Applications of Prime Numbers

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Möbius Function

In 1832, August Ferdinand Möbius introduced the Möbius function $\mu(n)$ [1], a multiplicative function as followed:

$$\mu(n) = \begin{cases} (-1)^{v(n)} & \text{if } n \text{ is square-free,} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $v(n)$ denotes the number of distinct prime divisors of n.
With this definition, we can show, for example, that $\mu(2021) = (-1)^2 = 1$, because $2021 = 43 * 47$ and therefore, a product of two distinct prime factors.

The following graph, in addition to future graphs, were created with the help of references. [2][3]

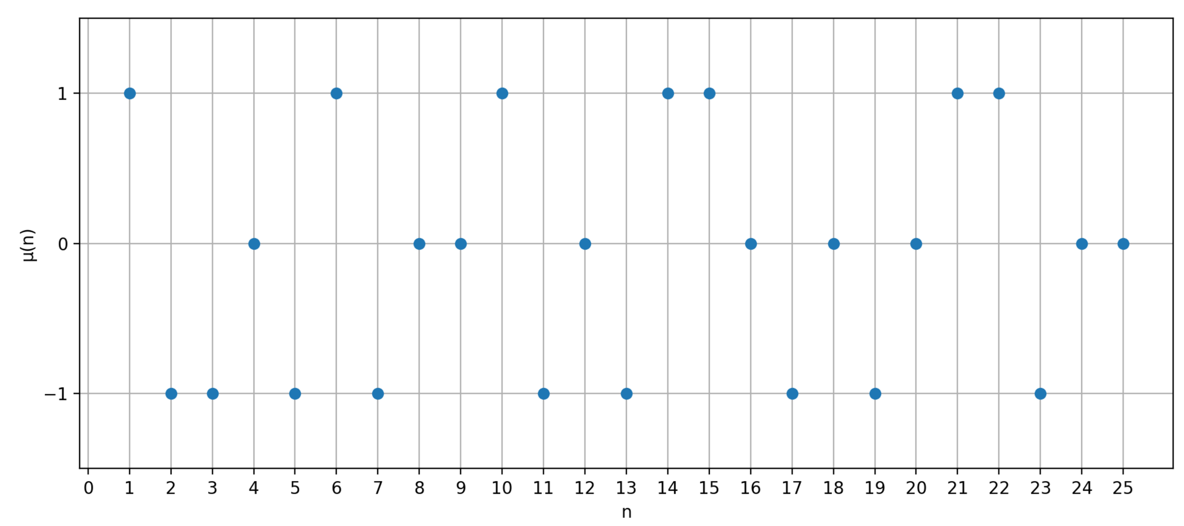

Fig. 1: Graph of Möbius Function $\mu(n)$ for $n \leq 25$.

Mertens Function

The Mertens function, named after Franz Mertens, is the partial sum of the Möbius function, defined as

$$M(x) = \sum_{k \leq x} \mu(k). \tag{2}$$

Thomas Joannes Stieltjes conjectured in 1885 that

$$M(x) = O(x^{\frac{1}{2}}) \tag{3}$$

In other words, he believed that the Mertens function was bounded by $\pm C\sqrt{n}$, where C is some constant. Known as the Mertens conjecture, it was later disproven by Andrew Odlyzko and Herman te Riele and revised to be

$$M(x) = O(x^{\frac{1}{2}+\epsilon}), \tag{4}$$

for any $\epsilon > 0$.

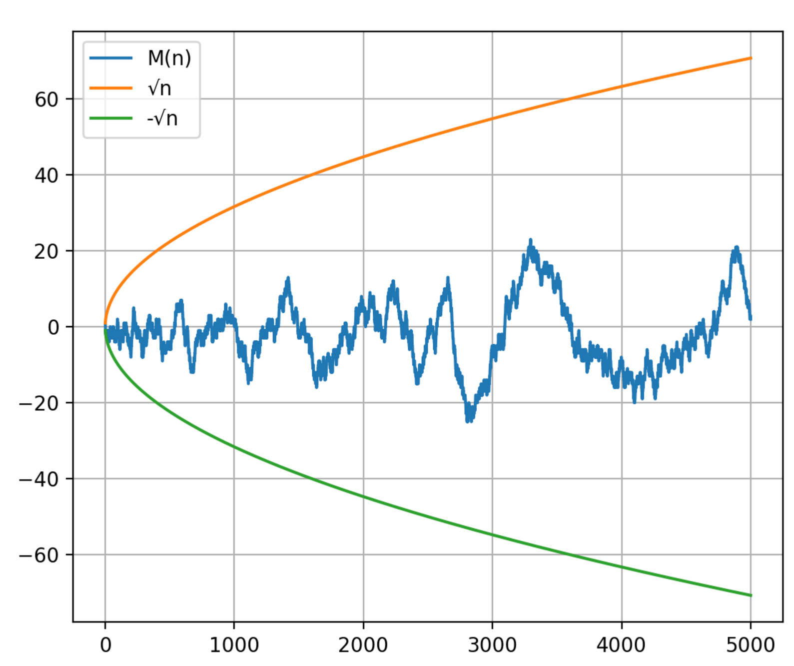

Fig. 2: Graph of Former Version of the Merten Conjecture (Disproven)

Connection to the Riemann Hypothesis

The Riemann Zeta Function $\zeta(s)$ is a function introduced by Leonhard Euler that can be expressed as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots \tag{5}$$

The Riemann Hypothesis is a conjecture stating that Riemann Zeta Function equals zero at very specific locations of $s = \sigma + it$. There exist (trivial) zeros that lie on negative even integers. However, we are more interested in the idea that every non-trivial zeros lie on the vertical line $\sigma = 1/2$. We can actually show that the modified Mertens conjecture (4) implies the Riemann Hypothesis with the following proof:
Recall the Mertens function $M(x)$ defined in (3). Then,

$$\int_1^{\infty} M(x)x^{-s-1}\,dx = \frac{1}{s\zeta(s)}, \quad \sigma > 1.$$

So, if the modified Mertens conjecture (4) is true, then then, by the p-test, the function $\frac{1}{\zeta(s)}$ converges for $\sigma > \frac{1}{2} + \epsilon$. Since $\frac{1}{\zeta(s)} < \infty$, we get that $\zeta(s) \neq 0$ for $\sigma > 1/2$, which implies the Riemann Hypothesis. □

The Prime Number Theorem

Let $\pi(x)$ be the exact value for the number of primes less than or equal to any given positive real number x. So then the Prime Number Theorem is a formula that gives an approximate value for $\pi(x)$ [4].

German mathematician Carl Friedrich Gauss informally conjectured that $\pi(x)$ is approximately $\frac{x}{ln(x)}$, more precisely, the Prime Number Theorem states:

$$\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{ln(x)}} = 1. \tag{6}$$

Though much about prime numbers is largely unknown, we do know that when we look at increasingly large numbers we will always find more primes. However, as we go further in size we will find progressively fewer amounts of primes. This translates to a curve that never flattens.

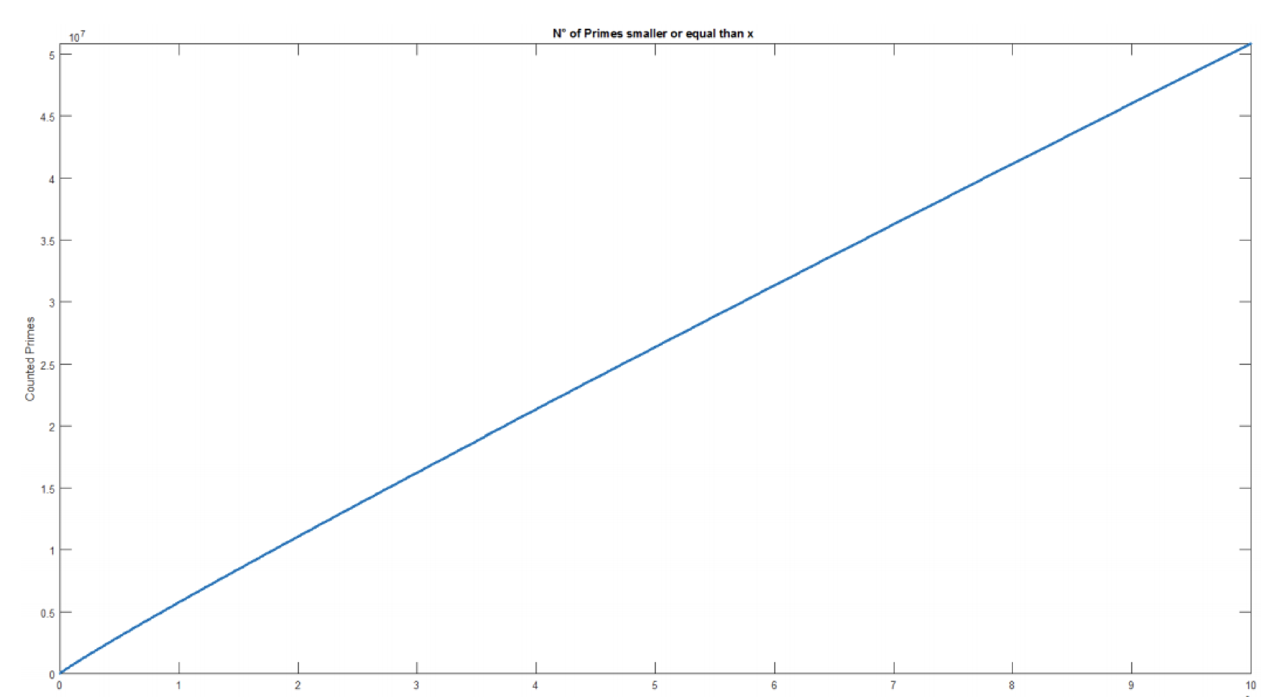

Fig. 3: Graph of Increasing Counted Primes: π for $0 \leq$x$\leq 10^9$

Graph taken from: https://math.uni.lu/eml/projects/reports/prime-distribution.pdf

Moreover, we also notice that in regards to the density of primes, we can see a similar relationship where when we start to search further for more primes, the density, $\frac{1}{ln(x)}$ continues to fall as well. Note that this is similar to the conjecture by Gauss.

The Prime Number Theorem (6), was later proved independently by Jacques Hadamard and Charles Jean de la Vallee Poussin in 1896.

Connection to the Prime Number Theorem

Recall from (5) that the Riemann Hypothesis is a conjecture stating that Riemann Zeta Function equals zero at very specific locations of $s = \sigma + it$. Now if we allow the function $P(x)$ to be:

$$P(x) = \sum_{n \leqslant x} \frac{\mu(n)}{n}. \tag{7}$$

It is known that the Riemann Hypothesis implies the Prime Number Theorem. From this statement, we have the following:

$$\int_1^{\infty} P(x)x^{-s-1}\,dx = \frac{1}{s\zeta(s+1)}, \quad \sigma > 0.$$

Noting that the bottom function, $s\zeta(s+1)$, is analytic when s equals zero, we can conclude that as $P(x) = O(x^{\frac{-1}{2}+\epsilon})$, this implies that $s\zeta(\sigma+1)$ converges for values σ greater than $\frac{-1}{2} + 2\epsilon$.

Now assuming that the hypothesis, $P(x) = O(x^{\frac{-1}{2}+\epsilon})$ is true, we can clearly see that this result implies the Riemann Hypothesis. So, as the Riemann Hypothesis implies the Prime Number Theorem, this ultimately leads us to the conclusion that $P(x) = O(x^{\frac{-1}{2}+\epsilon})$ implies the Prime Number Theorem. □

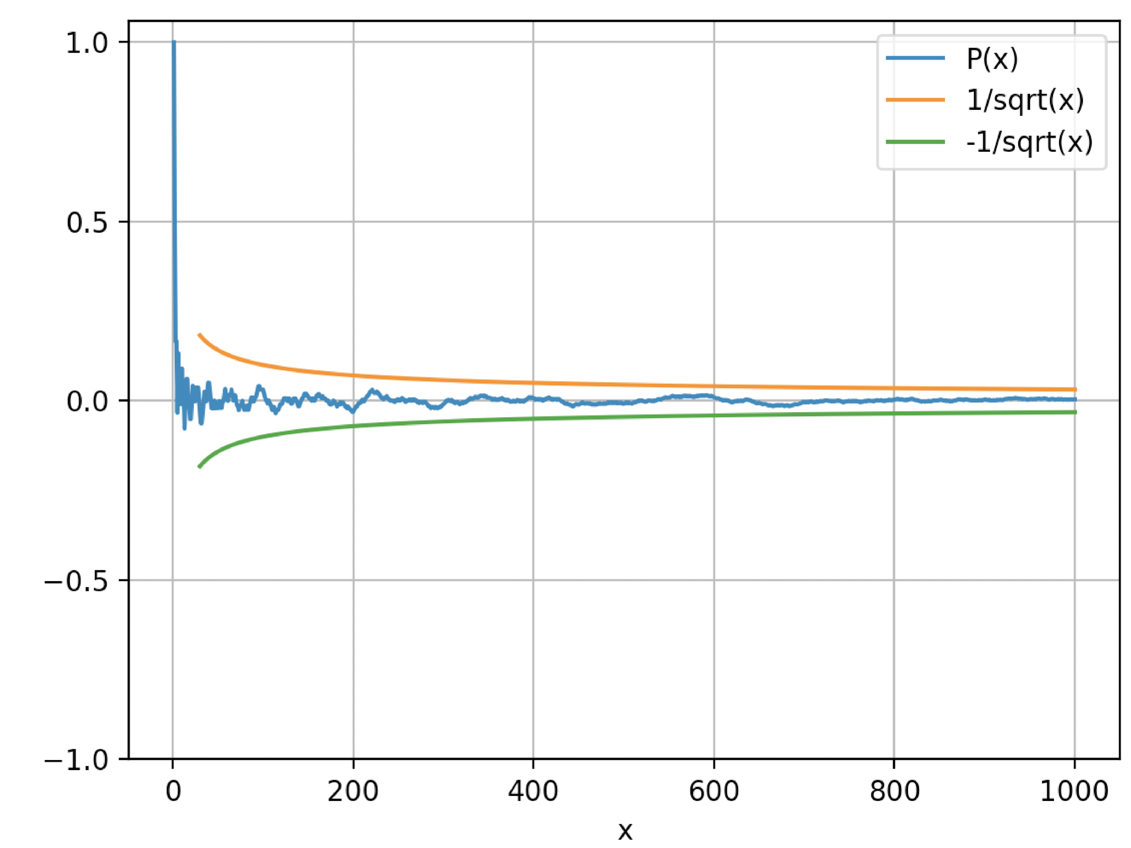

Fig. 4: Graph of $P(x)$ vs x and conjectural bounds for $\pi(x)$

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References

[1] Murty, M. R., Problems in Analytic Number Theory. 2nd ed, Springer, 2008.
[2] Program for Mobius Function:
https://www.geeksforgeeks.org/program-mobius-function/
[3] Tien Huynh, Jenny Do (2021), Program for Fig. 1, 2, 4:
https://github.com/jendo41170/DRP-2021-Mobius-Function
[4] Prime Number Theorem:
https://www.britannica.com/science/prime-number-theorem