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QUASI-FREE NUCLEON-NUCLEON SCATTERING

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Abstract: A simple approximation is described for the calculation of quasi-free † proton-proton scattering using distorted incoming and outgoing waves. The method is applied to Li^7. The results show that in this light nucleus the refraction by the collective nuclear potential does not destroy the strong connection between the angular correlation of the emerging protons and the momentum distribution of the knocked-out proton in the nuclear shell concerned.

1. Introduction

In recent papers [1,2]) (quoted hereafter as I and II) it was theoretically and experimentally shown that quasi-free † nucleon-nucleon scattering can be used as a tool for the investigation of the structure of nuclei. For the planning and the interpretation of future experiments, it seems desirable to perform a more detailed calculation of the cross sections expected in a typical case. In particular, it has not yet become clear how far multiple scattering in the nucleus destroys the direct connection between the angular correlations of the emerging protons and the momentum distribution of the struck proton in the nuclear shell concerned.

The present note describes a simple approximation for a calculation with distorted waves. The method, which is a generalization of an approximation used by Squires [3]) for inelastic scattering, is applied to the case of Li^7.

In the following, the essential contents of I and II are assumed to be known, and the symbols not newly defined have the earlier meaning ††.

2. The Approximation Method

The cross section for quasi-free proton-proton scattering, averaged over the orientations of the bombarded nucleus and summed over the orientations of the residual one, using the impulse approximation with distorted incoming and outgoing waves, is given by [1])

† This is perhaps a clearer expression than the usual name "quasi-elastic" scattering, which recently has also been used for inelastic scattering with a small energy loss.

†† In the right hand side of eq. (1) of I, a factor 2 should be inserted.

$$\frac{d\sigma}{dk_1 dk_2} = \frac{2\pi}{\hbar} \cdot \frac{1}{v_0} \cdot \sum_{f} \overline{\left|\sum_{m_s} g'_{flm_s} \cdot R_{12,03}\right|^2} \cdot \delta(E_1+E_2+E_n+E_b-E_0) \quad (1)$$

For g'_{flm} we take the relativistic generalization of its earlier definition:

$$g'_{flm}(\mathbf{k}) = A^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}}\int \psi_f^*(2,\ldots,A)\exp(-i\mathbf{k}\cdot\mathbf{r}_1)s_m^*(1)t_p^*(1)\psi_i(1,\ldots,A)$$
$$\cdot\exp\left[-i\sum_n \frac{E_n}{k_n\hbar^2c^2}\int V_n ds_n\right] d_1\ldots d_A. \quad (2)$$

Here $\hbar\mathbf{k}_n = \mathbf{p}_n$, iE_n/c^2 is the energy-momentum 4-vector of the proton with index n and the integrals in the exponent have to be taken along the classical path in the direction of $\mathbf{k}_n$ between infinity and the collision point. As in I (2) the horizontal bar in eq. (1) denotes the averaging over the orientations of the initial state. From these formulae the influence of the multiple scattering in the nucleus was in I roughly estimated by taking a constant average absorption and phase change for each shell involved, independent of the region in the nucleus where the collision takes place.

In the present note this approximation is improved by replacing the distorting exponential in eq. (2) by a non-constant analytic function of the coordinates of the collision point, which still allows a closed integration if harmonic oscillator wave functions are taken for the nuclear protons.

The distorting exponential is replaced by the function

$$D(x_1, x_2, x_3) = \exp[i(a+\sum_k b_k x_k+\sum_k\sum_l c_{kl}x_k x_l)][1-\exp(\alpha+\sum_k \beta_k x_k$$
$$+\sum_k\sum_l \gamma_{kl}x_k x_l)], \quad (3)$$

x_1, x_2 and x_3 representing the Cartesian x, y and z coordinates. The twenty real constants a, b_k, $c_{kl}(=c_{lk})$, d, β_k and $\gamma_{kl}(=\gamma_{lk})$ have to be suitably chosen and depend on the distorting potentials and on the momenta of the incoming and outgoing protons.

With this simple choice of a spin-independent distortion and harmonic oscillator wave functions, one may not hope to understand details of the experimental results, such as the deviation of the ratio of the $p_{\frac{3}{2}}$- and $p_{\frac{1}{2}}$-peaks in the oxygen spectrum [2]) from the value 2. But the distorting exponential is a smooth function and, apart from the spin dependence, one should be able to approximate it well inside of the nucleus by a suitably chosen $D(x_i)$. We therefore expect to find correctly the main modifications introduced by the use of distorted waves.

We first simplify and to a certain extent specialize eq. (1). In this expression the term $\sum_{m_s} g_{flm_s} R_{12,03}$ is equal to $(\sum_m |g'_{flm}|^2)^{\frac{1}{2}}$ times the matrix element for the cross section of the scattering process as considered hitherto, but in which the nuclear proton is replaced by a free proton with the spin wave function

$$\frac{\sum_m g'_{\mathrm{fl}m} s_m}{\sqrt{\sum_m |g'_{\mathrm{fl}m}|^2}}$$

and momentum $\hbar(\mathbf{k}_2+\mathbf{k}_1-\mathbf{k}_0)$. Quantizing the spin in the z-direction, the polarization vector of such a proton is given by

$$\mathbf{P}_{\mathrm{fl}} = \frac{\mathrm{Re}(g'^{*}_{\mathrm{fl}+} g'_{\mathrm{fl}-});\quad \mathrm{Im}(g'^{*}_{\mathrm{fl}+} g'_{\mathrm{fl}-});\quad |g'_{\mathrm{fl}+}|^2-|g'_{\mathrm{fl}-}|^2}{\sum_m |g'_{\mathrm{fl}m}|^2}. \tag{4}$$

Likewise,

$$\frac{\sum_{\mathrm{f}} \overline{|\sum_{m_3} g'_{\mathrm{fl}m_3} R_{12,03}|^2}}{\sum_{\mathrm{f}}\sum_m \overline{|g'_{\mathrm{fl}m}|^2}}$$

is reducible to the cross section for the scattering process in which the above-mentioned replacement is made for all orientations of the initial and final nuclear states. The "beam" replacing the nuclear proton is then defined by the polarization vector

$$\mathbf{P} = \frac{\sum_{\mathrm{f}}\sum_m \overline{|g'_{\mathrm{fl}m}|^2 \mathbf{P}_{\mathrm{fl}}}}{\sum_{\mathrm{f}}\sum_m \overline{|g'_{\mathrm{fl}m}|^2}}$$

$$= \frac{\mathrm{Re}\sum_{\mathrm{f}} \overline{(g'^{*}_{\mathrm{fl}+} g_{\mathrm{fl}-})};\quad \mathrm{Im}\sum_{\mathrm{f}} \overline{(g'^{*}_{\mathrm{fl}+} g'_{\mathrm{fl}-})};\quad \sum_{\mathrm{f}} \overline{|g'_{\mathrm{fl}+}|^2} - \sum_{\mathrm{f}} \overline{|g'_{\mathrm{fl}-}|^2}}{\sum_{\mathrm{f}}\sum_m \overline{|g'_{\mathrm{fl}m}|^2}} \tag{5}$$

and the momentum $\hbar(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_0)$. In case there exists a plane of mirror symmetry, $\mathbf{P}$ is perpendicular to it.

From now on, the special case is considered in which the momenta of the emerging protons are chosen to be of equal magnitude and directed in one plane with and at equal angles to the momentum of the bombarding particle. This case is not only experimentally interesting, but also the calculations simplify considerably; and, independently of their details, some general conclusions can be drawn. The origin of the Cartesian coordinate system is taken in the centre of mass of the bombarded nucleus, the z-axis in the direction of the incoming protons and the x-axis in the scattering plane. Assuming unpolarized incoming protons and target nuclei and parity conservation, there are two planes of mirror symmetry, and consequently $\mathbf{P}$ must vanish.

In the appendix the following result of the elimination of $R_{12,03}$ from eq. (1) for this case is derived:

$$\frac{d\sigma}{d\Omega_1 dE_1 d\Omega_2 dE_2} = \frac{4k_1^2}{\hbar v_0} \frac{p_1^2c^2 \sin^2\theta + m^2c^4}{2p_1^2c^2\sin^2\theta + m^2c^4} \frac{d\sigma_{fr}}{d\Omega'} \sum_f \sum_m \overline{|g'_{flm}|^2} \qquad (6)$$
$$\delta(E_1+E_2+E_n+E_b-E_0).$$

$d\sigma_{fr}/d\Omega'$ is the differential cross section for free protons with energies equal to $\sqrt{[p_1^2c^2\sin^2\theta+m^2c^4]}$ and scattering angles of 90°; all these quantities are taken in the centre of mass system.

In eq. (6) the dependence of the angular correlations on the properties of the target nucleus is contained only in $\sum_f\sum_m \overline{|g'_{flm}|^2}$, which is calculable with the help of approximation (3). Because of the symmetries of the problem, only $a, b_3, c_{11}, c_{22}, c_{33}, \alpha, \beta_3, \gamma_{11}, \gamma_{22}$, and γ_{33} in $D(x_i)$ have non-vanishing values.

3. The Example of Li^7

The results of this symmetric case are now applied to Li^7 as the target nucleus. It is assumed that its p-proton is *j-j*-coupled, and that a clean knocking-out of this proton leads to one $j = 0$ state of He^6. In reality, these assumptions are probably far from fulfilled and He^6 will be left in various excited states. Experimentally, the contributions of these states, all caused by p-shell collisions, are difficult to resolve and we expect that the sum of their cross sections should be given quite well by the corresponding one of our pure *j-j*-coupling model.

The non-vanishing parameters in $D(x_i)$ have been determined by fitting $D(x_i)$ to the distorting exponential for square well potentials. The values are taken equal at the origin, at the intersections of the z- and x-axis with the surface of the potentials and in the points of the y-axis with y-coordinates equal to the root mean square of the y-coordinates of the nucleons in the nucleus.

Of course, this determination of the parameters in $D(x_i)$ is rather arbitrary. If, for example, D is adjusted to the distorting exponential caused by Gaussian potentials, $1-|D|$ becomes larger at the origin and its width smaller. For the moment we shall not investigate the sensitivity of the final result of the calculation to the choice of the adjustment procedure and which choice is the most reasonable. Therefore, only the general trends of the results obtained are expected to be reliable.

The value of the factor in front of the δ-function in eq. (6) is plotted in figures 1a and 1b for collisions in the s- and the p-shells in Li^7 with bombarding energies of 180 and 400 MeV. The mean free paths in nuclear matter have been chosen as 2, 4, and 4×10^{-13} cm, and the real parts of the distorting square well potentials as 20, 15 and 5 MeV at 100, 200 and 400 MeV respectively. The classical frequency of the harmonic oscillator potential generating the nuclear wave functions is taken to be

$$\hbar\omega = 21.3 \text{ MeV},$$

resulting in a root mean square nuclear radius of $1.3 \times 7^{\frac{1}{3}} \times 10^{-13}$ cm, which is equal to that of the homogeneous sphere chosen as the boundary of the square well potentials. No momentum dependence of the oscillator potential has been assumed in this example. For $d\sigma_{fr}/d\Omega'$ at 90°, we have taken 4mb · ster^{-1} at all energies.

The dashed curves in the figures have been calculated with the constant reduction factor of I. The qualitative features of these curves are easy to understand. At 45° scattering angles the knocked-out protons are nearly in

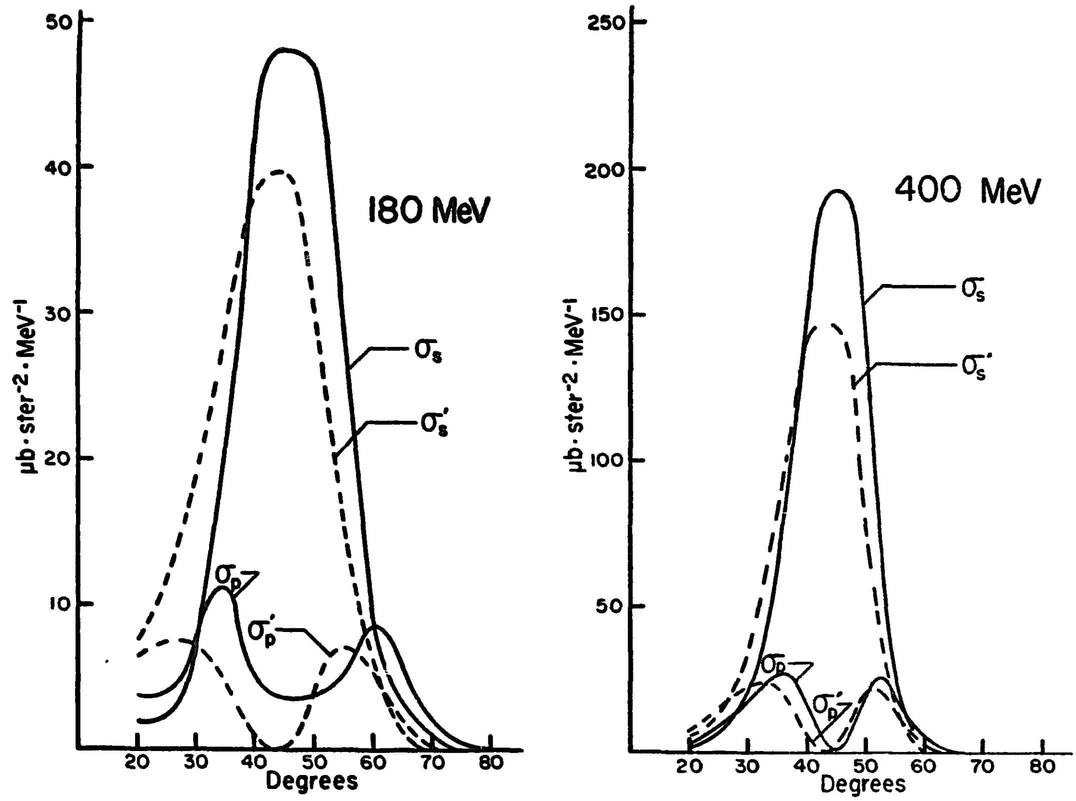

Fig. 1. The cross sections for symmetric coplanar quasi-free proton-proton scattering on Li^7 for 180 and 400 MeV bombarding energy, calculated with the present approximation method (full curves) and with a constant reduction factor (dashed curves).

rest in the nucleus. Here, the s-shell contribution has of course a maximum, and the one of the p-shell vanishes, its wave function in momentum space being antisymmetrical. The p-curve has maxima at angles roughly corresponding to the occurrence in the original nucleus of the most probable proton momenta opposite to, and in, the beam direction.

The full curves are based on the approximation described in the present paper. They show the same general behaviour as the dashed curves and the relation to the momentum distributions in the shells is still clear in a qualitative sense.

As remarked earlier the result of our example should not be taken too literally. It does seem likely that the approximations involved in the basic formulae (1) or (6), in the general form of (3), and even in the assumption of pure j-j coupling in Li^7 are sufficiently good for our present purpose of a rough quantitative description of the angular correlations. However, the determination of the parameters in $D(x_i)$ can certainly be improved considerably. Furthermore, the harmonic oscillator nuclear wave functions have too short tails at the nuclear surface, especially for the least bound particles. This will be of importance, because the strong nuclear absorption emphasizes the contributions of the collisions taking place near the nuclear surface. Also the amount of low momentum components must have been underestimated by taking harmonic oscillator wave functions. We hope to investigate these and related questions at a later time. It is to be expected that the p-shell results will be the most affected.

It is interesting to note that for the coplanar symmetric case only the p-shell particles contribute to $\sum_t \sum_m \overline{|g'_{t1m}|^2}$, whose wave functions contain the space wave function with orbital angular momentum component zero in the beam direction. The contributions of the other components vanish, because of the reflection symmetries of the situation. Consider now the absorptions along the classical paths of the bombarding and scattered protons for various collision points in the nucleus. Clearly, the collisions taking place on the largest distance from the z-axis, and especially those occurring farthest away from the scattering plane, result in the smallest total absorption. Because of the centrifugal force, the p-wave functions with angular momentum ± 1 along the beam direction have larger densities in these regions of the nucleus than the p-wave function with zero angular momentum z-component. This last function is in this respect quite equivalent to the s-wave function. Therefore, using oscillator functions, the intensity-increasing influence of the surface effect can only become strong for non-coplanar or asymmetric scattering, to which the other p-shell functions also contribute.

It is difficult to compare the experimental results of 180 MeV quasi-free proton-proton scattering on Li^7 as reported in II with the present calculations, because the experimental set-up allowed a large amount of non-coplanar correlation. The calculations indicate that measurements with improved angular resolution may be worthwhile, because the angular dependence of the correlation cross section seems to be markedly dependent on the momentum distributions in the shells concerned.

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Appendix

The reduction of $|\sum_{m_3} g'_{flm_3} R_{12,03}|^2$ to a free proton-proton cross section is accomplished by the following formula:

$$\frac{d\sigma^{\mathbf{P}}_{fr}}{d\mathbf{k}_1 d\mathbf{k}_2} = \frac{2\pi}{\hbar}\frac{1}{v_{03}} \frac{\overline{\sum_f |\sum_{m_3} g'_{flm_3} R_{12,03}|^2}}{\sum_f \sum_{m_3} |g'_{flm_3}|^2} \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_0-\mathbf{k}_3) \cdot \delta(E_1+E_2-E_0-E_2); \tag{a}$$

$d\sigma^{\mathbf{P}}_{fr}/d\mathbf{k}_1 d\mathbf{k}_2$ is the cross section for free proton-proton scattering corresponding to the nuclear case and is therefore taken in a coordinate system in which the momenta of the colliding protons with relative velocity v_{03} are $\hbar\mathbf{k}_0$ and $\hbar\mathbf{k}_3$ and the polarization vector of the proton with index 3 is equal to $\mathbf{P}$. As in I, a sufficiently smooth dependence of $R_{12,03}$ on the momenta has been assumed, because the values for the momenta in the free scattering are somewhat different from those in the nuclear problem, as the energy conservation is not the same for both cases. This assumption may be only roughly correct for scattering angles very different from 45 degrees, in which case the nuclear proton has a high momentum and the two energy conservations become rather different.

$d\sigma^{\mathbf{P}}_{fr}/d\mathbf{k}_1 d\mathbf{k}_2$ is not a directly experimentally known quantity, and therefore it will be expressed in terms of the centre of mass cross section $d\sigma^{\mathbf{P}}_{fr}/d\Omega'$. In the following, all primed quantities refer to the centre of mass system, which has the velocity v in the system used hitherto. We shall only consider the plane symmetric case, for which the index $\mathbf{P}$ may be omitted.

The Lorentz-transformation between the systems results in the following relations:

$$\frac{d\sigma_{fr}}{d\mathbf{k}_1 d\mathbf{k}_2} = \frac{dk'_{1z}}{dk_{1z}}\frac{dk'_{2z}}{dk_{2z}}\frac{d\sigma_{fr}}{d\mathbf{k}'_1 d\mathbf{k}'_2} = \left(1-\frac{v^2}{c^2}\right)^{-1}\left(1-\frac{vp_1\cos\theta}{E_1}\right)^2 \frac{d\sigma_{fr}}{d\mathbf{k}'_1 d\mathbf{k}'}, \tag{b}$$

θ being the angle between the directions of an outgoing proton and the bombarding proton.

The following formula, connecting $d\sigma_{fr}/d\mathbf{k}'_1 d\mathbf{k}'_2$ with the cross section $d\sigma_{fr}/d\Omega'$, can be verified by integrating both sides over $\mathbf{k}'_2$ and $|\mathbf{k}'_1|$:

$$\frac{d\sigma_{fr}}{d\mathbf{k}'_1 d\mathbf{k}'_2} = \frac{c^2\left(\frac{\mathbf{p}'^2_1}{E'_1}-\frac{\mathbf{p}'_1\cdot\mathbf{p}'_2}{E'_2}\right)}{|\mathbf{k}'_1|^3}\frac{d\sigma_{fr}}{d\Omega'}\delta(\mathbf{k}'_1+\mathbf{k}'_2-\mathbf{k}'_0-\mathbf{k}'_3)\,\delta(E'_1+E'_2-E'_0-E'_3).$$

Because $\mathbf{p}'_2 = -\mathbf{p}'_1$ one may write

$$\frac{d\sigma_{fr}}{d\mathbf{k}'_1 d\mathbf{k}'_2} = \frac{2c^2\hbar^3}{p'_1 E'_1}\frac{d\sigma_{fr}}{d\Omega'}\delta(\mathbf{k}'_1+\mathbf{k}'_2-\mathbf{k}'_0-\mathbf{k}'_3)\delta(E'_1+E'_2-E'_0-E'_3)$$
$$= \frac{2c^2\hbar^3}{p'_1 E'_1}\frac{d\sigma_{fr}}{d\Omega'}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_0-\mathbf{k}_3)\delta(E_1+E_2-E_0-E_3). \tag{c}$$

Inserting (c) in (b),

$$\frac{d\sigma_{fr}}{d\mathbf{k}_1 d\mathbf{k}_2} = \left(1-\frac{v^2}{c^2}\right)^{-1}\left(1-\frac{vp_1\cos\theta}{E_1}\right)^2 \frac{2c^2\hbar^3}{p'_1E'_1}\frac{d\sigma_{fr}}{d\Omega'} \cdot \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_0-\mathbf{k}_3)\delta(E_1+E_2-E_0-E_3). \tag{d}$$

From the Lorentz transformation to the centre of mass system, in which the scattered protons, because of the symmetry, move orthogonally to their original directions, it follows that

$$v = \frac{c^2p_1\cos\theta}{E'_1}, \qquad p'_1 = p_1\sin\theta, \qquad E'_1 = \sqrt{p_1{}^2c^2\sin^2\theta+m^2c^4}.$$

Equation (d) now becomes

$$\frac{d\sigma_{fr}}{d\mathbf{k}_1 d\mathbf{k}_2} = \left(1-\frac{c^2p_1{}^2\cos^2\theta}{E_1{}^2}\right)\frac{2c^2\hbar^3}{p_1\sin\theta\sqrt{p_1{}^2c^2\sin^2\theta+m^2c^4}}\frac{d\sigma_{fr}}{d\Omega'} \cdot \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_0-\mathbf{k}_3)\delta(E_1+E_2-E_0-E_3). \tag{e}$$

From (e) and (a) $\overline{\sum_f|\sum_{m_3} g'_{flm_3}R_{12,03}|^2}$ may be calculated and inserted in eq. (1) of the text, giving the result

$$\frac{d\sigma}{d\mathbf{k}_1 d\mathbf{k}_2} = \frac{v_{03}}{v_0}\left(1-\frac{c^2p_1{}^2\cos^2\theta}{E_1{}^2}\right)\frac{2c^2\hbar^3}{p_1\sin\theta\sqrt{p_1{}^2c^2\sin^2\theta+m^2c^4}} \cdot \frac{d\sigma_{fr}}{d\Omega'}\sum_f\sum_m\overline{|g'_{flm}|^2}\delta(E_1+E_2+E_n+E_b-E_3).$$

With

$$v_0 = c\sqrt{1-\frac{m^2c^4}{E_0{}^2}}, \quad v_{03} = \sqrt{1-\left(\frac{m^2c^4}{2p_1{}^2c^2\sin^2\theta+m^2c^4}\right)^2}$$

and using the relation

$$\frac{d\sigma}{d\Omega_1 dE_1 d\Omega_2 dE_2} = \frac{k_1k_2E_1E_2}{(\hbar c)^4}\frac{d\sigma}{d\mathbf{k}_1 d\mathbf{k}_2} = \frac{k_1{}^2E_1{}^2}{(\hbar c)^4}\frac{d\sigma}{d\mathbf{k}_1 d\mathbf{k}_2}$$

the expression (6) of the text is obtained.

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