Interpolation with Unequal Interval

We have

$$\phi(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \\ + \ldots\ldots + a_n(x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1}) \qquad (1)$$

Newton's equation of a function that passes through two points

(x_0, y_0) **and** (x_1, y_1) **is**

$$P(x) = a_0 + a_1(x - x_0)$$

$$P(x) = a_0 + a_1(x - x_0)$$

Set $x = x_0$

$$P(x_0) = y_0 = a_0$$

Set $x = x_1$

$$P(x_1) = y_1 = a_0 + a_1(x_1 - x_0)$$

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0}$$

Newton's equation of a function that passes through three points

(x_0, y_0) (x_1, y_1) **and** (x_2, y_2)

is

$$P(x) = a_0 + a_1(x - x_0)$$
$$+ a_2(x - x_0)(x - x_1)$$

To find a_2 **, set** $x = x_2$

$$P(x_2) = a_0 + a_1(x_2 - x_0)$$
$$+ a_2(x_2 - x_0)(x_2 - x_1)$$

$$a_2 = \frac{\dfrac{y_2 - y_1}{x_2 - x_1} - \dfrac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

Newton's equation of a function that passes through four points can be written by adding a fourth term .

$$P(x) = a_0 + a_1(x - x_0)$$
$$+ a_2(x - x_0)(x - x_1)$$
$$+ a_3(x - x_0)(x - x_1)(x - x_2)$$

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

The fourth term will vanish at all three previous points and, therefore, leaving all three previous coefficients intact.

Divided differences and the coefficients

The divided difference of a function, f

with respect to x_i **is denoted as** $f\left[x_i\right]$

It is called as *zeroth divided difference* and is simply the value of the function, f
at x_i

$$f\left[x_i\right] = f\left(x_i\right)$$

The divided difference of a function, f with respect to x_i and x_{i+1} called as the *first divided difference*, is denoted

$$f\left[x_i, x_{i+1}\right]$$

$$f\left[x_i, x_{i+1}\right] = \frac{f\left[x_{i+1}\right] - f\left[x_i\right]}{x_{i+1} - x_i}$$

The divided difference of a function, f with respect to x_i , x_{i+1} and x_{i+2} called as the *second divided difference*, is denoted as

$$f\left[x_i, x_{i+1}, x_{i+2}\right]$$

$$f\left[x_i, x_{i+1}, x_{i+2}\right] = \frac{f\left[x_{i+1}, x_{i+2}\right] - f\left[x_i, x_{i+1}\right]}{x_{i+2} - x_i}$$

The *third divided difference* with respect to

x_i , x_{i+1} , x_{i+2} **and** x_{i+3}

$$f\left[x_i, x_{i+1}, x_{i+2}, x_{i+3}\right]$$

$$= \frac{f\left[x_{i+1}, x_{i+2}, x_{i+3}\right] - f\left[x_i, x_{i+1}, x_{i+2}\right]}{x_{i+3} - x_i}$$

The coefficients of Newton's interpolating polynomial are:

$$a_0 = f[x_0]$$

$$a_1 = f[x_0, x_1]$$

$$a_2 = f[x_0, x_1, x_2]$$

$$a_3 = f[x_0, x_1, x_2, x_3]$$

$$a_4 = f[x_0, x_1, x_2, x_3, x_4]$$ **and so on.**

$f(x)$	First divided differences	Second divided differences	Third divided differences
$f[x_0]$			
	$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$		
$f[x_1]$		$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
	$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
$f[x_2]$		$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
	$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$		$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
$f[x_3]$		$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
	$f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3}$		$f[x_2, x_3, x_4, x_5] = \frac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$
$f[x_4]$		$f[x_3, x_4, x_5] = \frac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
	$f[x_4, x_5] = \frac{f[x_5] - f[x_4]}{x_5 - x_4}$		
$f[x_5]$			

Example

Find Newton's interpolating polynomial to approximate a function whose 5 data points are given below.

x	$f(x)$
2.0	0.85467
2.3	0.75682
2.6	0.43126
2.9	0.22364
3.2	0.08567

i	x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, \cdots, x_i]$	$f[x_{i-4}, \cdots, x_i]$
0	**2.0**	**0.85467**				
			-0.32617			
1	**2.3**	**0.75682**		**-1.26505**		
			-1.08520		**2.13363**	
2	**2.6**	**0.43126**		**0.65522**		**-2.02642**
			-0.69207		**-0.29808**	
3	**2.9**	**0.22364**		**0.38695**		
			-0.45990			
4	**3.2**	**0.08567**				

The 5 coefficients of the Newton's interpolating polynomial are:

$$a_0 = f\left[x_0\right] = 0.85467$$

$$a_1 = f\left[x_0, x_1\right] = -0.32617$$

$$a_2 = f\left[x_0, x_1, x_2\right] = -1.26505$$

$$a_3 = f\left[x_0, x_1, x_2, x_3\right] = 2.13363$$

$$a_4 = f\left[x_0, x_1, x_2, x_3, x_4\right] = -2.02642$$

$$
\begin{aligned}
P(x) = a_0 + a_1(x - x_0) \\
+ a_2(x - x_0)(x - x_1) \\
+ a_3(x - x_0)(x - x_1)(x - x_2) \\
+ a_4(x - x_0)(x - x_1)(x - x_2)(x - x_3)
\end{aligned}
$$

$$P(x) = 0.85467 - 0.32617(x - 2.0)$$
$$-1.26505(x - 2.0)(x - 2.3)$$
$$+2.13363(x - 2.0)(x - 2.3)(x - 2.6)$$
$$-2.02642(x - 2.0)(x - 2.3)(x - 2.6)(x - 2.9)$$

P(*x*) can now be used to estimate the value of the function f(*x*) say at *x* = 2.8.

$$P(2.8) = 0.85467 - 0.32617(2.8 - 2.0)$$
$$-1.26505(2.8 - 2.0)(2.8 - 2.3)$$
$$+2.13363(2.8 - 2.0)(2.8 - 2.3)(2.8 - 2.6)$$
$$-2.02642(2.8 - 2.0)(2.8 - 2.3)(2.8 - 2.6)(2.8 - 2.9)$$

$$f(2.8) \approx P(2.8) = 0.275$$

Interpolation with Unequal Intervals of Argument

- The interpolation formulas derived before are applicable only when the values of the functions are given at equidistant intervals.

- It is sometimes inconvenient, or even impossible, to obtain values of a function at equidistant values of its argument.

- Two formulas can be applicable on a functional values with unequal intervals of arguments.
  - Newton's formula
  - Lagrange's formula

Divided Difference

- The differences used in the Newton's formula are called divided differences.

- Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ for the values $x_0, x_1, x_2, \ldots, x_n$ of the independent variable x.

- First Order Difference is

$$\delta(x_1, x_0) = \frac{y_1 - y_0}{x_1 - x_0}, \quad \delta(x_2, x_1) = \frac{y_2 - y_1}{x_2 - x_1}, \delta(x_3, x_2) = \frac{y_3 - y_2}{x_3 - x_2}, \quad etc.$$

- Second Order Difference is

$$\delta(x_2, x_1, x_0) = \frac{\delta(x_2, x_1) - \delta(x_1, x_0)}{x_2 - x_0}$$

$$\delta(x_3, x_2, x_1) = \frac{\delta(x_3, x_2) - \delta(x_2, x_1)}{x_3 - x_1} \quad etc.$$

Divided Difference

- Third Order Difference is

$$\delta(x_3, x_2, x_1, x_0) = \frac{\delta(x_3, x_2, x_1) - \delta(x_2, x_1, x_0)}{x_3 - x_0}$$

$$\delta(x_4, x_3, x_2, x_1) = \frac{\delta(x_4, x_3, x_2) - \delta(x_3, x_2, x_1)}{x_4 - x_1}, etc$$

- Fourth Order Difference is

$$\delta(x_4, x_3, x_2, x_1, x_0) = \frac{\delta(x_4, x_3, x_2, x_1) - \delta(x_3, x_2, x_1, x_0)}{x_4 - x_0}, etc$$

- Note that the order of any divided difference is one less than the number of the arguments in it (e.g., 3rd order has 4 arguments).

- If (x_0, y_0), (x_1, y_1), (x_2, y_2) etc. are points on a curve, the first-order divided differences is the slope of the line through any two points.

Table of Divided Differences

x	y	δ	δ^2	δ^3	δ^4
x_0	y_0				
		$\delta(x_1, x_0)$			
x_1	y_1		$\delta(x_2, x_1, x_0)$		
		$\delta(x_2, x_1)$		$\delta(x_3, x_2, x_1, x_0)$	
x_2	y_2		$\delta(x_3, x_2, x_1)$		$\delta(x_4, x_3, x_2, x_1, x_0)$
		$\delta(x_3, x_2)$		$\delta(x_4, x_3, x_2, x_1)$	
x_3	y_3		$\delta(x_4, x_3, x_2)$		$\delta(x_{5,} x_4, x_3, x_2, x_1)$
		$\delta(x_4, x_3)$		$\delta(x_5, x_4, x_3, x_2)$	
x_4	y_4		$\delta(x_5, x_4, x_3)$		$\delta(x_{6,} x_5, x_4, x_3, x_2)$
		$\delta(x_5, x_4)$		$\delta(x_6, x_5, x_4, x_3)$	
x_5	y_5		$\delta(x_6, x_5, x_4)$		
		$\delta(x_6, x_5)$			
x_6	y_6				

Table of Divided Differences: Example

x	y	δ^1	δ^2	δ^3	δ^4	δ^5
-2	-729					
		418.5				
0	108		-95.7			
		-60		17.1		
3	-72		24		-2.9	
		60		-9		0.49
5	48		-39		2	
		-96		7		
7	-144		-4			
		-108				
8	-252					

For example,

$$\delta^2(x_3, x_2, x_1)$$

$$= \frac{\delta(x_3, x_2) - \delta(x_2, x_1)}{x_3 - x_1}$$

$$\frac{60 - (-60)}{5 - 0} = 24$$

$$\delta^5(x_5, x_4, x_3, x_2, x_1, x_0)$$

$$= \frac{\delta^4(x_5, x_4, x_3, x_2, x_1) - \delta^4(x_4, x_3, x_2, x_1, x_0)}{x_5 - x_0}$$

$$\frac{2 - (-2.9)}{8 - (-2)} = 0.49$$

Symmetry of Divided difference

In First Order

$$\delta(x_1, x_0) = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0 - y_1}{x_0 - x_1} = \delta(x_0, x_1)$$

Also

$$\delta(x_1, x_0) = \frac{y_1}{x_1 - x_0} - \frac{y_0}{x_1 - x_0} = \frac{y_1}{x_1 - x_0} + \frac{y_0}{x_0 - x_1}$$

Symmetry of Divided difference

In Second Order

$$\delta(x_2, x_1, x_0) = \frac{\delta(x_2, x_1) - \delta(x_1, x_0)}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2}{x_2 - x_1} + \frac{y_1}{x_1 - x_2} - \left(\frac{y_1}{x_1 - x_0} + \frac{y_0}{x_0 - x_1}\right)\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2}{x_2 - x_1} + y_1\left(\frac{1}{x_1 - x_2} - \frac{1}{x_1 - x_0}\right) - \frac{y_0}{x_0 - x_1}\right]$$

$$= \frac{y_2}{(x_2 - x_0)(x_2 - x_1)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_0}{(x_0 - x_1)(x_0 - x_2)}$$

Symmetry of Divided difference

In Third Order

$$\delta(x_3,x_2,x_1,x_0)=\frac{\delta(x_3,x_2,x_1)-\delta(x_2,x_1,x_0)}{x_3-x_0}$$

$$=\frac{1}{x_3-x_0}\left[\ \frac{y_3}{(x_3-x_1)(x_3-x_2)}+\frac{y_2}{(x_2-x_1)(x_2-x_3)}+\frac{y_1}{(x_1-x_2)(x_1-x_3)}\ \right]-$$

$$\left(\frac{y_2}{(x_2-x_0)(x_2-x_1)}+\frac{y_1}{(x_1-x_0)(x_1-x_2)}+\frac{y_0}{(x_0-x_1)(x_0-x_2)}\right)$$

$$\delta(x_3,x_2,x_1,x_0)=\frac{y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}+\frac{y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}+$$

$$\frac{y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}+\frac{y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

Symmetry of Divided difference

- The right hand side of the equations remains unchanged when any two values of x are interchanged and the corresponding y's are also interchanged.
- This means that a divided difference remains unchanged regardless of how much its arguments are interchanged. Thus

$$\delta(1, 5, 9) = \delta(5, 9, 1) = \delta(1, 9, 5) = \delta(5, 1, 9) \quad etc$$

Symmetry of Divided difference

We may therefore write,

$$\delta\left(x_n, x_{n-1}, \cdots, x_2, x_1, x_0\right) = \delta\left(x_0, x_1, x_2, \cdots, x_{n-1}, x_n\right) \qquad etc$$

It can be proven by mathematical induction that

$$\delta\left(x, x_0, x_1, x_2, \cdots, x_{n-1}, x_n\right) = \frac{y}{(x-x_0)(x-x_1)\cdots(x-x_n)} + \frac{y_0}{(x_0-x)(x_0-x_1)\cdots(x_0-x_n)}$$
$$+ \frac{y_1}{(x_1-x)(x_1-x_0)\cdots(x_1-x_n)} + \cdots + \frac{y_n}{(x_n-x)(x_n-x_0)\cdots(x_n-x_{n-1})}$$

Lagrange's Interpolation Formula

- This formula can be applied on non-equidistant variables.
- It is based on the theorem that the divided differences of a polynomial of the n^{th} degree are constant.
- Hence the $(n+1)^{th}$ divided differences of a polynomial of the n^{th} degree is zero.
- Let $f(x)$ denote a polynomial of the n^{th} degree which takes the values $y_0, y_1, y_2, \ldots, y_n$ when x has the values $x_0, x_1, x_2, \ldots, x_n$, respectively.
- Then the $(n+1)^{th}$ differences of this polynomial are zero.
- This formula is used to find the value of the independent variable corresponding to a given value of the function. (i.e., y can be expressed in terms of x and vice versa.)

Lagrange's Interpolation Formula

Therefore,

$$\delta\left(x, x_0, x_1, x_2, \cdots, x_{n-1}, x_n\right) = 0$$

$$\Rightarrow \frac{y}{(x-x_0)(x-x_1)\cdots(x-x_n)} + \frac{y_0}{(x_0-x)(x_0-x_1)\cdots(x_0-x_n)}$$

$$+ \frac{y_1}{(x_1-x)(x_1-x_0)\cdots(x_1-x_n)}$$

$$+ \cdots + \frac{y_n}{(x_n-x)(x_n-x_0)\cdots(x_n-x_{n-1})} = 0$$

Lagrange's Interpolation Formula

Transposing to the right-hand side all terms except the first, we have

$$\frac{y}{(x-x_0)(x-x_1)\cdots(x-x_n)} = \frac{y_0}{(x-x_0)(x_0-x_1)\cdots(x_0-x_n)}$$

$$+\frac{y_1}{(x-x_1)(x_1-x_0)\cdots(x_1-x_n)}$$

$$+\cdots\frac{y_n}{(x-x_n)(x_n-x_0)\cdots(x_n-x_{n-1})}$$

$$y = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)y_0}{(x-x_0)(x_0-x_1)\cdots(x_0-x_n)} + \frac{(x-x_0)(x-x_1)\cdots(x-x_n)y_1}{(x-x_1)(x_1-x_0)\cdots(x_1-x_n)}$$

$$+\cdots\frac{(x-x_0)(x-x_1)\cdots(x-x_n)y_n}{(x-x_n)(x_n-x_0)\cdots(x_n-x_{n-1})}$$

Lagrange's Interpolation Formula

Then,

$$y = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)\cdots(x - x_n)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\cdots(x_2 - x_n)} y_2$$

$$+ \cdots \frac{(x - x_0)(x - x_1)\cdots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1})} y_n$$

- This is the Lagrange's formula for interpolation.
- In this formula, $y = y_0, y_1, y_2, \ldots y_{\mathrm{n}}$ when $x = x_0, x_1, x_2, \ldots, x_n$, respectively.
- The vales of the independent variable may or may not be equidistant.

Lagrange's Interpolation Formula

- Since Lagrange's interpolation formula is merely a relation between two variables.
- Therefore, we can write a formula giving *x* as a function of *y*.
- Hence on interchanging *x* and *y* we get,

$$
\begin{aligned}
x = {} & \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}x_1 \\
& + \frac{(y-y_0)(y-y_1)(y-y_3)\cdots(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\cdots(y_2-y_n)}x_2 \\
& + \cdots \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)\cdots(y_n-y_{n-1})}x_n
\end{aligned}
$$

Class Work

The following table gives certain corresponding values of x and $\log_{10} x$. Compute the value of $\log_{10}$ (323.5) using Lagrange's interpolation formula.

x	321.0	322.8	324.2	325.0
$\log_{10} x$	2.50651	2.50893	2.51081	2.51188

Here x = 323.5, x_0 = 321.0, x_1 = 322.8, x_2 = 324.2, x_3 = 325.0....

Answer: 2.50987

Class Work

The following table gives the values of the probability integral

$$\left(\frac{2}{\sqrt{\pi}}\right)\int_0^x e^{-x^2}\,dx$$

corresponding to certain value of *x*.

For what value of x is this integral equal to 0.5?

$\left(\frac{2}{\sqrt{\pi}}\right)\int_0^x e^{-x^2}\,dx$	x
0.4846555	0.46
0.4937452	0.47
0.5027498	0.48
0.5116683	0.49

Answer: 0.476936

Class Work

Using Lagrange's interpolation formula, find the form of the function $y(x)$ from the following table.

x	0	1	3	4
y	-12	0	12	24

Answer: $y = (x-1)(x^2-5x+12) = x^3 - 6x^2 + 17x - 12$

Class Work

Find from Lagrange's interpolating polynomial of degree 2 approximating the function $y = ln\ (x)$ defined by the following table of values.

Hence, determine the value of ln (2.7)

x	2	2.5	3.0
$y = ln\ (x)$	0.69315	0.91629	1.09861

Answer: $y=-0.08164x^2+0.81366x-0.60761$

$ln\ 2.7 = 0.9941164$

Newton's General Interpolation Formula

- Newton's general formula for interpolation for unequal intervals of the argument can be derived by the following steps
  - Starting with any variable pair of values *x* and *y* and the pairs of given values.
  - Writing down the divided differences in ascending order.
  - Solving for *y* in successive steps until as many terms as desired are found.
  - Thus,

$$(a)\quad \frac{y-y_0}{x-x_0}=\delta(x,x_0)$$

$$(b)\quad \frac{\delta(x,x_0)-\delta(x_0,x_1)}{x-x_1}=\delta(x,x_0,x_1)$$

Newton's General Interpolation Formula

Similarly,

$$(c) \quad \frac{\delta(x, x_0, x_1) - \delta(x_0, x_1, x_2)}{x - x_2} = \delta(x, x_0, x_1, x_2)$$

$$(d) \quad \frac{\delta(x, x_0, x_1, x_2) - \delta(x_0, x_1, x_2, x_3)}{x - x_3} = \delta(x, x_0, x_1, x_2, x_3)$$

$$(e) \quad \frac{\delta(x, x_0, x_1, x_2, x_3) - \delta(x_0, x_1, x_2, x_3, x_4)}{x - x_4} = \delta(x, x_0, x_1, x_2, x_3, x_4)$$

$$(f) \quad \frac{\delta(x, x_0, x_1, x_2, x_3, x_4) - \delta(x_0, x_1, x_2, x_3, x_4, x_5)}{x - x_5} = \delta(x, x_0, x_1, x_2, x_3, x_4, x_5)$$

etc.

Newton's General Interpolation Formula

$$From \quad (a)\ \frac{y-y_0}{x-x_0} = \delta(x, x_0), \ \text{we can derive,}$$

$$(g) \quad y = y_0 + (x-x_0)\,\delta(x, x_0)$$

$$From \quad (b)\ \frac{\delta(x, x_0) - \delta(x_0, x_1)}{x-x_1} = \delta(x, x_0, x_1), \ \text{we can derive,}$$

$$\delta(x, x_0) = \delta(x_0, x_1) + (x-x_1)\,\delta(x, x_0, x_1)$$

Substituting into (g) *this value of* $\delta(x, x_0)$*, we get*

$$(h) \quad y = y_0 + (x-x_0)\left[\delta(x_0, x_1) + (x-x_1)\,\delta(x, x_0, x_1)\right]$$

$$= y_0 + (x-x_0)\,\delta(x_0, x_1) + (x-x_0)(x-x_1)\,\delta(x, x_0, x_1)$$

Newton's General Interpolation Formula

$$From \quad (c)\frac{\delta(x, x_0, x_1) - \delta(x_0, x_1, x_2)}{x - x_2} = \delta(x, x_0, x_1, x_2),$$

$$\delta(x, x_0, x_1) = \delta(x_0, x_1, x_2) + (x - x_2)\delta(x, x_0, x_1, x_2)$$

Substituting this into (h) *we get*

$$(i) \quad y = y_0 + (x - x_0)\delta(x_0, x_1)$$
$$+ (x - x_0)(x - x_1)\left[\delta(x_0, x_1, x_2) + (x - x_2)\delta(x, x_0, x_1, x_2)\right]$$

$$= y_0 + (x - x_0)\delta(x_0, x_1) + (x - x_0)(x - x_1)\delta(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\delta(x, x_0, x_1, x_2)$$

Newton's General Interpolation Formula

$$\text{From} \quad (d)\ \frac{\delta(x,x_0,x_1,x_2)-\delta(x_0,x_1,x_2,x_3)}{x-x_3}=\delta(x,x_0,x_1,x_2,x_3),$$

$$\delta(x,x_0,x_1,x_2)=\delta(x_0,x_1,x_2,x_3)+(x-x_3)\delta(x,x_0,x_1,x_2,x_3)$$

Substituting this into (i) *we get*

$$(j) \quad y=y_0+(x-x_0)\delta(x_0,x_1)+(x-x_0)(x-x_1)\delta(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)\begin{bmatrix}\delta(x_0,x_1,x_2,x_3)\\ +(x-x_3)\delta(x,x_0,x_1,x_2,x_3)\end{bmatrix}$$

$$=y_0+(x-x_0)\delta(x_0,x_1)+(x-x_0)(x-x_1)\delta(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)\delta(x_0,x_1,x_2,x_3)$$
$$+(x-x_0)(x-x_1)(x-x_2)(x-x_3)\delta(x,x_0,x_1,x_2,x_3)$$

Newton's General Interpolation Formula

$$\text{From} \quad (e) \frac{\delta(x, x_0, x_1, x_2, x_3) - \delta(x_0, x_1, x_2, x_3, x_4)}{x - x_4} = \delta(x, x_0, x_1, x_2, x_3, x_4),$$

$$\delta(x, x_0, x_1, x_2, x_3) = \delta(x_0, x_1, x_2, x_3, x_4) + (x - x_4)\delta(x, x_0, x_1, x_2, x_3, x_4)$$

Substituting this into (j) *we obtain*

$$\begin{aligned}(k) \quad y = {} & y_0 + (x - x_0)\delta(x_0, x_1) + (x - x_0)(x - x_1)\delta(x_0, x_1, x_2) \\ & + (x - x_0)(x - x_1)(x - x_2)\delta(x_0, x_1, x_2, x_3) \\ & + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\begin{bmatrix}\delta(x_0, x_1, x_2, x_3, x_4) \\ + (x - x_4)\delta(x, x_0, x_1, x_2, x_3, x_4)\end{bmatrix}\end{aligned}$$

$$\begin{aligned}= {} & y_0 + (x - x_0)\delta(x_0, x_1) + (x - x_0)(x - x_1)\delta(x_0, x_1, x_2) \\ & + (x - x_0)(x - x_1)(x - x_2)\delta(x_0, x_1, x_2, x_3) \\ & + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\delta(x_0, x_1, x_2, x_3, x_4) \\ & + (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)\delta(x, x_0, x_1, x_2, x_3, x_4)\end{aligned}$$

Newton's General Interpolation Formula

By continuing in this manner, or by mathematical induction, it can be proved that the general Newton formula with divided differences is

$$
\begin{aligned}
y = y_0 &+ (x-x_0)\delta(x_0,x_1) + (x-x_0)(x-x_1)\delta(x_0,x_1,x_2) \\
&+ (x-x_0)(x-x_1)(x-x_2)\delta(x_0,x_1,x_2,x_3) \\
&+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)\delta(x_0,x_1,x_2,x_3,x_4) \\
&\cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\delta(x_0,x_1,x_2,\cdots,x_n) \\
&\quad + (x-x_0)(x-x_1)\cdots(x-x_n)\delta(x,x_1,x_2,\cdots,x_n)
\end{aligned}
$$

The last term in this formula is the remainder term after n+1 terms, R_{n+1}. Hence,

$$
R_{n+1} = (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)\delta(x,x_1,x_2,\cdots,x_n)
$$

Newton's General Interpolation Formula Example

The following table gives certain corresponding values of x and $log_{10} x$. Find log_{10} (323.5) by Newton's General formula.

x	$log_{10} x$	δ^1	δ^2
322.8	2.50893		
		0.00134286	
324.2	2.51081		-0.00000244
		0.00133750	
325.0	2.51188		

$y = log_{10}(323.5)$

$=2.50893+(323.5-322.8)(0.00134286)+(323.5-322.8)(323.5-324.2)(-0.00000244)$

$=2.50893+0.000940+0.0000012$

$=2.50987$

Newton's General Interpolation Formula: Example

Find log_{10} (301) using Newton's formula

x	$log_{10}\, x$	δ^1	δ^2	δ^3
300	2.4771			
		0.00144		
304	2.4829		-2.3653E-06	
		0.00143		5.15322E-09
305	2.4843		-2.3292E-06	
		0.00142		
307	2.4871			

$$log_{10}(301) = 2.4771 + (1)\, 0.00144 + (1)(-3)(- -2.3653E\text{-}06)$$

$$+(1)(-3)(-4)(5.15322E\text{-}09)$$

$$= 2.478567$$

Class Work

Find the polynomial representation of $f(x)$ using the following table:

x	f(x)
-1	3
0	-6
3	39
6	822
7	1611

Solution:

x	f(x)	δ	δ^2	δ^3	δ^4
-1	3				
		-9			
0	-6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

$$f(x) = 3 + (x+1)(-9) + x(x+1)(6) + x(x+1)(x-3)(5) + x(x+1)(x-3)(x-6)$$

$$= x^4 - 3x^3 + 5x^2 - 6$$

Class Work

Using Newton's general interpolation formula, find from the following table the value of y for $x = 5.60275$

x	5.600	5.602	5.605	5.607	5.608
y	0.77556588	0.77682686	0.77871250	0.77996571	0.78059114

Lagrange's Interpolation formula

Let $y(x)$ be continuous and differentiable $(n+1)$ times in the interval (a, b). Given the $(n+1)$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ where the values of x need not necessarily be equally spaced, we wish to find a polynomial of degree n, say $L_n(x)$, such that

$$L_n(x_i) = y(x_i) = y_i, \quad i = 0, 1, \ldots, n \tag{3.28}$$

Before deriving the general formula, we first consider a simpler case, viz., the equation of a straight line (a linear polynomial) passing through two points (x_0, y_0) and (x_1, y_1). Such a polynomial, say $L_1(x)$, is easily seen to be

$$L_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$$= l_0(x) y_0 + l_1(x) y_1$$

$$= \sum_{i=0}^{1} l_i(x) y_i, \tag{3.29}$$

where

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad l_1(x) = \frac{x - x_0}{x_1 - x_0}. \tag{3.30}$$

From (3.30), it is seen that

$$l_0(x_0) = 1, \qquad l_0(x_1) = 0, \qquad l_1(x_0) = 0, \quad l_1(x_1) = 1.$$

These relations can be expressed in a more convenient form as

$$l_i(x_j) = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j. \end{cases} \tag{3.31}$$

The $l_i(x)$ in (3.29) also have the property

$$\sum_{i=0}^{1} l_i(x) = l_0(x) + l_1(x) = \frac{x - x_1}{x_0 - x_1} + \frac{x - x_0}{x_1 - x_0} = 1. \tag{3.32}$$

Equation (3.29) is the *Lagrange polynomial of degree one passing through* two points (x_0, y_0) and (x_1, y_1). In a similar way, the *Lagrange polynomial of degree two* passing through three points $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2) is written as

$$L_2(x) = \sum_{i=0}^{2} l_i(x)\, y_i$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2, \tag{3.33}$$

where the $l_i(x)$ satisfy the conditions given in (3.31) and (3.32).

To derive the general formula, let

$$L_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \tag{3.34}$$

be the desired polynomial of the nth degree such that conditions (3.28) (called the *interpolatory conditions*) are satisfied. Substituting these conditions in (3.34), we obtain the system of equations

$$\left.\begin{aligned}
y_0 &= a_0 + a_1x_0 + a_2x_0^2 + \cdots + a_nx_0^n \\
y_1 &= a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_nx_1^n \\
y_2 &= a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_nx_2^n \\
&\vdots \\
y_n &= a_0 + a_1x_n + a_2x_n^2 + \cdots + a_nx_n^n.
\end{aligned}\right\} \tag{3.35}$$

The set of Eqs. (3.35) will have a solution if

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} \neq 0. \tag{3.36}$$

The value of this determinant, called *Vandermonde's determinant*, is

$$(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)(x_1 - x_2)\ldots(x_1 - x_n)\ldots(x_{n-1} - x_n).$$

Eliminating $a_0, a_1, \ldots, a_n$ from Eqs. (3.34) and (3.35), we obtain

$$\begin{vmatrix} L_n(x) & 1 & x & x^2 & \cdots & x^n \\ y_0 & 1 & x_0 & x_0^2 & \cdots & x_0^n \\ y_1 & 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ y_n & 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = 0, \tag{3.37}$$

which shows that $L_n(x)$ is a linear combination of $y_0, y_1, y_2, \ldots, y_n$. Hence we write

$$L_n(x) = \sum_{i=0}^{n} l_i(x) y_i, \tag{3.38}$$

where $l_i(x)$ are polynomials in x of degree n. Since $L_n(x_j) = y_j$ for $j = 0, 1, 2, \ldots, n$, Eq. (3.32) gives

$$\left.\begin{aligned} l_i(x_j) &= 0 \qquad \text{if} \quad i \neq j \\ l_j(x_j) &= 1 \qquad \text{for all} \quad j \end{aligned}\right\},$$

which are the same as (3.31). Hence $l_i(x)$ may be written as

$$l_i(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}, \tag{3.39}$$

which obviously satisfies the conditions (3.31).

If we now set

$$\pi_{n+1}(x) = (x - x_0)(x - x_1)\dots(x - x_{i-1})(x - x_i)(x - x_{i+1})\dots(x - x_n), \quad (3.40)$$

then

$$\pi'_{n+1}(x_i) = \frac{d}{dx}[\pi_{n+1}(x)]_{x=x_i}$$

$$= (x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n) \quad (3.41)$$

so that (3.39) becomes

$$l_i(x) = \frac{\pi_{n+1}(x)}{(x - x_i)\,\pi'_{n+1}(x_i)}. \quad (3.42)$$

Hence (3.38) gives

$$L_n(x) = \sum_{i=0}^{n} \frac{\pi_{n+1}(x)}{(x - x_i)\,\pi'_{n+1}(x_i)} y_i, \quad (3.43)$$

which is called *Lagrange's interpolation formula*. The coefficients $l_i(x)$, defined in (3.39), are called *Lagrange interpolation coefficients*. Interchanging x and y in (3.43) we obtain the formula

$$L_n(y) = \sum_{i=0}^{n} \frac{\pi_{n+1}(y)}{(y-y_i)\,\pi'_{n+1}(y_i)} x_i, \tag{3.44}$$

which is useful for *inverse interpolation.*

Errors in Polynomial Interpolation

Let the function $y(x)$, defined by the $(n+1)$ points (x_i, y_i), $i = 0, 1, 2, \ldots, n$, be continuous and differentiable $(n+1)$ times, and let $y(x)$ be approximated by a polynomial $\phi_n(x)$ of degree not exceeding n such that

$$\phi_n(x_i) = y_i, \quad i = 0, 1, 2, \ldots, n \tag{3.1}$$

If we now use $\phi_n(x)$ to obtain approximate values of $y(x)$ at some points other than those defined by (3.1), what would be the accuracy of this approximation? Since the expression $y(x) - \phi_n(x)$ vanishes for $x = x_0, x_1, \ldots, x_n$, we put

$$y(x) - \phi_n(x) = L\pi_{n+1}(x), \tag{3.2}$$

where

$$\pi_{n+1}(x) = (x - x_0)(x - x_1)\ldots(x - x_n) \tag{3.3}$$

and L is to be determined such that Eq. (3.2) holds for any intermediate value of x, say $x = x'$, $x_0 < x' < x_n$. Clearly,

$$L = \frac{y(x') - \phi_n(x')}{\pi_{n+1}(x')}. \tag{3.4}$$

We construct a function $F(x)$ such that

$$F(x) = y(x) - \phi_n(x) - L\pi_{n+1}(x), \tag{3.5}$$

where L is given by Eq. (3.4) above,

It is clear that

$$F(x_0) = F(x_1) = \cdots = F(x_n) = F(x') = 0,$$

that is, $F(x)$ vanishes $(n+2)$ times in the interval $x_0 \le x \le x_n$; consequently, by the repeated application of Rolle's theorem (*see* Theorem 1.3, Section 1.2), $F'(x)$ must vanish $(n+1)$ times, $F''(x)$ must vanish n times, etc., in the interval $x_0 \le x \le x_n$. In particular, $F^{(n+1)}(x)$ must vanish once in the interval.

Let this point be given by $x = \xi$, $x_0 < \xi < x_n$. On differentiating (3.5) $(n+1)$ times with respect to x and putting $x = \xi$, we obtain

$$0 = y^{(n+1)}(\xi) - L(n+1)!$$

so that

$$L = \frac{y^{(n+1)}(\xi)}{(n+1)!}. \qquad (3.6)$$

Comparison of (3.4) and (3.6) yields the results

$$y(x') - \phi_n(x') = \frac{y^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x').$$

Dropping the prime on x', we obtain

$$y(x) - \phi_n(x) = \frac{\pi_{n+1}(x)}{(n+1)!}y^{(n+1)}(\xi), \quad x_0 < \xi < x_n, \qquad (3.7)$$

which is the required expression for the error.

Hermite Interpolation Formula

The interpolation formulae so far considered make use of only a certain number of function values. We now derive an interpolation formula in which both the function and its first derivative values are to be assigned at each point of interpolation. This is referred to as *Hermite's interpolation formula*. The interpolation problem is then defined as follows: Given the set of data points (x_i, y_i, y_i'), $i = 0, 1, \ldots, n$, it is required to determine a polynomial of the least degree, say $H_{2n+1}(x)$, such that

$$H_{2n+1}(x_i) = y_i \quad \text{and} \quad H'_{2n+1}(x_i) = y_i'; \qquad i = 0, 1, \ldots, n, \tag{3.49}$$

where the primes denote differentiation with respect to x. The polynomial $H_{2n+1}(x)$ is called *Hermite's interpolation polynomial*. We have here $(2n + 2)$ conditions and therefore the number of coefficients to be determined is $(2n + 2)$ and the degree of the polynomial is $(2n + 1)$. In analogy with the Lagrange interpolation formula (3.43), we seek a representation of the form

$$H_{2n+1}(x) = \sum_{i=0}^{n} u_i(x) y_i + \sum_{i=0}^{n} v_i(x) y_i', \tag{3.50}$$

where $u_i(x)$ and $v_i(x)$ are polynomials in x of degree $(2n+1)$. Using conditions (3.49), we obtain

$$\left.\begin{array}{ll} u_i(x_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}; & v_i(x) = 0, \text{ for all } i \\ u_i'(x) = 0, \text{ for all } i; & v_i'(x_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \end{array}\right\} \tag{3.51}$$

Since $u_i(x)$ and $v_i(x)$ are polynomials in x of degree $(2n+1)$, we write

$$u_i(x) = A_i(x)\,[l_i(x)]^2 \quad \text{and} \quad v_i(x) = B_i(x)\,[l_i(x)]^2, \tag{3.52}$$

where $l_i(x)$ are given by (3.42). It is easy to see that $A_i(x)$ and $B_i(x)$ are both linear functions in x. We therefore write

$$u_i(x) = (a_i x + b_i)\,[l_i(x)]^2 \quad \text{and} \quad v_i(x) = (c_i x + d_i)\,[l_i(x)]^2 \tag{3.53}$$

Using conditions (3.51) in (3.53), we obtain

$$\left.\begin{aligned} a_i x_i + b_i &= 1 \\ c_i x_i + d_i &= 0 \end{aligned}\right\} \tag{3.54a}$$

and

$$\left.\begin{aligned} a_i + 2l_i'(x_i) &= 0 \\ c_i &= 1. \end{aligned}\right\} \tag{3.54b}$$

From Eqs. (3.54), we deduce

$$\left.\begin{aligned} &a_i = -2l_i'(x_i), \quad b_i = 1 + 2x_i\ l_i'(x_i) \\ &c_i = 1, \qquad\qquad d_i = -x_i. \end{aligned}\right\} \tag{3.55}$$

Hence Eqs. (3.53) become

$$\begin{aligned} u_i(x) &= [-2x\, l_i'(x_i) + 1 + 2x_i l_i'(x_i)]\,[l_i(x)]^2 \\ &= [1 - 2\,(x - x_i)\, l_i'(x_i)]\,[l_i(x)]^2 \end{aligned} \tag{3.56a}$$

and

$$v_i(x) = (x - x_i)[l_i(x)]^2. \tag{3.56b}$$

Using the above expressions for $u_i(x)$ and $v_i(x)$ in (3.50), we obtain finally

$$H_{2n+1}(x) = \sum_{i=0}^{n} [1 - 2(x - x_i)\, l_i'(x_i)][l_i(x)]^2 y_i + \sum_{i=0}^{n} (x - x_i)[l_i(x)]^2 y_i', \tag{3.57}$$

which is the required *Hermite interpolation formula.*