Section-B

2021

5.(a) What is modeling? Compare between physical and mathematical more in brief.

- Modeling is the process of creating a simplified representation of a real-world system to understand its behavior, make predictions, or perform simulations. Models are used in various fields to study and analyze complex systems.
- Physical modeling involves creating a physical representation or prototype of a system to observe its behavior. This can be a scaled-down physical object or a mock-up of the system.
- Mathematical modeling, on the other hand, uses mathematical equations, relationships, and formulas to describe and simulate the behavior of a system. It involves creating a set of mathematical equations that represent the system's dynamics.

Comparison:

- Physical modeling uses tangible objects to represent a system, while mathematical modeling uses mathematical equations and abstract representations.
- Physical models are often limited in terms of the range of scenarios they can simulate, while mathematical models can be more versatile and used to simulate a wide range of conditions.
- Mathematical models can provide analytical solutions and are often more cost-effective compared to physical models.

Aspect	Physical	Mathematical
Nature	Concerned with tangible objects, real-world phenomena, and physical properties.	Abstract, dealing with concepts, symbols, and logical reasoning.
Representation	Uses real-world objects, measurements, and sensory inputs	Uses symbols, numbers, and mathematical notation.
Precision	Often subject to measurement limitations and uncertainty.	Precision is exact and depends on the level of abstraction.
Experimentation	Involves empirical observation, experimentation, and data collection.	Involves the manipulation of mathematical equations, proofs, and analysis.

Aspect	Physical	Mathematical
Predictive Power	Predicts real-world outcomes and behavior.	Predicts outcomes based on mathematical models and theories.
Subjective Interpretation	Interpretation can be influenced by human perception	Interpretation is based on logical principles and objective analysis
Certainty	Often involves uncertainty and probabilistic outcomes.	Provides a high degree of certainty when logical and mathematical rules are followed.
Empirical Evidence	Relies on empirical data, observations, and experiments.	Does not rely on empirical evidence but on logical reasoning
Examples	Physics, chemistry, biology, engineering, and other sciences.	Algebra, calculus, geometry, statistics, and other branches of mathematics.

(b) Is there a unique model of every system? Discuss about the tasks of deriving a model.

There is not necessarily a unique model for every system because the choice of a model depends on the purpose of modeling, the level of detail required, and the available data. Different models can be used to represent the same system, each serving a different purpose.

Tasks of deriving a model:

1. **Problem Definition:** Define the objectives and the scope of the modeling effort. Clearly state what you want to achieve through modeling.
2. **Data Collection:** Gather relevant data about the system you want to model. This may involve measurements, observations, and research.
3. **Model Selection:** Choose an appropriate modeling approach, which can be physical, mathematical, or a combination of both, based on the nature of the system and the available data.
4. **Formulation:** Develop the model by specifying the relationships, equations, and parameters that describe the system's behavior. In mathematical modeling, this involves writing down equations that represent the system's dynamics.
5. **Validation:** Validate the model by comparing its predictions or simulations to real-world data. Ensure that the model accurately reflects the system's behavior.
6. **Calibration:** Adjust the model's parameters to improve its accuracy, if necessary, based on the validation results.

7. **Simulation and Analysis:** Use the model to perform simulations or analyze its behavior to gain insights or make predictions about the system.
8. **Documentation and Reporting:** Document the model and the modeling process, and report the findings and results.

(c) What do you know about 'blobby objects'?

Blobby objects are a type of computer graphics representation used in 3D computer modeling and rendering. They are particularly useful for representing amorphous or shapeless objects that do not have well-defined surfaces, such as clouds, fire, smoke, and certain biological organisms.

Blobby objects are defined by a mathematical function that describes their internal structure and shape. Unlike traditional polygonal models that use meshes, blobby objects rely on implicit surfaces.

The key concept behind blobby objects is the "blob function" or "blob primitive," which defines how the object's density or intensity varies in 3D space. This function is typically Gaussian or other smooth mathematical functions.

Blobby objects allow for realistic rendering and animation of complex, organic shapes and are commonly used in computer graphics for special effects in movies, video games, and scientific visualization

6.(a) What is interpolation? Why is interpolation needed? 275

Interpolation is a mathematical and computational technique used to estimate values between known data points. It is used when you have a set of discrete data points and need to find values at points that lie within the range of the known data.

We specify a spline curve by giving a set of coordinate positions, called control points, which indicate the general shape of the curve. These coordinate positions are then fitted with piecewise-continuous, parametric polynomial functions in one of two ways. When polynomial sections are fitted so that all the control points are connected, as in Figure 1, the resulting curve is said to interpolate the set of control points.

FIGURE 1
A set of six control points interpolated with piecewise continuous polynomial sections.

Interpolation is needed for several reasons:

1. To estimate values at intermediate points between measured data points.
2. To create a smooth curve or function that fits the given data.
3. To make predictions or extrapolations based on available data.

4. To simplify data analysis and visualization by providing a continuous representation of discrete data.

Interpolation methods are commonly used **to digitize drawings or to specify animation paths**.

(b) What is a spline? What are the differences between interpolation and approximation splines?

A spline is a piecewise-defined continuous curve that is used in computer graphics, computer-aided design, and various applications where smooth and flexible curves are required. Splines are typically defined by a set of control points and mathematical functions that connect these points.

The main differences between interpolation and approximation splines are:

Interpolation splines pass through all the given data points, ensuring that the curve matches the data exactly. Approximation splines, on the other hand, do not necessarily pass through the data points but aim to approximate the overall shape of the data.

(c) Mention some of the cons of polynomial interpolation.

Polynomial interpolation is a method of approximating a function by fitting a polynomial to a set of data points. Here are some pros and cons of polynomial interpolation:

Pros:

- Polynomial interpolation is easy to understand and implement.
- It is computationally efficient for small datasets.
- It can be used to approximate any function over a finite interval.
- It is useful for generating smooth curves that pass through a set of data points.

Cons:

- Polynomial interpolation can be unstable for large datasets.
- It can produce oscillations or overshoots between data points, leading to inaccurate results.
- It is not suitable for extrapolation beyond the range of the data points.
- It can be sensitive to the choice of interpolation points.

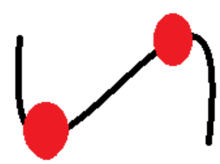
(d) What do you mean by Quadratic spline?

A spline is **a piecewise-defined curve** that is used to approximate or interpolate a set of points.

A quadratic spline is a type of spline where each **segment of the curve is defined by a quadratic polynomial** (i.e., a second-degree polynomial). This means that the curve is composed of connected quadratic segments.

It produces a smooth curve over the interval being studied

Quadratic splines are often used for approximating curves and are commonly used in computer-aided design (CAD) and computer graphics applications.

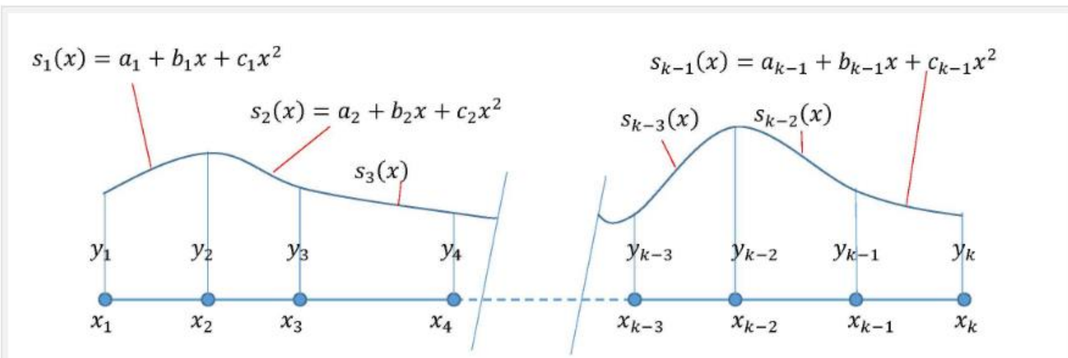


Equation:

$$a1\ x^2 + b1\ x + c1 = k1$$
$$a2\ x^2 + b2\ x + c2 = k2$$
$$a3\ x^2 + b3\ x + c3 = k3$$

https://engcourses-uofa.ca/books/numericalanalysis/piecewise-interpolation/quadratic-spline-interpolation/

$$
\begin{aligned}
a_1 + b_1 x_1 + c_1 x_1^2 &= y_1 \\
a_1 + b_1 x_2 + c_1 x_2^2 &= y_2 \\
a_2 + b_2 x_2 + c_2 x_2^2 &= y_2 \\
a_2 + b_2 x_3 + c_2 x_3^2 &= y_3 \\
&\vdots \\
a_{k-1} + b_{k-1} x_{k-1} + c_{k-1} x_{k-1}^2 &= y_{k-1} \\
a_{k-1} + b_{k-1} x_k + c_{k-1} x_k^2 &= y_k \\
b_1 + 2c_1 x_2 - b_2 - 2c_2 x_2 &= 0 \\
b_2 + 2c_2 x_3 - b_3 - 2c_3 x_3 &= 0 \\
&\vdots \\
b_{k-2} + 2c_{k-2} x_{k-1} - b_{k-1} - 2c_{k-1} x_{k-1} &= 0 \\
2c_1 &= 0
\end{aligned}
$$



7. (a) What is Hermite interpolation? Why do we use Hermite polynomials?

Hermite interpolation is a method of **interpolating data** points **using not only the function** values but **also their derivatives**. It is used to generate a curve that **not only passes through the data points** but also **has specified tangent or slope values** at those points.

Hermite polynomials are used because they provide a **high degree** of control over the shape of the interpolated curve. By specifying both **function values and derivatives,** you can create curves that have specific characteristics, **such as smooth transitions or sharp corners.**

Hermite polynomials are a set of orthogonal polynomials that have a variety of applications in mathematics, physics, and engineering. Some of the reasons why Hermite polynomials are used include:

- They are orthogonal with respect to the weight function e^(-x^2). This property makes them valuable for solving problems involving weighted integrals or inner products in mathematical physics and quantum mechanics.

- They are eigen functions of the quantum harmonic oscillator. This means that they can be used to solve the Schrödinger equation for the harmonic oscillator, which is an important model in quantum mechanics.

- They can be used to approximate functions. This is because they are a complete set of orthogonal polynomials, which means that any function can be expressed as a linear combination of Hermite polynomials.

- They can be used to interpolate data. This is because they are able to pass through any given set of data points.

Here are some specific examples of how Hermite polynomials are used:

- In signal processing, Hermite polynomials are used to design filters.

- In image processing, Hermite polynomials are used to smooth images and reduce noise.

- In statistics, Hermite polynomials are used to construct confidence intervals for parameters of interest.

- In physics, Hermite polynomials are used to describe the energy levels of atoms and molecules.

(b) Derive the expression for the Hermite bending functions to generate spline curve.

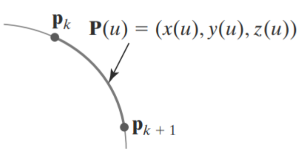


FIGURE 10
Parametric point function $\mathbf{P}(u)$ for a Hermite curve section between control points $\mathbf{p}_k$ and $\mathbf{p}_{k+1}$.

A **Hermite spline** (named after the French mathematician Charles Hermite) is an interpolating piecewise cubic polynomial with a specified tangent at each control point. Unlike the natural cubic splines, Hermite splines can be adjusted locally because each curve section depends only on its endpoint constraints.

If $\mathbf{P}(u)$ represents a parametric cubic point function for the curve section between control points $\mathbf{p}_k$ and $\mathbf{p}_{k+1}$, as shown in Figure 10, then the boundary conditions that define this Hermite curve section are

$$\begin{aligned} \mathbf{P}(0) &= \mathbf{p}_k \\ \mathbf{P}(1) &= \mathbf{p}_{k+1} \\ \mathbf{P}'(0) &= \mathbf{Dp}_k \\ \mathbf{P}'(1) &= \mathbf{Dp}_{k+1} \end{aligned} \tag{9}$$

with $\mathbf{Dp}_k$ and $\mathbf{Dp}_{k+1}$ specifying the values for the parametric derivatives (slope of the curve) at control points $\mathbf{p}_k$ and $\mathbf{p}_{k+1}$, respectively.

We can write the vector equivalent of Equations 8 for this Hermite curve section as

$$\mathbf{P}(u) = \mathbf{a}\,u^3 + \mathbf{b}\,u^2 + \mathbf{c}\,u + \mathbf{d}, \qquad 0 \le u \le 1 \tag{10}$$

where the x component of **P**(u) is $x(u) = a_x\,u^3 + b_x\,u^2 + c_x\,u + d_x$, and similarly for the y and z components. The matrix equivalent of Equation 10 is

$$\mathbf{P}(u) = [u^3 \quad u^2 \quad u \quad 1] \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} \tag{11}$$

and the derivative of the point function can be expressed as

$$\mathbf{P}'(u) = [3u^2 \quad 2u \quad 1 \quad 0] \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} \tag{12}$$

Substituting endpoint values 0 and 1 for parameter u into the preceding two equations, we can express the Hermite boundary conditions 9 in the matrix form

$$\begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{Dp}_k \\ \mathbf{Dp}_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} \tag{13}$$

Solving this equation for the polynomial coefficients, we get

$$\begin{aligned} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{Dp}_k \\ \mathbf{Dp}_{k+1} \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{Dp}_k \\ \mathbf{Dp}_{k+1} \end{bmatrix} \\ &= \mathbf{M}_H \cdot \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{Dp}_k \\ \mathbf{Dp}_{k+1} \end{bmatrix} \end{aligned} \tag{14}$$

where $\mathbf{M}_H$, the Hermite matrix, is the inverse of the boundary constraint matrix. Equation 11 can thus be written in terms of the boundary conditions as

$$\mathbf{P}(u) = [u^3 \quad u^2 \quad u \quad 1] \cdot \mathbf{M}_H \cdot \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{Dp}_k \\ \mathbf{Dp}_{k+1} \end{bmatrix} \tag{15}$$

Finally, we can determine expressions for the polynomial Hermite blending functions, $H_k(u)$ for $k = 0, 1, 2, 3$, by carrying out the matrix multiplications in Equation 15 and collecting coefficients for the boundary constraints to obtain the polynomial form

$$\begin{aligned} \mathbf{P}(u) &= \mathbf{p}_k(2u^3 - 3u^2 + 1) + \mathbf{p}_{k+1}(-2u^3 + 3u^2) + \mathbf{Dp}_k(u^3 - 2u^2 + u) \\ &\qquad + \mathbf{Dp}_{k+1}(u^3 - u^2) \\ &= \mathbf{p}_k H_0(u) + \mathbf{p}_{k+1} H_1 + \mathbf{Dp}_k H_2 + \mathbf{Dp}_{k+1} H_3 \end{aligned} \tag{16}$$

Figure 11 shows the shape of the four Hermite blending functions.

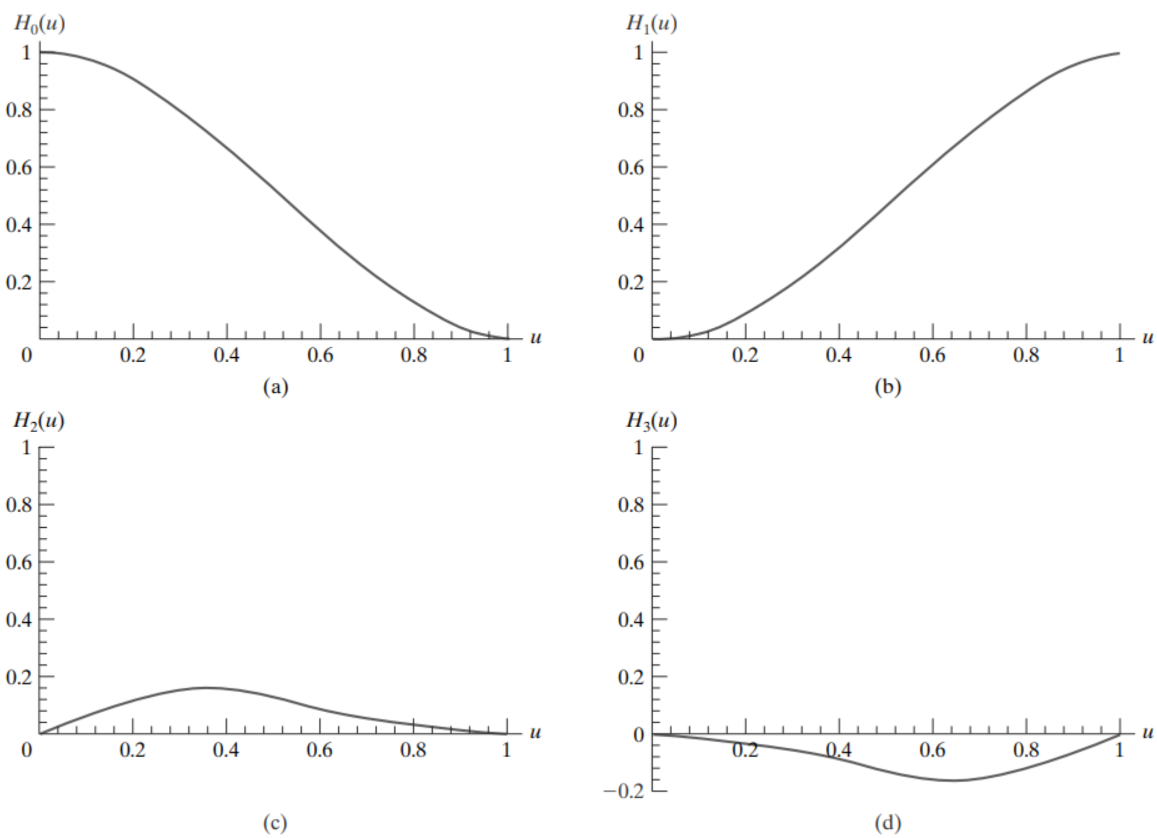


FIGURE 11
The Hermite blending functions.

(c) What are the limitations of Hermite curve?

Hermite curves can be sensitive to noise in the data. This is because they pass through all of the data points, including any outliers.

https://www.geeksforgeeks.org/hermite-curve-in-computer-graphics/

Application

The Hermite curve is used to interpolate sample points on a 2-D plane that results in a smooth curve, but not a free form, unlike the Bezier and B-spline curves. The most commonly used cubic spline is a 3-D planar curve.

Advantages

Hermite curves are easily calculatable. They are used to smoothly interpolate through control points. Understanding the mathematical background of Hermite curves will help us to understand the entire family of splines.

Disadvantages

- First-order derivatives are required; it is not suitable for a designer to provide first-order derivatives.
- There is no local control support.
- The order of the curve is constant in spite of the number of data points.
- Physically, the values of parameters that is the blending function H0, H1, H2, and H3 do not contain any meaning. If we know these values, we can't imagine the shape of the curve.

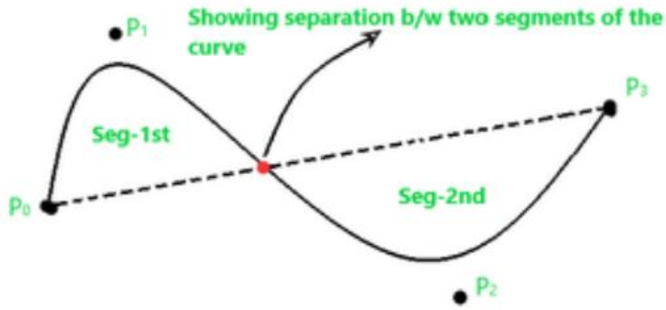
8. (a) Is Bezier curve an interpolation curve? Why B-spline curve better than Bezier curve?

Bezier curves are not interpolation curves by default. These are approximation curve. Meaning they do not necessarily pass through the control points (except for the start and end points) but are defined by them. Bezier curves provide control over the shape of the **curve without requiring it to go through specific points.**

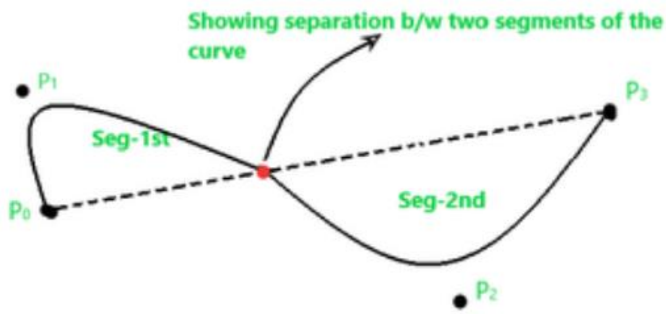
B-spline (Basis-spline) curves, on the other hand, can be used for interpolation and approximation. **B-splines offer more flexibility and control** as they are defined by a set of control points, and their shape can be adjusted without affecting their local control points.

B-Spline curve offers more control and flexibility than Bezier curve. It is possible to use lower degree curves and still maintain a large number of control points.

1. Change on one control point changes the segment only. i.e. local control over curve.
2. Used to draw open and closed curve.
3. Gives us polynomial of degree k-1
4. Allow change of number of control points without changing degree of Polynomial.



B-spline curve shape after changing the position of control point P_1 –



You can see in the above figure that only the **segment-1st** shape as we have only changed the control point P_1, and the shape of segment-2nd remains intact.

(b) What are the important properties of Bezier curve?

Important **Properties** Of Bézier Curves

1. The curve passes through the first, P_0 and last vertex points, P_n

   or first and last control points are the first and last point on the curve

   1. $P(0) = p_0$
   2. $P(1) = p_n$

   The points $\mathbf{P}_i$ are called *control points* for the Bézier curve.

   The polygon formed by connecting the Bézier points with lines, starting with $\mathbf{P}_0$ and finishing with $\mathbf{P}_n$, is called the *Bézier polygon*.

2. The convex hull(convex polygon boundary) of the Bézier polygon contains the Bézier curve.The curve lies within the convex hull of control points. Bézier blending functions are all positive and sum to 1

$$\sum_{k=0}^{n} BEZ_{k,n}(u) = 1$$

Means curve position is weighted sum of control points positions. It ensures that polynomial smoothly follows control points without oscillations.

(c) What are the uses of Bezier curve? What are the limitations of Bezier curve?

Uses of Bezier curves:

- Graphic design: Bezier curves are commonly used in graphic design software to create smooth and precise shapes, such as curves in fonts and vector graphics.
- Animation: They are used in computer animation to define the paths for the motion of objects.
- 3D modeling: Bezier surfaces and patches are used in 3D modeling software to create complex shapes.
- Robotics: In robotics, Bezier curves are used for defining motion trajectories.

<u>Limitations of the Bezier curve:</u> There are limitations of the Bezier curve.

It although unlimited but contains a finite number of control points.

There are many curves which cannot be completely expressed by Bezier curves.

Most cubic rational Bezier curves may be included in that class, and most curves of higher degree also.

It is liable to approximate in nature.

By means of an arbitrary number of points those could probably be approximated arbitrarily not exactly.

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