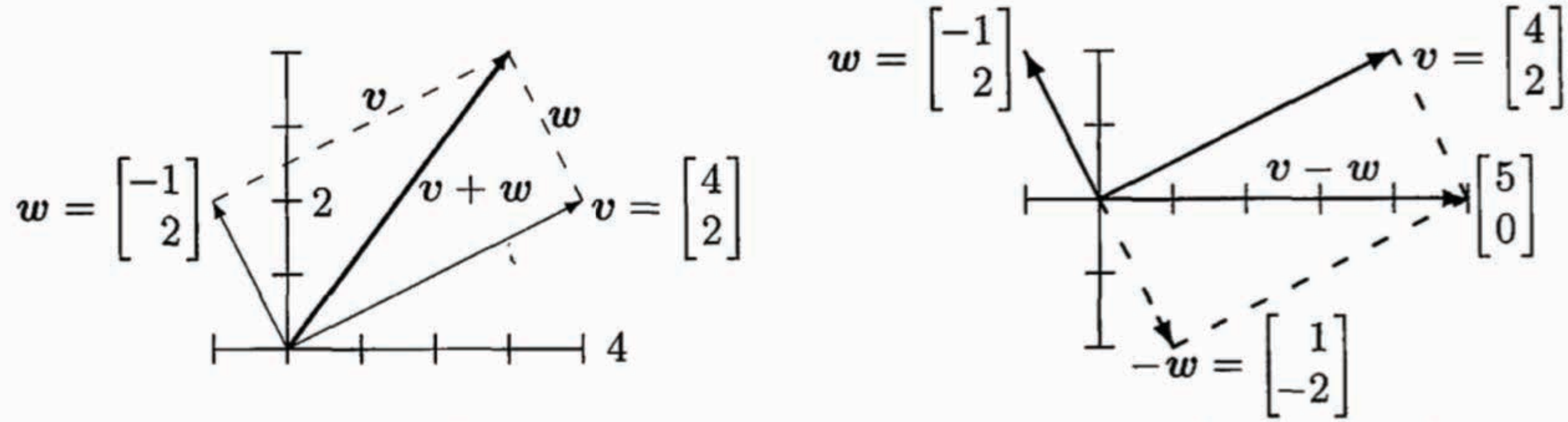
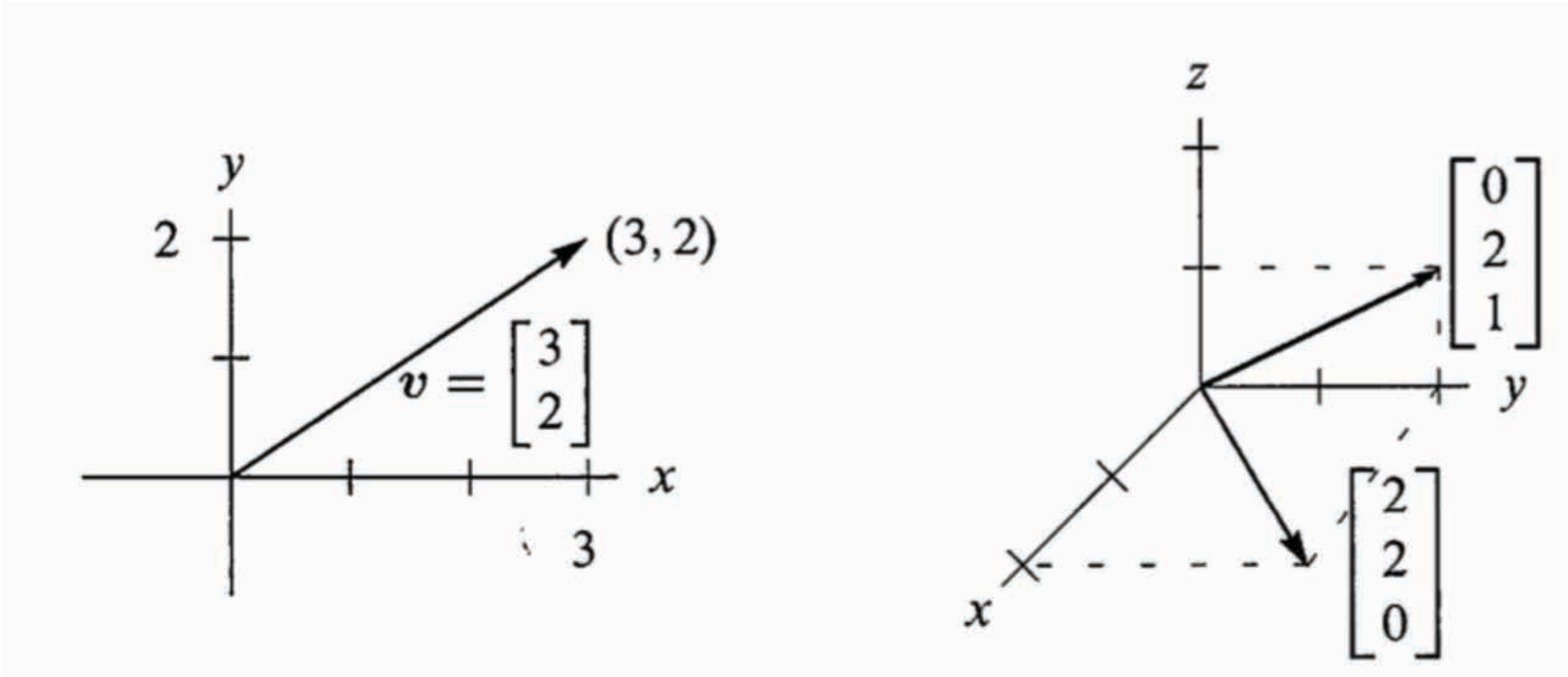
MATH-2241: Linear Algebra

Linear Algebra Basics

Course Teacher:
Md Rashed-Al-Mahfuz
Assistant Professor
Dept of Computer Science and Engineering
University of Rajshahi
Rajshahi-6205, Bangladesh



$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad v - w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$


y
2
(3, 2)
v = [3 2]
x
3
z
[0 2 1]
y
[2 2 0]
x

Linear Combination

First example $\quad \boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

Their linear combinations in three-dimensional space are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$:

Combinations $$c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}. \qquad (1)$$

Now something important: *Rewrite that combination using a matrix.* The vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ go into the columns of the matrix A. That matrix "multiplies" a vector:

Same combination is now A times x $$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}. \qquad (2)$$

Matrix times vector $$Ax = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}. \qquad (3)$$

Solving Linear Equations

Two equations	x	$-$	$2y$	$=$	1
Two unknowns	$3x$	$+$	$2y$	$=$	11

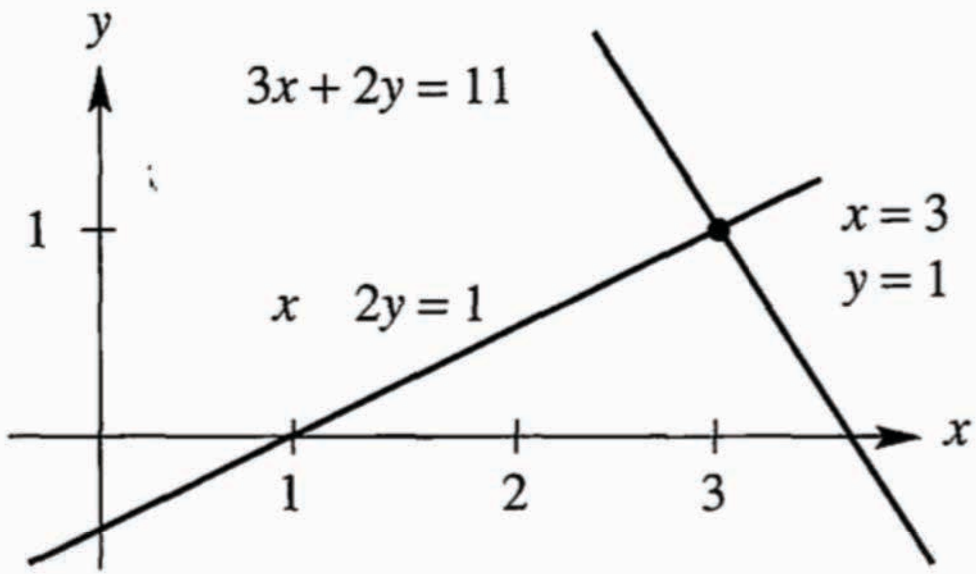


Figure 2.1: *Row picture*: The point (3, 1) where the lines meet is the solution.

Solving Linear Equations

Two equations	$x - 2y = 1$
Two unknowns	$3x + 2y = 11$

Turn now to the column picture. I want to recognize the same linear system as a "vector equation". Instead of numbers we need to see *vectors*. If you separate the original system into its columns instead of its rows, you get a vector equation:

Combination equals b $$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}=\boldsymbol{b}. \quad (2)$$

Solving Linear Equations

$$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}=\boldsymbol{b}.$$

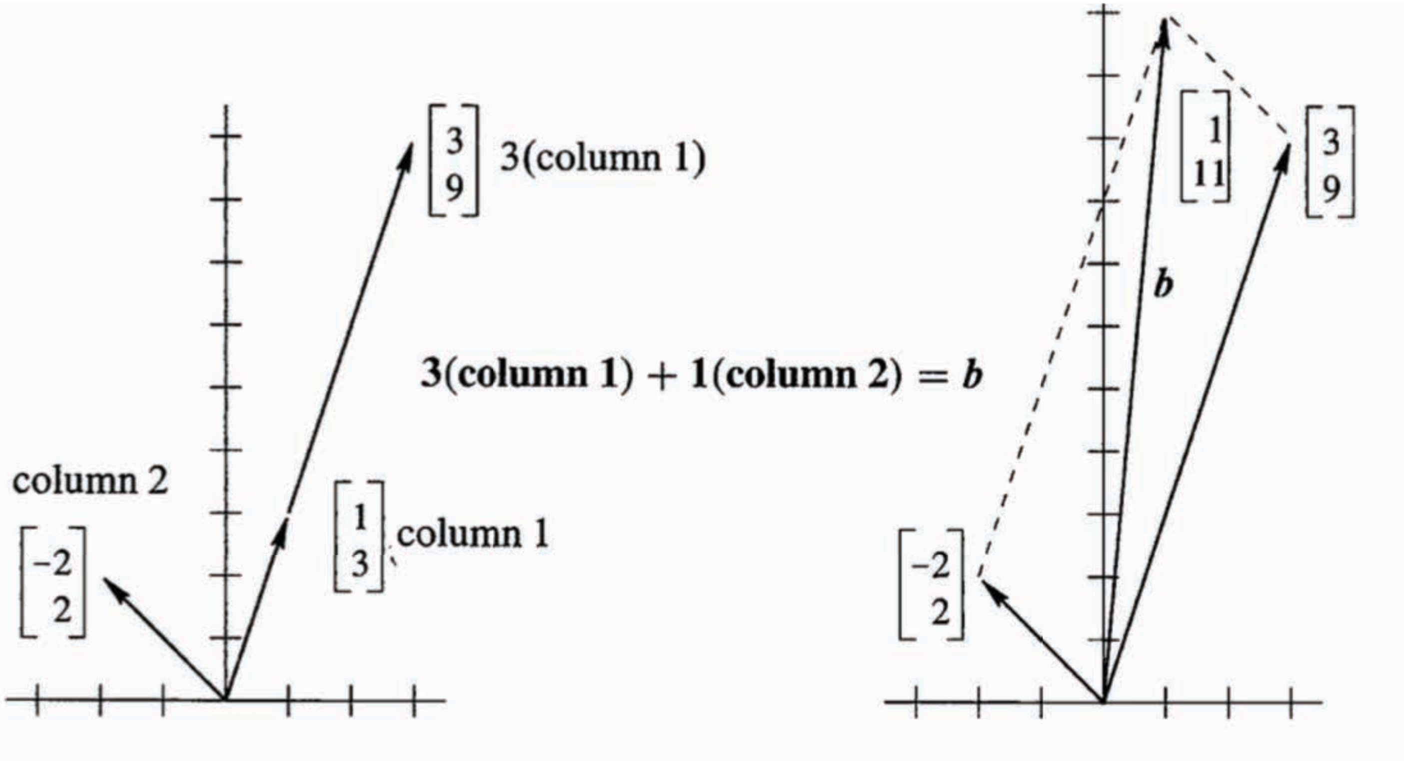


Figure 2.2: *Column picture*: A combination of columns produces the right side (1,11).

Linear combination $\quad 3\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}.$

Dot products with rows
Combination of columns $\quad A\boldsymbol{x}=\boldsymbol{b}$ is $\begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}.$

The Idea of Elimination

Before $\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$ **After** $\begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}$ (*multiply equation* 1 *by* 3) (*subtract to eliminate* $3x$)

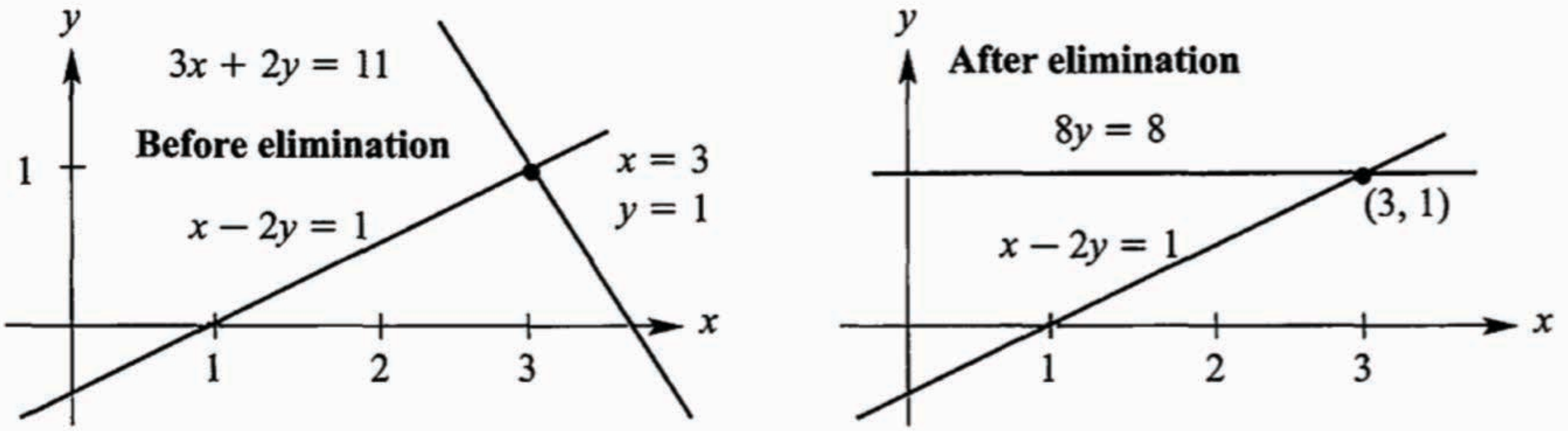


Figure 2.5: Eliminating x makes the second line horizontal. Then $8y = 8$ gives $y = 1$.

No solution

Example 1 ***Permanent failure with no solution.*** Elimination makes this clear:

$$\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 11 \end{aligned} \qquad \begin{array}{c} \text{Subtract 3 times} \\ \text{eqn. 1 from eqn. 2} \end{array} \qquad \begin{aligned} x - 2y &= 1 \\ \mathbf{0}y &= 8. \end{aligned}$$

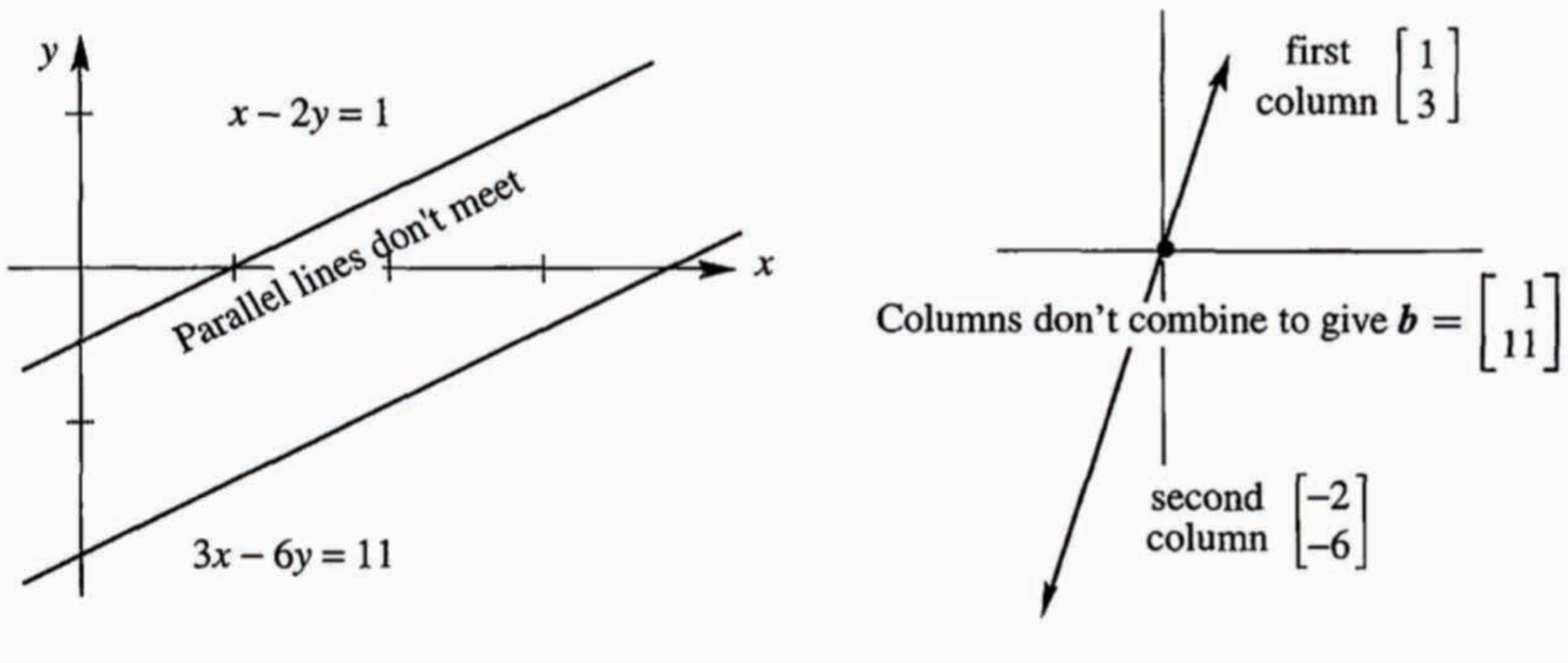


Figure 2.6: Row picture and column picture for Example 1: ***no solution***.

Many solutions

Example 2 ***Failure with infinitely many solutions. Change*** $\boldsymbol{b} = (1, 11)$ ***to*** $(1, 3)$.

$x - 2y = 1$	Subtract 3 times	$x - 2y = 1$	Still only
$3x - 6y = 3$	eqn. 1 from eqn. 2	$\mathbf{0}y = \mathbf{0}.$	**one pivot.**

Every y satisfies $0y = 0$. There is really only one equation $x - 2y = 1$. The unknown y is "***free***". After y is freely chosen, x is determined as $x = 1 + 2y$.

In the row picture, the parallel lines have become the same line. Every point on that line satisfies both equations. We have a whole line of solutions in Figure 2.7.

In the column picture, $\boldsymbol{b} = (1, 3)$ is now the same as column 1. So we can choose $x = 1$ and $y = 0$. We can also choose $x = 0$ and $y = -\frac{1}{2}$; column 2 times $-\frac{1}{2}$ equals $\boldsymbol{b}$. Every (x, y) that solves the row problem also solves the column problem.