

University of Rajshahi
Department of Computer Science and Engineering
B.Sc. (Engg.) Part-2 Even Semester Examination-2018
Course: MATH2241 (Linear Algebra)
Marks: 52.50 Time: 3 Hours

[N.B. Answer SIX questions taking THREE from each section.]

Section-A

1. (a) Define vector space and its subspace with example. Let $V = \mathbb{R}^3$ and let $W = \{(x, y, z) : x + y + z = 0\}$. Prove that W is a subspace of V . 2.75
 (b) Prove that vector space V is the direct sum of its subspaces U and W , if and only if
 (i) $V = U + W$ and (ii) $U \cap W = \{0\}$. 3
 (c) Show that $(1,1,1), (0,1,1)$ and $(0,1,-1)$ generate \mathbb{R}^3 . 3
2. (a) Define linearly independent set. Let V be the vector space of 2×2 matrices over \mathbb{R} . Are the matrices $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ linearly dependent? 3
 (b) Define basis and dimension of vector space. Determine whether or not $(1,1,2), (1,2,5)$ and $(5,3,4)$ form a basis for the vector space \mathbb{R}^3 . 3
 (c) Let U and W be the following subspaces of \mathbb{R}^4 : $U = \{(a, b, c, d) : b + c + d = 0\}$, $W = \{(a, b, c, d) : a + b = 0, c = 2d\}$. Find the dimension and a basis of (i) U , (ii) W . 2.75
3. (a) Define co-ordinate of a vector. Find the co-ordinate of the polynomial $v = 2t^2 - 5t + 6$ where v is the vector in $V = \{at^2 + bt + 6 : a, b, c \in \mathbb{R}\}$ which generated by the polynomial $e_1 = 1, e_2 = t - 1$ and $e_3 = (t - 1)^2$. 3.25
 (b) Define linear mapping with example. Let $F: V \rightarrow U$ be a linear mapping. Show that the kernel of F is a subspace of V . 3
 (c) Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by $(x, y, z) = 8x - 3y + 5z$. Is the map F linear? 2.5
4. (a) Let $F: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a linear mapping defined by $F(x, y, z, t) = (x - y + z + t, x + 2z - t, x + y + 3z - 3t)$. Find a basis and the dimension of the kernel F . 3
 (b) Define linear operator. Let T be the linear operator on \mathbb{R}^2 defined by $T(3,1) = (2, -4)$ and $T(1,1) = (0,2)$. Find $T(a,b)$ and $T(6,5)$. 2.75
 (c) Define a matrix representation of linear operator. Find the matrix represented by the linear operator \mathbb{R}^2 defined by $T(x,y) = (3x - 2y, 3x + 5y)$ relative to the basis $\{e_1 = (1,2), e_2 = (-3,0)\}$. 3

Section-B

5. (a) Let T be the linear operator on \mathbb{R}^3 defined by $T(x, y, z) = (2y + z, x - 4y, 3x)$ 4
 i) Find the matrix of T in the basis $\{f_1 = (1, 2, 1), f_2 = (1, 2, 0), f_3 = (3, 0, 0)\}$
 ii) Verify that $[T]_f[v]_f = [T(v)]_f$, for any $v \in \mathbb{R}^3$.
- (b) Define transition matrix. Consider the following bases of \mathbb{R}^3 : $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ and $\{f_1 = (1, 1, 1), f_2 = (1, 1, 0), f_3 = (1, 0, 0)\}$. 4.75
 i) Find the transition matrix P from $\{e_i\}$ to $\{f_i\}$.
 ii) Find the transition matrix Q from $\{f_i\}$ to $\{e_i\}$.
 iii) Verify that $Q = P^{-1}$.
6. (a) Define eigenvalue and eigenvector of a matrix A . Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ be a matrix. Find all eigenvalues and eigenvectors of A . 2.75
 (b) Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ be a matrix. Find an invertible matrix P such that $P^{-1}AP$ is diagonal. 3
 (c) Suppose λ is an eigenvalue of an invertible operator T . Show that λ^{-1} is an eigenvalue of T^{-1} . 3
7. (a) Define linear functional. Let $V = \{a + bt : a, b \in \mathbb{R}\}$ consider $\phi_1 : V \rightarrow \mathbb{R}$ and $\phi_2 : V \rightarrow \mathbb{R}$ be defined $\phi_1(f(t)) = \int_0^1 f(t)dt$ and $\phi_2(f(t)) = \int_0^2 f(t)dt$. Find the basis $\{v_1, v_2\}$ of V which is dual to $\{\phi_1, \phi_2\}$. 3
 (b) Define dual space and annihilator. Let W be a subset of a vector space V . Show that W^0 is a subspace of V^* . 2.75
 (c) Let W be a subspace of \mathbb{R}^4 spanned by $v_1 = (1, 2, -3, 4)$ and $v_2 = (0, 1, 4, -1)$. Find a basis of the annihilator of W . 3
8. (a) Define inner product. Verify that the following is an inner product in \mathbb{R}^2
 $\langle u, v \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ where $u = (x_1, x_2)$, $v = (y_1, y_2)$. 3
 (b) Define orthogonal vector. Find a unit vector orthogonal to $v_1 = (1, 1, 2)$ and $v_2 = (0, 1, 3)$. 2.75
 (c) Define orthonormal set. Find an orthonormal basis of the subspace W of \mathbb{C}^3 spanned by $v_1 = (1, i, 0)$ and $v_2 = (1, 2, 1 - i)$. 3

University of Rajshahi
Department of Computer Science and Engineering
B.Sc. (Engg.) Part-2 Even Semester Examination -2017
Course: MATH2241 (Linear Algebra)
Full Marks: 52.5 Time: 3:00 hours

[N.B. Answer any six questions taking THREE from each of the groups]

Section-A

1. a) Define subspace of a vector space. If U and W be subspace of vector space V . Show that $U \cap W$ is a subspace of V . 2.5
- b) Let U and W be the subspaces of $V = \mathbb{R}^3$ defined by $U = \{(a, b, c) | a = b = c\}$ and $W = \{(0, b, c)\}$. Show that $V = U \oplus W$. 3.0
- c) Prove that $L(S)$ is the intersection of all subspaces of V containing S . 3.25

2. a) Define basis and dimension of a vector space. Determine whether or not the following form a basis for the vector space \mathbb{R}^3 : $(1,1,1), (1,2,3)$ and $(2, -1, 1)$. 3.0
- b) Let U and W be the subspaces of \mathbb{R}^4 generated by $\{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$ and $\{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$ respectively. Find $\dim(U+W)$. 3
- c) Let v_1, v_2, \dots, v_m be independent vectors and suppose $U = a_1v_1 + a_2v_2 + \dots + a_mv_m$ where the a_i are scalars. Show that the above representation of U is unique. 2.75

3. a) Define a linear mapping and kernel. Let f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find $(gof)(4)$ and $(fog)(4)$. Also find formula for fog and gof . 3.25
- b) Define image of a linear mapping. Let $F: V \rightarrow U$ be a linear mapping. Show that image of F is a subspace of U . 3
- c) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear mapping defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$. Find a basis and the dimension of (i) Image of T and (ii) kernel of T . 2.5

4. a) For any operators $S, T \in A(V)$. Show that $[ST]_e = [S]_e [T]_e$ where $\dim V = 2$. 3.25
- b) Find the matrix representation of the linear operator T on \mathbb{R}^3 relative to the usual basis $\{e_1 = (1,0,0), e_2 = (0,1,0) \text{ and } e_3 = (0,0,1)\}$, where $T(x, y, z) = (2y + z, z - 4y, 3x)$. 3
- c) Find the trace of T on \mathbb{R}^3 , where $T(x, y, z) = (x, y + 2z, 3x + 4y - 5z)$.

Section-B

3.25

5. a) Define eigenvalue and eigenvectors of a matrix A . Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ be a matrix. Find all eigenvalues and eigenvectors of A . 3.00
- b) Find the minimum polynomial of the matrix $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ 2.50
- c) Suppose λ is an eigenvalue of an invertible operator T . Show that λ^{-1} is an eigenvalue of T^{-1} . 2.50
6. a) Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear mapping defined by $F(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$. Find the matrix of F in the following bases of \mathbb{R}^3 and \mathbb{R}^2 : $\{f_1 = (1, 1, 1), f_2 = (1, 1, 0), f_3 = (1, 0, 0)\}$ $\{g_1 = (1, 3), g_2 = (2, 5)\}$ 3.00
- b) Show that similar matrices have the same eigenvalues. 3.00
- c) Find all eigenvalues and a basis of each eigenspace of the operator $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (2x + y, y - z, 2y + 4z)$ 2.75
7. a) Define linear functional and dual space. Consider the following basis of \mathbb{R}^3 : $\{v_1 = (1, -1, 3), v_2 = (0, 1, -1), v_3 = (0, 3, -2)\}$. Find the dual basis $\{\phi_1, \phi_2, \phi_3\}$. 4.75
- b) Define annihilator of a set W of a vector space V . Suppose V has a finite dimension and W is a subspace of V , show that $\dim W + \dim W^\circ = \dim V$. 4
8. a) Define an inner product space. Give an example of it. 2.00
- b) Let $\{v_1 = (1, 1, 1), v_2 = (0, 1, 1), v_3 = (0, 0, 1)\}$ be the basis of \mathbb{R}^3 . Applying Gram-Schmidt orthogonalization process, find the orthogonal basis of \mathbb{R}^3 . 4.75
- c) Prove $|(u, v)| = \|u\| \|v\|$ 2.00

University of Rajshahi
Department of Computer Science and Engineering
B.Sc. (Engg.) Part-II (Even Semester) Examination-2016
Course: MATH2241 (Linear Algebra)

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 Dept. of CSE, Computer Science &
 Engineering, University of Rajshahi,*

Full Marks: 52.5

Time: 3 Hours

[Answer any Six (06) questions taking Three (03) questions from each part]

Part-A

1. (a) Define a subspace of a vector space. Let $V = \mathbb{R}^3$, and let $W = \{(x,y,z): x + y + z = 0\}$. Prove that W is a subspace of V . 2.5
- (b) Define a vector space. Prove that vector space V is the direct sum of its subspace U and W , if and only if (i) $V = U + W$ and (ii) $U \cap W = \{0\}$. 3.25
- (c) Write the matrix $E = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$ as a linear combination of $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$. 3

2. (a) Let u , v , and w be linearly independent vectors. Prove that $u + v$, $u - v$, and $u - 2v + w$ are also linearly independent. 2
- (b) Suppose U and W are finite dimensional subspaces of a vector space V . Prove that $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$. 3.75
- (c) Find the coordinate vectors of $v = (3,1,-4)$ relative to the basis $f_1 = (1,1,1)$, $f_2 = (0,1,1)$, and $f_3 = (0,0,1)$. 2.75

3. (a) Define: a linear mapping; Kernel and Image of a linear mapping. Suppose $F: V \rightarrow U$ is a linear mapping. Prove that (i) the image of F is a subspace of U , and (ii) the kernel of F is a subspace of V . 3
- (b) Find $T(a,b,c)$ where $T: \mathbb{R}^3 \rightarrow \mathbb{R}$ is defined by $T(1,1,1) = 3$, $T(0,1,-2) = 1$, and $T(0,0,1) = -2$. 3
- (c) Show that the mapping $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $F(x,y) = (x+y, x)$ is linear. 2.75

4. (a) Consider the bases $\{e_1 = (1,0), e_2 = (0,1)\}$ and $\{f_1 = (1,3), f_2 = (2,5)\}$ of \mathbb{R}^2 .
 - i. Find the transition matrix P from $\{e_i\}$ to $\{f_i\}$. 2.25
 - ii. Show that $[T]_f = P^{-1}[T]_e P$ for the linear operator on \mathbb{R}^2 defined by $T(x,y) = (2y, 3x-y)$. 3
- (b) Let T be the linear operator on \mathbb{R}^2 defined by $T(x,y) = (4x-2y, 2x+y)$. Verify that $[T]_f[v]_f = [T(v)]_f$ for any vector $v \in \mathbb{R}^2$. 3.5

Part-B

5. (a) Define characteristic equation. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$. 3
- (b) If v_1, v_2, \dots, v_n be nonzero eigenvectors of an operator T belonging to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then prove that v_1, v_2, \dots, v_n are linearly independent. 3
- (c) If A and B are n -square matrices, then show that AB and BA have same eigenvalues. 2.75
6. (a) Define minimal polynomial. Find the minimal polynomial of $A = \begin{bmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{bmatrix}$. 3
- (b) Let x and y be eigenvectors belonging to distinct eigenvalues λ_1 and λ_2 of a symmetric matrix A . Prove that x and y are orthogonal. 2.75
- (c) State Cayley-Hamilton theorem. Using Cayley-Hamilton theorem, find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$. 3
7. (a) Find a basis of the annihilators W^0 of the subspace W of \mathbb{R}^4 spanned by $v_1 = (1, 2, -3, 4)$ and $v_2 = (0, 1, 4, -1)$. 3
- (b) Consider the basis $\{v_1 = (2, 1), v_2 = (3, 1)\}$ of \mathbb{R}^2 . Find the dual basis $\{\phi_1, \phi_2\}$. 3
- (c) Suppose V has finite dimension. Show that if $v \in V, v \neq 0$, then there exists a $\varphi \in V'$ such that $\varphi(v) \neq 0$. 2.75
8. (a) Define an inner product space. State and prove Cauchy Schwartz inequality in an inner product space. 4.5
- (b) Use Gram-Schmidt orthogonalization process to transform the basis $\{v_1 = (1, 1, 1), v_2 = (0, 1, 1), v_3 = (0, 0, 1)\}$ of \mathbb{R}^3 into an orthonormal basis $\{u_i\}$. 4.25

University of Rajshahi

Department of Computer Science and Engineering

B.Sc. (Engg.) Part-II (Even Semester) Examination-2015

MATH2241 (Linear Algebra)

Full Marks: 52.5

Time: 3 Hours

[Answer any six (06) questions taking three questions from each part]

Part-A

1. (a) Let U and W be the subspaces of \mathbb{R}^3 defined by $U = \{(a, b, c) | a = b = c\}$ and $W = \{(0, b, c)\}$. Show that $\mathbb{R}^3 = U \oplus W$. 2.75
- (b) Let $V = \{f | f: \mathbb{R} \rightarrow \mathbb{R}\}$ be the vector space. Show that W is a subspace of V where W consists of all even functions. 3
- (c) Show that $(1, 1, 1), (0, 1, 1)$ and $(0, 1, -1)$ generate \mathbb{R}^3 . 3

2. (a) Let $V = \{at^2 + bt + c | a, b, c \in \mathbb{R}\}$ be the vector space. The polynomials $e_1 = 1$, $e_2 = t - 1$ and $e_3 = (t - 1)^2$ form a basis for V . Let $v = 2t^2 - 5t + 6$. Find $[v]_e$. 3
- (b) Let $\{e_i\}$ and $\{f_i\}$, $i = 1, 2$ be the bases of a vector space V . Suppose $f_1 = a_1 e_1 + a_2 e_2$ and $f_2 = b_1 e_1 + b_2 e_2$. Let P be the matrix whose rows are the coordinate vectors of f_1 and f_2 respectively relative to the basis $\{e_i\}$: $P = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$. Show that for any vector $v \in V$, $[v]_f P = [v]_e$. 3
- (c) Let $W = \{(a, b, c, d) | a = d, b = 2c\}$ be subspace of \mathbb{R}^4 . Find a basis and dimension of W . 2.75

3. (a) Suppose $F: V \rightarrow U$ and $G: V \rightarrow U$ are linear mappings over a field K . The mappings $F + G: V \rightarrow U$ and $kF: V \rightarrow U$ be defined by $(F + G)(u) = F(v) + G(v)$ and $(kF)(v) = kF(v)$, $k \in K$. Show that $(F + G)$ and (kF) are linear. 3
- (b) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T(3, 1) = (2, -4)$ and $T(1, 1) = (0, 2)$. Find $T(7, 4)$. 3
- (c) Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $G: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and $H: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by $F(x, y, z) = (x + y + z, x + y)$, $G(x, y, z) = (2x + z, x + y)$ and $H(x, y, z) = (2y, x)$. Show that $F, G, H \in (\mathbb{R}^3, \mathbb{R}^2)$ are linearly independent. 2.75

4. (a) For any operator $S, T \in A(V)$ show that $[ST]_e = [S]_e[T]_e$ where $\dim V = 2$. 2.75
- (b) Find the matrix representation of the linear operator T on \mathbb{R}^3 relative to the usual basis $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)\}$ where $T(x, y, z) = (2y + z, z - 4y, 3x)$. 3
- (c) Show that all the matrices similar to an invertible matrix are invertible. 3

Part-B

5. (a) Define eigenvalue and eigenvectors of a matrix A . Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ be a matrix. Find all eigenvalues and eigenvectors of A . 3
- (b) Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ be a matrix. Find an invertible matrix P such that $P^{-1}AP$ is diagonal. 2.75

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- (c) Find all eigenvalues and a basis of each eigenspace of the operator $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (2x + y, y - z, 2y + 4z)$. 3
6. (a) State Cayley-Hamilton Theorem. Verify Cayley-Hamilton Theorem for the matrix 3
 $A = \begin{pmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{pmatrix}$.
- (b) Show that a matrix A and its transpose A' have the same characteristics polynomial. 2.75
- (c) Find the minimum polynomial $m(t)$ of $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix}$. 3
7. (a) Find a polynomial for which the matrix $A = \begin{pmatrix} 2 & 3 & -2 \\ 0 & 5 & 4 \\ 1 & 0 & -1 \end{pmatrix}$ is a root and verify. 3
- (b) Suppose λ is an eigenvalue of an invertible operator T . Show that λ^{-1} is an eigenvalue of T^{-1} . 2.75
- (c) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. Show that A and B have different characteristic polynomial. 3
8. (a) Define W^+ . Show that W^+ is a subspace of a vector space V .
 (b) Let $\{v_1 = (1, 1, 1), v_2 = (0, 1, 1), v_3 = (0, 0, 1)\}$ be the basis of \mathbb{R}^3 . Find the orthogonal basis of \mathbb{R}^3 . 3
 (c) Find an orthogonal transformation of co-ordinates which diagonalizes the quadratic form $q(x, y) = 2x^2 - 4xy + 4y^2$. 2.75

University of Rajshahi

Department of Computer Science & Engineering

2nd Year 2nd Semester Examination 2014

Course: MATH-2241 (Linear Algebra)

Full Marks: 52.5 Duration: 3(Three) Hours

Answer 06(Six) questions taking any 03(Three) questions from each part

Part-A

1. a) Define a vector space. Let V be the set of ordered pairs of real numbers. Show that V is not a vector space over \mathbb{R} with addition in V and scalar multiplication on V defined by $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (a, kb)$. 2.75
b) Define linear combination of vectors. Write u as a linear combination of polynomials $v = 2t^2 + 3t - 4$ and $w = t^2 - 2t - 3$, where $u = 4t^2 - 6t - 1$. 3
c) Define linear span. Show that the xy -plane in \mathbb{R}^3 is generated by $u = (2, -1, 0)$ and $v = (1, 3, 0)$. 3

2. a) Define basis and dimension of a vector space. Determine whether or not the following form a basis for the vector space \mathbb{R}^3 : $(1, 1, 2)$, $(1, 2, 5)$ and $(5, 3, 4)$. 3
b) Define linear independent vectors of a vector space. Determine whether the vectors $(1, -2, 4, 1)$, $(2, 1, 0, -3)$, $(3, -6, 1, 4)$ in \mathbb{R}^4 are linearly dependent or independent. 3
c) Let V be the vector space of 2×2 symmetric matrices over K . Show that $\dim V=3$. 2.75

3. a) Define $\ker F$ and $\text{Im } F$ of a linear mapping $F: V \rightarrow U$. Show that $\text{Im } F$ is a subspace of U . 3
b) Determine whether or not the mapping F is linear, where $F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $F(x, y, z) = (x + 1, y + z)$. 3
c) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear mapping defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$. Find a basis of the dimension of the kernel of T . 2.75

4. a) For any operators $S, T \in A(V)$, show that $[T + S]_e = [T]_e + [S]_e$, where $\dim V=3$. 3
b) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T(x, y) = (2x - 3y, x + 4y)$. Find $[T]_f^g$ and $[T]_g^f$, where $\{g_1 = (1, 0), g_2 = (0, 1)\}$ and $\{f_1 = (1, 3), f_2 = (2, 5)\}$ are the bases of \mathbb{R}^2 . 3
c) Define matrix representation of T relative to the bases $\{f_i\}$ and $\{g_i\}$. Find the transition matrix from the bases $\{e_1 = (1, 0), e_2 = (0, 1)\}$ and $\{f_1 = (1, 1), f_2 = (-1, 0)\}$. 2.75

Part-B

5. a) Define eigenvalues and eigenvectors of a matrix A . Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ be a matrix. Show that A is diagonalizable matrix. 3
b) State Cayley-Hamilton theorem. Verify the Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 2 & 5 \\ 1 & -3 \end{pmatrix}$. 3
c) Find the minimum polynomial $m(t)$ of $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix}$. 2.75

6. a) Define linear functional on a vector space V . Further, define dual space and dual basis of V . 3
b) Let $\{v_1 = (2, 1), v_2 = (3, 1)\}$ be the basis of \mathbb{R}^2 . Find the dual basis $\{\phi_1, \phi_2\}$ of \mathbb{R}^2 . 2.75
c) Define an annihilator. Let W be a subset of a vector space V . Let V^* be the dual space of V and W^0 be the annihilator of W . Prove that W^0 is a subspace of V^* . 3

7. a) Let W be the subspace of \mathbb{R}^4 spanned by $v_1 = (1, 2, -3, 4)$ and $v_2 = (0, 1, 4, -1)$. Find the basis of the annihilator of W . 2.75
b) Define inner product space and normed space. Establish the relation $\|u\| = \sqrt{\langle u, u \rangle}$. 3
c) For any vectors $u, v \in V$, prove that $|\langle u, v \rangle| \leq \|u\| \|v\|$. 3

8. a) Define orthogonal vectors and orthogonal complement of a subset of a vector space. Let W be a subset of a vector space V . Define $W^\perp = \{v \in V : \langle v, w \rangle = 0, \forall w \in W\}$. Then show that W^\perp is a subspace of V .

3

b) Define orthonormal set of vectors. Let $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ be the usual basis of \mathbb{R}^3 . Show that $\{e_1, e_2, e_3\}$ is an orthonormal basis of \mathbb{R}^3 .

2.75

c) Let $\{v_1 = (1, 1, 1), v_2 = (0, 1, 1), v_3 = (0, 0, 1)\}$ be the basis of \mathbb{R}^3 . Applying Gram-Schmidt orthogonalization process, find the orthonormal basis of \mathbb{R}^3 .

3

University of Rajshahi
Department of Computer Science and Engineering
B. Sc. (Engg.) Part-II Even Semester Examination 2013
Course: CSE-2241 (Linear Algebra) MATH2241
Full Marks: 52.5 Time: 3 Hours
[Answer any six questions taking any 3 from each Group]

Group A

1. a) Define a Subspace of a Vector space. 3.75
 Let $V = \mathbb{R}^3$. Show that $W = \{(a,b,c); a+b+c=0\}$ is a subspace of V .
- b) Let $V = \mathbb{R}^3$. Show that $W = \{(a,b,c); a^2+b^2+c^2 \leq 1\}$ is not a subspace of V . 2.5
- c) Let U and W be subspaces of a vector space V . Prove that $U \cap W$ is a subspace of V . 2.5

2. a) Define sum of two Subspaces of a Vector space. Let U and W be subspaces of a vector space V . Prove that $U+W$ is a subspace of V . 3
- b) Prove that the vector space V is the direct sum of its subspaces U and W if and only if
 (i) $V = U+W$ and (ii) $U \cap W = \{0\}$. 3
- c) Let V be the vector space of n -square matrices over the field \mathbb{R} . Let U and W be the Subspaces of Symmetric and Antisymmetric matrices respectively. Show that $V = U \oplus W$. 2.75

3. a) Find a basis and dimension of the subspace W of \mathbb{R}^3 , where 5
 (i) $W = \{(a, b, c) : a+b+c = 0\}$
 (ii) $W = \{(a, b, c) : a=b=c\}$
- b) Define direct sums. Show that the vector space $V=\mathbb{R}^3$ is not the direct sum of the subspaces $U = \{(a, b, 0) : a, b \in \mathbb{R}\}$ and $W = \{(0, b, c) : b, c \in \mathbb{R}\}$ 3.75

4. a) Define linear mapping. Show that every linear mapping takes the zero vector into the zero vector. 2.75
- b) Determine whether the following are linear mapping or not: 3
 (i) $F(x, y) = (x+y, x)$
 (ii) $F(x, y) = (x+1, y)$
 Where $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$
- c) Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear mapping for $F(1,0) = (2,3)$ and $F(0,1) = (1,4)$. 3
 Find a formula for F ; that is, find $F(a,b)$.

Group B

5. a) Let V be of finite dimension, and let $F: V \rightarrow U$ be a linear mapping with image U and kernel W . Prove that: 3
 $\dim U + \dim W = \dim V$
- b) Let V be a vector space of 2×2 matrices over \mathbb{R} and let $M = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$. 2.75

Let $F: V \rightarrow V$ be the linear map defined by $F(A) = AM - MA$. Find a basis and dimension of the Kernel W of F .

c) Let $\{e_1, e_2, \dots, e_n\}$ be a basis of V .

For any operators $S, T \in A(V)$, prove that $[ST]_e = [S]_e [T]_e$

3

6. a) Define matrix representation of a Linear Operator. Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear operator defined by $F(x,y) = (2x+3y, 4x-5y)$.

Find the matrix representation of F relative to the basis

$$S = \{u_1, u_2\} = \{(1,2), (2,5)\}$$

b) Consider the following basis of \mathbb{R}^2

$$E = \{e_1, e_2\} = \{(1,0), (0,1)\}$$

$$S = \{u_1, u_2\} = \{(1,3), (1,4)\}$$

5

(i) Find the change-of-basis matrix P from the basis E to S .

(ii) Find the change-of-basis matrix Q from the basis S to E .

(iii) Find the co-ordinate of $V = (5, -3)$ relative to S .

7. a) Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Find all eigenvalues of A and the corresponding eigenvectors.

3

Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

b) Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

3

c) Show that similar matrices have the same eigenvalues.

2.75

8. a) Define an inner product space. Let A be a real $n \times n$ positive definite matrix. Prove that the function $\langle u, v \rangle = u^T A v$ is an inner product on \mathbb{R}^n .

4

b) Prove that the symmetric matrix $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ is positive definite if and only if $a > 0$, $d > 0$ and $|A| = ad - b^2 > 0$.

2.75

c) Let $u = (x_1, x_2)$ and $v = (y_1, y_2)$ belong to \mathbb{R}^2 . Verify that the function $f(u, v) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$ defines an inner product on \mathbb{R}^2 .

2

University of Rajshahi
Department of Computer Science and Engineering
B.Sc(Engg) 2nd year 2nd semister, 2012

Course- MATH2241 (Linear Algebra)

Time: 4 Hours

**Property of Seminar Library Marks :52.5
 Dept. of Computer Science &
 Engineering
 University of Rajshahi.**

(Answer SIX questions taking any THREE from each group.)

$$\frac{3}{4} \times 6 = 52 \frac{1}{2}$$

Part-A

1. (a) Define subspace of a vector space V. Let V be a vector space over a field K. Then prove that a subset W of V is a subspace of V if and only if $\alpha u + \beta v \in W$, for all $u, v \in V, \alpha, \beta \in K$. 4 $\frac{3}{4}$
- (b) What do you mean by a linear combination of vectors? Express (1, -2, 5) as a linear combination of vectors (1, 1, 1), (1, 2, 3) and (-1, -5, 1) in \mathbb{R}^3 . 4
2. (a) Define linearly dependent set of vectors. Show that the set of vectors (1, -2, 1), (2, 1, -1) and (7, -4, 1) in \mathbb{R}^3 are linearly dependent. 3
- (b) Define basis of a vector space. Show that the set of vectors {(1, 1, 2), (1, 2, 5), (5, 3, 4)} is not a basis for the vector space \mathbb{R}^3 . 3
- (c) Let v_1, v_2, \dots, v_m be independent vectors and suppose $u = a_1v_1 + a_2v_2 + \dots + a_mv_m$, where the a_i are scalars. Show that the above representation of u is unique. 2 $\frac{3}{4}$
3. (a) Define linear mapping. Show that the mapping $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $F(x,y) = (x+y, x)$ is linear. 2 $\frac{3}{4}$
- (b) Define kernel of a linear mapping. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear mapping defined by $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$. Find a basis and the dimension of kernel of T. 3
- (c) Define image of a linear mapping. Let $F: V \rightarrow U$ be a linear mapping, then show that the image of F is a subspace of U. 3
4. (a) Define a linear operator and then define its matrix representation. Let $\{e_1, e_2, \dots, e_n\}$ be a basis of V and let T be any operator on V. Then prove that $[T]_e[v]_e = [T(v)]_e$ for any vector $v \in V$. 4 $\frac{3}{4}$
- (b) Let T be the linear operator on \mathbb{R}^3 defined by $T(x,y,z) = (2y+z, x-4y, 3x)$. Find the matrix representation of T in the basis $\{f_1 = (1,1,1), f_2 = (1,1,0), f_3 = (1,0,0)\}$. 4

Part-B

5. (a) Define a transition matrix. Consider the following basis of \mathfrak{N}^2 : $\{e_1 = (1,0), e_2 = (0,1)\}$ and $\{f_1 = (1,2), f_2 = (2,3)\}$. Find the transition matrix P from $\{e_i\}$ to $\{f_i\}$ and then verify that $[v]_e = P[v]_f$ for any vector $v \in \mathfrak{N}^2$.

(b) Define the followings:

Similar matrices, Trace of a matrix A and Diagonalizable operator T.

$1\frac{3}{4}$

- (c) Let V be a vector space of 2×2 matrices over \mathfrak{N} and let $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Let T be linear operator on V defined by $T(A) = MA$, where $A \in V$. Find the trace of T.

4

6. (a) Show that every matrix is a zero of its characteristic polynomial.

- (b) Find the minimum polynomial $m(t)$ of

$4\frac{3}{4}$

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix}$$

7. (a) Define a linear functional on a vector space V and then define the dual space of V. Also define the dual basis of V^* . Consider the following basis of

\mathfrak{N}^3 : $\{v_1 = (1, -1, 3), v_2 = (0, 1, -1), v_3 = (0, 3, -2)\}$. Find the dual basis $\{\phi_1, \phi_2, \phi_3\}$.

4

- (b) Let V be the vector space of polynomials over \mathfrak{N} of degree ≤ 2 . Let ϕ_1, ϕ_2 and ϕ_3 be the linear functionals on V defined by $\phi_1(f(t)) = \int_0^1 f(t) dt$, $\phi_2(f(t)) = f'(1)$, $\phi_3(f(t)) = f(0)$ where $f(t) = a + bt + ct^2 \in V$ and $f'(t)$ denotes the derivative of $f(t)$. Find the basis $\{f_1(t), f_2(t), f_3(t)\}$ of V which is dual to $\{\phi_1, \phi_2, \phi_3\}$.

$2\frac{3}{4}$

- (c) Define an annihilator of a subset W of a vector space V. Let W^0 be the set of all annihilators of W. Then prove that W^0 is a subspace of V^* .

2

8. (a) Define an inner product. Space V over K. Verify that the following is an inner product in \mathfrak{N}^2 :

$$\langle u, v \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2, \text{ where } u = (x_1, x_2), v = (y_1, y_2).$$

4

- (b) Define orthogonal vectors and orthogonal complements. Let W be a subset of a vector space V. Show that the orthogonal complements, denoted by W^\perp , is a subspace of V.

$2\frac{3}{4}$

- (c) Express the norm of a vector u in terms of its inner product. Find the norm of

$v = (3,4) \in \mathfrak{N}^2$ with respect to :

2

(i) The usual inner product;

(ii) The inner product in $v = (3,4)$ by the relation $\langle u, v \rangle = x_1y_1 - x_2y_1 - x_1y_2 + 3x_2y_2$ as $u = (x_1, x_2), v = (y_1, y_2)$.