

# Relativistic Description of Two-body Scattering Reactions

Gashaw Adera (gashaw@aims.ac.za)  
African Institute for Mathematical Sciences (AIMS)

Supervised by Dr. Brandon Van Der Ventel  
University of Stellenbosch

June 4, 2007

# Abstract

In this paper, two-body elastic scattering is treated in the relativistic quantum mechanics framework. By using lowest-order Feynman diagrams, detailed derivations of the invariant matrix element and hence the differential cross-section for unpolarized electron-proton scattering are made. First, the proton is approximated as a spin- $\frac{1}{2}$  point particle which allows a consistent quantum electrodynamical description of the scattering process. The Feynman rules and trace algebra have been employed in constructing the relativistic quantum mechanical expression of invariant amplitude. Moreover, by using a suitable basis for the second rank tensor, the hadronic tensor for point proton is generalised to include the electromagnetic form factors which lead us to treat the proton as an extended object within the finite volume. The calculated differential cross section for a point proton is compared to the Rutherford and Mott predictions at laboratory angles between  $0^\circ$  and  $180^\circ$  and initial electron energy between 1 MeV and 1 GeV by using numerical simulations. The results are plotted against scattering angle in the laboratory frame. In the simulation it is shown that the calculated differential cross-section agrees with the Mott prediction. It also shows the expected deviation from the Rutherford prediction.

# Contents

<b>Abstract</b>	<b>i</b>
<b>List of Figures</b>	<b>iv</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Kinematics of Scattering Processes</b>	<b>2</b>
2.1 Kinematics of Non-relativistic Scattering . . . . .	2
2.1.1 Galilean Transformation . . . . .	2
2.2 The Kinetic Energies in Laboratory and Center-of-mass Frames . . . . .	4
2.2.1 Scattering Cross-section in Laboratory and Center-of-mass Frames . . . .	6
2.3 Kinematics of a Relativistic Particle . . . . .	7
2.3.1 Energy-momentum Four-vector . . . . .	7
2.3.2 Lorentz Transformation (Boost) . . . . .	8
2.3.3 Mandelstam Variables . . . . .	10
<b>3 Relativistic Dirac Equation</b>	<b>12</b>
3.1 The Klein-Gordon ( $\mathcal{KG}$ ) Equation . . . . .	12
3.2 Dirac Free Particle Equation . . . . .	13
3.3 Free Particle Solution of Dirac Equation . . . . .	14
3.3.1 Normalisation of Dirac Spinor . . . . .	15
<b>4 Quantum Electrodynamics of Electron-proton Elastic Scattering</b>	<b>16</b>
4.1 Feynman Diagrams . . . . .	16
4.1.1 Components of Feynman Diagram . . . . .	16
4.2 Feynman Rules for Tree Diagram . . . . .	17
4.3 Fermi's Golden Rule . . . . .	18
4.3.1 Transition Rate . . . . .	18
4.3.2 Cross-Section for Two-body Scattering . . . . .	19
4.4 Elastic Scattering of Electron from a Point Proton . . . . .	19

4.4.1	Evaluation of Invariant Amplitude of Unpolarised Electron-proton Scattering	20
4.5	Proton Form Factors and Electron-proton Elastic Scattering . . . . .	24
<b>5</b>	<b>Simulation, Discussion and Conclusion</b>	<b>28</b>
5.1	Simulation and Discussion . . . . .	28
5.2	Conclusion . . . . .	30
<b>A</b>	<b>Background Mathematics</b>	<b>31</b>
A.1	The Dirac Delta Function . . . . .	31
A.2	Relativistic Notation and Four-vector Formalism . . . . .	32
A.3	Dirac Algebra of Gamma Matrix and Trace Theorems . . . . .	33
A.3.1	Pauli Matrices . . . . .	33
A.3.2	Dirac Matrices . . . . .	33
A.3.3	Trace Theorems . . . . .	34
<b>B</b>	<b>The Python Code</b>	<b>36</b>
	<b>Bibliography</b>	<b>40</b>

# List of Figures

2.1	The kinematics of two-body elastic collision $A + B \longrightarrow C + D$ process in (b) as viewed from the lab system; (b) as viewed from the C.M. frame. . . . .	2
2.2	Illustration of the geometrical representation $\mathbf{v}_L$ and $\mathbf{v}$ with directions $(\theta_L, \phi_L)$ and $(\theta, \phi)$ . . . . .	3
2.3	A two-body elastic collision $A + B \rightarrow C + D$ (a) in the lab frame (b) in the centre-of-mass frame . . . . .	5
2.4	Illustration of scattering from a target into a solid angle element $d\Omega_L = \sin \theta_L d\theta_L d\phi_L$	6
4.1	(a) Wavy line representing the outgoing photons and (b) straight lines representing the incoming fermion. If we change the position of the dots to the other ends, we get lines that represent (a) the incoming photon and (b) the outgoing fermion. We use dots to represent lepton-photon vertices . . . . .	16
4.2	Primitive diagram that illustrates the emission of photon from electron . . . . .	17
4.3	The lowest-order tree diagram . . . . .	17
4.4	The lowest-order Feynman diagram for electron-proton elastic scattering. Each factor of the invariant amplitude are associated to the corresponding element of the Feynman diagram. . . . .	20
5.1	Comparison of the calculated cross-section with (a) Rutherford differential cross-section and (b) Mott differential cross-section for 10 MeV, 50 MeV, and 90 MeV incident energies of electron. . . . .	28
5.2	Comparison of the calculated cross-section with (a) Rutherford differential cross-section and (b) Mott differential cross-section for 100 MeV, 150 MeV, 350 MeV and 700 MeV incident energies of electron. . . . .	29
5.3	(a) Angular distribution of differential cross-section versus energy of the initial electron. (b) Simulation for energy the scattered electron versus the scattering angle at incident energies 100 MeV, 150 MeV, 200 MeV, and 250 MeV . . . . .	29

# 1. Introduction

The study of elementary particles and their interaction has brought about a better understanding of our universe from sub-nuclear scales to cosmological scales. Scattering experiments have made possible the discovery of fundamental constituents of matter and the kind of interactions they exhibit. In other words, scattering experiments form the basis of the study of the structure of particles. There are four fundamental interactions observed in nature: gravitational, electromagnetic, weak and strong nuclear forces. The electromagnetic interaction is mediated by the exchange of virtual photons. The weak nuclear interaction is responsible for beta decay and the exchanged particles are  $W^\pm$  and  $Z^0$ . For the strong nuclear interaction between quarks, the exchanged particles are gluons.

The theories that describe electromagnetic, weak nuclear and strong nuclear interactions are Quantum Electrodynamics (QED), the theory of Glashow, Weinberg and Salam (GWS theory), and Quantum Chromodynamics (QCD), respectively. It can be seen that fundamental particles have been classified based on their response to these interactions as well as their structure. The two main groups are hadrons and leptons. Hadrons experience all four the fundamental forces and they are composite particles. Leptons do not experience the strong force, and they are elementary particles, for example the electrons. The particles which mediate the different forces are generally called gauge bosons [Ryd96].

Scattering is a process in which two or more particles from a distant past momentarily come together and interact by exchanging energy and momentum via the field quanta and then separate from one another to a distant future. However, this essay is aimed at describing the scattering process of two fermions, namely, the electron and the proton, which interact electromagnetically by exchanging a single virtual photon, which has a non-zero  $q^2$  [Ron94]. The particles scattered off the target emerge with a new state which contains important information regarding the target particle. We in turn use the information to investigate the structure and properties of the target particle from which the incident particle is scattered. The quantity that is used to characterise the scattering process is called the scattering cross-section, which is the measure of the tendency for the incoming particle to scatter in certain direction after the interaction.

The main focus of this study is the derivation of the invariant amplitude and the differential cross-section of unpolarized electron-proton elastic scattering by using Feynman diagrams. A study of this kind is of paramount importance in comparing the contemporary theories of Nuclear and Particle Physics with scattering experiments. In the second chapter non-relativistic and relativistic kinematics of elastic scattering process are discussed by using the collinear reference frames. Since e-p scattering is due to electromagnetic interaction, our study is made using the framework of Quantum Electrodynamics which describes elementary particles as the quantum states of electromagnetic field. The Klein-Gordon and Dirac equations for relativistic particles are also briefly presented in Chapter 3. In the fourth chapter more attention is given to the derivations of the scattering cross-section and the invariant amplitude. Firstly we consider the proton to be a structureless fermion. As the next step we take into account the proton structure by means of the form factor description. Simulations and interpretation of our results as well as inference of the study are contained in the fifth chapter.

## 2. Kinematics of Scattering Processes

### 2.1 Kinematics of Non-relativistic Scattering

The kinematics of a system of particles are derived based on energy and momentum conservation laws [BJ95]. The derivation of the differential cross-section requires kinematic descriptions. Thus we set up the kinematic framework by first defining the frames of reference that are essential in the study of scattering in general, in particular elastic scattering in which the total energy and momentum are conserved.

To begin with, we consider the elastic collision of two particles  $A$  and  $B$  having masses  $m_A$  and  $m_B$ . The laboratory (lab) frame of reference is the coordinate system in which the target particle  $B$  is initially at rest. The centre-of-mass (C.M.) frame of reference is a system in which the centre of mass is always at rest before and after collision [Joa84]. In the non-relativistic regime, we relate energies, scattering angles and cross-sections measured in lab frame to their C.M. counterparts using a Galilean transformation,  $\mathcal{GT}$ .

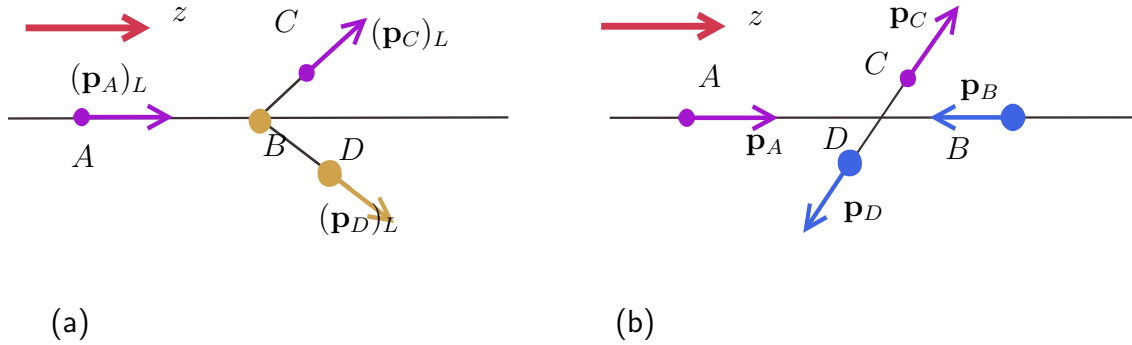


Figure 2.1: The kinematics of two-body elastic collision  $A + B \longrightarrow C + D$  process in (a) as viewed from the lab system; (b) as viewed from the C.M. frame.

#### 2.1.1 Galilean Transformation

For a system of two non-relativistic particles  $A$  and  $B$ , the space-time four-vector coordinates with respect to the lab and the C.M. frames can be given as  $(t_L, x_L, y_L, z_L)$  and  $(t, x, y, z)$ , respectively. For convenience, we choose the  $z$ -axis as the incident direction of particle  $A$  (see figure 2.1). A centre of mass of the two-body system always retains its uniform motion relative to any inertial frame. If we assume that  $\mathbf{V}_L = V_L \hat{z}$  be the velocity of the centre of mass with respect to the lab frame, then the explicit equations of the Galilean transformation from the lab frame to the C.M. frame can be written as

$$t = t_L; \quad x = x_L; \quad y = y_L; \quad z = z_L - V_L t_L \quad (2.1)$$

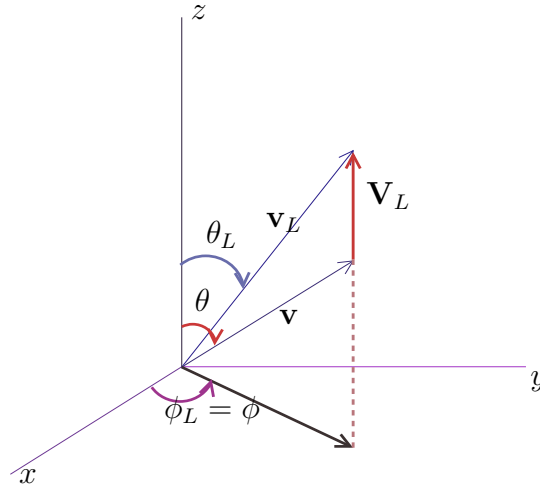


Figure 2.2: Illustration of the geometrical representation  $\mathbf{v}_L$  and  $\mathbf{v}$  with directions  $(\theta_L, \phi_L)$  and  $(\theta, \phi)$

Similarly, we can go from the C.M. frame to the lab frame by only changing  $-V_L$  to  $+V_L$

$$t_L = t; \quad x_L = x; \quad y_L = y; \quad z_L = z + V_L t \quad (2.2)$$

Here, we can also relate the velocity in the lab frame to its counterpart in the C.M. frame. If we denote the velocities of the incident particle in the collision process by  $\mathbf{v}_L$  and  $\mathbf{v}$  with the corresponding directions specified by  $(\theta_L, \phi_L)$  and  $(\theta, \phi)$  in the lab and C.M. frames, respectively, they are related as

$$\mathbf{v}_L = \mathbf{v} + \mathbf{V}_L \quad (2.3)$$

So by simply referring to figure 2.2 and applying vectorial analysis, one can get the relation between the lab frame angle  $\theta_L$  and the C.M. frame angle  $\theta$ . That is,

$$\tan \theta_L = \frac{\sin \theta}{(\cos \theta + \tau)} \quad (2.4)$$

or

$$\cos \theta_L = \frac{\sin \theta}{(\sin^2 \theta + (\cos \theta + \tau)^2)^{\frac{1}{2}}} = \frac{\sin \theta}{(1 + 2\tau \cos \theta + \tau^2)^{\frac{1}{2}}} \quad (2.5)$$

where

$$\tau = \frac{V_L}{|\mathbf{v}|} \quad (2.6)$$



## 2.2 The Kinetic Energies in Laboratory and Center-of-mass Frames

The total momentum of the system of, say, two particles  $A$  and  $B$  in the lab frame is equal to the velocity  $\mathbf{V}_L$  of the centre of mass of the system times the total mass  $M = m_A + m_B$ , that is

$$\mathbf{P}_L = M\mathbf{V}_L \quad (2.7)$$

The initial kinetic energy in lab system is given by

$$T_{iL} = \frac{(\mathbf{p}_A)_L^2}{2m_A} = \frac{\mathbf{P}_L^2}{2m_A} \quad (2.8)$$

where  $(\mathbf{p}_A)_L$  is the momentum of the incident particle  $A$  in the lab frame.

We can establish the relation between the kinetic energies in the lab and the C.M. systems by using the relative momentum, which is invariant under the Galilean transformation, and the reduced mass of the system of colliding particles [Joa84].

If we consider a system of two particles 1 and 2, the relative momentum  $\mathbf{p}$  in any inertial frame can be defined as

$$\mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2} \quad (2.9)$$

and the reduced mass  $\mu$  is defined as

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad (2.10)$$

where  $m_1$  and  $m_2$  are the respective masses and  $\mathbf{p}_1$  and  $\mathbf{p}_2$  the corresponding momenta of the two particles. Thus in the C.M. frame  $\mathbf{p}_i$  is given by

$$\mathbf{p}_i = \mathbf{p}_A = -\mathbf{p}_B \quad (2.11)$$

Then the initial kinetic energy  $T_i$  in the C.M. frame is given by

$$T_i = \frac{\mathbf{p}_A^2}{2m_A} + \frac{\mathbf{p}_B^2}{2m_B} = \frac{\mathbf{p}_i^2}{2\mu_i} \quad (2.12)$$

where  $\mathbf{p}_A$  and  $\mathbf{p}_B$ , respectively, are the three-momenta of particles  $A$  and  $B$ , which are situated in the initial channel, with  $\mu_i$  as their reduced mass. The relative momentum  $\mathbf{p}_i$  is

$$\mathbf{p}_i = \frac{m_B}{m_A + m_B}(\mathbf{p}_A)_L \quad (2.13)$$

Therefore, by plugging Eq.(2.13) into Eq.(2.12) and recalling Eq.(2.8), the relation between kinetic energy  $T_i$  in the C.M. system and its counterpart  $(T_i)_L$  in the lab system of the initial channel for the collision process of particles  $A$  and  $B$  can be set up as

$$T_i = \frac{m_B}{m_A + m_B}(T_i)_L \quad (2.14)$$

We can get the same expression for  $T_i$  by taking out the kinetic energy of the centre of mass from the total kinetic energy available in lab system for its motion is irrelevant in the description of scattering process [BJ95]. That is,

$$T_i = (T_i)_L - \frac{\mathbf{P}_L^2}{2M} \quad (2.15)$$

where  $M$  is the total mass  $m_A + m_B$  of the two-body system.

Based on figure 2.3 which is the illustration of the binary rearrangement collision,  $A+B \rightarrow C+D$ , we can also set up the relation between the final kinetic energies  $(T_f)_L$  and  $T_f$  in the two frames for particles  $C$  and  $D$  of the final channel. That is

$$(T_f)_L = \frac{(\mathbf{p}_C)_L^2}{2m_C} + \frac{(\mathbf{p}_D)_L^2}{2m_D} \quad (2.16)$$

and the total momentum in the lab system is

$$\mathbf{P}_L = (\mathbf{p}_C)_L + (\mathbf{p}_D)_L \quad (2.17)$$

where  $(\mathbf{p}_C)_L$  and  $(\mathbf{p}_D)_L$ , respectively, are momenta of particle  $C$  and  $D$  in the lab frame.

Here again, the  $\mathcal{GT}$  invariant relative momentum  $\mathbf{p}_f$  of the final channel is defined in the same way as Eq.(2.11), that is

$$\mathbf{p}_f = \mathbf{p}_C = -\mathbf{p}_D \quad (2.18)$$

and

$$\mathbf{p}_f = \frac{m_D(\mathbf{p}_C)_L - m_C(\mathbf{p}_D)_L}{m_C + m_D} \quad (2.19)$$

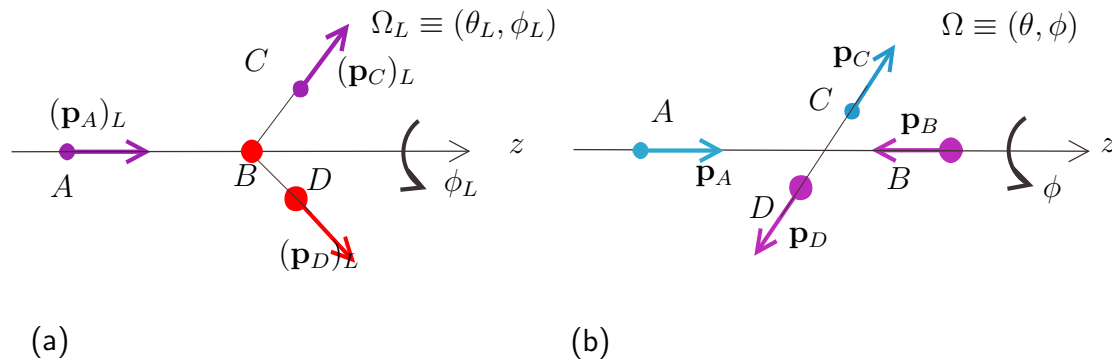


Figure 2.3: A two-body elastic collision  $A + B \rightarrow C + D$  (a) in the lab frame (b) in the centre-of-mass frame

Thus the final kinetic energy in the C.M. system is

$$T_f = \frac{\mathbf{p}_C^2}{2m_C} + \frac{\mathbf{p}_D^2}{2m_D} = \frac{\mathbf{p}_f^2}{2\mu_f} \quad (2.20)$$

where  $\mu_f$  the reduced mass of particles  $C$  and  $D$ .

By applying the same definition as that of Eq.(2.15), we can relate the final channel kinetic energies available in the lab and the C.M. systems as

$$T_f = (T_f)_L - \frac{(\mathbf{P}_L^2)}{2M} \quad (2.21)$$

### 2.2.1 Scattering Cross-section in Laboratory and Center-of-mass Frames

Before we proceed to the kinematics of cross-section, its better to have an idea of what cross-section refers to. Cross-section is a quantity that provides the information regarding the interaction in the scattering process. That is, based on the available initial and final states of colliding particles, it tells the probability for a particular scattering to take place. Note that this does not refer to the geometry of the target particle, rather it refers to the effective area over which two or more particles interact such that they make a transition from an initial state to a final state. So it represents the intrinsic scattering probability of the particular process [HM84].

Here we define a differential cross-section  $d\sigma$  in laboratory frame in terms of number of particles appearing in the initial and final states of the scattering process. If we consider a particle scattered off a target, which is placed at the origin of the lab frame, and then passes through small surface element subtending a solid angle element  $d\Omega_L = \sin \theta_L d\theta_L d\phi_L$  at a polar angle  $\theta_L$  and azimuthal angle  $\phi_L$  with respect to the incident direction (see figure (2.4), then the differential cross-section is

$$d\sigma = \frac{N_s(\theta_L, \phi_L) d\Omega}{N_b \cdot N_t} \quad (2.22)$$

where  $N_b$  is incident flux,  $N_t$  is number of particles placed at the target, and  $N_s(\theta_L, \phi_L)$  is the flux of the scattered particle through a unit solid angle  $d\Omega$  about the direction  $(\theta_L, \phi_L)$ .

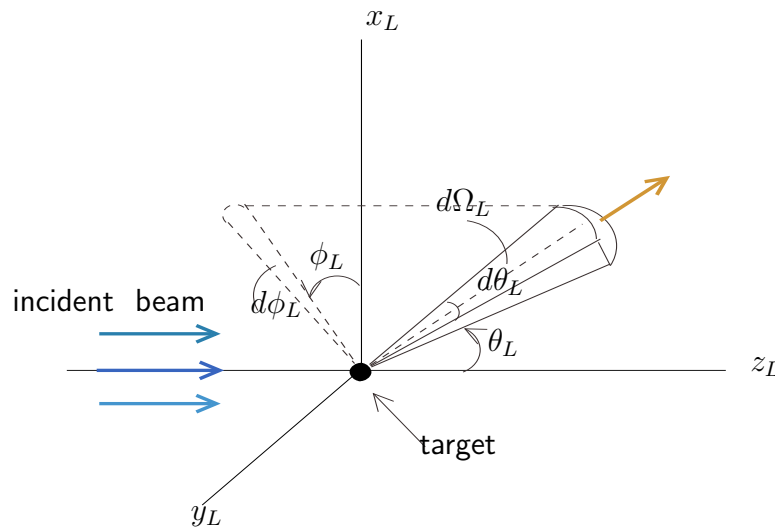


Figure 2.4: Illustration of scattering from a target into a solid angle element  $d\Omega_L = \sin \theta_L d\theta_L d\phi_L$

In Eq.(2.22) we can see that the same number of particles scattered into solid angle  $d\Omega$  about the direction  $(\theta, \phi)$  in the C.M. frame as are scattered into  $d\Omega_L$  about  $(\theta_L, \phi_L)$  in the lab frame [Joa84]. So for the scattered particle  $C$  in the reaction  $A + B \rightarrow C + D$  we have that,

$$\frac{d\sigma_C}{d\Omega_L}(\theta_L, \phi_L)d\Omega_L = \frac{d\sigma_C}{d\Omega}(\theta, \phi)d\Omega \quad (2.23)$$

or

$$\frac{d\sigma_C}{d\Omega_L}(\theta_L, \phi_L) \sin \theta_L d\theta_L d\phi_L = \frac{d\sigma_C}{d\Omega}(\theta, \phi) \sin \theta d\theta d\phi \quad (2.24)$$

Scattering cross-section has dimensions of area-cm<sup>2</sup>. More conveniently we use “barns”. 1 barn = 10<sup>-24</sup>cm<sup>2</sup>; differential cross-sections  $d\sigma/d\Omega$  are given in barns per steradian [Gri87]. If we refer back to figure (2.2) we notice that  $d\phi_L = d\phi$ , then for the scattered particle  $C$  Eq.(2.5) and (2.6) can be rewritten as

$$\cos \theta_L = \frac{\cos \theta + \tau}{(1 + 2\tau_C \cos \theta + \tau_C^2)^{\frac{1}{2}}} \quad (2.25)$$

with

$$\tau_C = \frac{V_L}{|\mathbf{v}_C|} \quad (2.26)$$

Then by substituting Eq.(2.25) into Eq.(2.24) we get

$$\frac{d\sigma_C}{d\Omega_L}(\theta_L, \phi_L) = \frac{(1 + \tau_C^2 + 2\tau_C \cos \theta)^{\frac{3}{2}}}{|1 + \tau_C \cos \theta|} \frac{d\sigma_C}{d\Omega}(\theta, \phi) \quad (2.27)$$

The same is true for particle  $D$  which is also scattered in the direction of  $(\pi - \theta, \pi + \phi)$  in the C.M. frame.

## 2.3 Kinematics of a Relativistic Particle

### 2.3.1 Energy-momentum Four-vector

From now on we use relativistic notations and four-vector formalism briefly discussed in appendix A.2. For any relativistic massive particle its energy  $E$  and three-momentum  $\mathbf{p}$  are related as

$$E^2 = m^2 c^4 + \mathbf{p}^2 c^2 \quad (2.28)$$

where  $m$  is rest-mass of the particle and  $c$  is the speed of light. The relativistic three-momentum and energy for a particle with 3-vector velocity are given by

$$\mathbf{p} = m\gamma\mathbf{v} \quad (2.29)$$

and

$$E = m\gamma c^2 \quad \text{or} \quad \frac{E}{c} = m\gamma c \quad (2.30)$$

where  $\gamma$  is Lorentz factor which we define later.

Thus the energy-momentum four-vector  $p^\mu$  of the particle is defined as

$$p^\mu = \left( \frac{E}{c}, \mathbf{p} \right) \quad (2.31)$$

which gives

$$p^2 = p \cdot p = p^\mu g_{\mu\nu} p^\nu = p^\mu p_\mu = \frac{E^2}{c^2} - \mathbf{p} \cdot \mathbf{p} = m^2 c^2 \quad (2.32)$$

where  $g^{\mu\nu}$  is the metric tensor with zero off-diagonal and  $(+1, -1, -1, -1)$  diagonal elements (see appendix A.2). In natural units

$$p^2 = E^2 - \mathbf{p}^2 = m^2 \quad (2.33)$$

Besides, the relativistic energy of the particle is related to its kinetic energy  $T$  by

$$E = T + m = \frac{\mathbf{p}^2}{2m} + m \quad (2.34)$$

We often use the notation either  $p \cdot x$  or  $p_\mu x^\mu$  to represent the scalar product or the contraction,

$$p \cdot x = p_\mu x^\mu = Et - \mathbf{p} \cdot \mathbf{x} \quad (2.35)$$

Note also that we use either  $p$  or  $p^\mu$  interchangeably to represent the four-vector momentum.

### 2.3.2 Lorentz Transformation (Boost)

In Minkowski space, in which special relativity is well defined, we have to take into account the distortions of the longitudinal space and the time components of the four-vectors, such as space-time coordinates of a point or an event  $x^\mu = (t, \mathbf{x})$  and energy-momentum four-vector  $p^\mu = (E, \mathbf{p})$ , in the transformation from one inertial frame to another. For the study of scattering processes, the laboratory frame and the centre-of-mass frame are the two collinear frames we choose to transform kinematic and dynamic quantities.

If we consider the C.M. frame which is moving with uniform velocity  $\mathbf{V}_L = V_L \hat{z}$  along  $z$ -axis with respect to the lab frame, then we use the Lorentz transformation ( $\mathcal{LT}$ ) to relate the coordinates of four-vectors such as,  $x_L^\mu = (t, \mathbf{x}_L)$  and  $p_L^\mu = (E_L, \mathbf{p}_L)$  in the lab frame to their corresponding four-vectors  $x^\mu = (t, \mathbf{x})$  and  $p^\mu = (E, \mathbf{p})$  in the C.M. frame. In matrix form, the Lorentz transformation of space-time coordinate of a given event is written as

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t_L \\ x_L \\ y_L \\ z_L \end{pmatrix} \quad (2.36)$$

or

$$x^\mu = \Lambda_\nu^\mu x_L^\nu \quad (2.37)$$

where  $\Lambda_\nu^\mu$  is the  $4 \times 4$  matrix in Eq.(2.36). The corresponding Lorentz transformation of the space-time coordinates from the C.M. frame to the lab frame is

$$x_{L\nu} = \Lambda_\nu^\mu x_\mu \quad (2.38)$$

where  $\gamma = \left(1 - \frac{V_L^2}{c^2}\right)^{-\frac{1}{2}}$  is called Lorentz factor and  $\beta = \frac{V_L}{c}$ . Notice that we are still working in natural units.

The energy-momentum four-vector also transforms in the same manner. That is,

$$p^\mu = \Lambda_\nu^\mu p_L^\nu \quad \text{and} \quad p_{L\nu} = \Lambda_\nu^\mu p_\mu \quad (2.39)$$

If we consider a particle that moves along  $z$ -axis, the second expression of Eq.(2.39) can explicitly be written as

$$\begin{pmatrix} E_L \\ 0 \\ 0 \\ p_{zL} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ 0 \\ 0 \\ p_z \end{pmatrix} \quad (2.40)$$

But the total three-vector momentum  $\mathbf{p}$  in the C.M. system is always zero; and this implies that  $p_z = 0$ . From this argument we get that  $E_L = \gamma E$  and  $p_{Lz} = \gamma\beta E$  from which we obtain

$$V_L = \frac{p_{Lz}}{E_L} \quad (2.41)$$

For the two-particle scattering case mentioned earlier, the  $V_L$  becomes

$$V_L = \frac{p_{Lz}}{E_L + m} \quad (2.42)$$

where  $m$  is the mass of the target particle initially at rest in the lab frame.

In what follows, we see how the three-vector velocity transforms. For convenience, we constrain the particle to  $xz$ -plane. To find the velocity that are related in terms of Lorentz transformation we need invariant time with respect to which we differentiate the coordinates of the space-time of the particle. We use the proper time recorded in the rest-mass frame of the particle to differentiate the coordinates of space-time. Then we get

$$(v_z)_L = \frac{v_z + V_L}{(1 + V_L v_z)} \quad \text{and} \quad (v_x)_L = \frac{v_x}{\gamma(1 + V_L v_z)} \quad (2.43)$$

By using Eq.(2.43) and referring back to figure(2.2), we get

$$\tan \theta_L = \frac{\sin \theta}{\gamma(\cos \theta + \tau)} \quad (2.44)$$

where

$$\tau = \frac{V_L}{|\mathbf{v}|} \quad (2.45)$$

Relativistic quantum mechanics is in favour of quantities that are invariant under Lorentz transformation. Scalar product of a contravariant four-vector with its covariant counterpart is Lorentz invariant. Crucial examples are,

$$x_\mu x^\mu = (t^2 - \mathbf{x} \cdot \mathbf{x}) \quad (2.46)$$

$$p_\mu p^\mu = (E^2 - \mathbf{p} \cdot \mathbf{p}) \quad (2.47)$$

### 2.3.3 Mandelstam Variables

We introduce three Lorentz invariant variables  $s$ ,  $t$ , and  $u$  which are used to describe the reaction kinematics of two-body scattering; they are defined as,

$$s = (p_A + p_B)^2 = (p_C + p_D)^2 \quad (2.48)$$

$$t = (p_A - p_C)^2 = (p_B - p_D)^2 \quad (2.49)$$

$$u = (p_A - p_D)^2 = (p_B - p_C)^2 \quad (2.50)$$

Their dependence on one another is constrained by the relation

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2 \quad (2.51)$$

which can be proved by using the mass-shell condition  $p_A^2 = m_A^2$ ,  $p_B^2 = m_B^2$ ,  $p_C^2 = m_C^2$ , and  $p_D^2 = m_D^2$ . The variable  $s$  is equivalent to the square of energy available in the C.M. system and  $t$  is the square of the four-momentum transfer.

In the C.M. frame we can redefine the initial and the final relative three-momenta  $\mathbf{p}_i$  and  $\mathbf{p}_f$  in terms of  $s$  which is expressed in the C.M. frame as

$$s = (E_A + E_B)^2 = \left( \sqrt{(m_A^2 + |\mathbf{p}_i|^2)} + \sqrt{(m_B^2 + |\mathbf{p}_i|^2)} \right)^2, \quad (2.52)$$

that is

$$|\mathbf{p}_i| = \frac{1}{2\sqrt{s}} \sqrt{(s - m_A^2 - m_B^2)^2 - 4m_A^2 m_B^2} = \frac{1}{2\sqrt{s}} \sqrt{(\lambda(s, m_A^2, m_B^2))} \quad (2.53)$$

$$|\mathbf{p}_f| = \frac{1}{2\sqrt{s}} \sqrt{(s - m_C^2 - m_D^2)^2 - 4m_C^2 m_D^2} = \frac{1}{2\sqrt{s}} \sqrt{(\lambda(s, m_C^2, m_D^2))} \quad (2.54)$$

where Eq.(2.54) is obtained from  $s = (E_C + E_D)^2$  and we have introduced the “triangle function”,

$$\lambda(x, y, z) = (x - y - z)^2 - 4yz. \quad (2.55)$$

Thus the energies can be expressed as

$$E_A = \frac{1}{2\sqrt{s}}(s + m_A^2 - m_B^2), \quad E_B = \frac{1}{2\sqrt{s}}(s - m_A^2 + m_B^2) \quad (2.56)$$

and

$$E_C = \frac{1}{2\sqrt{s}}(s + m_C^2 - m_D^2), \quad E_D = \frac{1}{2\sqrt{s}}(s - m_C^2 + m_D^2) \quad (2.57)$$

We can also get expressions for the other two Mandelstam variables  $t$  and  $u$

$$\begin{aligned} t &= (p_A - p_C)^2 = p_A^2 + p_C^2 - 2p_A \cdot p_C \\ &= p_A^2 + p_C^2 - 2(E_A E_C - |\mathbf{p}_A| |\mathbf{p}_C| \cos \theta) \\ &= -4|\mathbf{p}|^2 \sin^2 \left( \frac{\theta}{2} \right) \end{aligned} \quad (2.58)$$

and in a similar manner

$$u = -4|\mathbf{p}|^2 \cos^2 \left( \frac{\theta}{2} \right) \quad (2.59)$$

where we have set  $m_A = m_C$ ,  $m_B = m_D$ , and  $|\mathbf{p}_f| = |\mathbf{p}_i| = |\mathbf{p}|$ .

Finally, the differential cross-sections for relativistic collision transform between the collinear frames in the same way as we have seen for the non-relativistic case in section 2.2.1. But we have to take into account the relativistic correction. Then for particle  $C$  in the final channel, the transformation relation is given by

$$\frac{d\sigma_C}{d\Omega_L}(\theta_L, \phi_L) = \frac{[\gamma^2(\cos \theta + \tau_C)^2 + \sin^2 \theta_C]^{\frac{3}{2}}}{\gamma |1 + \tau_C \cos \theta|} \frac{d\sigma_C}{d\Omega}(\theta, \phi) \quad (2.60)$$

where  $\gamma = \left(1 - \frac{V_L^2}{c^2}\right)^{-\frac{1}{2}}$ . The same holds for particle  $D$  that appears in the final channel.



### 3. Relativistic Dirac Equation

In non-relativistic quantum mechanics, the Schrödinger equation is the fundamental differential equation that governs the behavior of quantum particles. For relativistic quantum mechanics the Klein-Gordon and Dirac equations describe non-interacting spin-0 and spin- $\frac{1}{2}$  particles, respectively. However, when pair-production becomes kinematically allowed, then a simple one-particle differential equation is no longer valid, and we have to use quantum field theory as the theoretical framework [Gri87].

#### 3.1 The Klein-Gordon ( $\mathcal{KG}$ ) Equation

Beginning with the relativistic energy and three-momentum relation  $E^2 = m^2c^4 + \mathbf{p}^2c^2$  we can obtain the  $\mathcal{KG}$  equation by using the corresponding quantum mechanical hermitian operators  $i\frac{\partial}{\partial t}$  and  $-i\vec{\nabla}$  of  $E$  and  $\mathbf{p}$ , respectively. By introducing the wavefunction  $\psi$  the Klein-Gordon equation is written as

$$\frac{\partial^2}{\partial t^2}\psi = (\vec{\nabla}^2 - m^2)\psi \quad (3.1)$$

where  $m$  is the rest-mass energy of the particle in natural units. In relativistic notation Eq.(3.1) is rewritten as

$$(p^\mu p_\mu - m^2)\psi = 0 \quad (3.2)$$

where  $p_\mu = (E, -\mathbf{p}) = i\partial_\mu = i(\frac{\partial}{\partial t}, \vec{\nabla})$  and  $p^\mu = (E, \mathbf{p}) = i\partial^\mu = i(\frac{\partial}{\partial t}, -\vec{\nabla})$  [Ryd96], hence

$$p^\mu p_\mu = -(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2) \quad (3.3)$$

The continuity equation for the  $\mathcal{KG}$  equation can be obtained from Eq.(3.1) and written as

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad (3.4)$$

where the probability density  $\rho$  and the probability current density  $\vec{J}$  are:

$$\rho = i \left( \psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t} \psi \right) \quad (3.5)$$

and

$$\vec{J} = -i \left( \psi^* \vec{\nabla} \psi - \vec{\nabla} \psi^* \psi \right) \quad (3.6)$$

where  $\psi^*$  is the complex conjugate of  $\psi$ . Since the  $\mathcal{KG}$  equation is second order in  $t$ , it has got positive and negative energy solutions and corresponding positive and negative probability densities. However, the negative probability density does not make sense for it violates the laws of statistical mechanics. This means that it is always expected to be scalar quantity. Dirac successfully derived the relativistic wave equation which is first order in both  $t$  and  $\mathbf{x}$  and with positive definite probability density [BJ95].

## 3.2 Dirac Free Particle Equation

The Dirac equation is written as

$$\begin{aligned} i\frac{\partial\psi}{\partial t} &= [-i(\alpha_1\frac{\partial}{\partial x^1} + \alpha_2\frac{\partial}{\partial x^2} + \alpha_3\frac{\partial}{\partial x^3}) + \beta m]\psi \\ &= (-i\vec{\alpha} \cdot \vec{\nabla} + \beta m)\psi \end{aligned} \quad (3.7)$$

In order to determine what the coefficients  $\alpha_i$  and  $\beta$  stand for, we can take the square of the operators in Dirac's equation and compare it with  $\mathcal{KG}$  equation. The Dirac equation with squared operator should be equivalent to the  $\mathcal{KG}$  equation. So we can find the conditions set on  $\alpha_i$  and  $\beta$ . Squaring the operators on both sides gives

$$\begin{aligned} \left(i\frac{\partial}{\partial t}\right)^2 \psi &= (-i\vec{\alpha} \cdot \vec{\nabla} + \beta m)(-i\vec{\alpha} \cdot \vec{\nabla} + \beta m)\psi \\ &= -\sum_{i=1}^3 \alpha_i^2 \frac{\partial^2}{(\partial x^i)^2} \psi - \sum_{i,j=1, i>j}^3 (\alpha_i \alpha_j + \alpha_j \alpha_i) \frac{\partial^2}{(\partial x^i \partial x^j)} \\ &\quad -im \sum_{i=1}^3 (\alpha_i \beta + \beta \alpha_i) \frac{\partial}{\partial x_i} + \beta^2 m^2 \psi \end{aligned} \quad (3.8)$$

So we assume that  $\psi$  in Eq.(3.7) must satisfy the  $\mathcal{KG}$  equation,

$$-\frac{\partial^2}{\partial t^2} \psi = (-\vec{\nabla}^2 + m^2)\psi \quad (3.9)$$

Then the conditions imposed on  $\alpha_i$  and  $\beta$  due to the above requirement are

$$\begin{aligned} \alpha_i \alpha_j + \alpha_j \alpha_i &= 0 \quad i, j = 1, 2, 3 \quad i \neq j \\ \alpha_i \beta + \beta \alpha_i &= 0 \quad i, j = 1, 2, 3 \\ \alpha_i^2 &= \beta^2 = I \quad i = 1, 2, 3 \end{aligned} \quad (3.10)$$

Using this set of equations we can construct an anticommutation relation [BJ95]

$$\{\alpha_\mu, \alpha_\nu\} = \alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 2\delta_{\mu\nu} I, \quad \nu, \mu = 0, 1, 2, 3 \quad (3.11)$$

where  $\alpha_0 = \beta$ . For the coefficients  $\alpha_i$  and  $\beta$  to satisfy the anticommutation relation given in Eq.(3.11), they must possibly be square matrices with the minimum even dimension of 4. Thus the familiar choices for the  $\alpha_i$ 's and  $\beta$  are the block matrices given by

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (3.12)$$

where  $I$  is the  $2 \times 2$  identity matrix,  $0$  is also  $2 \times 2$  matrix of zero elements, and  $\sigma_i$  are  $2 \times 2$  Pauli matrices (see appendix A.3). As an immediate consequence, the wavefunction  $\psi$  must be a four-component column matrix,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad (3.13)$$

By using Einstein summation notation, the Lorentz covariance form of the Dirac equation is written as

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad (3.14)$$

where  $\gamma^\mu$  is a Dirac gamma matrix which replace  $\beta$  and  $\alpha_i$ , and is defined as

$$\gamma^0 = \beta; \quad \gamma^k = \beta\alpha^k, \quad k = 1, 2, 3 \quad (3.15)$$

or equivalently

$$\gamma^\mu = (\gamma^0, \vec{\gamma}) = (\beta, \beta\vec{\alpha}) \quad (3.16)$$

Similarly, the gamma matrices satisfy the anticommutation relation [BJ95]

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I \quad (3.17)$$

and  $\gamma^0 \gamma^0 = -\gamma^i \gamma^i = I$ .

The continuity equation for the Dirac equation becomes

$$\partial_\mu j^\mu = 0 \quad (3.18)$$

where  $j^\mu = (\rho, \vec{J}) = \bar{\psi} \gamma^\mu \psi$  is a four-vector current, and we have used  $\bar{\psi} = \psi^\dagger \gamma^0$  which is Lorentz invariant. Thus the probability density becomes

$$\rho = \psi^\dagger \psi = \sum_{n=1}^4 |\psi_n|^2 \quad (3.19)$$

which is a positive (scalar) density as it is expected to be. If we multiply  $j^\mu$  by the electron charge  $-e$  it can be regarded as the electric current density or current-charge four-vector of the electron. That is,

$$j_e^\mu = -e \bar{\psi} \gamma^\mu \psi \quad (3.20)$$

### 3.3 Free Particle Solution of Dirac Equation

We begin by choosing a solution to be a plane wavefunction which has a form

$$\psi(x, p, s) = N e^{-ip \cdot x} u(p, s) \quad (3.21)$$

where  $p \cdot x = p^\mu x_\mu = Et - \mathbf{p} \cdot \mathbf{x}$  is a contraction of four-vectors  $p$  and  $x$ ,  $s$  is the spin state of the particle and  $u(p, s)$  is four-component column matrix and is called the Dirac spinor; it can be written in terms of two two-component spinors,

$$u(p, s) = C \begin{pmatrix} \chi \\ \phi \end{pmatrix} \quad (3.22)$$

where  $N$  and  $C$  are normalisation factors. It is worth noting the context in which we have used the Mandelstam variable  $s$  and the spin state  $s$  in order to avoid confusion. By plugging Eq.(3.22)

into Eq.(3.21) and then Eq.(3.21) into Eq.(3.7) as well as recalling the  $2 \times 2$  block expressions of  $\beta$  and  $\alpha'_i$ s we get

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} mI & \sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & -mI \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} \quad (3.23)$$

From which we obtain positive energy and negative energy solutions  $u(p, s)$  and  $v(p, s)$  given by

$$u(p, s) = C \begin{pmatrix} \chi^s \\ \frac{\sigma \cdot \mathbf{p}}{E + m} \chi^s \end{pmatrix} \quad \text{and} \quad v(p, s) = C \begin{pmatrix} \frac{\sigma \cdot \mathbf{p}}{E + m} \phi^s \\ \phi^s \end{pmatrix} \quad (3.24)$$

where in the two expressions  $\chi^s$  and  $\phi^s$  each represents either of the two possible spin states  $\pm \frac{1}{2}$  of a particle, that is either spin up  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  or spin down  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

In terms of gamma matrices and free particle spinors, the Dirac equation can be rewritten as

$$(\gamma^\mu p_\mu - m)u(p, s) = 0 \quad (3.25)$$

Here we introduce the slash notation. If  $A$  is a four-vector then

$$\gamma \cdot A = \gamma^\mu A_\mu \equiv \not{A} \quad (3.26)$$

Thus Eq.(3.25) becomes

$$(\not{p} - m)u(p, s) = 0 \quad (3.27)$$

Moreover, the electron current can also be rewritten as  $j_e^\mu = -e\bar{u}\gamma^\mu u$ , which also satisfies the current conservation law  $\partial_\mu j_e^\mu = 0$ . The Dirac equation for a spin- $\frac{1}{2}$  particle, say, an electron interacting with external electromagnetic field having a four-vector potential  $A^\mu = (\phi, \mathbf{A})$ , contains information regarding the intrinsic magnetic moment  $-e\frac{\sigma}{2m}$  of the electron that gives rise to magnetic interaction. Here  $\phi$  and  $\mathbf{A}$  are the scalar and vector potentials of the field, respectively.

### 3.3.1 Normalisation of Dirac Spinor

For the Dirac equation of the free particle, we can choose the plane wave solution of the form

$$\psi = Nu(p, s)e^{-ip \cdot x}, \quad s = 1, 2 \quad (3.28)$$

In order to determine the normalization factors, we use the covariant normalization [DD64]

$$\bar{u}(p, s)u(p, s) = 1 \quad (3.29)$$

or equivalently

$$u^\dagger(p, s)u(p, s) = \frac{E}{m} \quad (3.30)$$

So by using this convention we get

$$N = \sqrt{\frac{m}{EV}} \quad \text{and} \quad C = \sqrt{\frac{E}{m}} \quad (3.31)$$

where  $V$  is the normalization volume. This convention is especially useful when we deal with a system of massive particles. Moreover, it helps to get Lorentz invariant expression of the differential cross-section.

# 4. Quantum Electrodynamics of Electron-proton Elastic Scattering

In this chapter, more emphasis is given to the dynamic description of the scattering process. Electromagnetic Interaction between spin- $\frac{1}{2}$  Dirac particles is well described by the QED. In the study of two particle scattering, we need to know the rate of transition from the initial state to the final state. One way of addressing this problem is evaluating the Feynman diagram in order to find the invariant amplitude, which contains all the dynamics of the scattering under question. In other words, the physics of any scattering processes can be extracted from the corresponding Feynman diagram.

## 4.1 Feynman Diagrams

In Quantum Electrodynamics, we can make the calculations of the relativistic scattering process easier by employing the Feynman diagrams ( $\mathcal{FD}$ ). A Feynman diagram is a graphical representation of a scattering process [Ron94]. It gives schematic description of the scattering processes that arise from the interaction of particles.

In this study, we focus only on the lowest-order Feynman diagram which is often called as the tree diagram to represent the electron-proton elastic scattering due to the electromagnetic interaction. The interaction takes place via force carrier called virtual or “off-mass shell” photon which is specified by four-momentum transfer  $q^2 \neq 0$ . The photon is spin-1 particle.

### 4.1.1 Components of Feynman Diagram

Feynman diagrams are treated in Minkowski space-time. Basically, we need three elements to construct  $\mathcal{FD}$  for any lowest-order scattering in  $\mathcal{EM}$  fields: Straight lines for representing the free incoming and outgoing particles, wavy lines for virtual photon, and vertices where two incoming and outgoing solid lines and a single wavy line meet at the moment of interaction (see figure 4.1).

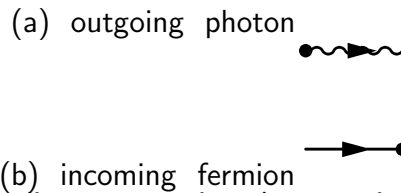


Figure 4.1: (a) Wavy line representing the outgoing photons and (b) straight lines representing the incoming fermion. If we change the position of the dots to the other ends, we get lines that represent (a) the incoming photon and (b) the outgoing fermion. We use dots to represent lepton-photon vertices

The building block for constructing  $\mathcal{FD}$  of scattering, such as tree diagrams and other higher order Feynman diagrams, contains two straight (external) lines, a single wavy (internal) line and a vertex, which is a point where these three lines meet and such a graph is often called a primitive diagram (see figure 4.2). In  $\mathcal{EM}$  two-body interactions, however, we need at least two particle-photon vertices, four incoming and outgoing solid lines and a single wavy line; such a diagram is called the lowest-order  $\mathcal{FD}$  or tree diagram (see figure 4.3).

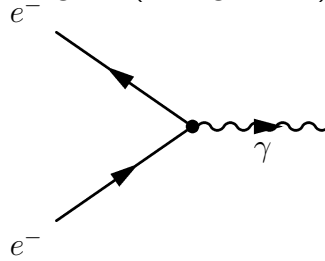


Figure 4.2: Primitive diagram that illustrates the emission of photon from electron

In evaluating the cross-section, we take our first step based on the assumption that both the electron and proton are  $\text{spin-}\frac{1}{2}$  elementary particles.

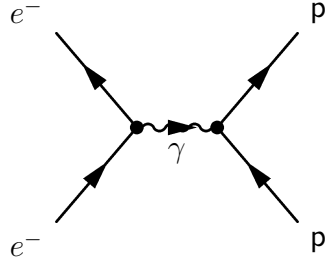


Figure 4.3: The lowest-order tree diagram

## 4.2 Feynman Rules for Tree Diagram

By using Feynman rules, we can extract multiplicative factors from a  $\mathcal{FD}$  representing a particular scattering process in order to construct expressions for the invariant amplitude and the scattering cross-section which describe the process. The  $\mathcal{FD}$  is drawn such that each of its elements corresponds to a factor in the express of invariant scattering amplitude.

If we consider  $\mathcal{FD}$  for fermion-fermion scattering due to  $\mathcal{EM}$  interaction, the external (incoming and outgoing) lines are associated with Dirac spinors

$$u(p, s) \quad (\text{for incoming particle}) \quad (4.1)$$

$$\bar{u}(p, s) \quad (\text{for outgoing particle}) \quad (4.2)$$

where  $p$  and  $s$ , respectively, are four-momenta and spin states ( $\pm\frac{1}{2}$ ) of a particle under study. The internal (wavy) line representing a virtual photon has got a propagator,

$$\frac{-ig^{\mu\nu}}{q^2} \quad (4.3)$$

associated with it, where  $g^{\mu\nu}$  is the metric tensor and  $q^2$  is the square of four-momentum transfer. With each point vertex of  $\mathcal{FD}$  we associate a factor

$$-ig\gamma^\mu \quad (4.4)$$

where  $g$  is a coupling constant. For  $\mathcal{EM}$  interaction it is the charge of the positron  $e = \sqrt{2\pi\alpha}$  where  $\alpha = 1/137$  is the fine-structure constant. Besides that, the four-momentum must be conserved at the vertices [Gri87]. In Quantum Electrodynamics, we represent a vertex by a dot if the incoming and outgoing particles are point-like, otherwise, we use a "blob".

## 4.3 Fermi's Golden Rule

Expressions of cross-section have two basic ingredients: the invariant amplitude and phase space factor, which contains all kinematic aspects of scattering process [Gri87]. Here we develop the important expression for transition rate and cross-section using Fermi's golden rule.

### 4.3.1 Transition Rate

To begin with, we consider the scattering process of two particles  $1 + 2 \rightarrow 1' + 2'$ . The transition rate per unit volume, from the initial state to the final state [AH03], is given by

$$W_{fi} = \frac{|\mathcal{T}_{fi}|^2}{VT} \quad (4.5)$$

where  $T$  is time interval of observation,  $V$  is the spatial volume of the interaction region and  $|\mathcal{T}_{fi}|^2$  is a transition probability which is defined in terms of the transition matrix element  $\mathcal{M}_{fi}$  as

$$\mathcal{T}_{fi} = -iN_1N_2N_{1'}N_{2'}(2\pi)^4\delta^4(p_1 + p_2 - p'_1 - p'_2)\mathcal{M}_{fi} \quad (4.6)$$

Thus

$$W_{fi} = \frac{(N_1N_2N_{1'}N_{2'})^2[(2\pi)^4\delta^4(p_1 + p_2 - p'_1 - p'_2)]^2|\mathcal{M}_{fi}|^2}{VT} \quad (4.7)$$

Now we use the delta function property in the well defined spacial volume  $V$  and time interval  $T$  in space-time [GR03]

$$\begin{aligned} [(2\pi)^4\delta^4(p_1 + p_2 - p'_1 - p'_2)]^2 &= (2\pi)^4\delta(0)(2\pi)^4\delta^4(p_1 + p_2 - p'_1 - p'_2) \\ &= VT(2\pi)^4\delta^4(p_1 + p_2 - p'_1 - p'_2) \end{aligned} \quad (4.8)$$

then we get

$$W_{fi} = (N_1N_2N_{1'}N_{2'})^2(2\pi)^4\delta^4(p_1 + p_2 - p'_1 - p'_2)|\mathcal{M}_{fi}|^2 \quad (4.9)$$

It is worth noting that in our derivations thus far, we have indirectly used Dirac's plane wave-functions which represent particles in Quantum Field Theory.

### 4.3.2 Cross-Section for Two-body Scattering

We now define the differential cross-section  $d\sigma$  in terms of the transition rate per unit volume [HM84], that is

$$d\sigma = \frac{W_{fi}}{\text{initial flux}} (\text{number of final states}) \quad (4.10)$$

Experiments, however, measure the differential cross-section for scattering into a particular solid angle  $d\Omega$  in the momentum space interval  $d^3\mathbf{p}'_1 d^3\mathbf{p}'_2$ , where  $\mathbf{p}'_1$  and  $\mathbf{p}'_2$  are the final three-vector momenta of the electron and the proton, respectively [Ryd96].

Initial flux is the product of incident flux and the density of the target particle and hence given by

$$\frac{|\mathbf{v}_1 - \mathbf{v}_2|}{V^2} = \frac{1}{V^2} \frac{((p_1 \cdot p_2)^2 - m_1^2 m_2^2)^{\frac{1}{2}}}{E_1 E_2} \quad (4.11)$$

where  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are the velocities,  $E_1$  and  $E_2$  are energies and  $m_1$  and  $m_2$  are masses of the incident and the target particles, respectively. The number of final states in momentum space interval  $d^3\mathbf{p}'_1 d^3\mathbf{p}'_2$  is given by

$$\frac{V}{(2\pi)^3} d^3\mathbf{p}'_1 \frac{V}{(2\pi)^3} d^3\mathbf{p}'_2 \quad (4.12)$$

and thus the differential cross-section for a transition from initial state to final state is expressed as

$$\begin{aligned} d\sigma &= \frac{E_1 E_2 V^2}{((p_1 \cdot p_2)^2 - m_1^2 m_2^2)^{\frac{1}{2}}} W_{fi} \frac{V}{(2\pi)^3} d^3\mathbf{p}'_1 \frac{V}{(2\pi)^3} d^3\mathbf{p}'_2 \\ &= \frac{((p_1 \cdot p_2)^2 - m_1^2 m_2^2)^{\frac{1}{2}}}{m_1 m_2} |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2) \\ &\quad \times \frac{m_1}{(2\pi)^3} \frac{d^3\mathbf{p}'_1}{E'_1} \frac{m_2}{(2\pi)^3} \frac{d^3\mathbf{p}'_2}{E'_2} \end{aligned} \quad (4.13)$$

In a more convenient and compact form

$$d\sigma = \frac{4m_1 m_2}{((p_1 \cdot p_2)^2 - m_1^2 m_2^2)^{\frac{1}{2}}} |\mathcal{M}_{fi}|^2 d\text{Lips}(s; p'_1, p'_2) \quad (4.14)$$

where

$$d\text{Lips}(s; p'_1, p'_2) = (2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2) \frac{m_1}{(2\pi)^3} \frac{d^3\mathbf{p}'_1}{2E'_1} \frac{m_2}{(2\pi)^3} \frac{d^3\mathbf{p}'_2}{2E'_2}, \quad (4.15)$$

is the Lorentz invariant phase space factor for a system of two particles and  $s = (p_1 + p_2)^2 = (p'_1 + p'_2)^2$  is a Mandelstam variable. We notice that other factors in Eq.(4.14) are also Lorentz invariant.

## 4.4 Elastic Scattering of Electron from a Point Proton

In this section we derive the invariant amplitude  $\mathcal{M}_{fi}$  by putting together the factors that we associated with  $\mathcal{FD}$  earlier in section 4.2; we also take the initial and final spin states into consideration.



### 4.4.1 Evaluation of Invariant Amplitude of Unpolarised Electron-proton Scattering

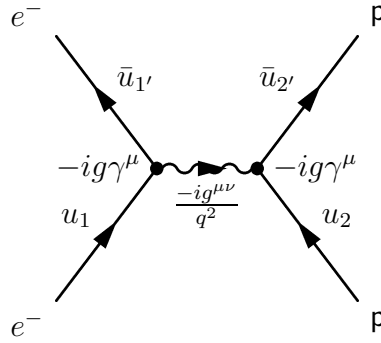


Figure 4.4: The lowest-order Feynman diagram for electron-proton elastic scattering. Each factor of the invariant amplitude are associated to the corresponding element of the Feynman diagram.

The process can be labelled as

$$e + p \rightarrow e + p \quad (4.16)$$

$$1 + 2 \rightarrow 1' + 2' \quad (4.17)$$

The initial and final free electron and free proton are described by wave functions of the form

$$\psi = \sqrt{\frac{m}{EV}} u(p, s) e^{-ip \cdot x} \quad (4.18)$$

$$\bar{\psi} = \sqrt{\frac{m}{EV}} \bar{u}(p, s) e^{+ip \cdot x} \quad (4.19)$$

We start with the general way of expressing the invariant amplitude from the  $\mathcal{FD}$  based on the Feynman rules. By referring to the Feynman diagram in figure 4.4, we begin by writing down the quantum mechanical expression of  $\mathcal{M}_{fi}$ . That is

$$\mathcal{M}_{fi} = \langle \text{Final States} | \text{Interaction Operator} | \text{Initial States} \rangle = \langle P_f, S_f | \hat{\mathcal{O}} | P_i, S_i \rangle \quad (4.20)$$

where  $P_i$  and  $P_f$  are the total 4-momenta and  $S_i$  and  $S_f$  are the total spins states of the electron and the proton before and after scattering, respectively, in the scattering process. By using the Kronecker (tensor) product we can rewrite Eq.(4.20) as

$$\mathcal{M}_{fi} = \frac{g^2}{q^2} [\bar{u}(p'_1, s'_1) \otimes \bar{u}(p'_2, s'_2)] [\gamma_\mu \otimes \gamma^\mu] [u(p_1, s_1) \otimes u(p_2, s_2)] \quad (4.21)$$

Now we use the property of the Kronecker product for matrices, say,  $A$ ,  $B$ ,  $C$ , and  $D$

$$(A \otimes B)(C \otimes D) = AC \otimes BD \quad (4.22)$$

Thus we get

$$\mathcal{M}_{fi} = \frac{g^2}{q^2} [\bar{u}_1' \gamma_\mu u_1] [\bar{u}_2' \gamma^\mu u_2] \quad (4.23)$$

where we have used a shorthand notation  $u_i = u(p_i, s_i)$ . This expression reminds us that  $\mathcal{M}_{fi}$  is the product of the electron and the proton currents and also note that  $\bar{u}_{1'}\gamma_\mu u_1$  is simply a complex number. The complex conjugate of  $\mathcal{M}_{fi}$  then becomes

$$\begin{aligned}\mathcal{M}_{fi}^* &= \frac{g^2}{q^2} [\bar{u}_{1'}\gamma_\mu u_1]^* [\bar{u}_{2'}\gamma^\mu u_2]^* \\ &= \frac{g^2}{q^2} [\bar{u}_1\bar{\gamma}_\nu u_{1'}] [\bar{u}_2\bar{\gamma}^\nu u_{2'}] \\ &= \frac{g^2}{q^2} [\bar{u}_1\gamma_\nu u_{1'}] [\bar{u}_2\gamma^\nu u_{2'}]\end{aligned}\quad (4.24)$$

where we have used the fact that the complex conjugate of any scalar number is equal to its hermitian conjugate and with the help of the gamma matrices obey such properties as  $\overline{\gamma^\mu} = \gamma^\mu$ ,  $\gamma^{0\dagger} = \gamma^0$ , and  $(\gamma^0)^2 = I$ . Hence the square of the invariant amplitude becomes

$$|\mathcal{M}_{fi}|^2 = \mathcal{M}_{fi}^* \mathcal{M}_{fi} = \left(\frac{g^2}{q^2}\right)^2 [(\bar{u}_{1'}\gamma_\mu u_1)(\bar{u}_{2'}\gamma^\mu u_2)][(\bar{u}_1\gamma_\nu u_{1'})(\bar{u}_2\gamma^\nu u_{2'})] \quad (4.25)$$

For unpolarised incoming and outgoing particles, however, we have to average over the initial spin states and sum over the final spin states in order to get the spin-averaged invariant amplitude. Then it becomes independent of spin states [Bra05, Gin04, Gri87]. That is,

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{4} \sum_{s_1, s_2, s'_1, s'_2} |\mathcal{M}_{fi}|^2 \quad (4.26)$$

After plugging Eq.(4.25) into Eq.(4.26) and collecting together factors that belong to the same particle, we may get a compact expression

$$\overline{|\mathcal{M}_{fi}|^2} = \left(\frac{g^2}{q^2}\right)^2 L_{\mu\nu} W^{\mu\nu} \quad (4.27)$$

where

$$L_{\mu\nu} = \frac{1}{2} \sum_{s_1, s'_1} [(\bar{u}_{1'}\gamma_\mu u_1)(\bar{u}_1\gamma_\nu u_{1'})], \quad (4.28)$$

$$W^{\mu\nu} = \frac{1}{2} \sum_{s_2, s'_2} [(\bar{u}_{2'}\gamma^\mu u_2)(\bar{u}_2\gamma^\nu u_{2'})] \quad (4.29)$$

are lepton and hadronic tensors, respectively.

For convenience, let's first evaluate  $W^{\mu\nu}$ . Here we rather choose to write it by using matrix indices as

$$W^{\mu\nu} = \frac{1}{2} \sum_{s_2, s'_2} [(\bar{u}_{2'})_\alpha (\gamma^\mu)_{\alpha\rho} (u_2)_\rho (\bar{u}_2)_\varepsilon (\gamma^\nu)_{\varepsilon\eta} (u_{2'})_\eta] \quad (4.30)$$

So now we are dealing with elements of matrices, thus we can change their position without affecting the expression. That is

$$W^{\mu\nu} = \frac{1}{2} \sum_{\eta} [(u_{2'})_\eta (\bar{u}_{2'})_\alpha (\gamma^\mu)_{\alpha\rho} (u_2)_\rho (\bar{u}_2)_\varepsilon (\gamma^\nu)_{\varepsilon\eta}] \quad (4.31)$$

As we can see in Eq.(4.31), the spin sum is reduced to matrix trace. We also introduce the positive energy projection operator

$$\Lambda(p, s)_{\alpha\beta} = \sum_s u(p, s)_\alpha \bar{u}(p, s)_\beta = \frac{(\not{p} + m)}{2m} \quad (4.32)$$

Thus

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{2} \sum_{s_2} \Lambda(p'_2, s'_2)_{\eta\alpha} \gamma^\mu_{\alpha\rho} \sum_{s'_2} \Lambda(p_2, s_2)_{\rho\epsilon} \gamma^\nu_{\epsilon\eta} \\ &= \frac{1}{2} \sum_{\eta} \left( \frac{\not{p}'_2 + m_2}{2m_2} \right)_{\eta\alpha} \gamma^\mu_{\alpha\rho} \left( \frac{\not{p}_2 + m_2}{2m_2} \right)_{\rho\epsilon} \gamma^\nu_{\epsilon\eta} \\ &= \frac{1}{8m_2^2} \sum_{\eta} ((\not{p}'_2 + m_2) \gamma^\mu (\not{p}_2 + m_2) \gamma^\nu)_{\eta\eta} \\ &= \frac{1}{8m_2^2} \text{Tr}[(\not{p}'_2 + m_2) \gamma^\mu (\not{p}_2 + m_2) \gamma^\nu] \end{aligned} \quad (4.33)$$

The method of reducing the spin sum to matrix trace is commonly called as Casimir's trick [Gri87]. Now we are left only with the evaluation of the trace without using the explicit forms of Dirac spinors and gamma matrices. By applying trace theorems of gamma matrices (see appendix A.3),

$$\begin{aligned} \text{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu} \\ \text{Trace of any product of an odd number of } \gamma_\mu \text{'s is zero} \\ \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\beta \gamma^\alpha] &= 4[g^{\mu\nu} g^{\beta\alpha} - g^{\mu\beta} g^{\nu\alpha} + g^{\mu\alpha} g^{\nu\beta}] \end{aligned} \quad (4.34)$$

then the hadronic tensor becomes

$$\begin{aligned} W^{\mu\nu} &= \frac{4}{8m_2^2} \left( \text{Tr}[\not{p}'_2 \gamma^\mu \not{p}_2 \gamma^\nu] + \frac{m_2^2}{4} \text{Tr}[\gamma^\mu \gamma^\nu] \right) \\ &= \frac{1}{2m_2^2} ([p_2^\mu p_2^\nu + p_2'^\mu p_2'^\nu - g^{\mu\nu} (p_2 \cdot p_2')] + m_2^2 g^{\mu\nu}) \\ &= \frac{1}{2m_2^2} [p_2^\mu p_2'^\nu + p_2'^\mu p_2^\nu + g^{\mu\nu} (m_2^2 - p_2 \cdot p_2')] \end{aligned} \quad (4.35)$$

It is simple to show that  $g^{\mu\nu} (m_2^2 - p_2 \cdot p_2') = \frac{q^2}{2} g^{\mu\nu}$ .

Similarly the electron tensor can be written as

$$L_{\mu\nu} = \frac{1}{2m_1^2} [(p_1)_\mu (p'_1)_\nu + (p_1)_\nu (p'_1)_\mu - g_{\mu\nu} (p_1 \cdot p'_1 - m_1^2)] \quad (4.36)$$

After performing the contraction of lepton and hadronic tensors, we get

$$|\overline{\mathcal{M}}_{fi}|^2 = \left( \frac{e^4}{q^4} \right) \frac{1}{2m_1^2 m_2^2} [(p_1 \cdot p_2)(p'_1 \cdot p'_2) + (p_1 \cdot p'_2)(p_2 \cdot p'_1) - m_1^2 p_2 \cdot p'_2 - m_2^2 p_1 \cdot p'_1 + 2m_2^2 m_1^2] \quad (4.37)$$

This expression reminds us that the spin-averaged amplitude depends only on the initial and the final momenta of the electron and the proton.

Having calculated the square of the invariant amplitude, we proceed to evaluate the differential cross-section. The e-p scattering experiments are often carried out in the setting where the proton is initially at rest. As a consequence, we choose to work in the lab frame. Thus the four-momenta are given by

$$p_2 = (E_2, \mathbf{0}) = (m_2, \mathbf{0}), \quad p_1 = (E_1, \mathbf{p}_1), \quad p'_1 = (E'_1, \mathbf{p}'_1) \quad (4.38)$$

We notice that  $\mathbf{p}'_1$  and  $\mathbf{p}'_2$  are not independent variables; they appeared in Eq.(4.13) as integration variables. So in order to restrict their values we have to integrate over  $d^3\mathbf{p}'_1$  and  $d^3\mathbf{p}'_2$ . However, we do not integrate over  $d\Omega$  for we want to obtain the angular distribution of the differential cross-section. Moreover, we have to express  $d\mathbf{p}'_1$  in terms of spherical coordinates in momentum space as

$$d^3\mathbf{p}'_1 = |\mathbf{p}'_1|^2 d|\mathbf{p}'_1| d\Omega_e = |\mathbf{p}'_1| E'_1 dE'_1 d\Omega_e \quad (4.39)$$

$$\Rightarrow \frac{d^3\mathbf{p}'_1}{E'_1} = |\mathbf{p}'_1| dE'_1 d\Omega_e \quad (4.40)$$

where we have used the relativistic energy and momentum relation  $|\mathbf{p}'_1|^2 + m_1^2 = E_1'^2$ . The subscript  $e$  is used because we prefer to look only at the scattered electron.

Based on Eq.(4.38) in the lab frame the incoming flux factor is simplified as

$$\frac{m_1 m_2}{((p_1 \cdot p_2)^2 - m_1^2 m_2^2)^{\frac{1}{2}}} = \frac{m_1 m_2}{(m_2^2 (E_1^2 - m_1^2))^{\frac{1}{2}}} = \frac{m_1}{|\mathbf{p}'_1|} \quad (4.41)$$

Thus the invariant differential cross-section becomes

$$d\bar{\sigma}_e = \frac{1}{|\mathbf{p}_1|} \int \frac{m_1^2 m_2}{2\pi^2} |\overline{\mathcal{M}_{fi}}|^2 \delta^4(p_1 + p_2 - p'_1 - p'_2) |\mathbf{p}'_1| dE'_1 \frac{d^3\mathbf{p}'_2}{E'_2} d\Omega_e \quad (4.42)$$

Note that we have used the same symbol  $d\bar{\sigma}_e$  before and after integration [Gri87]. The three dimensional delta function enables us to integrate over  $d^3\mathbf{p}'_2$  which is evaluated at  $\mathbf{p}'_2 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1$  that gives rise to the three-vector momentum conservation and after integration we are left with

$$d\bar{\sigma}_e = \frac{1}{|\mathbf{p}_1|} \int \frac{m_1^2 m_2}{(2\pi)^2} |\overline{\mathcal{M}_{fi}}|^2 \delta \left( E_1 + E_2 - E'_1 - \sqrt{(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1)^2 + m_2^2} \right) \frac{|\mathbf{p}'_1| dE'_1 d\Omega_e}{E'_2} \quad (4.43)$$

Here by reminding ourselves that we are working in the lab frame where the proton was initially at rest (i.e.,  $\mathbf{p}_2 = 0$ ), hence we write  $E'_2$  as

$$E'_2(E'_1) = \left( E_1^2 + E_1'^2 - 2m_1^2 - 2\sqrt{E_1^2 - m_1^2} \sqrt{E_1'^2 - m_1^2} \cos \theta + m_2^2 \right)^{\frac{1}{2}} \quad (4.44)$$

In Eq.(4.44)  $E'_2(E'_1)$  is just to indicate the dependence of  $E'_2$  on  $E'_1$ . Then

$$d\bar{\sigma}_e = \frac{1}{|\mathbf{p}_1|} \int \frac{m_1^2 m_2}{(2\pi)^2} |\overline{\mathcal{M}_{fi}}|^2 [f(E'_1)] h(E'_1) dE'_1 d\Omega_e \quad (4.45)$$

where

$$f(E'_1) = E_1 + E_2 - E'_1 - \left( E_1^2 - 2m_1^2 + m_2^2 + E_1'^2 - 2\sqrt{E_1^2 - m_1^2} \sqrt{E_1'^2 - m_1^2} \cos \theta \right)^{\frac{1}{2}} \quad (4.46)$$

and  $h(E'_1) = \frac{\sqrt{E_1'^2 - m_1^2}}{E_2'(E'_1)}$ . Now if we introduce the property of Dirac delta function (see appendix A.1):

$$\delta[f(x - x_o)] = \delta(x - x_o) \left| \frac{df(x)}{dx} \right|_{x=x_o}^{-1} \quad \text{and} \quad \int h(x) \delta[f(x)] = h(x_o) \left| \frac{df(x)}{dx} \right|_{x=x_o}^{-1} \quad (4.47)$$

where  $x_o$  is a root of  $f(x) = 0$ . Eq.(4.45) demands the first derivative as well as the zero of  $f(E'_1)$ . Obviously,  $f(E'_1) = 0$  does not only enforce the conservation of energy but also it does fix  $\mathbf{p}'_1$  and hence  $\mathbf{p}'_2$ . If we denote  $E_1^*$  as the positive energy solution for  $f(E'_1) = 0$ , after some steps we can write

$$E_1^* = \frac{2AB - 2A^3 + C[B^2 - 2A^2B + A^4 + m_1^2(C^2 - 4A^2)]^{\frac{1}{2}}}{(C^2 - 4A^2)} \quad (4.48)$$

where we have denoted  $A = E_1 + E_2$ ,  $B = E_1^2 - 2m_1^2 + m_2^2$  and  $C = 2\sqrt{E_1^2 - m_1^2} \cos \theta$ ; and

$$\frac{df(E'_1)}{dE'_1} = -1 - \frac{\left[ E'_1 \left( 2 - C(E_1'^2 - m_1^2)^{-\frac{1}{2}} \right) \right]}{2 \left[ E_1'^2 + B - C\sqrt{E_1'^2 - m_1^2} \right]^{\frac{1}{2}}} \quad (4.49)$$

As we can see Eq.(4.48) relates  $E'_1$  to the initial electron energy and scattering angle. Note also that Eq.(4.49) is evaluated at  $E_1^*$ . Thus the differential cross-section given in Eq.(4.43) for the electron scattered through the solid angle element  $d\Omega_e$  becomes

$$\frac{d\bar{\sigma}_e}{d\Omega_e} = \frac{m_1^2 m_2}{(2\pi)^2} \frac{1}{\sqrt{E_1^2 - m_1^2}} \frac{|\mathcal{M}_{fi}|^2}{\left| \frac{df(E'_1)}{dE'_1} \right|_{E'_1=E_1^*}} \frac{h(E_1^*)}{\left| \frac{df(E'_1)}{dE'_1} \right|_{E'_1=E_1^*}} \quad (4.50)$$

Thus Eq.(4.50) is the one we compare to the Mott cross-section for the electron scattered from the point proton. Apparently, since most scattering experiments use unpolarized incident and target particles, we can also compare the outcomes of our calculation to the results of those appropriate experiments. But in this project the comparison is made only with the Rutherford and the Mott predictions for the scattering from a stationary point charge. The Mott cross-section is given by

$$\frac{d\sigma_e}{d\Omega_e} = \frac{\alpha^2}{4|\mathbf{p}_1|^2 \beta^2 \sin^4\left(\frac{\theta}{2}\right)} \left( 1 - \beta^2 \sin^2\left(\frac{\theta}{2}\right) \right) \quad (4.51)$$

where  $\beta = \frac{|\mathbf{p}_1|}{E_1}$  in natural units. It reduces to the Rutherford formula in the limit  $\beta \rightarrow 0$ .

## 4.5 Proton Form Factors and Electron-proton Elastic Scattering

The real proton is not a point particle and that is why Mott cross-section does not fit to experimental data for high energy probe by electrons [CR56, Jaf98]. So this suggests that we need

to add some other information regarding the proton structure to our calculations. In the previous section, we constructed the spin-averaged lepton and hadronic tensors  $L_{\mu\nu}$  and  $W^{\mu\nu}$  for the unpolarized e-p scattering in the framework of QED. Unlike the lepton tensor for the electron-photon vertex, the hadronic tensor for proton-photon vertex is not completely understood and specified by QED. In this section, therefore, we generalize from point proton to composite proton by writing down the most general form of the second-rank tensor. So it is at this point that the concept of form factors enter the model.

Because of the lack of a full understanding of the structure of hadrons, QED instead uses symmetry principles: Lorentz invariance and  $\mathcal{EM}$  current conservation (gauge invariance) to obtain the general form of  $W^{\mu\nu}$  [AH03]. So we use form factors to model the spacial extension of the particle whenever our knowledge about its internal structure is insufficient. From now on, we consider the proton as having structure in the finite volume. The hadronic tensor (given in Eq.(4.35)), which was extracted from the Feynman diagram (figure 4.4), does not take the structure of the proton into account; and it can also be rewritten as

$$W^{\mu\nu} = \frac{1}{2m_2^2} \left( p_2^\mu p_2'^\nu + p_2'^\mu p_2^\nu + \frac{q^2}{2} g^{\mu\nu} \right) \quad (4.52)$$

As we can see it takes three arguments  $p_2$ ,  $p_2'$ , and  $q$ . But they are not independent variable because  $p_2' = p_2 + q$ . Thus we notice that the hadronic tensor is the function of only two independent variables  $p_2$  and  $q$ . That is,  $W^{\mu\nu} \rightarrow W^{\mu\nu}(p_2, q)$ . Moreover, based on Eq.(4.33) we can deduce that the unknown quantity  $W^{\mu\nu}(p_2, q)$  is certainly a second rank tensor. So in order to obtain the general form for the hadronic tensor, we must expand  $W^{\mu\nu}$  in terms of the basis constructed from the two independent four-vector variables  $p_2$  and  $q$  and metric tensor  $g^{\mu\nu}$  [AH03]. The available basis to build second-rank tensor are  $g^{\mu\nu}$ ,  $p_2^\mu p_2^\nu$ ,  $q^\mu q^\nu$ ,  $p_2^\mu q^\nu + q^\mu p_2^\nu$ , and  $p_2^\mu q^\nu - q^\mu p_2^\nu$ . We note that the first four basis are symmetric and parity invariant, whereas, the last one is anti-symmetric (i.e.  $p_2^\nu q^\mu - q^\nu p_2^\mu = -(p_2^\mu q^\nu - q^\mu p_2^\nu)$  swapping indices does not preserve the original expression). Thus the most general hadronic tensor may be written as

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{m_2^2} p_2^\mu p_2^\nu + \frac{W_3}{m_2^2} (p_2^\mu q^\nu - q^\mu p_2^\nu) + \frac{W_4}{m_2^2} (p_2^\mu q^\nu + q^\mu p_2^\nu) + \frac{W_5}{m_2^2} q^\mu q^\nu \quad (4.53)$$

where  $W_i$  are unknown functions of  $q^2$ . For convenience, by collecting symmetric terms and anti-symmetric terms together, we can rewrite Eq.(4.53) as

$$W^{\mu\nu} = W_S^{\mu\nu} + W_A^{\mu\nu} \quad (4.54)$$

Thus

$$L_{\mu\nu} W^{\mu\nu} = L_{\mu\nu} (W_S^{\mu\nu} + W_A^{\mu\nu}) \quad (4.55)$$

For an unpolarized electron,  $L_{\mu\nu}$  is completely symmetric. Since the contraction of symmetric tensor and anti-symmetric tensor is zero, we drop the second term at the right hand side of eq.(4.55). In other words, the anti-symmetric term  $\frac{W_3}{m_2^2} (p_2^\mu q^\nu - q^\mu p_2^\nu)$  does not contribute to  $|\overline{\mathcal{M}}_{fi}|^2$  [Gri87, HM84]. Then the possible form for  $W^{\mu\nu}$  is

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{m_2^2} p_2^\mu p_2^\nu + \frac{W_4}{m_2^2} (p_2^\mu q^\nu + q^\mu p_2^\nu) + \frac{W_5}{m_2^2} q^\mu q^\nu \quad (4.56)$$

These four functions are not independent [Gri87]. It is known that the  $W^{\mu\nu}$  is the product of two proton currents; and consequently it satisfies the current conservation law at the hadronic vertex (or it is gauge invariant). That is,

$$q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0 \quad (4.57)$$

We can find their relation by plugging Eq.(4.56) into Eq.(4.57). Hence

$$q_\mu W^{\mu\nu} = -W_1 q^\nu + (p_2 \cdot q) \frac{W_2}{m_2^2} q^\nu + \frac{W_4}{m_2^2} ((p_2 \cdot q) q^\nu + q^2 p_2^\nu) + q^2 \frac{W_5}{m_2^2} q^\nu = 0 \quad (4.58)$$

Now by factoring  $p^\nu$  and  $q^\nu$  out, we get

$$\left( p_2 \cdot q \frac{W_2}{m_2^2} + q^2 \frac{W_4}{m_2^2} \right) p^\nu + \left( -W_1 + p_2 \cdot q \frac{W_4}{m_2^2} + q^2 \frac{W_5}{m_2^2} \right) q^\nu = 0 \quad (4.59)$$

Eq.(4.59) holds if and only if the coefficients of  $p^\nu$  and  $q^\nu$  vanish for  $p^\nu$  and  $q^\nu$  are independent four-vectors. That is,

$$\begin{aligned} p_2 \cdot q \frac{W_2}{m_2^2} + q^2 \frac{W_4}{m_2^2} &= 0 \\ -W_1 + p_2 \cdot q \frac{W_4}{m_2^2} + q^2 \frac{W_5}{m_2^2} &= 0 \end{aligned} \quad (4.60)$$

which imply that

$$W_4 = -\frac{p_2 \cdot q}{q^2} W_2 \quad \text{and} \quad W_5 = \frac{m_2^2 W_1}{q^2} + \left( \frac{p_2 \cdot q}{q^2} \right)^2 W_2 \quad (4.61)$$

Then substituting Eq.(4.61) back into Eq.(4.56) we get

$$W^{\mu\nu} = W_1 \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2}{m_2^2} \left( p_2^\mu p_2^\nu - \frac{p_2 \cdot q}{q^2} (p_2^\mu q^\nu + q^\mu p_2^\nu) + \left( \frac{p_2 \cdot q}{q^2} \right)^2 q^\mu q^\nu \right) \quad (4.62)$$

The expression inside the parenthesis of the second term of Eq.(4.62) can be factorized into the form  $(p_2^\mu - A q^\mu)(p_2^\nu - B q^\nu)$ . So after expanding this form and making a comparison with the coefficient of  $W_2$  in Eq.(4.62) we find that  $A = B = \frac{p_2 \cdot q}{q^2}$ . Thus

$$W^{\mu\nu} = W_1 \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2}{m_2^2} \left( p_2^\mu - \frac{p_2 \cdot q}{q^2} q^\mu \right) \left( p_2^\nu - \frac{p_2 \cdot q}{q^2} q^\nu \right) \quad (4.63)$$

Had the anti-symmetric term not been dropped, we would have used  $q_\nu W^{\mu\nu} = 0$  to show that  $W_3 = 0$  which would have supported our previous argument. By using the lepton tensor in Eq.(4.36), we contract the lepton tensor and the hadronic tensor, and by applying the current conservation law  $q^\mu L_{\mu\nu} = q^\nu L_{\mu\nu} = 0$ , the invariant amplitude becomes

$$\begin{aligned} |\overline{\mathcal{M}}_{fi}|^2 &= \left( \frac{e^2}{q^2} \right)^2 L_{\mu\nu} W^{\mu\nu} \\ &= \left( \frac{e^2}{q^2} \right)^2 \left[ W_1 (p_1 \cdot p_1' - 2m_1^2) + \frac{W_2}{m_2^2} \left( (p_1 \cdot p_2)(p_2 \cdot p_1') + \frac{q^2}{4} m_2^2 \right) \right] \end{aligned} \quad (4.64)$$

Since we are working in the lab frame, by applying Eq.(4.38), we finally obtain

$$|\overline{\mathcal{M}_{fi}}|^2 = \frac{e^4}{q^4} \frac{E_1 E'_1}{m_1^2} \left( 2W_1 \sin^2 \left( \frac{\theta}{2} \right) + W_2 \cos^2 \left( \frac{\theta}{2} \right) \right) \quad (4.65)$$

where  $E'_1 = E_1 / (1 + (2E_1/m_2) \sin^2(\theta/2))$  and we neglect the electron rest-mass  $m_1$  in the numerators. Thus the differential cross-section can be expressed as

$$\left( \frac{d\sigma_e}{d\Omega_e} \right)_{lab} = \left( \frac{d\sigma_e}{d\Omega_e} \right)_{Mott} \frac{E'_1}{E_1} \left( W_2 + 2W_1 \tan^2 \left( \frac{\theta}{2} \right) \right) \quad (4.66)$$

This is known as Rosenbluth formula [GR03, Gri87]. Thus  $W_1(q^2)$  and  $W_2(q^2)$  are form factors. The measured quantity from the scattering experiment is the differential cross section. So by undertaking a series of e-p elastic experiment one can determine these form factors at various  $q^2$ . This is done by varying  $E_1$  and  $\theta$  for a fixed value of  $q^2$  and carrying out the same procedure for other values of  $q^2$ . Then we plot  $\frac{(d\sigma_e/d\Omega_e)_{lab}}{(d\sigma_e/d\Omega_e)_{Mott}}$  against  $\tan^2 \left( \frac{\theta}{2} \right)$  which is a straight line with slope  $2W_1$  and intersects the axis of  $\frac{(d\sigma_e/d\Omega_e)_{lab}}{(d\sigma_e/d\Omega_e)_{Mott}}$  at  $W_2$ . This may require extrapolation. Actually, these two form factors can be combined to give the electric and magnetic form factors which, respectively, are used to describe the spacial distribution of the charge and the magnetic moment of the proton [BJ95].



# 5. Simulation, Discussion and Conclusion

## 5.1 Simulation and Discussion

For the e-p elastic scattering, we compare the derived differential cross-section to Rutherford and Mott cross-sections by regarding the target proton as the Dirac point particle. In fact, this treatment holds in the framework of Quantum Electrodynamics in which the structure of the hadron is unknown. Rutherford prediction does not take into account the spin of the particles. So in order to get the Rutherford cross-section for e-p scattering we have to assume that both are spin-0 particles and with relativistic correction it is given by [GR03, Ter06]

$$\frac{d\sigma_e}{d\Omega_e} = \frac{\alpha^2}{4|\mathbf{p}_1|^2\beta^2 \sin^4\left(\frac{\theta}{2}\right)} \quad (5.1)$$

where  $E_1$  is the energy of incident electron in the lab frame. Eq.(5.1) only tells us that the scattering is from the point-charge target. The Mott cross-section is a modification of Rutherford cross-section for the scattering from the point-like target by replacing spin-0 by spin- $\frac{1}{2}$  and was given by Eq.(4.51).

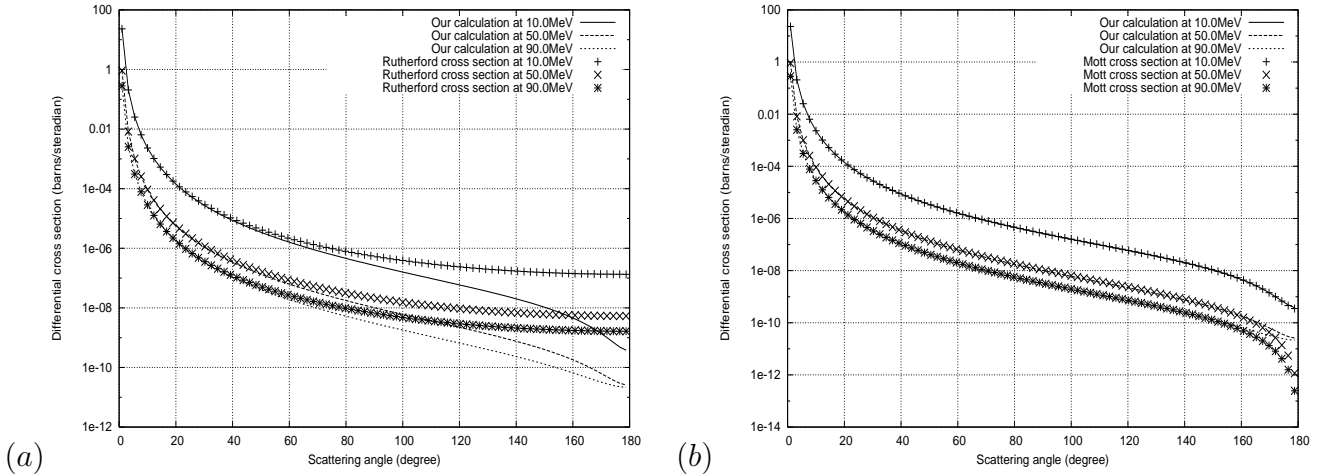


Figure 5.1: Comparison of the calculated cross-section with (a) Rutherford differential cross-section and (b) Mott differential cross-section for 10 MeV, 50 MeV, and 90 MeV incident energies of electron.

In our simulations, we use the final result of our calculation of differential cross-section written in Eq.(4.50). For the invariant amplitude we take the expression given in Eq.(4.37). Since we are working in the lab frame, we also use the values given for momenta in Eq.(4.38) as inputs in our Python piece of code. The study of the e-p elastic process is done in the energy range  $m_1 < E_1 < m_2$  of the incident electron. The mass of electron is  $m_1 \approx 0.50$  MeV and the mass of the proton is  $m_2 \approx 939.0$  MeV. Moreover, we choose to work in natural units so that mass, momentum and energy have the same units, MeV. The unit of space-time coordinates is  $(\text{MeV})^{-1}$ . Note also that  $|\mathcal{M}_{fi}|^2$  and  $d\bar{\sigma}_e/d\Omega_e$  are evaluated at  $E'_1 = E_1^*$  given in Eq.(4.48) to

obey the requirement of energy conservation. The units of differential cross-section and scattering angle are barns per steradian and degree, respectively. We fix the initial energy of electron, in order to observe the dependence of differential cross-section on the scattering angle and the vice versa.

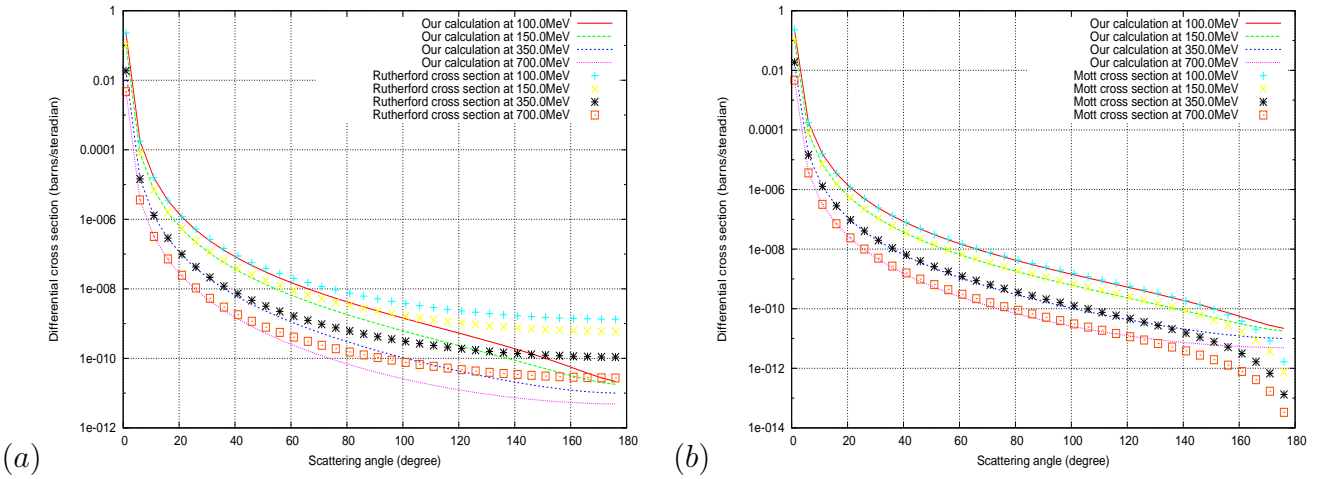


Figure 5.2: Comparison of the calculated cross-section with (a) Rutherford differential cross-section and (b) Mott differential cross-section for 100 MeV, 150 MeV, 350 MeV and 700 MeV incident energies of electron.

As we can see from figure 5.1 (a) and 5.2 (a), the calculated cross-section shows deviation from the Rutherford cross-section, whereas, in figure 5.1 (b) and 5.2 (b) our result coincides with the Mott cross-section as expected. But for large  $E_1$ , our results separates from Mott cross section at larger angles closer to  $180^\circ$ . As  $\theta$  increases  $d\bar{\sigma}_e/d\Omega_e$  decreases.

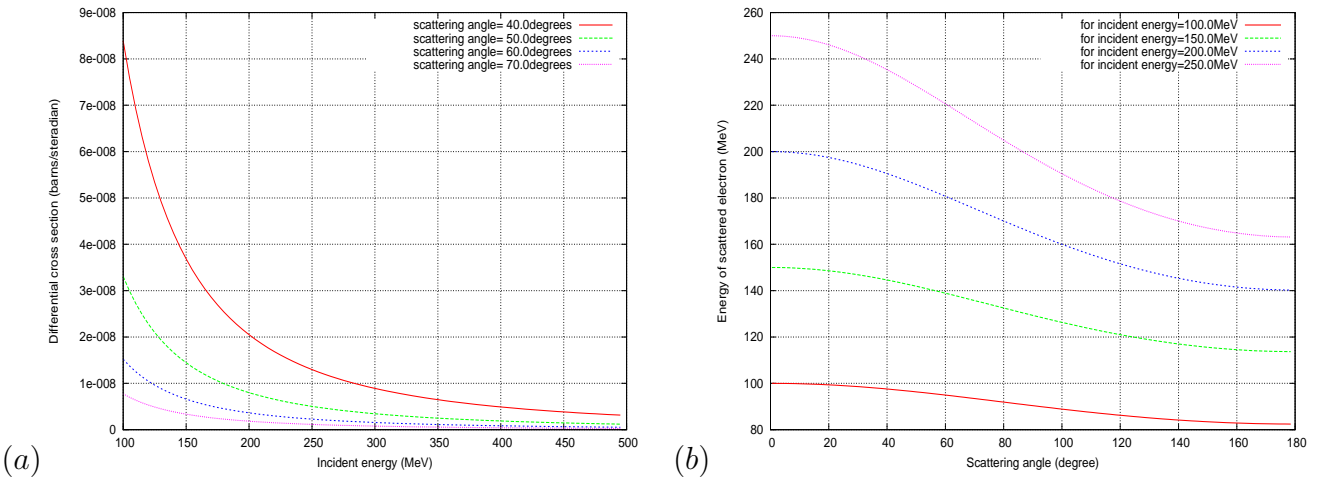


Figure 5.3: (a) Angular distribution of differential cross-section versus energy of the initial electron. (b) Simulation for energy the scattered electron versus the scattering angle at incident energies 100 MeV, 150 MeV, 200 MeV, and 250 MeV

Figure 5.3 (a) is the illustration of the dependence of differential cross-section on the initial electron beam energy. The simulation is done for  $\theta = 40, 50, 60$ , and  $70$  degrees. Figure

5.3 (b) is the plot of the energy of scattered electrons as a function of scattering angle which was calculated before in chapter four (see Eq.(4.48)). For a particular incident electron energy, as the scattering angle increases, the scattered electron energy decreases. In other words, this means that whenever there is relatively high four-momentum transfer from electron to the proton during the moment of interaction, the electron scattered at relatively large angle. Note that the maximum scattering angle is  $180^\circ$ .

## 5.2 Conclusion

In the relativistic scattering process, the Feynman diagram and the Feynman rules are the powerful tools used to evaluate the invariant amplitude which is proportional to the scattering cross-section. The result of this essay is found to be consistent with the Mott cross-section for the electron scattered from the spin- $\frac{1}{2}$  point-like Dirac proton particularly when incident electron energy,  $E_1$  is much less than the rest-mass of the proton,  $m_2$ . It was also shown that the scattering cross-section depends on the two independent quantities: the energy of incident beam of electron and the scattering angle. In addition, in the lab frame, once the initial state four-vector momenta and the scattering angle are known, it is possible to determine the final state four-vector momenta. The other important outcome is that a relatively large number of electrons are scattered at small scattering angles. As the scattering angle increases at a fixed value of  $E_1$ , the number of scattered particles decreases. Also, at a particular scattering angle, as the incident beam energy increases, the differential cross-section decreases.

Moreover, by applying symmetric principle, one can obtain more general way of describing the internal structure of the proton from the study the e-p elastic scattering. However, the detailed information of the structure of the proton can be found from the QCD treatment of e-p inelastic scattering. Therefore, this understanding can be extended to study scattering processes involving more than two particles in the final states. Finally, the writer wants to undertake further theoretical and experimental study of high energy electron-proton elastic and inelastic scattering in order to investigate the quark model of nucleons that describes the proton as the bound state of three quarks.

# Appendix A. Background Mathematics

## A.1 The Dirac Delta Function

The Dirac delta function,  $\delta(x)$ , is an infinitely high, infinitesimally narrow spike at the origin [Gri87], with area of unity; mathematically it is defined as

$$\delta(x - x_o) = \begin{cases} \infty & \text{if } x = x_o \\ 0 & \text{if } x \neq x_o \end{cases} \quad (\text{A.1})$$

and

$$\int_{-\infty}^{\infty} \delta(x - x_o) dx = 1 \quad (\text{A.2})$$

## Properties

1). It follows from the definition that for any ordinary function  $f(x)$

$$\int_{-\infty}^{\infty} f(x) \delta(x - x_o) dx = f(x_o) \int_{-\infty}^{\infty} \delta(x - x_o) dx = f(x_o) \quad (\text{A.3})$$

and we can also have

$$\int_{x_1}^{x_2} f(x) \delta(x - x_o) dx = \begin{cases} f(x_o) & \text{if } x_1 < x_o < x_2 \\ 0 & \text{if } x_o > x_2 \text{ or } x_o < x_1 \end{cases} \quad (\text{A.4})$$

The three dimensional delta function  $\delta^3(\vec{x})$  can be defined as

$$\delta^3(\vec{x}) = \delta(x) \delta(y) \delta(z), \quad (\text{A.5})$$

and for any function  $f(\vec{x}) = f(x, y, z)$  integrated over a volume  $V$  gives

$$\int_V f(\vec{x}) \delta^3(\vec{x} - \vec{x}_o) = \begin{cases} f(\vec{x}_o) & \text{if } \vec{x}_o \text{ lies inside } V \\ 0 & \text{if } \vec{x}_o \text{ lies outside } V \end{cases} \quad (\text{A.6})$$

2). If  $k$  is a non-zero real number, then  $\int \delta(kx) dx = \frac{1}{|k|} \int \delta(x) dx$  and for any arbitrary function  $f(x)$  we have

$$\int f(x) \delta(k(x - x_o)) dx = \frac{f(x_o)}{|k|} \int \delta(x - x_o) dx = \frac{f(x_o)}{|k|} \quad (\text{A.7})$$

3). If function  $g(x)$  is an argument of delta function and if  $x_o$  is the solution for  $g(x) = 0$ , then it follows Eq.(A.7) that

$$\int \delta[g(x)] dx = \int \frac{1}{|g'(x)|_{x=x_o}} \delta(x - x_o) dx = \frac{1}{|g'(x_o)|} \quad (\text{A.8})$$

and for any continuous function  $f(x)$  at  $x = x_o$

$$\int f(x) \delta[g(x)] dx = f(x_o) \int \frac{1}{|g'(x)|_{x=x_o}} \delta(x - x_o) dx = \frac{f(x_o)}{|g'(x_o)|} \quad (\text{A.9})$$

## A.2 Relativistic Notation and Four-vector Formalism

Any point or event in Minkowski space or space-time is specified by  $x^\mu$ ,  $\mu = 0,1,2,3$  and explicitly written as [Bra05]

$$\begin{aligned} x^\mu &= (x^0, x^1, x^2, x^3) = (ct, x, y, z) \\ &= (ct, \vec{x}) \end{aligned} \quad (\text{A.10})$$

Any mathematical object  $a^\mu$  is called a four-vector. Four-vector, with upper index, is called contravariant and one with, lower index, is called covariant vector; and the contravariant is related to its covariant counterpart through the metric tensor  $g_{\mu\nu}$  as

$$x_\mu = \sum_\nu g_{\mu\nu} x^\nu; \quad x^\mu = \sum_\nu g^{\mu\nu} x_\nu \quad (\text{A.11})$$

where the Minkowski metric is defined as

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{A.12})$$

Here we introduce Einstein summation convention that drops the summation symbol for expressions with repeated indices. That is  $\sum_\mu g_{\mu\nu} x^\mu \rightarrow g_{\mu\nu} x^\mu$ . Thus Eq.(A.11) rewritten as

$$x_\mu = g_{\mu\nu} x^\nu; \quad x^\mu = g^{\mu\nu} x_\nu \quad (\text{A.13})$$

Minkowski metric  $g_{\mu\nu}$  is defined by the relation

$$g^{\mu\nu} g_{\mu\alpha} = \delta_\alpha^\nu I \quad (\text{A.14})$$

where  $\delta_\alpha^\nu$  is the Kronecker delta and  $I$  is  $4 \times 4$  identity metric. It is worth noting that here also we used the summation convention.

If  $a^\mu$  and  $b^\mu$  are four-vectors, their scalar product is given by

$$a \cdot b = (a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3) = g_{\mu\nu} a^\nu b^\mu = a_\mu b^\mu \quad (\text{A.15})$$

Similarly, the scalar product of  $x_\mu$  with itself is

$$x \cdot x = g_{\mu\nu} x^\nu x^\mu = (c^2 t^2 - \vec{x} \cdot \vec{x}) \quad (\text{A.16})$$

The covariant gradient is obtained by differentiating with respect to contravariant four-vector  $x^\mu$ . That is,

$$\begin{aligned} \partial_\mu &= \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial ct}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \\ &= \left( \frac{\partial}{\partial ct}, \vec{\nabla} \right) \end{aligned} \quad (\text{A.17})$$

and

$$\partial^\mu = g^{\mu\nu} \partial_\nu = \left( \frac{\partial}{\partial ct}, -\vec{\nabla} \right) \quad (\text{A.18})$$

By taking their scalar product we get

$$\square = \partial^\mu \partial_\mu = \left( \frac{\partial^2}{c^2 \partial t^2} - \vec{\nabla}^2 \right) \quad (\text{A.19})$$

called the d'Alembertian operator.

## A.3 Dirac Algebra of Gamma Matrix and Trace Theorems

### A.3.1 Pauli Matrices

These are the three Hermitian, unitary  $2 \times 2$  matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.20})$$

**product rules**

$$\begin{aligned} \sigma_1^2 = \sigma_2^2 = \sigma_3^2 &= I \\ \{\sigma_i, \sigma_j\} &= 2\delta_{ij} \quad (\text{anticommutation relation}) \end{aligned} \quad (\text{A.21})$$

### A.3.2 Dirac Matrices

These are four unitary traceless  $4 \times 4$  matrices

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \quad (\text{A.22})$$

Here  $I$  is  $2 \times 2$  identity matrix and  $0$  is  $2 \times 2$  matrix of zero. and these matrices satisfy anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I \quad (\text{A.23})$$

where  $g^{\mu\nu}$  is the metric tensor. We notice that  $g_{\mu\nu} g^{\mu\nu} = 4$ . Now we define the fifth Dirac metric out of the four ones

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (\text{A.24})$$

$(\gamma^0)^2 = (\gamma^5)^2 = I$ , and  $(\gamma^k)^2 = -I$ , where  $k = 1, 2, 3$ . And  $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ . Other properties of gamma matrices are

$$\overline{\gamma^\mu} = \gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu; \quad \overline{i\gamma^5} = \gamma^0(\gamma^5)^\dagger\gamma^0 = i\gamma^5; \quad \overline{\gamma^\mu\gamma^5} = \gamma^0(\gamma^5)^\dagger(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu\gamma^5 \quad (\text{A.25})$$

### A.3.3 Trace Theorems

Here we recall the slash notation  $\gamma \cdot a = \not{a}$  we have introduced in chapter three.

#### Theorem 1

$$\text{Tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu} \quad (\text{A.26})$$

Proof:

we use the anticommutation relation  $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$ .

$$\begin{aligned} \text{Tr}[\gamma^\mu \gamma^\nu] &= \frac{1}{2} \text{Tr}[\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu] \\ &= g^{\mu\nu} \text{Tr}[I] \\ &= 4g^{\mu\nu} \end{aligned} \quad (\text{A.27})$$

And

$$\text{Tr}[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta] = 4(g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha}) \quad (\text{A.28})$$

Proof: We use the the anticommutation relation  $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$ .

$$\begin{aligned} \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta] &= \text{Tr}[(2g^{\mu\nu} I - \gamma^\nu \gamma^\mu) \gamma^\alpha \gamma^\beta] \\ &= 2g^{\mu\nu} \text{Tr}[\gamma^\alpha \gamma^\beta] - \text{Tr}[\gamma^\nu \gamma^\mu \gamma^\alpha \gamma^\beta] \\ &= 8g^{\mu\nu} g^{\alpha\beta} - \text{Tr}[\gamma^\nu \gamma^\mu \gamma^\alpha \gamma^\beta] \end{aligned} \quad (\text{A.29})$$

We do the same for  $\text{Tr}[\gamma^\nu \gamma^\mu \gamma^\alpha \gamma^\beta]$ . Now we apply the anticommutation relation for  $\gamma^\mu \gamma^\alpha$  also, then we get

$$\begin{aligned} \text{Tr}[\gamma^\nu \gamma^\mu \gamma^\alpha \gamma^\beta] &= \text{Tr}[\gamma^\nu (2g^{\mu\alpha} I - \gamma^\alpha \gamma^\mu) \gamma^\beta] \\ &= 8g^{\mu\alpha} g^{\nu\beta} - \text{Tr}[\gamma^\nu \gamma^\alpha \gamma^\mu \gamma^\beta] \end{aligned} \quad (\text{A.30})$$

Once again by using  $\gamma^\mu \gamma^\beta + \gamma^\beta \gamma^\mu = 2g^{\mu\beta} I$ , we get

$$\text{Tr}[\gamma^\nu \gamma^\alpha \gamma^\mu \gamma^\beta] = 8g^{\mu\beta} g^{\nu\alpha} - \text{Tr}[\gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\mu] \quad (\text{A.31})$$

By putting all these together, we get

$$\text{Tr}[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta] = 8g^{\mu\nu} g^{\alpha\beta} - 8g^{\mu\alpha} g^{\nu\beta} + 8g^{\mu\beta} g^{\nu\alpha} - \text{Tr}[\gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\mu] \quad (\text{A.32})$$

By recalling the cyclic property, we get

$$\text{Tr}[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta] = 4(g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha}) \quad (\text{A.33})$$

#### Theorem 2

$$\text{The trace of an odd number of } \gamma \text{ matrices is zero.} \quad (\text{A.34})$$

Proof: For n odd,

$$\begin{aligned} \text{Tr}[\not{a}_1 \cdots \not{a}_n] &= \text{Tr}[\not{a}_1 \cdots \not{a}_n \gamma^5 \gamma^5] \\ &= \text{Tr}[\gamma^5 \not{a}_1 \cdots \not{a}_n \gamma^5] \\ &= (-1)^n \text{Tr}[\not{a}_1 \cdots \not{a}_n \gamma^5 \gamma^5] \\ &= 0 \end{aligned} \quad (\text{A.35})$$

**Theorem 3**

For any arbitrary four-vectors  $a$  and  $b$ ,

$$\text{Tr}[\not{a}\not{b}] = 4a \cdot b \quad (\text{A.36})$$

Proof:

$$\begin{aligned} \text{Tr}[\not{a}\not{b}] &= \text{Tr}[\not{b}\not{a}] = \frac{1}{2} \text{Tr}[\not{a}\not{b} + \not{b}\not{a}] \\ &= \frac{1}{2} \text{Tr}[a_\mu b_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)] \\ &= \frac{1}{2} \text{Tr}[a_\mu b_\nu (2g^{\mu\nu})] \\ &= \text{Tr}[a \cdot b] \\ &= a \cdot b \text{Tr}[I] \\ &= 4a \cdot b \end{aligned} \quad (\text{A.37})$$

**Theorem 4**

For any arbitrary four-vectors  $a$ ,  $b$ ,  $c$  and  $d$ ,

$$\text{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)] \quad (\text{A.38})$$

Proof: From theorem 3 we have  $\not{a}\not{b} + \not{b}\not{a} = 2a \cdot b$ . By using this and the cyclic property  $\text{Tr}[\not{a}_1 \cdots \not{a}_n] = \text{Tr}[\not{a}_n \not{a}_1 \cdots \not{a}_{n-1}]$

$$\text{Tr}[\not{a}\not{b}\not{c}\not{d}] = 2a \cdot b - \text{Tr}[\not{b}\not{c}\not{d}\not{a}] \quad (\text{A.39})$$

Repeating this trick a number of times gives

$$\begin{aligned} \text{Tr}[\not{a}\not{b}\not{c}\not{d}] &= 2(a \cdot b)4(c \cdot d) - 2((a \cdot b)4(a \cdot c))\text{Tr}[\not{b}\not{c}] + \text{Tr}[\not{b}\not{c}\not{d}\not{a}] \\ &= 8(a \cdot b)(c \cdot d) - 8(a \cdot c)(b \cdot d) + 8(b \cdot c)(a \cdot d) - \text{Tr}[\not{b}\not{c}\not{d}\not{a}] \end{aligned} \quad (\text{A.40})$$

The rest is trivial. By using Eq.(A.28) we can also show that

$$\text{Tr}[\not{a}\gamma^\mu \not{b}\gamma^\nu] = 4(a^\mu b^\nu + a^\nu b^\mu - a \cdot b g^{\mu\nu}) \quad (\text{A.41})$$



# Appendix B. The Python Code

```
#!/usr/lib/python
from Numeric import *
import math
import Gnuplot
#####
#This function returns the scalar product two four-vectors by using metric tensor
#####
def metric_tensor():#Eq.(A.12)
    m_tensor=zeros([4,4])
    for i in range(len(m_tensor[0])):
        for j in range(len(m_tensor[0])):
            if i == j:
                if i==0:
                    m_tensor[i][j]=1
                else:
                    m_tensor[i][j]=-1
            else:
                m_tensor[i][j]
    return m_tensor

def contractionM(x, y):
    y_transpos = reshape(y, (y.shape[1], y.shape[0]))
    y_con = matrixmultiply(metric_tensor(), y_transpos)
    result = matrixmultiply(x, y_con)[0][0]
    return result
#####
#This function returns the square of four-momentum transfer  $q^2$ 
#####
def Four_momentum_square(a,b):
    q = a-b
    q2 = contractionM(q,q)
    return q2
#####
#The Invariant Amplitude (matrix element)
#####
def matrix_element():#Eq.(4.37)
    ga = Four_momentum_square(p1i, p1f)
    sta0 = ga**2 ; sta1 = (4*math.pi*alpha)**2
    sta2 = 2*(m1*m2)**2; sta3 = (p3p4)*(p1p2)
    sta4 = (p2p3)*(p1p4); sta5 = m2**2*(p1p3)
    sta6 = m1**2*(p2p4)
    ME2 = sta1*(sta3 +sta4 - sta5 -sta6 + sta2)/(sta0*sta2)
    return ME2
#####
```

---

```

#The Differentail Cross-Section (our result)
#*****
def Scattering_cross_section():#Eq.(4.50)
    car0 = (m1**2*m2)/(2*math.pi)**2
    car1 = math.sqrt(E1i**2 - m1**2)
    car2 = matrix_element()
    H = math.sqrt(E1f**2 - m1**2)/E2f
    Df = abs(-(2*E1f-(cc*E1f)/math.sqrt(E1f**2-m1**2))
            /(2*math.sqrt(-cc*math.sqrt(E1f**2-m1**2)+E1f**2+bb))-1)#Eq.(4.49)
    Cross_section = (car0*car2*H)/(Df*car1)
    return Cross_section
#*****
#This one returns the Rutherford's cross-section
#*****
def Rutherford_cs():#Eq.(5.1)
    rr = alpha=1.0/137; ad = arad/2
    dd = 4*(E1i**2 - m1**2)**2*(math.sin(ad))**4
    ruth = (rr**2*E1i**2)/dd
    return ruth
#*****
#This one returns the Mott cross-section
#*****
def Mott_sc():#Eq.(4.51)
    mo2 = Rutherford_cs()*(1-((E1i**2 - m1**2)/(E1i**2))
            *(math.sin(arad/2))**2)
    return mo2
#*****
#This one returns all the outputs such as differential cross-sections, energy
#of the scattered electron, the four-momenta. Few modifiacation is required.
#we choose to work in natural units
#*****
gn = Gnuplot.Gnuplot(persist=1)
m1 = 0.50#MeV
m2 = 939.0#MeV
alpha = 1.0/137#fine structure constant
Start = 10.0
grand0 = []; grandM = []; grandR = []
for E1i in arange(Start, 1000.0, 80.0):
    plix = 0.0; pliy = 0.0; pliz = math.sqrt(E1i**2-m1**2)
    E2i = m2; p2ix = p2iy = p2iz = 0.0
    p1i = array([[E1i, plix, pliy, pliz]])
    p2i = array([[E2i, p2ix, p2iy, p2iz]])
    list_scato = []; list_scatoR = []; list_scatoM=[];list_scatoE=[]
    for angle in arange(1.0,180.0, 2.25):
        arad = angle*math.pi/180.0
        aa = E1i + E2i; bb = E1i**2 - 2*m1**2 + m2**2
        cc = 2*math.sqrt(E1i**2 - m1**2)*math.cos(arad)

```

---

```

#E1f=(cc*math.sqrt((cc**2-4*aa**2)*m1**2+bb**2-2*aa**2*bb+aa**4)
      +2*aa*bb-2*aa**3)/(cc**2-4*aa**2)#this does't obey conservation
E1f = -(cc*math.sqrt((cc**2-4*aa**2)*m1**2+bb**2-2*aa**2*bb+aa**4)
      -2*aa*bb+2*aa**3)/(cc**2-4*aa**2)#Eq. (4.48)
E2f = math.sqrt(bb + E1f**2 - cc*math.sqrt(E1f**2 - m1**2))
p1fx = math.sqrt(E1f**2 - m1**2)*math.sin(arad); p1fy = 0
p1fz = math.sqrt(E1f**2 - m1**2)*math.cos(arad)
p1f = array([[E1f, p1fx, p1fy, p1fz]])
p2f = p1i+p2i-p1f#conservation of for momentum at the calculated E1f.
p2fx = p2f[0][1]; p2fy = p2f[0][2]; p2fz = p2f[0][3]
p2f = array([[E2f, p2fx, p2fy, p2fz]])
p1p2 = contractionM(p1i, p2i); p1p3 = contractionM(p1i, p1f)
p1p4 = contractionM(p1i, p2f); p2p3 = contractionM(p2i, p1f)
p2p4 = contractionM(p2i, p2f); p3p4 = contractionM(p1f, p2f)
our_cs = Scattering_cross_section()#Our calculation
ruth = Rutherford_cs()#Rutherford cross section
mott = Mott_sc()#Mott cross section
list_scato.append([angle, our_cs])
list_scater.append([angle, ruth])
list_scaterm.append([angle, mott])

pdata0 = Gnuplot.Data(list_scato, with="lines", title =
                      'Our calculation at '+str(E1i)+"MeV")
pdataM = Gnuplot.Data(list_scaterm, with="points", title =
                      'Mott cross section at '+str(E1i)+"MeV")
pdataR = Gnuplot.Data(list_scater, with="points", title =
                      'Rutherford cross section at '+str(E1i)+"MeV")

grand0.append(pdata0)
grandM.append(pdataM)
grandR.append(pdataR)

gn('set xlabel "Scattering angle (degree)"'); gn('set logscale y')
gn('set grid')
gn('set ylabel "Differential cross section (barns/steradian)"')
gn.plot(grand0[0],grand0[1],grand0[2],grand0[4])
gn.replot(grandM[0],grandM[1],grandM[2],grandM[4])
#gn.replot(grandR[0],grandR[1],grandR[2],grandR[4])

```

# Acknowledgement

First of all, I would like to lift the name of my lord on high. He is my shepherd. Jesus has led and comforted my life with his mighty hands and heavenly blesses and enabled me to bring this project to light. He is the prince of peace, the everlasting father, and the almighty God. His kingdom shall not end.

I am grateful to the genuine supervision of Dr. Brandon Van der Ventel, who spent his invaluable time and effort guiding me in the proper direction as well as developed a strong confidence in me to do my level best. Without him all my effort would have been in vain.

I would like to give many thanks to all AIMS staffs especially Henry Amuasi who committed himself to share my burdens throughout this project and gave me constructive comments and suggestions, Jan Groenewald who helped me a lot in fixing the problem with latex and also Ambrose Chongo who edited the earlier draft of this paper. I also appreciate Dr. Sam Webster who tutored me for more than a semester in a very friendly way.

I would also like to express my heart felt gratitude to the founders of AIMS especially Prof. Neil Turok and Prof. Fritz Hahne through whom the widows of opportunity opened for African students.

Finally, I am indebted to Girma Goro and all AIMS2006/7 students who have shown me the spirit of friendship, forgiveness and given me courage during my stay at AIMS.

# Bibliography

- [AH03] I.J.R. Aitchison and A.J.G. Hey, *Gauge theories in particle physics*, Institute of Physics, London, 2003.
- [BJ95] W.E. Burcham and M. Jobes, *Nuclear and particle physics*, Longman, 1995.
- [Bra05] Andreas Brandhuber, *Relativistic quantum mechanics*, <http://www.strings.ph.qmul.ac.uk/~andreas/RQM/rqm.html>, 2005.
- [CR56] E.E. Chambert and R.Hofstadter, Phys. Rev. **103** (1956).
- [DD64] Bjorken J. D and Drell, *Relativistic quantum mechanics*, McGraw-Hill, 1964.
- [Gin04] Douglas M. Gingrich, *Feynman rules for tree graphs*, <http://www.phys.ualberta.ca>, 2004.
- [GR03] W. Greiner and J. Reinhardt, *Quantum electrodynamics*, Springer, 2003.
- [Gri87] David Griffiths, *Introduction to elementary particles*, Wiley, 1987.
- [HM84] Francis Halzen and Alan D. Martin, *Quarks and leptons: an introductory course to modern particle physics*, wiley, 1984.
- [Jaf98] Robert L. Jaffe, hep-ph/9811327 **2** (1998).
- [Joa84] C. J. Joachain, *Quantum collision theory*, Elsevier Science, North Holland, 1984.
- [McM93] Sara McMury, *Quantum mechanics*, Addison-Wesely, 1993.
- [Ron94] William B. Ronlick, *The fundamental particles and their interaction*, 1994.
- [Ros50] M.N. Rosenbluth, Phys. Rev. **79** (1950).
- [Ryd96] Lewis H. Ryder, *quantum field theory*, Cambridge University Press, 1996.
- [Ter06] Dimitri Terryn, *Scattering cross sections*, <http://www.stringsschool.blogspot.com/2006/06/scattering-cross-sections>, 2006.