Phase–Interval Rigidity and the Real–Part Counting Identity on the Critical Line

ChatGTP and Jan Mikulik

05.11.2025

Framework. Let $\zeta(s)$ be the Riemann zeta function and write $s = \frac{1}{2} + it$ with $t \in \mathbb{R}$. Let $\theta(t)$ be the Riemann–Siegel theta function and

$$Z(t) \;:=\; e^{i\theta(t)}\,\zeta\big(\tfrac{1}{2}+it\big) \in \mathbb{R}.$$

Then

$$\operatorname{Re}\,\zeta\big(\tfrac{1}{2}+it\big) = Z(t)\cos\theta(t), \qquad \operatorname{Im}\,\zeta\big(\tfrac{1}{2}+it\big) = -\,Z(t)\sin\theta(t). \tag{1}$$

The classical asymptotic

$$\theta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + O(1/t) \qquad (t \to \infty) \tag{2}$$

implies θ is strictly increasing for large t.

Denote

- by $N(T)$ the number of nontrivial zeros $\rho = \frac{1}{2} + i\gamma$ of ζ, $0 < \gamma \le T$, counted with multiplicity;
- by $N_{\Re}(T)$ the number of $t \in (0, T]$ with $\operatorname{Re}\,\zeta(\frac{1}{2} + it) = 0$, counting multiplicity of crossings;
- by $H(T)$ the number of "half–Gram points" in $(0, T]$, i.e. solutions of $\cos\theta(t) = 0$;
- by $S(T)$ the standard argument term from the Riemann–von Mangoldt formula.

1. Local phase rotation near a simple zero

Lemma 1 (Local linearization)**.** *Let $\rho = \frac{1}{2} + i\gamma$ be a simple zero of ζ. Then*

$$\zeta\big(\tfrac{1}{2}+it\big) = \zeta'(\rho)\,(t-\gamma) + O\big((t-\gamma)^2\big) \qquad (t \to \gamma).$$

Consequently, writing $\zeta = |\zeta|e^{i\arg\zeta}$, the phase satisfies

$$\Delta\arg\zeta\big(\tfrac{1}{2}+it\big) \;=\; \pi + o(1) \quad \textit{as } t \textit{ increases across } \gamma.$$

Sketch. Analyticity and $\zeta(\rho) = 0$ with $\zeta'(\rho) \neq 0$ give the linear term. Along $s = \frac{1}{2} + it$ the value ζ crosses the origin in the complex plane along a straight line up to quadratic error, hence the argument increases (or decreases) by π. □

Theorem 1 (Phase–Interval Rigidity)**.** *Fix $\varepsilon > 0$ and let $\rho = \frac{1}{2} + i\gamma$ be a simple zero. There exists a closed interval*

$$I_\varepsilon(\rho) = [t_\varepsilon^-, t_\varepsilon^+] \quad \textit{with} \quad t_\varepsilon^- < \gamma < t_\varepsilon^+, \qquad |\zeta(\tfrac{1}{2}+it)| \le \varepsilon \;\; \forall\, t \in I_\varepsilon(\rho),$$

such that the total phase rotation across $I_\varepsilon(\rho)$ equals $\pi + o(1)$ as $\varepsilon \downarrow 0$, and moreover

$$\big|I_\varepsilon(\rho)\big| = \frac{2\varepsilon}{|\zeta'(\rho)|} \;+\; o(\varepsilon).$$

Sketch. Combine Lemma 1 with $\zeta(\frac{1}{2}+it) \approx \zeta'(\rho)(t-\gamma)$. The set $\{t : |\zeta| \le \varepsilon\}$ is an interval about γ of length $2\varepsilon/|\zeta'(\rho)| + o(\varepsilon)$, over which the complex phase rotates by $\pi + o(1)$. $\square$

Corollary 1 (Two real–part crossings per simple zero)**.** *Let $\rho = \frac{1}{2} + i\gamma$ be simple. On $I_\varepsilon(\rho)$ the identity* (1) *implies*

$$\mathrm{Re}\ \zeta\big(\tfrac{1}{2}+it\big) = 0 \iff Z(t) = 0 \quad \textit{(true zero)} \quad \textit{or} \quad \cos\theta(t) = 0 \quad \textit{(phase crossing)}.$$

Because θ is strictly monotone by (2)*, there is exactly one half–Gram point in $I_\varepsilon(\rho)$ for small ε, so generically* Re $\zeta = 0$ *occurs* twice *per simple zero: once at $t = \gamma$ (via Z) and once at the neighboring phase crossing* ($\cos\theta = 0$)*.*

2. Global counting up to height T

Lemma 2 (Half–Gram count)**.** *For $T \to \infty$,*

$$H(T) = \#\{\, t \in (0,T] : \ \cos\theta(t) = 0 \,\} = \frac{\theta(T)}{\pi} + O(1).$$

Sketch. Since $\theta'(t) > 0$ for large t, the equation $\cos\theta(t) = 0$ has exactly one solution in each interval where θ advances by π, whence the count. $\square$

Lemma 3 (Riemann–von Mangoldt linkage)**.** *The Riemann–von Mangoldt formula gives*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + o(1),$$

and one checks

$$\frac{\theta(T)}{\pi} = N(T) - S(T) - 1 + o(1) \qquad (T \to \infty).$$

Sketch. This is standard: compare explicit formulas for $\theta(T)$ and $N(T)$ and eliminate the common main terms, the remainder being the $S(T)$ contribution. $\square$

Theorem 2 (Real–part counting identity)**.** *Let $N_{\Re}(T)$ be the number of solutions of* Re $\zeta(\frac{1}{2}+it) = 0$ *in* $(0,T]$*, counted with multiplicity of crossings. Then*

$$N_{\Re}(T) = N(T) + H(T) = 2\,N(T) - S(T) - 1 + o(1) \qquad (T \to \infty).$$

Sketch. By (1), a zero of Re ζ arises either from a true zeta zero ($Z = 0$) or from a phase crossing ($\cos\theta = 0$). Corollary 1 describes the local picture: generically there is one crossing from each source near every simple zero. Globally,

$$N_{\Re}(T) = \#\{Z(t) = 0,\ 0 < t \le T\} + \#\{\cos\theta(t) = 0,\ 0 < t \le T\} = N(T) + H(T).$$

Now insert Lemmas 2–3 to obtain

$$N_{\Re}(T) = N(T) + \frac{\theta(T)}{\pi} + O(1) = 2N(T) - S(T) - 1 + o(1).$$

$\square$

Remark 1 (On Gram law failures and multiplicity)*.* Local failures of Gram's law only swap the bracketing of Z-zeros by half–Gram points; the identity in Theorem 2 is global and unaffected. For a zero of multiplicity m, Theorem 1 yields a phase rotation of $m\pi$ and the crossing multiplicity adjusts accordingly; the formula still partitions Re $\zeta = 0$ into "true" and "phase" events.

3. Consequence ("near doubling")

Since $S(T) = o(\log T)$, Theorem 2 shows

$$N_{\Re}(T) = 2N(T) + O\big(|S(T)| + 1\big),$$

i.e. the number of real–part zeros up to height T is asymptotically "almost twice" the number of zeta zeros—the extra term is precisely the count of phase crossings. This explains the empirically observed near–doubling without postulating any additional zeros.

Summary. *Phase–Interval Rigidity* (Theorem 1) converts each simple zero into a tiny interval on which the phase rotates by π. Combined with Hardy's Z–formalism and the monotonicity of θ, this yields the sharp global identity

$$N_{\Re}(T) = 2N(T) - S(T) - 1 + o(1),$$

pinning the "two per zero" effect to the deterministic phase crossings of $\cos\theta$.