(z-2)}{(z-2)(z+1)}\,dz$$

$$= \frac{1}{6}\int \frac{dz}{z-2} - \frac{1}{6}\int \frac{dz}{z+1}$$

$$= \frac{1}{6}\left[\log(z-2) - \log(z+1)\right]$$

$$= \frac{1}{6}\left[\log(x^2-2) - \log(x^2+1)\right].$$

Page-150 : 29(1)/ $\int \frac{dx}{1+3e^x+2e^{2x}}$

put $e^x = z$ $\therefore x = \log z$ $\therefore dx = \frac{dz}{z}$.

$$\therefore I = \int \frac{dz}{z(1+3+2z^2)}$$

Let, $\frac{1}{z(1+z)(1+2z)} = \frac{A}{z} + \frac{B}{1+z} + \frac{C}{1+2z}$

$$\therefore 1 = A(1+z)(1+2z) + Bz(1+2z) + Cz(1+z).$$

putting $z = 0, -1, -\frac{1}{2}$, we get $A = 1,\ B = 1,\ C = -4$.

$$\therefore I = \int \frac{dz}{z} + \int \frac{dz}{1+z} - 4\int \frac{dz}{1+2z}$$

$$= \log z + \log(1+z) - \frac{4}{2}\log(1+2z)$$

$$= \log e^x + \log(1+e^x) - 2\log(1+2e^x)$$

$$= \log e^x + \log(1+e^x) - 2\log(1+2e^x).$$

$$= x + \log(1+e^x)$$

11) $\int \frac{e^x dx}{e^x - 3e^{-x} + 2}$

put, $e^x = z$ ; $\therefore e^x dx = dz$

$\therefore I = \int \frac{dz}{z - \frac{3}{z} + 2} = \int \frac{z\,dz}{z^2 + 2z - 3}$

Let, $\frac{z}{z^2+2z-3} = \frac{z}{(z-1)(z+3)} = \frac{A}{z-1} + \frac{B}{z+3}$

$\therefore z = A(z+3) + B(z-1)$

Putting, $z = 1$, $A = 1/4$, $B = 3/4$

$\therefore I = \frac{1}{4}\int \frac{dz}{z-1} + \frac{3}{4}\int \frac{dz}{z+3}$

$= \frac{1}{4}\log(z-1) + 3/4 \log(z+3)$

$= \frac{1}{4}\log(e^x - 1) + 3/4 \log(e^x + 3)$.

# Chapter – 6

## Irrelational Functions:

Page – 158 Ex: 2/ ***

$$\int \frac{dx}{(x^2-2x+1)\sqrt{(x^2-2x+3)}}$$

$$= \int \frac{dx}{(x-1)^2 \sqrt{\{(x-1)^2+2\}}}$$

$$= \int \frac{dz}{z^2\sqrt{z^2+2}} \quad \left[\text{putting } (x-1) = z\right]$$

putting $z = \sqrt{2}\tan\theta$

$$\therefore \; I = \int \frac{\sqrt{2}\sec^2\theta \, d\theta}{2\tan^2\theta \cdot \sqrt{2}\sec\theta}$$

$$= \frac{1}{2}\int \operatorname{cosec}\theta \cot\theta \, d\theta$$

$$= -\frac{1}{2}\operatorname{cosec}\theta.$$

since, $\tan\theta = \frac{1}{\sqrt{2}} z$, $\operatorname{cosec}\theta = \frac{\sqrt{(z^2+2)}}{z}$

$$\therefore \; I = -\frac{1}{2} \cdot \frac{\sqrt{(z^2+2)}}{z}$$

$$= -\frac{1}{2} \frac{\sqrt{x^2-2x+3}}{x-1}.$$

# Chapter – 6

## Irrelational Functions:

Page – 158 Ex: 2 /***

$$\int \frac{dx}{(x^2-2x+1)\sqrt{(x^2-2x+3)}}$$

$$= \int \frac{dx}{(x-1)^2\sqrt{\{(x-1)^2+2\}}}$$

$$= \int \frac{dz}{z^2\sqrt{z^2+2}} \quad [\text{putting } (x-1)=z]$$

putting $z=\sqrt{2}\tan\theta$

$$\therefore I = \int \frac{\sqrt{2}\sec^2\theta\, d\theta}{2\tan^2\theta \cdot \sqrt{2}\sec\theta}$$

$$= \frac{1}{2}\int \operatorname{cosec}\theta \cot\theta\, d\theta$$

$$= -\frac{1}{2}\operatorname{cosec}\theta.$$

Since, $\tan\theta = \frac{1}{\sqrt{2}}z$, $\operatorname{cosec}\theta = \frac{\sqrt{(z^2+2)}}{z}$

$$\therefore I = -\frac{1}{2}\cdot\frac{\sqrt{(z^2+2)}}{z}$$

$$= -\frac{1}{2}\frac{\sqrt{x^2-2x+3}}{x-1}.$$

Page-160 ~~*~~ (i) !

$$\int \frac{\sqrt{x-x^2}}{x^3}\,dx$$

Chapter - 6

$$= \int \frac{\sqrt{1-x}}{x^{5/2}}\,dx$$

In this form, $m = -5/2$, $n = 1$, $p = \frac{r}{s} = 1/2$

$$\therefore \frac{m+1}{n} + \frac{r}{s} = -3/2 + 1/2 = -1,$$

put $1-x = z^2x$; $\therefore x = \frac{1}{1+z^2}$, $dx = \frac{-2z}{(1+z^2)^2}dz$.

$$\therefore I = -2\int \frac{z\sqrt{x}\cdot z\,dz}{x^{5/2}(1+z^2)^2}$$

$$= -2\int \frac{z^2\,dz}{x^2(1+z^2)^2}$$

$$= -2\int \frac{z^2(1+z^2)^2}{(1+z^2)^2}\,dz = -2\frac{z^3}{3} + c$$

$$= -2/3\,\frac{(x-x^2)^{3/2}}{x^3} + c.$$

*(ii)

$$\int \frac{\sqrt{x}\sqrt{1-2x}}{x^4}$$

8(1))

$$\int \frac{dx}{(x^2+1)\sqrt{x^2+4}}$$

put $x = 2\tan\theta$

$\therefore dx = 2\sec^2\theta \, d\theta$

$$I = \int \frac{2\sec^2\theta \, d\theta}{(4\tan^2\theta + 1)\, 2\sec\theta} = \int \frac{\sec\theta \, d\theta}{1 + 4\tan^2\theta}$$

$$= \int \frac{\cos\theta \, d\theta}{\cos^2\theta + 4\sin^2\theta} = \int \frac{\cos\theta \, d\theta}{3\sin^2\theta + 1}$$

$$= \frac{1}{3}\int \frac{\cos\theta \, d\theta}{\sin^2\theta + 1/3}$$

put $u = \sin\theta$ $\therefore du = \cos\theta \, d\theta$.

$$\therefore I = 1/3 \int \frac{du}{u^2 + 1/3} = 1/3 \cdot \sqrt{3} \tan^{-1}(u\sqrt{3}) + c$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}(\sin\theta \cdot \sqrt{3}) + c$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{\sqrt{3}x}{\sqrt{x^2+4}}\right) + c$$

$$\left[\because \tan\theta = \frac{x}{2}, \ \sin\theta = \frac{x}{\sqrt{x^2+4}}\right]$$

* 11) $\int \frac{dx}{(x^2-1)\sqrt{x^2-9}}$

put $x = 3\sec\theta$

Then, $dx = 3\sec\theta\tan\theta\, d\theta$

$$I = \int \frac{3\sec\theta\tan\theta\, d\theta}{(9\sec^2\theta-1)\, 3\tan\theta}$$

$$= \int \frac{\cos\theta\, d\theta}{(9-\cos^2\theta)}$$

$$= \int \frac{\cos\theta\, d\theta}{8+\sin^2\theta}$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{\sin\theta}{2\sqrt{2}}\right) + C$$

Here, $\cos\theta = 3/x$ $\therefore$ $\sin\theta$ $\frac{\sqrt{x^2-9}}{x}$,

# Chapter – 8

Definite Integrals (without using the propertise the definite integrals):

Page – 213 ***Ex: 3/ $\displaystyle\int_{\alpha}^{\beta}\sqrt{(x-\alpha)(\beta-x)}\,dx$

put, $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$dx = -2\alpha\cos\theta\sin\theta\,d\theta + 2\beta\sin\theta\cos\theta\,d\theta$

$= 2(\beta-\alpha)\cos\theta\sin\theta\,d\theta.$

$x-\alpha = \alpha\cos^2\theta + \beta\sin^2\theta - \alpha$

$= -\alpha(1-\cos^2\theta) + \beta\sin^2\theta$

$= (\beta-\alpha)\sin^2\theta$

$\beta - x = \beta - \alpha\cos^2\theta - \beta\sin^2\theta$

$= \beta(1-\sin^2\theta) - \alpha\cos^2\theta$

$= (\beta-\alpha)\cos^2\theta.$

when, $x=\alpha$, $(\beta-\alpha)\sin^2\theta = 0$

$\sin\theta = 0$, $\beta \neq \alpha$ $\therefore \theta = 0$

Similarly, when $x=\beta$, $(\beta-\alpha)\cos^2\theta = 0.$

$\therefore \cos\theta = 0 \quad \therefore \theta = \pi/2$

$$\therefore I = 2(\beta-\alpha)^2 \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

Now,

$$\sin^2\theta\cos^2\theta = \frac{1}{4}\cdot 4\sin^2\theta\cos^2\theta$$
$$= \frac{1}{4}\sin^2 2\theta$$
$$= \frac{1}{8}\cdot 2\sin^2 2\theta$$
$$= \frac{1}{8}(1-\cos 4\theta)$$

$$I = 2(\beta-\alpha)^2\cdot\frac{1}{8}\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta$$
$$= \frac{1}{4}(\beta-\alpha)^2\left[\theta - \frac{\sin 4\theta}{\theta}\right]_0^{\pi/2}$$
$$= \frac{1}{4}(\beta-\alpha)^2\cdot\pi/2$$
$$= \frac{1}{8}\pi(\beta-\alpha)^2.$$

Page-214 Ex:2/ **

$$\int_{\alpha}^{\beta} \frac{dx}{\sqrt{\{(x-\alpha)(\beta-x)\}}}$$

put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$dx = -2\alpha\cos\theta\sin\theta\, d\theta + 2\beta\sin\theta\cos\theta\, d\theta$

$= 2(\beta-\alpha)\cos\theta\sin\theta\, d\theta$

$x-\alpha = \beta\sin^2\theta - \alpha(1-\cos^2\theta)$

$= (\beta-\alpha)\sin^2\theta$

$\beta - x = \beta(1-\sin^2\theta) - \alpha\cos^2\theta$

$= (\beta-\alpha)\cos^2\theta$

When $x=\alpha$, $(\beta-\alpha)\sin^2\theta = 0$.

$\therefore \sin\theta = 0$, $\beta \neq \alpha$ $\therefore \theta = 0$.

Similarly, when $x=\beta$, $(\beta-\alpha)\cos^2\theta = 0$

$\therefore \cos\theta = 0$ ; $\therefore \theta = \frac{1}{2}\pi$

$$\therefore I = \int_0^{\pi/2} \frac{2(\beta-\alpha)\cos\theta\sin\theta\, d\theta}{\sqrt{(\beta-\alpha)^2\cos^2\theta\sin^2\theta}}$$

$$= \int_0^{\pi/2} \frac{2(\beta-\alpha)\cos\theta\sin\theta\, d\theta}{(\beta-\alpha)\sin\theta\cos\theta} = \int_0^{\pi/2} 2\,d\theta =$$

$$= 2\left[\theta\right]_0^{\pi/2} = 2\cdot\frac{\pi}{2} = \pi.$$

Ex: 51*

$$\int_0^{\pi/2} \frac{dx}{(1-2x^2)\sqrt{(1-x^2)}} = \frac{1}{2}\log(2+\sqrt{3}) \text{ show that.}$$

put $x = \sin\theta$ ; $dx = \cos\theta \, d\theta$

$x=0$, $\theta = 0$

and, $x = 1/2$, $\theta = \pi/6$

$$\therefore I = \int_0^{\pi/6} \frac{\cos\theta \, d\theta}{(1-2\sin^2\theta)\sqrt{(1-\sin^2\theta)}}$$

$$= \int_0^{\pi/6} \frac{d\theta}{(1-2\sin^2\theta)}$$

$$\boxed{\begin{aligned} &\int \sec x \, dx \\ &= \log(\sec x + \tan x) \\ &= \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \end{aligned}}$$

$$= \int_0^{\pi/6} \frac{d\theta}{\cos 2\theta} = \int_0^{\pi/6} \sec 2\theta \, d\theta$$

$$= \frac{1}{2}\Big[\log(\sec 2\theta + \tan 2\theta)\Big]_0^{\pi/6}$$

$$= \frac{1}{2}\left[\log\left(\sec\frac{\pi}{3} + \tan\pi/3\right)\right]$$

$$= \frac{1}{2}\log(2+\sqrt{3}).$$

Page – 220 16(1)* / $\int_1^2 \sqrt{(x-1)(2-x)}\, dx = \frac{1}{8}\pi$

put $x = 2\sin^2\theta + \cos^2\theta$

$dx = (4\sin\theta\cos\theta - 2\sin\theta\cos\theta)\, d\theta$

$= 2\sin\theta\cos\theta\, d\theta$

when, $x - 1 = 2\sin^2\theta + \cos^2\theta - 1$

$= 2\sin^2\theta - \sin^2\theta$

$= \sin^2\theta$

$2 - x = 2 - 2\sin^2\theta - \cos^2\theta$

$= 2\cos^2\theta - \cos^2\theta$

$= \cos^2\theta$

when, $x = 1$, $\theta = 0$

$x = 2$, $\theta = \pi/2$

$I = \int_0^{\pi/2} 2\sin^2\theta\cos^2\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} 4\sin^2\theta\cos^2\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\, d\theta$

$= \frac{1}{4}\int_0^{\pi/2} 2\sin^2 2\theta\, d\theta$

$= \frac{1}{4}\int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta$

$= \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$

$= \frac{1}{4} \cdot \pi/2$

$= \pi/8.$

ii) $\displaystyle\int_{8}^{15} \frac{dx}{(x-3)\sqrt{(x+1)}} = \frac{1}{2}\log 5/3$

put, $x+1 = z^2$

$dx = 2z\,dz$

$x = 8 \;,\; z = 3$

$x = 15 \;,\; z = 4$

$$\therefore I = \int_{3}^{4} \frac{2z\,dz}{(z^2-1-3)z}$$

$$= \int_{3}^{4} \frac{2\,dz}{z^2-4}$$

$$= 2\int_{3}^{4} \frac{dz}{z^2-2^2}$$

$$= 2\left[\log\left(\frac{z-2}{z+2}\right)\right]_{3}^{4}$$

$$= \frac{2}{2.2}\left[\log\frac{2}{6} - \log\frac{1}{5}\right]$$

$$= \frac{1}{2}\left[\log 1/3 - \log 1/5\right]$$

$$= 1/2 \log 5/3 .$$

Page-221 (25*) /

$$\int_0^{\frac{1}{4}\pi} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx = \pi/4$$

$$= \int_0^{\frac{1}{4}\pi} \frac{2\sin x \cos x}{\sin^4 x + \cos^4 x}$$

$$= \int_0^{\pi/4} \frac{2\sin x \cos x \cdot \frac{1}{\cos^4 x}}{\tan^4 x + 1}$$

$$= \int_0^{\pi/4} \frac{2\tan x \sec^2 x}{\tan^4 x + 1}$$

Let $\tan^2 x = z$ $\therefore 2\tan x \sec^2 x \, dx = dz$,

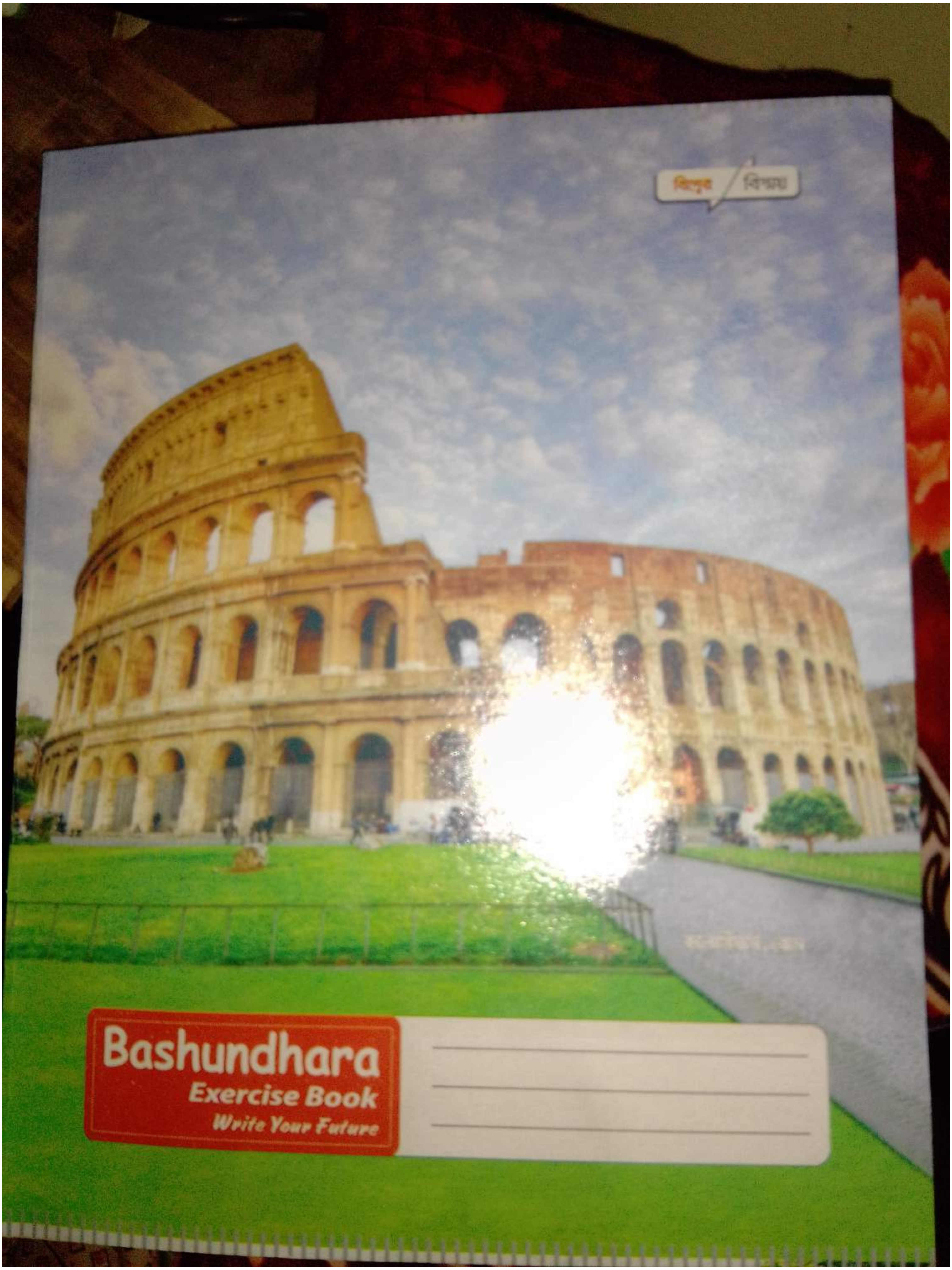
$$\therefore I = \int_0^{\pi/4} \frac{dz}{z^2+1}$$

$$= \left[\tan^{-1} z\right]_0^{\pi/4}$$

$$= \left[\tan^{-1}(\tan^2 x)\right]_0^{\pi/4}$$

$$= \left[x\right]_0^{\pi/4}$$

$$= \pi/4.$$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# Definite Integrals – II

Note : 1 (i) Beta functions / 1st Eulerian Integral:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \;; \quad m, n > 0 \quad [\beta(m,n) = \beta(n,m)]$$

(ii) Gamma functions / 2nd Eulerian Integral:

$$\Gamma n = \int_0^\alpha e^{-x} x^{n-1}\,dx \;; \quad n > 0 .$$

(2) $$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

(3) $$\Gamma n+1 = n\Gamma n = \lfloor n$$

(4) $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{\Gamma \frac{p+1}{2} \, \Gamma \frac{q+1}{2}}{\Gamma \frac{p+q+2}{2}} .$$

Page-238 : 17(i)) $\displaystyle\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx = \frac{8}{315}$

Now, $\displaystyle\int_0^{\pi/2} \sin^4 x \cos^4 x \, dx$

$$= \int_0^{\pi/2} \sin^4 x \left(1-\sin^2 x\right)^2 \cos x \, dx$$

$$= \int_0^1 z^4\left(1-x^2\right)^2 dx \quad \left[z = \sin x\right]$$

$$= \left[\frac{z^5}{5} - \frac{2z^7}{7} + \frac{z^9}{9}\right]_0^1$$

$$= \frac{1}{5} - 2/7 + 1/9$$

$$= \frac{8}{315}$$

Page-239 : 18(i)) $\displaystyle\int_0^1 x^3(1-x)^3 dx = \frac{1}{140}$

Now, $\displaystyle\int_0^1 \left(x^3 - 3x^4 + 3x^5 - x^6\right) dx$

$$= \left(\frac{1}{4}x^4 - \frac{3}{5}x^5 + \frac{3}{6}x^6 - \frac{1}{7}x^7\right)_0^1$$

$$= \frac{1}{4} - \frac{3}{5} + 2 - 1/7 = \frac{1}{140}.$$

(ii) $\int_0^1 x^3(1-x^2)^{5/2}\,dx = \frac{2}{63}$

put $x=\sin^2\theta$ $\therefore dx = 2\sin\theta\cos\theta\,d\theta$.

$\theta=0,\ x=0;$

$\theta=\frac{1}{2}\pi,\ x=1.$

$$\therefore I = \int_0^{\pi/2} \sin^3\theta\cdot\cos^5\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^3\theta\cos^6\theta\,d\theta$$

$$= \int_0^{\pi/2} \cos^6\theta\left(1-\cos^2\theta\right)\cdot\sin\theta\,d\theta$$

$$= \int_0^1 u^6(1-u^2)\,du \quad [u=\cos\theta]$$

$$= \left[\frac{u^7}{7}-\frac{u^9}{9}\right]_0^1$$

$$= \frac{1}{7}-\frac{1}{9}$$

$$= \frac{2}{36}.$$

Page-235: Ex(1)/ $\int_0^1 x^6\sqrt{(1-x^2)}\,dx$

$x = \sin\theta \quad \therefore dx = \cos\theta\, d\theta$

$1-x^2 = \cos^2\theta$

$x=0 \;,\; \theta = 0$

$x = 1 \;,\; \theta = \pi/2$

$$\therefore I = \int_0^{\pi/2} \sin^6\theta \cos^2\theta\, d\theta = \frac{1\cdot 3\cdot 5\cdot 1}{2\cdot 4\cdot 6\cdot 8\cdot 1}\cdot\frac{\pi}{2}$$

$$= \frac{5\pi}{256}.$$

Page-236: Ex-2/ $\int_0^1 x^2(1-x)^{3/2}\,dx$

$x = \sin^2\theta \quad \therefore dx = 2\sin\theta\cos\theta\, d\theta$

$x = 0, 1$ we have. $\theta = 0, \pi/2$.

$$\therefore I = 2\int_0^{\pi/2} \sin^5\theta\cos^4\theta\, d\theta$$

$$= 2\cdot\frac{2\cdot 4}{5\cdot 7\cdot 9}$$

$$= \frac{16}{315}.$$

Page-200 Ex: 2/ Find the value of $\int_0^1 x^2 dx$.

Solⁿ: $\int_0^1 x^2 dx$ exists since $x^2$ is continuous in $[0, 1]$

From the definition, $\int_0^1 x^2 dx = \lim_{h \to 0} \sum (rh)^2$, where $nh=1$

$$= \lim_{h\to 0} h\left[1^2h^2 + 2^2h^2 + \cdots + n^2h^2\right]$$

$$= \lim_{h\to 0} \left[h^3\left(1^2+2^2+\cdots+n^2\right)\right]$$

$$= \lim_{h\to 0} h^3 \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{6} \lim_{h\to 0} \left(2n^3h^3 + 3n^2h^2 \cdot h + nh \cdot h^2\right)$$

$$= \frac{1}{6} \cdot \lim_{h\to 0} \left(2+3h+h^2\right), \text{ since } nh = 1,$$

$$= \frac{1}{6} \cdot 2$$

$$= 1/3$$

Define $\int_a^b f(x)\,dx$ :/ Define integration as the limit of a sum !

Let $f(x)$ be a real valued continuous function in $[a,b]$, where $a<b$. Again, let the internal $[a,b]$ be divided into $n$ equal sub-intervals, each of length $h$, by the points $a+h$, $a+2h$, ... + $a+(n-1)h$.

$a$ —— $a+h$ —— $a+2h$ ........ $a+(n-1)h$ —— $a+nh=b$

we now consider the sum $S$ given by,

$$S = h\left[f(a)+f(a+h)+\cdots+f(a+\overline{n-1}h)\right]$$

$$= \sum_{n=0}^{n-1} f(a+nh) \text{ ——— } (1)$$

Now, if $n\to\infty$, then $h\to 0$. Then (1) $\Rightarrow$

$\lim\limits_{h\to 0} h \sum\limits_{r=0}^{n+1} f(a+rh)$, the value of this limit is called the definite integral of $f(x)$ in $[a,b]$ and which is denoted by the symbol $\int_a^b f(x)$

This $\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{h\to 0}^{n-1} f(a+rh)$, ~~where n~~

where $nh = b-a$ —— (2)

If we put $a=0,\ b=1$ then $nh=1$

$2 \Rightarrow \int_0^1 f(x)$.

Page-202 Ex:4/ $\int_a^b \sin x\,dx = \cos a - \cos b$.

$\int_a^b \sin x\,dx$ which exists since $\sin x$ is continuous in $[a,b]$.

$$\int_a^b \sin x\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} \sin(a+rh) \quad , \text{ where } nh = b-a.$$

$$= \lim_{h\to 0} h \left[\sin a + \sin(a+h) + \sin(a+2h) + \cdots \text{ to } n \text{ terms}\right]$$

$$= \lim_{h\to 0} h \left\{\sin a + (n-1)\frac{h}{2}\right\} \frac{\sin \frac{1}{2}nh}{\sin \frac{1}{2}h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}h}{\sin\frac{1}{2}h}\, 2\sin\frac{1}{2}nh \,.\, \sin\left\{a+(n-1)\frac{h}{2}\right\}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}h}{\sin\frac{1}{2}h}\left[\cos\left(a-\frac{1}{2}h\right) - \cos\left\{a+(2n-1)\frac{h}{2}\right\}\right]$$

$$= \lim_{h \to 0} \left[\cos\left(a - \frac{1}{2}h\right) - \cos\left(a + nh - \frac{1}{2}h\right)\right],$$

since $\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1,$

$$= \lim_{\theta \to 0} \left[\cos\left(a - \frac{1}{2}h\right) - \cos\left(b - \frac{1}{2}h\right)\right], \text{ since } a + nh = b.$$

$$= a \cos a - \cos b.$$

Page-212: 1(III)] $\int_0^1 x^3 dx$

$$= \lim_{h \to 0} h \sum_{r=0}^{n} (rh)^3, \text{ where } nh = 1 - 0 = 1$$

$$= \lim_{h \to 0} h^4 \left(1^3 + 2^3 + \cdots + n^3\right)$$

$$= \lim_{h \to 0} h^4 \left\{\frac{n(n+1)}{2}\right\}^2$$

$$= \frac{1}{4} \cdot \lim_{h \to 0} \left\{(nh \cdot nh + h)\right\}^2$$

$$= \frac{1}{4}.$$

(iv) $\int_0^1 (ax+b)\, dx$

$$= \lim_{h\to 0} h \sum_{r=1}^{n} (arh+b), \quad \text{where } nh = 1-0 = 1$$

$$= \lim_{h\to 0} h \left[(ah+b) + (2ah+b) + \cdots + (nah+b)\right]$$

$$= \lim_{h\to 0} h \left[(1+2+3+\cdots+n)ah + nb\right]$$

$$= \lim_{h\to 0} h \left[\frac{n(n+1)ah}{2} + nb\right]$$

$$= \lim_{h\to 0} h \left[\frac{nh(nh+h)a}{2} + bnh\right]$$

$$= \lim_{h\to 0} \left[\frac{a(1+h)}{2} + b\right]$$

$$= \frac{a}{2} + b.$$

\# Formula: $\sin\alpha + \sin(\alpha+\beta) + \cdots + \sin\{\alpha + (n-1)\beta\}$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\beta/2} \sin \alpha + (n-1)\beta.$$

v) $\int_0^{\pi/2} \sin x \, dx$

By the definition of integration as the limit of sum, we can write, $a=0$, $b=\pi/2$.

$$\int_{a=0}^{b=\pi/2} \sin x\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} \sin(rh), \text{ where } nh = \pi/2$$

$$= \lim_{h\to 0} h \left[\sin 0 + \sin h + \sin 2h + \sin 3h + \dots + \sin(n-1)h\right]$$

$$= \lim_{h\to 0} h \cdot \frac{\sin h \cdot \frac{h}{2}}{\sin h/2} \left\{\sin 0 + (n-1)\cdot h/2\right\}$$

$$= \lim_{h\to 0} 2 \cdot \frac{h/2}{\sin h/2} \lim_{h\to 0} \frac{nh}{2} \sin\left(\frac{nh}{2} - \frac{h}{2}\right)$$

$$= 2 \cdot 1 \lim_{h\to 0} \sin \pi/4 \lim_{h\to 0} \sin\left(\pi/4 - h/2\right)$$

$$= 2 \sin \pi/4 \cdot \sin \pi/4$$

$$= 2 \sin^2 \pi/4$$

$$= 2 \cdot 1/2$$

$$= 1.$$

vii) $\int_0^1 \sqrt{x}\, dx$

Now, $I = x_1\sqrt{x_1} + (x_2 - x_1)\sqrt{x_2} + (x_3 - x_2)\sqrt{3} + \cdots,$

where, $x_1, (x_2 - x_1)$, etc $\to 0$.

Then putting $x_r = r^2h^2$, we get.

$$I = h^2\sqrt{h^2} + \{(2h)^2 - h^2\}.2h + \{(3h)^2 - (2h)^2\}3h + \cdots, \ h \to 0$$

$$= h^3 \sum_{r=1}^{n} r\{r^2 - (r-1)^2\}, \ n^2h^2 = 1, \ n \to \infty$$

$$= h^3 \sum_{r=1}^{n} (2r^2 - r), \ h \to 0$$

$$= h^3 \left[\frac{1}{3}n(n-1)(2n+1) - \frac{1}{2}n(n+1)\right], \ nh \to 1, \ h \to 0 = 2/3.$$

Page-216 Ex: 1/ $\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right\}$

Dividing the numerator and denominator of each term of the above series by $n$, the given series becomes,

$$=\lim_{n\to\infty}\left\{\frac{\frac{1}{n}}{1+\frac{1}{n}}+\frac{\frac{1}{n}}{1+\frac{2}{n}}+\cdots+\frac{\frac{1}{n}}{1+\frac{n}{n}}\right\}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}$$

$$=\lim_{h\to 0}h\sum_{r=1}^{n}\frac{1}{1+rh}\qquad\left[\text{putting } h=\frac{1}{n}\right]$$

$$=\int_0^1\frac{1}{1+x}\,dx$$

$$=\Big[\log(1+x)\Big]_0^1$$

$$=\log(1+1)$$

$$=\log 2.$$

Page- 222 : 29(ii) / $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1/n}{1+r^2/n^2} \qquad [h\to 0,\ nh=1]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+r^2/n^2}$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \left[\tan^{-1}x\right]_0^1 = \pi/4.$$

iii) $\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^3}{n^3+2^3}+\cdots\cdots+\frac{n^2}{2n^3}\right]$

$$= \lim_{n\to\infty}\frac{n^2}{n^3}\left[\frac{\left(\frac{1}{n}\right)^2}{1+\left(\frac{1}{n}\right)^3}+\frac{\left(2/n\right)^2}{1+\left(2/n\right)^3}+\cdots+\frac{\left(n/n\right)^2}{1+\left(n/n\right)^3}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3}$$

$$= \lim_{h\to 0} h \sum_{r=1}^{n} \frac{(rh)^2}{1+(rh)^3} \quad [h \to 0,\ nh = 1]$$

$$= \int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\Big[\log(1+x^3)\Big]_0^1$$

$$= \frac{1}{3}\log 2.$$

iv) $$\lim_{n\to\infty}\left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \cdots + \frac{n^2}{(n+2)^3} + \cdots + \frac{1}{8n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n^2}{(n+0)^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \cdots + \frac{n^2}{(n+n)^3}\right]$$

$$= \lim_{n\to\infty} \frac{n^2}{n^3}\left[\frac{1}{(1+\frac{0}{n})^3} + \frac{1}{(1+1/n)^3} + \cdots + \frac{1}{(1+r/n)^3}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{n} \frac{1}{(1+r/n)^3}$$

$$= \lim_{h\to 0} h \sum_{r=0}^{n} \frac{1}{1+(rh)^3} \quad [h \to 0,\ nh = 1]$$

$$= \int_0^1 \frac{dx}{(1+x)^3} = \left[-\frac{1}{2(1+x)^2}\right]_0^1$$

$$= -\frac{1}{2\cdot 2^2} + \frac{1}{2\cdot 1^2} = \frac{1}{2} - \frac{1}{8} = 3/8.$$

viii) $\lim_{n\to\infty} \left\{ \left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right) \right\}^{1/n}$

$$\therefore \log S = \lim_{n\to\infty} \frac{1}{n}\left[\log\left(1+\frac{1^2}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\log\left(1+\frac{n^2}{n^2}\right)\right]$$

$$= \lim_{n\to 0} \frac{1}{n}\sum_{r=1}^{n} \log\left(1+\frac{r^2}{n^2}\right)$$

$$\log S = \int_0^1 \log(1+x^2)\,dx$$

$$= \left[\log(1+x^2)\,x\right]_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot 2x\cdot x\,dx$$

$$= \log 2 - 2\int_0^1 \frac{(1+x^2)-1}{1+x^2}\,dx$$

$$= \log - 2 + \pi/2 .$$

ix) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n+r}{n^2+r^2}$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{1+\frac{r}{n}}{1+\left(\frac{r}{n}\right)^2}$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} \frac{1+rh}{1+(rh)^2} \quad [\text{where } nh = 1]$$

$$= \int_0^1 \frac{1+x}{1+x^2}\, dx$$

$$= \int_0^1 \frac{dx}{1+x^2} + \frac{1}{2}\int_0^1 \frac{2x\,dx}{1+x^2}$$

$$= \left[\tan^{-1} x + \frac{1}{2}\log(1+x^2)\right]_0^1$$

$$= \tan^{-1} 1 + \frac{1}{2}\log 2$$

$$= \pi/4 + \frac{1}{2}\log 2.$$

xi) $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}\right]$

~~L=~~ ~~Lim~~ Given expression,

$$u = \lim_{n\to\infty} \sum_{r=0}^{2n} \frac{1}{n+r}$$

$$= \lim_{n\to\infty}\left[\sum_{r=0}^{n} \frac{1}{n+r} + \sum_{r=n+1}^{2n} \frac{1}{n+r}\right]$$

Let $r = n+s$ in the second series, then,

$$u = \lim_{n\to\infty}\left[\sum_{r=0}^{n} \frac{1}{n+r} + \sum_{s=1}^{n} \frac{1}{2n+s}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=0}^{n}\frac{1}{1+\frac{r}{n}} + \frac{1}{n}\sum_{s=1}^{n}\frac{1}{2+s/n}\right]$$

$$= \lim_{h\to 0}\left[h\sum_{r=0}^{n}\frac{1}{1+rh} + h\sum_{s=1}^{n}\frac{1}{2+sh}\right] \quad [\text{where } nh=1]$$

$$= \int_0^1 \frac{dx}{1+x^2} + \int_0^1 \frac{dx}{2+x}$$

$$= \Big[\log(1+x) + \log(2+x)\Big]_0^1$$

$$= \Big[\log\{(1+x)(2+x)\}\Big]_0^1$$

$$= \log 6 - \log 2$$

$$= \log 6/2$$

$$= \log 3.$$

□ Statement of Fundamental Theorem of Integral Calculus:

If $f(x)$ is integrable in $[a,b]$ and if there exists a function $\phi(x) = f(x)$ in $(a,b)$, then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

V.V.I. *****

□ General properties of Definit integral:

1. $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Proof: By the fundamental theorem of integral calculus, $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$ if $\phi'(x) = f(x)$ in $[a,b]$

$$= -[\phi(a) - \phi(b)]$$

$$= \int_b^a f(x)\,dx.$$

| Calculus:

there

[a,b]

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad , \; a < c < b$

** 4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ ***

proof: $\int_0^{\pi/2} \sin x\,dx = \int_0^{\pi/2} \sin\left(\frac{\pi}{2}-x\right) dx$

$= \int_0^{\pi/2} \cos x\,dx$

** 5. $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx \quad$ if $f(a+x) = f(x)$.

proof: $\int_0^{4\pi} \sin^8 x\,dx = 4\int_0^{\pi} \sin^8 x\,dx \quad [\because \sin(\pi+x) = \sin x]$

6. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

7. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x)\,dx = f(x) \\ & \text{if } f(2a-x)\,dx = -f(x) \end{cases}$

proof: $\int_0^{\pi} \sin x\,dx = \int_0^{2\cdot\pi/2} \sin x\,dx = 2\int_0^{\pi/2} \sin x\,dx \quad [\sin(\pi-x) = \text{Si}$

$$\int_0^{\pi} \cos x\,dx = \int_0^{2\cdot\pi/2} \cos x\,dx = 0 \qquad \left[\cos\left(2\cdot\tfrac{\pi}{2} - x\right) = -\cos x\right]$$

***8.** $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_0^{a} \{f(x) + f(-x)\}\,dx$

$$= 2\int_0^{a} f(x)\,dx \quad \begin{cases} \text{if } f(-x) = f(x) \\ \text{if } f(-x) = f(x) \end{cases}$$

Proof:
$$\int_{-2}^{2} x^9 (1-x^2)^7\,dx$$

$$= \int_{-2}^{2} f(x)\,dx \qquad f(x) = x^9(1-x^2)^7$$
$$\therefore f(x) = -x^9(1-x^2)^7$$

$$= \int_{-2}^{2} f(x)\,dx = \int_0^{2} \{f(x) + f(-x)\}\,dx$$

$$= \int_0^{2} [f(x) - f(x)]\,dx$$

$$= 0.$$

Page-229 Ex: 1/ $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}\,dx = \frac{\pi}{4}$.

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\left\{\sin\left(\frac{\pi}{2}-x\right)\right\}}+\sqrt{\left\{\cos\left(\frac{\pi}{2}-x\right)\right\}}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{(\cos x)}+\sqrt{(\sin x)}}\,dx$$

$$\therefore 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{(\cos x)}+\sqrt{(\sin x)}}\,dx$$

$$= \int_0^{\pi/2} dx$$

$$= \left[x\right]_0^{\pi/2}$$

$$= \pi/2$$

$$\therefore I = \pi/4$$

Ex: 2/ $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2.$

put $x = \tan\theta \quad \therefore dx = \sec^2\theta\, d\theta$

when, $x=0$ , $\theta = 0$;

$x = 1$, $\theta = \pi/4$.

$$\therefore I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$= \int_0^{\pi/4} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\}d\theta$$

Now, $\displaystyle 1+\tan\left(\frac{\pi}{4}-\theta\right) = 1 + \frac{1-\tan\theta}{1+\tan\theta}$

$$= \frac{2}{1+\tan\theta}$$

$$\therefore I = \int_0^{\pi/4} \log\frac{2}{1+\tan\theta}\,d\theta$$

$$= \int_0^{\pi/4} \{\log 2 - \log(1+\tan\theta)\}\,d\theta$$

$$= \int_0^{\pi/4} \log 2\theta - \int_0^{\pi/4}\{\log(1+\tan\theta)\}\,d\theta$$

$$= \frac{1}{4}\pi.\log 2 - I$$

$\Rightarrow I + I = \frac{1}{4}\pi . \log 2$

$\Rightarrow 2I = \frac{1}{4}\pi . \log 2$

$\therefore I = \frac{\pi}{8} . \log 2 .$

## VIII (B)

Page-237: 2*/

$$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$$

Let, $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ ——— (1)

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} \text{ ——— (2)}$$

Now, (1) + (2) $\Rightarrow$ $2I = \int_0^{\pi/2} dx$

$$= [x]_0^{\pi/2}$$

$$= \pi/2$$

$\therefore I = \pi/4 .$

**7.** $\int_0^{\pi} x \log \sin x \, dx = \frac{1}{2}\pi^2 \log \frac{1}{2}$.

$$L.H.S \equiv I = \int_0^{\pi} (\pi - x) \log \sin(\pi - x) \, dx$$

$$= \int_0^{\pi} (\pi - x) \log \sin x \, dx$$

$$= \pi \int_0^{\pi} \log \sin x \, dx - I$$

$$\therefore 2I = \pi \int_0^{\pi} \log \sin x \, dx$$

$$\therefore I = 2 \cdot \frac{1}{2}\pi \int_0^{\pi/2} \log \sin x \, dx \qquad [f(2a-x) = f(x)]$$

$$= \pi \cdot \frac{1}{2}\pi \log \frac{1}{2}$$

$$= \frac{1}{2}\pi^2 \log \frac{1}{2}$$

$$= R.H.S.$$

15) $\displaystyle\int_0^{\frac{1}{4}\pi} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log 2.$

Let, $\displaystyle I = \int_0^{\frac{1}{4}\pi} \log(1+\tan\theta)\, d\theta$

$\displaystyle = \int_0^{\pi/4} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\} d\theta$

Now, $\displaystyle 1+\tan\left(\frac{\pi}{4}-\theta\right) = 1+\frac{1-\tan\theta}{1+\tan\theta}$

$\displaystyle = \frac{2}{1+\tan\theta}$

$\displaystyle \therefore I = \int_0^{\pi/4} \log\frac{2}{1+\tan\theta}\, d\theta$

$\displaystyle = \int_0^{\pi/4} \{\log 2 - \log(1+\tan\theta)\}\, d\theta$

$\displaystyle = \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4}\{\log(1+\tan\theta)\}$

$\displaystyle = \frac{1}{4}\pi . \log 2 - I$

$\displaystyle \therefore 2I = \frac{1}{4}\pi . \log 2$

$\displaystyle \therefore I = \frac{1}{8}\pi . \log 2 .$

Page-239 : 20(i)* $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$

$\therefore I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}$

$= \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$

$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$

$= \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x} - I$

$\therefore 2I = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$

$= \pi\left[-\tan^{-1}(\cos x)\right]_0^{\pi}$

$= \pi\left[-\tan^{-1}(-1) + \tan^{-1}1\right]$

$= \pi\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \pi\cdot\frac{\pi}{2}$

$= \frac{\pi^2}{2}$

$\therefore I = \frac{\pi^2}{4}.$

11) $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$

let, $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$

$$= \int_0^{\pi/2} \frac{\sin^2(\frac{\pi}{2}-x)}{\sin(\frac{\pi}{2}-x)+\cos(\frac{\pi}{2}-x)} dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 x\, dx}{\cos x + \sin x} dx$$

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{dx}{\cos x + \sin x}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin(x+\pi/4)}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}} \left[\log \tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} \left[\log \tan \frac{3\pi}{8} - \log \tan \frac{\pi}{8}\right]$$

$$= \frac{1}{\sqrt{2}} \log \frac{\tan \frac{3\pi}{8}}{\tan \frac{\pi}{8}}$$

$$= \frac{1}{\sqrt{2}} \log \frac{2\sin \frac{3\pi}{8} \cos \frac{\pi}{8}}{2\cos \frac{3\pi}{8} \sin \frac{\pi}{8}}$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sin \frac{\pi}{2} + \sin \frac{\pi}{4}}{\sin \frac{\pi}{2} - \sin \frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} \log \frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$= \frac{1}{\sqrt{2}} \log (\sqrt{2}-1)^2 = \frac{2}{\sqrt{2}} \log (\sqrt{2}+1)$$

$$\therefore I = \frac{1}{\sqrt{2}} \log (\sqrt{2}+1).$$

(iii) $\displaystyle\int_0^{\pi} \frac{x \tan x \, dx}{\sec x + \tan x} = \frac{1}{2}\pi(\pi-2).$

Let, $\displaystyle I = \int_0^{\pi} \frac{x \tan x \, dx}{\sec x + \tan x}$

$$= \int_0^{\pi} \frac{(\pi-x)\tan(\pi-x)\, dx}{\sec(\pi-x) + \tan(\pi-x)}$$

$$= \int_0^{\pi} \frac{-(\pi-x)\tan x}{-\sec x - \tan x}\, dx = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x}\, dx - I$$

$$\therefore 2I = \pi \int_0^{\pi} \frac{\tan x \, dx}{\sec x + \tan x}$$

$$= \pi \int_0^{\pi} \tan x(\sec x - \tan x)\, dx \quad \left[\because \sec^2 x - \tan^2 x = 1\right]$$

$$= \pi \int_0^{\pi} (\sec x \tan x - \tan^2 x)\, dx$$

$$= \pi \int_0^{\pi} (\sec x \tan x - (\sec^2 x + 1)\, dx$$

$$= \pi \left[\sec x - \tan x + x\right]_0^{\pi}$$

$$= \pi\left[-1 + \pi - 1\right] = \pi(\pi - 2)$$

$$\therefore I = \frac{\pi}{2}(\pi - 2).$$

(iv) $\int_0^{\pi/2} \frac{x\,dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1)$.

Let, $I = \int_0^{\pi/2} \frac{x\,dx}{\sin x + \cos x} = \int_0^{\pi/2} \frac{(\frac{\pi}{2}-x)\,dx}{\sin(\frac{\pi}{2}-x) + \cos(\frac{\pi}{2}-x)}$

$= \int_0^{\pi/2} \frac{(\pi/2 - x)\,dx}{\cos x + \sin x}$

$= \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\cos x + \sin x} - I$

$\therefore 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\cos x + \sin x}$

$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}$

$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin(x + \pi/4)}$

$= \frac{\pi}{2\sqrt{2}} \left[\log \tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right]_0^{\pi/2}$

$= \frac{\pi}{2\sqrt{2}} \left[\log \tan \frac{3\pi}{8} - \log \tan \frac{\pi}{8}\right]$

$$= \frac{\pi}{2\sqrt{2}} \log \frac{\tan\frac{3\pi}{8}}{\tan\frac{\pi}{8}}$$

$$= \frac{\pi}{2\sqrt{2}} \log \frac{2\sin\frac{3\pi}{8}\cos\frac{\pi}{8}}{2\cos\frac{3\pi}{8}\sin\pi/8}$$

$$= \frac{\pi}{2\sqrt{2}} \log \frac{\sin\frac{\pi}{2} + \sin\frac{\pi}{4}}{\sin\frac{\pi}{2} - \sin\frac{\pi}{4}}$$

$$= \frac{\pi}{2\sqrt{2}} \log \frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{\pi}{2\sqrt{2}} \log \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$= \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1)^2$$

$$= \frac{2\pi}{2\sqrt{2}} \log(\sqrt{2}+1)$$

$$\therefore I = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1)$$

viii) $$\int_0^{\frac{1}{4}\pi} \frac{x\,dx}{1+\cos 2x+\sin 2x} = \frac{\pi}{16}\log 2.$$

let, $$I = \int_0^{\pi/4} \frac{x\,dx}{1+\cos 2x+\sin 2x}$$

$$= \int_0^{\pi/4} \frac{\left(\frac{\pi}{4}-x\right)dx}{1+\cos 2\left(\frac{\pi}{4}-x\right)+\sin 2\left(\frac{\pi}{4}-x\right)}$$

$$= \int_0^{\pi/4} \frac{\left(\frac{\pi}{4}-x\right)dx}{1+\sin 2x+\cos 2x}$$

$$= \frac{\pi}{4}\int_0^{\pi/4} \frac{dx}{1+\sin 2x+\cos 2x} - I$$

$$\therefore 2I = \frac{\pi}{4}\int_0^{\pi/4} \frac{dx}{2\cos^2 x+2\cos x\sin x}$$

$$= \frac{\pi}{8}\int_0^{\pi/4} \frac{\sec^2 x\,dx}{1+\tan x}$$

$$= \frac{\pi}{8}\Big[\log(1+\tan x)\Big]_0^{\pi/4}$$

$$= \frac{\pi}{8}\log 2$$

$$\therefore I = \frac{\pi}{16}\log 2.$$

## Chapter-7

### Integration by successive Reduction:

Reduction Function: The formula in which a certain involving come parameters connected with some integer is of lower is called a Reduction formula.

1) Reduction formula for $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$

11) Reduction formula for $\int_0^{\pi/2} \sin^n x\,dx$ or $\int_0^{\pi/2} \cos^n x\,dx$

using this formula evaluate, $\int_0^{\pi/2} \cos^9 x\,dx$.

Page-190 Ex: 6

$$I_9 = \int_0^{\pi/2} \cos^9 x\,dx = \frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{7}\cdot\frac{2}{3}\cdot 1$$

$$= \frac{128}{315}.$$

***

Page-231/ Reduction formula:

Let, $I_n = \int \sin^n x\,dx$

$$= \int \sin^{n-1} x \sin x\,dx$$

$$= \sin^{n-1} x \int \sin x\,dx + (n-1)\int \sin^{n-2} x \cdot \cos^2 x\,dx$$

$$I_n = \sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^{n}x\,dx$$

$$I_n = -\sin^{n-1}x\cos x + (n-1)\, I_{n-2} - (n-1) I_n$$

$$\therefore I_n = \frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n} I_{n-2} \quad \text{--- (1)}$$

This is the reduction formula.

Formula for $I_n = \int \sin^n x\,dx$.

we rewrite (1),

$$\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} dx$$

$$\therefore \int_0^{\pi/2} \sin^n x\,dx = -\left[\frac{\sin^{n-1}x\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2} \quad \text{--- (2)}$$

This is the reduction formula for $\int_0^{\pi/2} \sin^n x\,dx$

we set $I_n = \int_0^{\pi/2} \sin^n x\,dx$

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^n\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \cos^n x\,dx$$

Replacing $n$ by $n-2$, $n-4$ etc, successively, then

From (2) we have,

$$J_{n-2} = \frac{n-3}{n-2} J_{n-4}$$

$$J_{n-4} = \frac{n-5}{n-4} J_{n-6} \text{, etc.}$$

Then, $J_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-6} \cdots \frac{3}{4} \cdot \frac{1}{2} J_0$ if $n$ is even.

$$= \frac{n-1}{n} \cdot \frac{n-5}{n-6} \cdots \frac{4}{5} \cdot \frac{2}{3} J_1 \text{, if } n \text{ is odd} \quad \text{——} (3)$$

$$I_n = \int_0^{\pi/2} \sin^n dx = \int_0^{\pi/2} \cos^n x \, dx$$

Now, $J_0 = \int_0^{\pi/2} dx = \pi/2$

$$J_1 = \int_0^{\pi/2} \sin x \, dx = 1$$

where, $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$

Equation (3) is called the walli's formula.

Page- 181 : Ex: 1/

① $\int \tan^5 x\,dx$.

~~Formula~~: $\int \tan^n x\,dx = \int \tan^{n-2} x\,dx \cdot \tan^2 x\,dx$

$$I_n = \int \tan^{n-2} x \cdot (\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$\therefore I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}.$$

$$\therefore I_5 = \frac{\tan^{5-1} x}{5-1} - I_3$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + I_1$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \log \sec x.$$

(ii) $\int \tan^6 x\,dx$

$$\Rightarrow I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$I_6 = \frac{1}{5}\tan^5 x - I_4$$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x - I_2$$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x.$$

Page-174: Obtain Reduction Formula: $\boxed{\text{V.V.I}}$

$$I_{m,n} = \int \sin^m x \cos^n x \, dx$$

$$= \int \cos^{n-1} x \sin^m x \cos x \, dx$$

$$= \cos^{n-1} x \int \sin^m x \underset{(d\sin x)}{\cos x \, dx} + (n-1) \int \left\{ \cos^{n-2} x \sin x . \int \sin^m x \underset{(d\sin x)}{\cos x \, dx} \right\} dx$$

$$= \cos^{n-1} x \cdot \frac{\sin^{m+1} x}{m+1} + \frac{(n-1)}{m+1} \int \cos^{n-2} x \sin^{m+2} x \, dx \quad \Big| \; x\int dx = \frac{x^2}{2}$$

$$= \frac{\cos^{n-1} x . \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} \sin^m x (1 - \cos^2 x) dx$$

$$= \frac{\cos^{n-1} x . \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

This is the reduction Formula,

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} \cdot I_{m,n-2}$$

Reduction formula for $\int \cos^2 x \sin^4 x \, dx$

$$I_{4,2} = \frac{\cos x \sin^5 x}{6} + \frac{1}{6} \cdot I_{4,0}.$$

Now, $I_{4,0} = \int \sin^4 x \cos x \, dx = \frac{1}{4} \int (2\sin^2 x) \, dx$

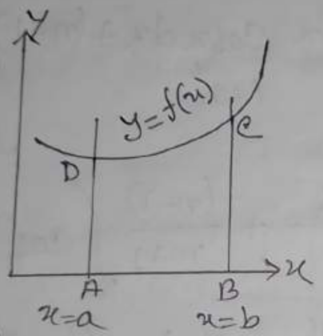
$$= \frac{1}{4} \left[ \frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8} \right] + c$$

chapter – 8 VIII (c) $\longrightarrow$ Reduction formula.

## Chapter – 10 (Important Chapter for exam)

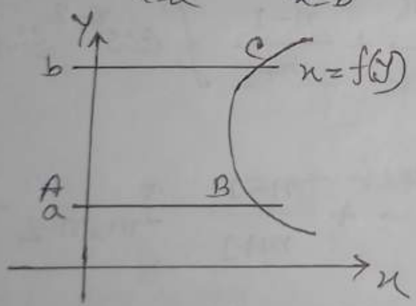
Determination of area of plane curves:

Note: ①



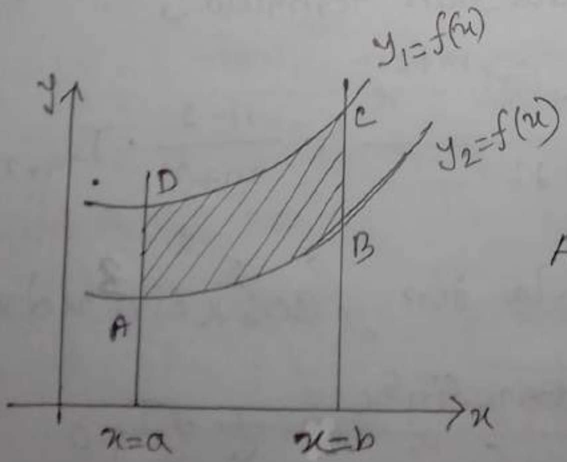

In fig. ① area of $ABCD = \int_{x=a}^{x=b} y\,dx$.

②




In fig. ② area of $ABC = \int_{y=a}^{b} x\,dy$

③

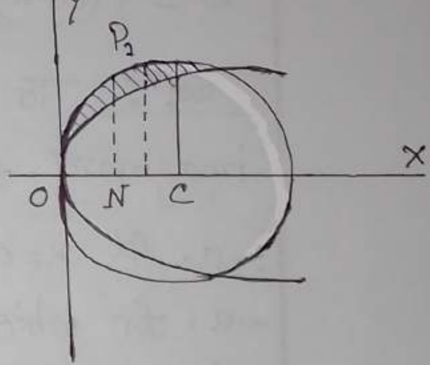



In fig. ③ area of $ABCD = \int_a^b y_1\,dx - \int_a^b y_2\,dx$

$= \int_a^b (y_1 - y_2)\,dx$

Page-324 Ex. 1/ Find the area above the x-axis, included between the parabola $y^2=ax$ and the circle $x^2+y^2=2ax$.

Solⁿ: The abscissa of the common point of the curve $y^2=ax$ and $x^2+y^2=2ax$ are given by $x^2+ax=2ax$. when $x=0$ and $x=a$.




We are thus to find out the area between the curves and the ordinates $x=0$ and $x=a$ above the x-axis.

The required area is therefore,

$$\int_0^a (y_1-y_2)\,dx \quad \text{where, } y_1^2=2ax-x^2 \text{ and } y_2^2=ax.$$

$$=\int_0^a\left(\sqrt{2ax-x^2}-\sqrt{ax}\right)dx$$

Now, putting $x=2a\sin^2\theta$.

$$\int_0^a\sqrt{2ax-x^2}\,dx=\int_0^{\pi/4}2a\sin\theta\cos\theta\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$=a^2\int_0^{\pi/4}(1-\cos 4\theta)\,d\theta$$

$$=a^2\left[\theta-\frac{\sin 4\theta}{4}\right]_0^{\pi/4}=\frac{\pi}{4}a^2$$

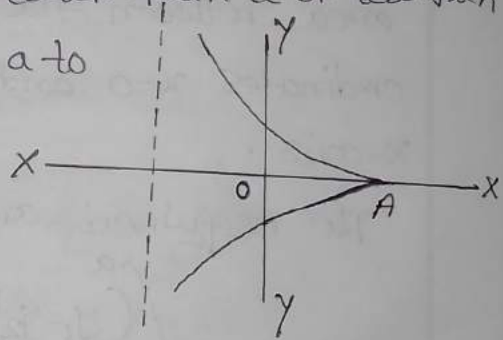
also, $\int_0^a\sqrt{ax}\,dx=\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^a=\frac{2}{3}a^2.$

$\therefore$ The required area is $= \frac{\pi}{4}a^2 - \frac{3}{2}a^2$

$$= a^2\left(\frac{\pi}{4} - \frac{2}{3}\right).$$

Page-326 Ex: 3/ Find the area between the curves $y^2 = \{(a-x)^3/(a+x)\}$ and the asymptote.

Soln: To trace the curve, we notice that $y$ is imaginary for values of $x$ greater than $a$ or less than $-a$. At $x=a$, $y=0$, and from $a$ to $-a$, for ~~which~~ each value of $x$, $y$ has two equal and opposite values, tending to $\pm\infty$ as approches $-a$. At $x=a$,




the x-axis touches both the branches. The figure is symmetrical about the x-axis.

The required area between the curve and its asymptotes is,

$$2\int_{-a}^{a} y\,dx = 2\int_{-a}^{a}\sqrt{\frac{(a-x)^3}{a+x}}\,dx$$

and substituting $z$ for $a+x$ this reduces to

$$2\int_{a}^{2a}(2a-z)\sqrt{\frac{2a-z}{z}}\,dz$$

$$=2\int_{0}^{\pi/2} 2a\cos^2\theta\,\frac{\cos\theta}{\sin\theta}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$\left[\text{where, } z=2a\sin^2\theta\right]$$

$$=16a^2\int_{0}^{\pi/2}\cos^4\theta\,d\theta$$

$$=16a^2\,\frac{1\cdot 3}{2\cdot 4}\cdot\pi/2$$

$$=3\pi a^2.$$

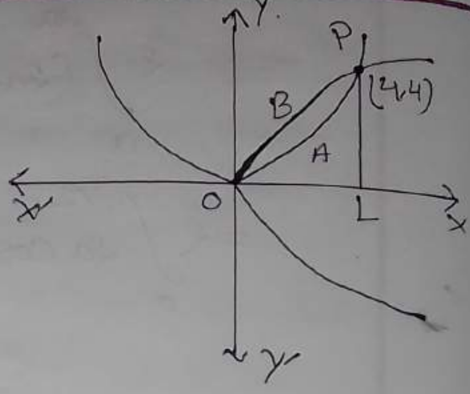
Page-345 Ex: 3 / (i) Find the bounded by the parabolas $x^2=4y$ and $y^2=4x$.

(ii) show that the area between $y=x^2$ and $x=y^2$ is $1/3$ sq. unit.

Soln: (i) The parabolas $x^2=4y$ and $y^2=4x$ intersect at $O(0,0)$ and $P(4,4)$.

$\therefore$ Required Area = Area OBPLO − Area OAPLO

$$= \int_0^4 2x^{1/2}\,dx - \int_0^4 \frac{1}{4}x^2\,dx$$




$$= 2\cdot\frac{2}{3}\left[x^{3/2}\right]_0^4 - \frac{1}{4}\cdot\frac{1}{3}\left[x^3\right]_0^4$$

$$= \frac{32}{3} - \frac{16}{3}$$

$$= \frac{16}{3} \text{ sq. units.}$$

ii) Here, the parabolas intersect at (0,0) and (1,1).

Required area $= \int_0^1 x^{1/2}\,dx - \int_0^1 x^2\,dx$

$$= \frac{2}{3}\left[x^{3/2}\right]_0^1 - \frac{1}{3}\left[x^3\right]_0^1$$

$$= 2/3 - 1/3$$

$$= 1/3 \text{ sq. unit.}$$

V.V.I

Page-347 (10) / Find the area of the segment cut off from $y^2 = 4x$ by the line $y = x$.

Sol$^{n}$: $y^2 = 4x$ ———— (i)

$y = x$ ———— (ii)

solving (i) + (ii) we get $x = 0, 4$.

For $\cancel{x \neq 0}$ , $\cancel{y \neq 0}$ $\quad x = 0$ , $y = 0$

$\cancel{x = 4}$ , $\cancel{y =}$ $\quad x = 4$ , $y = 4$

and at the point whose abscissa $= 4$.

let, $y_1^2 = 4x$ and $y_2 = x$.

$\therefore$ The required area $= \int_0^4 (y_1 - y_2)\,dx$

$$= \int_0^4 \left(\sqrt{4x} - x\right) dx$$

$$= \left[\frac{4}{3} x^{3/2} - \frac{1}{2} x^2\right]_0^4$$

$$= 8/3.$$

V.V.I

11/ Find the area bounded by the curve $y^2 = x^3$ and the line $y = 2x$.

$$y^2 = x^3 \text{ ——— } (i)$$

$$y = 2x \text{ ——— } (ii)$$

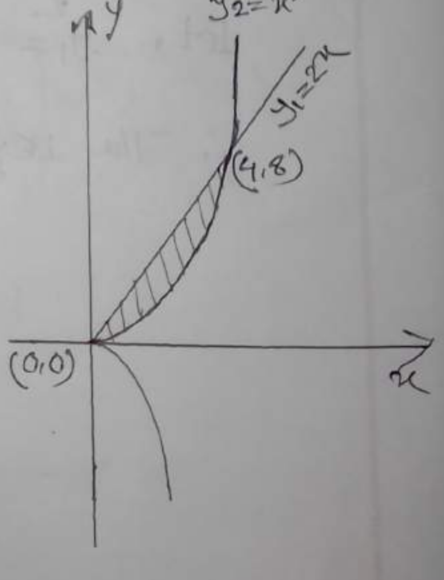
Solving (i) + (ii) we get, $x = 0, 4$.

For, $x = 0$, $y = 0$

$x = 4$, $y = 8$

$\therefore$ point of intesection of (i) + (ii) are $(0,0)\,(4,8)$.

let, $y_1 = 2x$ and $y_2^2 = x^3$




$\therefore$ Required area $= \int_0^4 (y_1 - y_2)\,dx$

$$= \int_0^4 \left(2x - x^{3/2}\right) dx$$

$$= \left[x^2 - \tfrac{2}{5}x^5\right]_0^4$$

$$= \frac{16}{5}$$

V.V.I

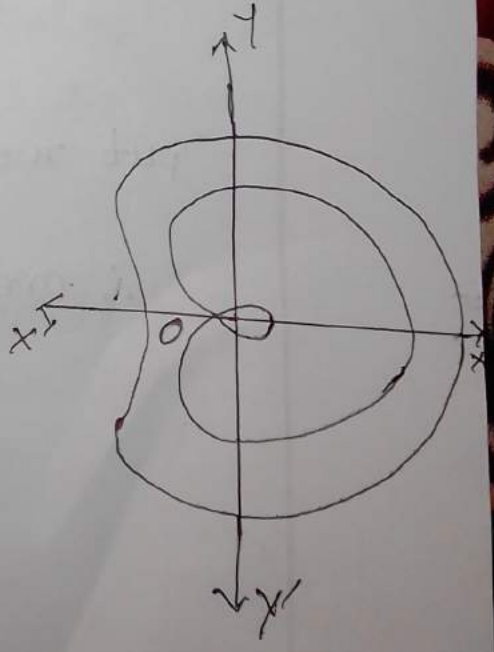
Page-348: 17(i) Find the area of the following curves $a^2y^2 = a^2x^2 - x^4$ $(a>0)$.

From the given equation, $y = \pm(x/a)\sqrt{a^2-x^2}$.

The curve connot proceed beyond $x = \pm a$. Also tangents at the origin are given by $a^2y^2 = a^2x^2$. From a rough sketch it now appears that the curve contains of 2 loops with the node at the origin. ~~Future~~ Further, ~~A=4~~ from symmetry,

$$A = 4(\text{area in 1st quadrant}) = 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{x}{a}\sqrt{a^2-x^2}\,dx$$




put $x = a\sin\theta$

Then, $A = 4a^2\int_0^{\pi/2} \sin\theta\cos^2\theta\,d\theta$

$$= 4a^2\left[-\tfrac{1}{3}\cos^3\theta\right]_0^{\pi/2}$$

$$= 4a^2 \cdot \tfrac{1}{3}$$

$$= \frac{4}{3}a^2$$

18(ii) / Find the area of the loop of each of the following curve $ay^2 = x^2(a-x)$.

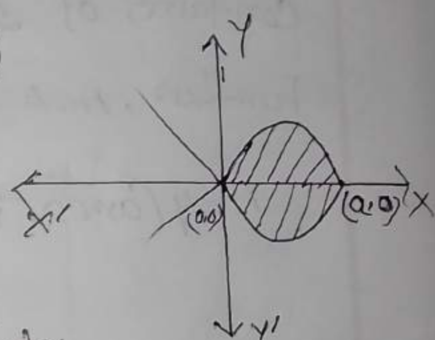
$$y^2 = \frac{x^2(a-x)}{a}$$

$$\Rightarrow y = \pm \frac{x\sqrt{(a-x)}}{\sqrt{a}} \quad \text{--- (1)}$$

(1) is symmetric about $x$-axis

$y=0$ , $x = a, 0$

$\therefore$ intersection point $(0,0), (a,0)$




Required area $= 2\int_0^a y\,dx$

$$= \frac{2}{\sqrt{a}}\int_0^a x\sqrt{a-x}\,dx$$

put $x = a\sin^2\theta$

$$\therefore \text{area} = 4a^2\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$$

$$= \frac{8}{15}a^2$$

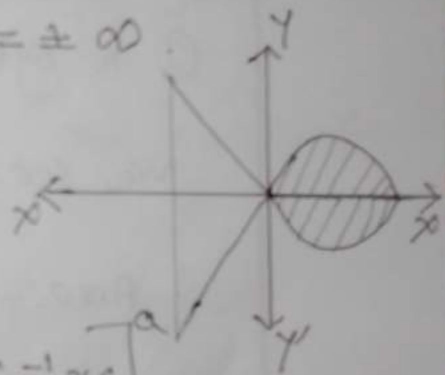
V.V.I
P(1)/ loops Find the area of the loop on one of two of following curves $x(x^2+y^2)=a(x^2-y^2)$.

$$x(x^2+y^2)=a(x^2-y^2)$$

$$\Rightarrow y = \pm \frac{x\sqrt{a-x}}{\sqrt{a+x}} \quad \text{—— (1)}$$

(1) is symmetric about y axis. for (0,0) (a,0).

we see that for, $x=-a$, $y=\pm\infty$.




$$\therefore \text{Area} = 2\int_0^a y\,dx = 2\int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx$$

$$= 2\left[\left(\frac{1}{2}x-a\right)\sqrt{a^2-x^2} - \frac{1}{2}a^2\sin^{-1}x/a\right]_0^a$$

$$= 2\left(-\frac{1}{2}a^2 \cdot \frac{1}{2}\pi + a^2\right)$$

$$= 2a^2\left(1-\frac{1}{4}\pi\right)$$

V.V.I

20(ii) / Find the whole area included between each of the following curves and its asymptotes

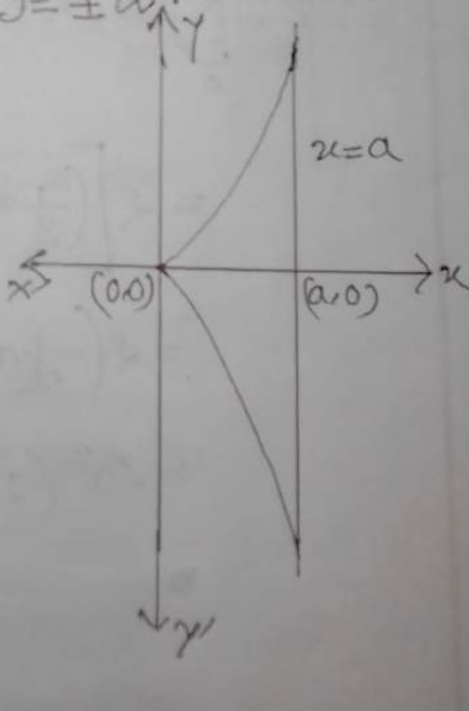
$$y^2(a-x) = x^3.$$

$$y^2 = \frac{x^3}{a-x}$$

$$y = \pm \frac{x^{3/2}}{\sqrt{a-x}} \quad \text{——— (1)}$$

(1) is symmetric for $y=0$, $x=0$ at $(0,0)$ $(a,0)$.

we see that for $x=a$, $y=\pm\infty$.




$$\therefore \text{Area} = 2\int_0^a y\,dx$$

$$= 2\int_0^a \frac{x^{3/2}}{\sqrt{a-x}}\,dx$$

$$= 4a^2 \int 4 \sin^4\theta\, d\theta$$

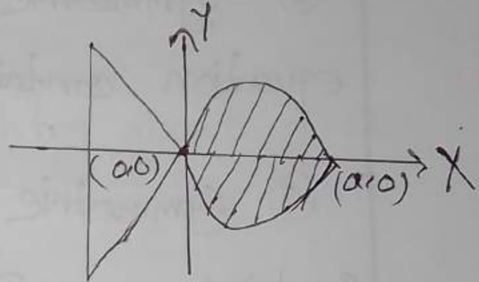
$$= \tfrac{3}{4} a^2 \pi$$

V.V.I

(iii) $y^2(a-x)=x^2(a+x)$

Solⁿ: $y=\pm x\sqrt{\dfrac{a+x}{a-x}}$ ——— (1)

(1) is symmetric about $x$-axis for $x=a$, $y=\pm\infty$ at $(0,0)$, $(a,0)$.




$$\therefore \text{Area} = 2\int_0^a y\,dx$$

$$= 2\int_0^a x\sqrt{\frac{a+x}{a-x}}\,dx \quad , \text{ put } x=a\cos\phi.$$

$$= 2\int_{\pi/2}^{0} a\cos\phi\cot\tfrac{1}{2}\phi(-a\sin\phi)\,d\phi$$

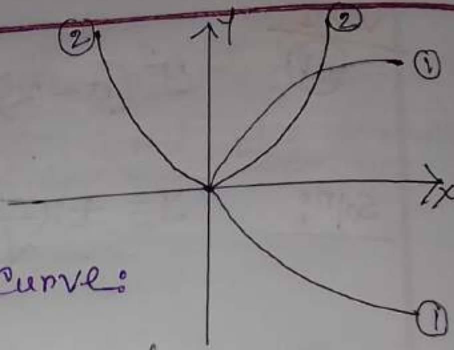
$$= 2a^2\int_0^{\pi/2}\cos\phi\,(1+\cos\phi)\,d\phi$$

$$= 2a^2\left[\sin\phi+\tfrac{1}{2}\phi+\tfrac{1}{2}\sin\phi\cos\phi\right]_0^{\pi/2}$$

$$= 2a^2\left(1+\tfrac{1}{4}\pi\right)$$

$y^2 = 4ax$

$x^2 = 4ay$




Technique to trach the given Curve:

① Symmetric th about x axis if the given equation contains even power of y on it.

② Symmetric about y axis if the given equation Contains even power of x on it.

③ If on interchanging x and y, the given equation doesn't change, the given curve is symmetric about the line $y=x$.

④ If we put $-x$ for y and $-y$ for x the given equation doesn't change, the curve is symmetric about the line $y=-x$.

⑤ If we put x for $-y$ and y for $-y$ the given equation doesn't change the curve is symmetric about opposite quardrants.

# Chapter-12

## Volumes and Surface area:

A: 1. Volume of the solid revolving about $x$-axis

$$= \pi \int_{x_1}^{x_2} y^2 dx$$

2. Volume of the solid revolving about $y$-axis

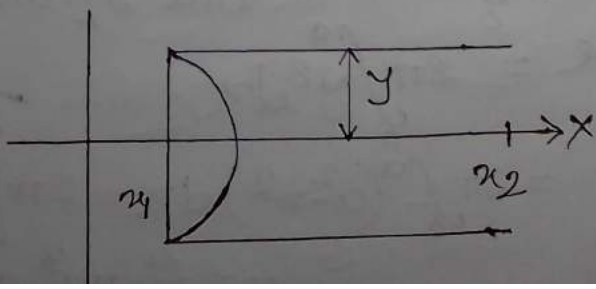
$$= \pi \int_{y_1}^{y_2} x^2 dy$$

B: 1. Surface area for A(1),

$$S = 2\pi \int_{S_1}^{S_2} y\,ds = 2\pi \int_{S_1}^{S_2} y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

2. Surface area for A(2),

$$S = 2\pi \int_{S_1}^{S_2} x\,ds = 2\pi \int_{S_1}^{S_2} x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dx$$




$$S = 2\pi \int_{S_1}^{S_2} y\,dx$$

$$S \cdot \int^{x_2} y^2 dx$$

Page-386 Ex: 2/ The circle $x^2+y^2=a^2$ revolves round the x-axis. Find the surface area and the volume of the whole surface generated.

Solⁿ: $x^2+y^2=a^2$

$$\Rightarrow y=\sqrt{a^2-x^2}$$

$$\frac{dy}{dx}=-\frac{x}{\sqrt{a^2-x^2}}$$

$$\therefore \left(\frac{ds}{dx}\right)^2=1+\left(\frac{dy}{dx}\right)^2$$

$$=1+\frac{x^2}{a^2-x^2}$$

$$=\frac{a^2}{a^2-x^2}$$

Required surface area $S=4\pi\int_0^a y\left(\frac{ds}{dx}\right)dx$

$$=2\pi\int_0^a\sqrt{a^2-x^2}\cdot\frac{a}{\sqrt{a^2-x^2}}\,dx$$

$$=4\pi a\int_0^a dx=4\pi a^2 \text{ sq. unit}$$

Required volume $=2\pi\int_0^a y^2dx$

$$=2\pi\int_0^a(a^2-x^2)\,dx=2\pi\left[a^2x-\frac{1}{3}x^3\right]_0^a$$

$$=\frac{4}{3}\pi a^3 \text{ cubic units.}$$

Ex:3/ Find the volume generated by the revolution about $x$-axis of the area bounded by the loop of the curve $y^2 = x^2(2-x)$.

Soln: Equation to the curve is $y^2 = x^2(2-x)$.

Since, $y=0$ when $x=0, 2$, for the loop of the curve, $x$ varies from 0 to 2.

$$\text{Required surface area} = \pi \int_0^2 y^2 \, dx$$

$$= \pi \int_0^2 x^2(2-x) \, dx$$

$$= \pi \left[ \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^2 \cancel{dx}$$

$$= \frac{4\pi}{3}.$$

# Important Math for Differentiation

1. $f(x) = x - [x] \longrightarrow$ greatest integer not greater than 0

$f(x) = |x+1| + |x|$

2. $(\delta, \epsilon)$ Definition $\longrightarrow$ Definition of limit and hence defined continuity.

3. Distinguis between $\lim_{x \to a} f(a)$ and $f(a)$.

4. Define derivative at a point

$f(x) = |x| \begin{cases} x \\ 0 \\ -x \end{cases}$

$f(x) = x|x| \begin{cases} x^2 \\ 0 \\ -x^2 \end{cases}$

5. Every differentiateable function in continuous but converse is not true $\rightarrow$ Proof.

6. $f(x) = \begin{cases} 1 & x < 0 \\ 1 + \sin x & 0 < x < \frac{\pi}{2} \\ 2 + \left(\pi - \frac{\pi}{2}\right)^2 & \cancel{3/2}\ \pi/2 \leq 0 \end{cases}$

$f(x) = \begin{cases} 3 + 2x & -3/2 < x \leq 0 \\ 3 - 2x & 0 < x \leq 3/2 \end{cases}$

7. $y = \cos^{-1}\frac{1-x^2}{1+x^2}$, $\frac{dy}{dz} = ?$ ~~put~~ $z = \tan^{-1}\frac{2x}{1-x^2}$

$\frac{dz}{dy} = ?$

8. $y = (\sin^{-1}x)^2$

$\downarrow$

$(1-x^2)y_{n+2} - (2x+1)xy_{n+1} - x^2y_n = 0$

9. Leventh's Theorem: $y = a\log(n/x) + b\log(\log x)$

$$\lim_{x \to 0}\left(\frac{\sin x}{x}\right)^{1/x^3}$$

10. Geometrical theorem of mean value significance.

11. Roll's value Theorem:

$f(x) = x^2 - 6x + 8 \quad [2, 4]$

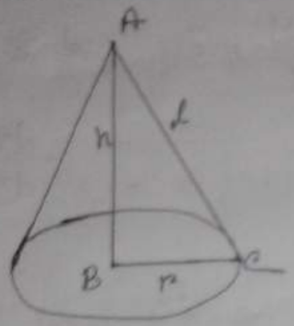
$f(x) = x^2 - 5x + 8 \quad [1, 4]$

$f(x) = x(x-1)(x-2)$

12. $f(b) - f(a) = (b-a)\, f'(a)$, $a = 0$, $b = 1/2$

13. Define maxima or minima.

বইয়ের Page-343 : Exp-17, 32(i).




$S = \pi \times \cdots \times \cdots$

$= \pi \times BC \times AC$

$v = \frac{1}{3} \pi r^2 h$

$= \sin^{-1} \frac{1}{\sqrt{3}}$

14. Define partial derivative (chapter-2):

chap-2: Ex: 4, 5, Exp: 16, 18, 17.

15. Define relation between tangent and Normal.

16. Define pedal equation in a curve.

page → 447 : 13 (1, 11) → V.V.I

Asmptotes → Exmples.

# Important type of Math for Integration

1. $\displaystyle\int \frac{dx}{x+\sqrt{x-1}}$

2. $\displaystyle\int \frac{dx}{\sqrt{x^2-7x+12}}$

3. $\displaystyle\int \frac{dx}{3+4\sin x}$

5. $\displaystyle\int \frac{dx}{(x^2-2x+1)\sqrt{x^2-2x+3}}$

6. $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx \rightarrow$ v.v.I.

7. $\displaystyle\int_2^4 \frac{x-3}{\sqrt{(5-x)(x-1)}}\,dx$

8. $\displaystyle\int_{-a}^{a} ax\sqrt{x^2-a^2}\,dx$

9. * $I_n = \int \sin^n x\,dx \longrightarrow$ find the reduction formula

10. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx \longrightarrow$ using summation of integrals....

11. $\displaystyle\lim_{x\to\infty}\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$.

12. Chapter-10 : Exercise → **10, **11, **17(i), **19(i), **20(i)
Page – 345

13. General properties of definite integrals → V.V.I.

14. $\int \sqrt{(x-\alpha)(\beta-x)}\, dx$

15. $\int \sqrt{(x-1)(2-x)}\quad dx$

16. $\int \sqrt{(x-1)(3-1)}\ dx$

17. $\int_{\alpha}^{\beta} \frac{dx}{\sqrt{(x-1)(3-x)}}$ , $\int_{1}^{2} \frac{dx}{\sqrt{(x-1)(2-x)}}$

put $x = \alpha \cos^2\theta + \beta \sin^2\theta$

chapter-3 : ~~page-34~~ 34(i, ii).

18. Summation of series

19. $\int_{0}^{1} x^3(1-x^5)$