Z transform and ROC of infinite duration sequence, properties of z-transform, inversion of the z transform.

2020

5. a) Define Z-transform. What is ROC? 2.75

Answer:

Z-transform:

The method to analysis and represent a discrete time signal of LTI(Linear time invariant) in the frequency domain is z-transform. It can be considered as a discrete time equivalent of the Laplace transformation.

Basic of z-transform:

Z-transform of x(n) will convert the time domain signal of x(n) to z domain signal x(z) where the signal becomes the function of a complex variable z.

x(n) = Discrete time signal
x(z) = z-transform of x(n)

The z-transform of x(n) has 2 conditions:

i) Bilateral or two-sided z-transform of a discrete time signal or a sequence x(n):
$x(z) = \sum_{-\propto}^{\propto} x(n)z^{-n}$ z is a complex variable

Symbolically z[x(n)]

ii) One sided or unilateral z-transform of a discrete time sequence x(n):
$x(z) = \sum_{0}^{\propto} x(n)z^{-n}$

ROC(Region of Convergence):

The set of all values of z for which X(z) converges and attains a finite value is called Region of Convergence.

b) Find the Z-transform and ROC of the following finite duration signal: 3

i) $x_1(n) = \{1, \underset{\uparrow}{2}, 3, 4, 5\}$

ii) $x_2(n) = \{0, 0, 1, 2, 3, 4, 5, 1\}$

Answer:

i) Here we can see that the arrow is below 2, so we will consider it as x(0) for calculation. We have both sided discrete signal.

X(z) = z{x(n)}

$= \sum_{-\propto}^{\propto} x(n)z^{-n}$

$= \sum_{-1}^{3} x(n)z^{-n}$

$= 1z^{1} + 2z^{0} + 3z^{-1} + 4z^{-2} + 5z^{-3}$

$= z + 2 + \frac{3}{z} + \frac{4}{z^2} + \frac{5}{z^3}$

Here we can see for z=0 X(z) = ∝ and for z=∝ , X(z) = ∝

So, ROC on entire z plane except z=0 and z=∝

ii) $x_2(n)$ = {0, 0, 1, 2, 3, 4, 5, 1}

Since there is no arrow sign given we consider as right sided discrete signal.

X(z) = z{x(n)}

$= \sum_{0}^{\propto} x(n)z^{-n}$

$= \sum_{0}^{7} x(n)z^{-n}$

$= 0z^{0} + 0z^{-1} + 1z^{-2} + 2z^{-3} + 3z^{-4} + 4z^{-5} + 5z^{-6} + 1z^{-7}$

$= 0 + 0 + 1/z^{2} + 2/z^{3} + 3/z^{4} + 4/z^{5} + 5z^{6} + 1/z^{7}$

Here we can see for z=0, X(z) = ∝

So, ROC for z-plane for all values of z but z≠0

2019

4.

(a) Why Z-transform? How do differ Z-transform with Fourier transform? 2.75

Answer:

Z-transform is mainly a mathematical tool for analyzing digital signal processing systems. It allows solving difference equations for discrete type systems.

So, to analyze and interact with digital signals of Linear Time-invariant system, we require z-transform.

Fourier transform is used for continuous functions. The Fourier transform of a function f(t) is calculated as the integral of f(t) * e^(-j*w*t) from -infinity to infinity.

This operation is very useful in telecommunications, music processing etc. because it has a physical interpretation: it gives the strength of f (t) components at various frequencies. For this reason, the Fourier Transform of f (t) is called the “frequency domain representation” of f (t).

Fourier Transform is a restricted type of Laplace transform.

Z-transform is used only with discrete sets of numbers i.e. {a1, a2, …}, not continuous functions of t. The sequence can be infinite. The z-transform of a[n] is defined as the sum of a[n] * z^ (-n). The limits of summation can be finite or infinite (-infinity to +infinity or 0 to infinity). Z-transform is extensively used in digital control and digital signal processing.

For both transforms, one has to establish convergence (of the integral or infinite series). For practical signals (functions) this is not usually a problem.

(b) Define ROC. Write down some properties of ROC. 3

Answer:

The set of points in z-plane for which the Z-transform of a discrete-time sequence x(n), i.e., X(z) converges is called the region of convergence (ROC) of X(z).

Properties of ROC:

1. The ROC of the Z-transform is a ring or disc in the z-plane centered at the origin.
2. The ROC of the Z-transform cannot contain any poles.
3. The ROC of Z-transform of an LTI stable system contains the unit circle.
4. If x(n) is a finite duration causal sequence (i.e., right-sided sequence), then its ROC is the entire z-plane except at z = 0.
5. If x(n) is a finite duration anti-causal sequence (i.e., left sided sequence), then its ROC is the entire z-plane except at z=∞.
6. If x(n) is a finite duration two-sided sequence, then its ROC is the entire z-plane except at z = 0 and z=∞.

(c)Determine the z-transform of the signal 3

x(n) = 0.5u(n) + 0.75u(n-1)

Answer:

$$x(n) = 0.5\,u(n) + 0.75\,u(n-1).$$

$$\cancel{X(x(n)}$$

$$X(z) = \sum_{-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{-\infty}^{\infty} \left(0.5u(n) + 0.75u(n-1)\right) z^{n}$$

$$= \sum_{-\infty}^{\infty} 0.5\,u(n)\, z^{n} + \sum_{-\infty}^{\infty} 0.75\, u(n-1)\, z^{-n}$$

$$= \sum_{0}^{\infty} 0.5\, z^{n} + \sum_{1}^{\infty} 0.75\, z^{n}$$

$$= 0.5 \sum_{0}^{\infty} z^{n} + 0.75 \sum_{1}^{\infty} z^{n}$$

$$= 0.5 \sum_{0}^{\infty} (z^{1})^{n} + 0.75 \left\{ \sum_{0}^{\infty} z^{n} - 1 \right\}$$

$$= 0.5 \frac{1}{1-z^{1}} + 0.75 \left\{ \frac{1}{1-z^{1}} - 1 \right\}$$

$$= \frac{0.5}{1-\frac{1}{z}} + 0.75 \left(\frac{1}{1-\frac{1}{z}} - 1 \right)$$

$$= \frac{0.5}{\frac{z-1}{z}} + 0.75 \left(\frac{z}{z-1} - 1 \right)$$

$$= \frac{0.5z}{z-1} + 0.75 \cdot \frac{\cancel{z} - \cancel{z} + 1}{z-1}$$

$$= \frac{0.5z}{z-1} + \frac{0.75}{z-1}$$

$$= \frac{0.5z + 0.75}{z-1}$$

2018

4. a) What is z-transform? (2020 5a) 1.75

b) Find the z-transform of the sequence x(n) = $((\frac{1}{3})^n\ u(-n))$ 3

Answer:

We know x(z) = $\sum_{-\propto}^{\propto} x(n)z^{-n}$

$= \sum_{-\propto}^{\propto}(\frac{1}{3})^n\ u(-n)z^{-n}$

$=\sum_{-\propto}^{0}(\frac{1}{3})^n\ z^{-n}$

$=\sum_{-\propto}^{0}(\frac{1}{3}z^{-1})^n$

$=\sum_{-\propto}^{0}(\frac{1}{3z})^n$

$=\sum_{\propto}^{0}(\frac{1}{3z})^{-k}$

$=\sum_{0}^{\propto}(\frac{1}{3z})^{k}$

$= \frac{1}{1-\frac{1}{3z}}$ $[\because \sum_{0}^{\propto}(\mathrm{a})^n = \frac{1}{1-a}]$

$= \frac{3z}{3z-1}$

> u(-n) = 1 ; n ≤ 0
> 0 ; n>0
>
> x(n) = $\left(\frac{1}{3}\right)^n .1; n \le 0$
> 0 ; n > 0
> Let, k = -n, n= -∝ then k = ∝
> n=0 then k = 0
> and n = -k

c) Which approaches are available to invert the z-transform to recover the sequence x(n). 4

Answer: Methods to Find Inverse Z-Transform:

When the analysis is needed in discrete format, we convert the frequency domain signal back into discrete format through inverse Z-transformation. We follow the following three ways to determine the inverse Z-transformation.

1. Long Division Method
2. Partial Fraction expansion method
3. Residue or Contour integral method

Long Division Method: In this method, the Z-transform of the signal x Z can be represented as the ratio of polynomial as: x(z)=N(Z)/D(Z) Now, if we go on dividing the numerator by denominator, then we will get a series as: x(z)=x(0)+x(1)Z^{-1} +x(2) Z^{-2}+.........

The above sequence represents the series of inverse Z-transform of the given signal forn≥0 and the above system is causal. However for n<0 the series can be written as:

$x(z)=x(-1)\,Z^1 + x(-2)\,Z^2 + x(-3)\,Z^3 + \ldots\ldots$

$$F(z)=\frac{z}{z-0.5}$$

$$\begin{array}{r l}
 & \underline{1+0.5z^{-1}+0.25z^{-2}+\cdots} \\
z-0.5 \big) & z \\
 & \underline{z-0.5} \\
 & 0.5 \\
 & \underline{0.5-0.25z^{-1}} \\
 & 0.25z^{-1} \\
 & \underline{0.25z^{-1}-0.125z^{-2}} \\
 & \vdots
\end{array}$$

$$F(z)=1+0.5z^{-1}+0.25z^{-2}+\cdots$$

$$f=\{1, 0.5, 0.25, \cdots\}$$

$$f[k]=0.5^k$$

Partial Fraction Expansion Method:

Here also the signal is expressed first in N z/D z form. If it is a rational fraction it will be represented as follows:

$x(z)=b_0+b_1Z^{-1}+b_2Z^{-2}+\ldots\ldots\ldots+b_mZ^{-m})/(a_0+a_1Z^{-1}+a_2Z^{-2}+\ldots\ldots\ldots+a_nZ^{-N})$

The above one is improper when m<n and $a_n \neq 0$. If the ratio is not proper i.e. Improper, then we have to convert it to the proper form to solve it.

$$H(z) = \frac{z^2 + 2z}{z^2 - 3z + 2}$$

$$\Rightarrow H(z) = \frac{z(z+2)}{z^2 - 3z + 2}$$

$$\Rightarrow \frac{H(z)}{z} = \frac{z+2}{z^2 - 3z + 2}$$

$$= \frac{z+2}{(z-1)(z-2)}$$

$$= \frac{A}{z-1} + \frac{B}{z-2}$$

$$\because \frac{A(z-2) + B(z-1)}{z^2 - 3z + 2} = \frac{z+2}{z^2 - 3z + 2}$$

$$A(z-2) + B(z-1) = z+2$$

Putting $z = 2$,
$B = 4$

Putting $z = 1$, $A(1-2) = 1+2$
$-A = 3$
$\therefore A = -3$

$$\therefore \frac{-3}{z-1} + \frac{4}{z-2}$$

$$\frac{H(z)}{z} = \frac{-3}{z-1} + \frac{4}{z-2}$$

$$H(z) = -3\frac{z}{z-1} + 4\frac{z}{z-2}$$

we apply this when polynomial is equal / numerator has greater degree.

Step 1: $\frac{H(z)}{z}$
Step 2: Partial fraction
Step 3: Send z to right
Step 4: Take I.Z.T

Apply I.Z.T,
$H(n) = -3u(n) + 4\,2^n u(n)$

Here it is right sided sequence.

$H(0) = -3 + 4 = 1$
$H(1) = -3 + 4 \cdot 2^1 = 5$
$H(2) = -3 + 4 \cdot 2^2 = 13$

Interm of samples,
$= \{1, 5, 13, \ldots\}$

Digital Products

Residue or Contour Integral Method:

In this method, we obtain inverse Z-transform x (n) by summing residues of $[X(Z)\, Z^{n-1}]$ at all poles. Mathematically, this may be expressed as

$$x(n) = \sum_{all\ poles\ X(z)} residues\ of\ |\ x(z) Z^{n-1}|$$

Here, the residue for any pole of order m at $z=\beta$ is

$$Residues = \frac{1}{(m-1)!}\lim_{Z\to\beta}\{\frac{d^{m-1}}{dZ^{m-1}}\{(z-\beta)^m X(z)Z^{n-1}\}$$

$x(n) = \sum$ Residues of $x(z) \times X(z)\, z^{n-1} \Big|$ at pole of Residue

example: $X(z) = \dfrac{z^{18}}{(z-1/2)(z-1)(z-4)}$; $x(z)$ converges for $|z| = 1$. Find $x(-16)$.

$$x(n) = \left(z-\tfrac{1}{2}\right) \times \frac{z^{18} z^{n-1}}{(z-\frac{1}{2})(z-1)(z-4)}\Bigg|_{z=\frac{1}{2}} + (z-1)\frac{z^{18} z^{n-1}}{(z-1/2)(z-1)(z-4)}\Bigg|_{z=1} + (z-4)\cdot\frac{z^{18} z^{n-1}}{(z-\frac{1}{2})(z-1)(z-4)}\Bigg|_{z=4}$$

$$= \frac{z^{18} z^{n-1}}{(z-1)(z-4)}\Bigg|_{z=\frac{1}{2}} + \frac{z^{18} z^{n-1}}{(z-\frac{1}{2})(z-4)}\Bigg|_{z=1} + \frac{z^{18} z^{n-1}}{(z-\frac{1}{2})(z-1)}\Bigg|_{z=4}$$

Give, $x(z)$ converges at $|z| = 1$.

$$\therefore x(n) = \frac{z^{18} z^{n-1}}{(z-\frac{1}{2})(z-4)}\Bigg|_{z=1} = \frac{1^{18}\, 1^{n-1}}{(1-\frac{1}{2})(1-4)} = \frac{1^{n-1}}{-\frac{3}{2}}$$

$$x(-16) = \frac{1^{-16-1}}{-\frac{3}{2}} = \frac{1}{1^{17}} \times \left(-\frac{2}{3}\right) = -\frac{2}{3}$$ Ans.

2017

4. a) What are the properties of Region of Convergence? (2019-4b) 1.75

b) Find the z-transform of the sequence x(n) = $((\frac{1}{3})^{n-1}\ u(n-1))$ 3

Answer:

We know x(z) = $\sum_{-\infty}^{\infty} x(n) z^{-n}$

= $\sum_{-\infty}^{\infty} (\frac{1}{3})^{n-1}\ u(n-1) z^{-n}$

u(n-1) = 1 ; n ≥ 1
0 ; $n \le 0$

x(n) = $\left(\frac{1}{3}\right)^{n}.1; n \le 0$
0 ; n > 0

$$=\sum_1^{\infty}\left(\frac{1}{3}\right)^{n-1} z^{-n}$$

$$=\sum_1^{\infty}\left(\frac{1}{3}\right)^{n} z^{-n} \left(\frac{1}{3}\right)^{-1}$$

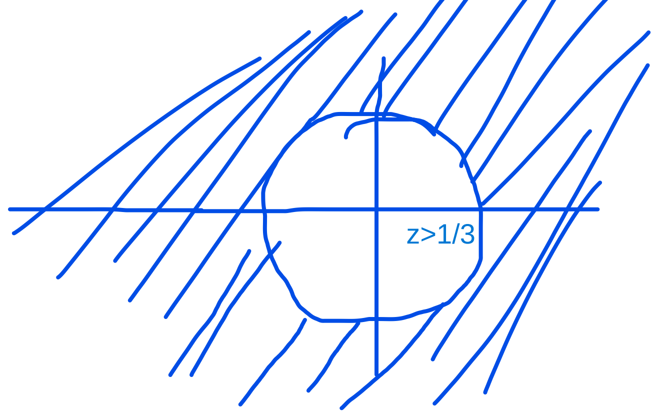
$$=3\left\{\sum_0^{\infty}\left(\frac{1}{3}\ z^{-1}\right)^{n} - 1\right\}$$

$$= 3\left(\frac{1}{1-\frac{1}{3z}} - 1\right) \quad \left[\sum_0^{\infty}(a)^n = \frac{1}{1-a}\right]$$

$$= 3\left(\frac{3z}{3z-1} - 1\right)$$

$$= 3\left(\frac{3z-3z+1}{3z-1}\right)$$

$$= \frac{3*1}{3z-1} = \frac{3}{3z-1}$$



c) Define inverse z-transform. Find the inverse z-transform of 4

$$X(z) = \frac{z+0.2}{(z+0.5)(z-1)}, |z| > 1$$

Answer:

$$X(z) = \frac{z+0.2}{(z+0.5)(z-1)} \quad |z| > 1$$

$$\frac{X(z)}{z} = \frac{z+0.2}{z(z+0.5)(z-1)}$$

$$= \frac{A}{z} + \frac{B}{z+0.5} + \frac{C}{z-1}$$

$$A(z+0.5)(z-1) + Bz(z-1) + C(z)(z+.5) = z+0.2$$

$$z=0, \quad A(0.5)(-1) = 0.2$$

$$A = \frac{0.2}{-0.5} = -0.4$$

$$z=-0.5, \quad B(-0.5)(-0.5-1) = -0.5+0.2$$

$$B(0.75) = -0.3$$

$$B = -0.4$$

$$z=1, \quad C(1)(1+0.5) = 1+0.2$$

$$C = \frac{1.2}{1.5} = 0.8$$

$$\therefore \frac{X(z)}{z} = \frac{-0.4}{z} + \frac{-0.4}{z+0.5} + \frac{0.8}{z-1}$$

$$X(z) = -0.4 - 0.4\frac{z}{z+0.5} + 0.8\frac{z}{z-1}$$

$$= -0.4\delta(n) - 0.4(0.5)^n u(n) + 0.8u(n)$$

Inverse Z-transform: The procedure for transforming from the z-domain to the time domain is called the inverse z-transform.

The signal can be converted from time domain into z domain with the help of z transform (ZT). Similar way the signal can be converted from z domain to time domain with the help of inverse z transform (IZT).

$x(n) \xrightarrow{z} X(z)$

$X(z) \xrightarrow{IZT} x(n)$

2016

5. a) Explain inverse z-transform. (2017-4c) 2

b) Determine the inverse z-transform $X(z) = \frac{z^2+z}{(z-1)(z-3)}, and\ ROC : |z| > 3$ 2.75

Answer:

$$\frac{z^2+z}{(z-1)(z-3)}$$

$$\frac{X(z)}{z} = \frac{z+1}{(z-1)(z-3)}$$

$$\frac{z+1}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$z+1 = A(z-3) + B(z-1)$$

For $z = 1 \Rightarrow 1+1 = A(1-3)$

$$\Rightarrow A = \frac{2}{-2} = -1.$$

For $z = 3 \Rightarrow 3+1 = B(3-1)$

$$\frac{4}{2} = B \Rightarrow B = 2$$

$$\therefore \frac{X(z)}{z} = \frac{-1}{z-1} + \frac{2}{z-3}$$

$$X(n) = -\frac{z}{z-1} + \frac{2z}{z-3}$$

$$= -1^n u(n) + 2(3)^n u(n)$$

No need to change

Right → $\therefore$ ROC $|z| > 3$

c) What are the properties of region of Convergence? (2019-4b). Find the z-transformed ROC of the finite signal of the sequence x(n) = {2, 4, $\underset{\uparrow}{5}$, 7,0, 1} 4

Answer:

X(z) = z{x(n)}

$= \sum_{-\propto}^{\propto} x(n)z^{-n}$

$$= \sum_{-2}^{3} x(n) z^{-n}$$
$$= 2z^{-(-2)} + 4z^{-(-1)} + 5z^{-0} + 7z^{-1} + 0z^{-2} + 1z^{-3}$$
$$= 2z^{2} + 4z + 5 + \frac{7}{\mathrm{z}} + \frac{1}{z^{3}}$$

Here we can see for z=0, x(z) = ∝ and for z=∝ , X(z) = ∝

So, ROC on entire z plane except z=0 and z=∝

Q: What are some real life applications of Z transforms?

Z transform is used in many applications of mathematics and signal processing. The lists of applications of z transform are:-

-Uses to analysis of digital filters.

-Used to simulate the continuous systems.

-Analyze the linear discrete system.

-Used to finding frequency response.

-Analysis of discrete signal.

-Helps in system design and analysis and also checks the systems stability.

-For automatic controls in telecommunication.

-Enhance the electrical and mechanical energy to provide dynamic nature of the system.

If we see the main applications of z transform then we find that it is analysis tool that analyze the whole discrete time signals and systems and their related issues. If we talk the application areas of

This transform wherever it is used, they are:-

-Digital signal processing.

-Population science.

-Control theory.

-Digital signal processing