Elastic and Inelastic Stability of Beams

Jen-Hsuan Hsiao
j3hsiao@eng.ucsd.edu

University of California, San Diego

Introduction

To discuss the structural stability of beams we are referring to the theory of buckling, that is to say, we are looking at the axial compressive loads (critical loads) that columns are subjected to. In this report we first look at the elastic stability of ideal slender columns under pinned ends. We find the critical loads through the equilibrium method including the higher buckling loads cases. Second, to discuss the buckling of columns with different end constraints, especially clamped-free and clamped-clamped cases, we need to find the Euler equation, the boundary conditions equations, and the general solution. However, not all columns are ideal slender columns. Last, we shed light on the inelastic buckling for columns with intermediates slenderness ratios and discuss the Tangents-Modulus Formula for dealing with inelastic buckling.

1 Elastic Buckling

When slender columns are subjected to critical axial compressive loads and kept increasing, the column is in an unstable equilibrium, the columns is like to buckle and experience sudden lateral deflection before yielding. The compressive loads that cause the columns to buckle are the critical loads P_{cr}, which depends on the modulus of elasticity E, moment of inertia I of the cross section, and the effective length Le(depends on the boundary conditions) of the columns. Since the theory is mostly based on ideal slender columns, it is important to summarize their properties:

1. Ideal columns are made of elastic-perfectly plastic materials and perfectly straight.
2. The compressive load lies along its central longitudinal axis and no bending moment or lateral force is considered.
3. The weight of the column is neglected.
4. Uniform cooling were conducted such that the column is free of residual stresses.
5. The effect of temperature is neglected.
6. The effect of long loading periods is neglected, to be more specific, the model is assumed to be a time invarient system.

1.1 Elastic Buckling For Columns With Pinned Ends

At this point we assume the column is made of homogeneous material for the simplification of the model. Since ideal column tends to remain straight under all range of axial load, it is important to apply a small lateral force to the column at a certain critical load when the column is in unstable equilibrium.

We first consider the problem of a pinned ends column (Fig. 1). We then select the lower part of the column (Fig. 2) and write down the equilibrium in terms of the balance of the moment around point A.

$$M(x) + Py = 0 \tag{1}$$

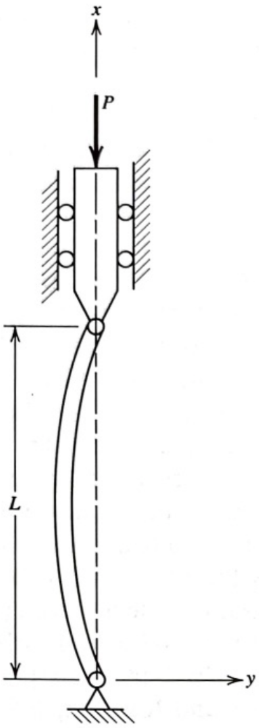

Figure 1: Column with pinned ends. Reproduced from Advanced mechanics of materials

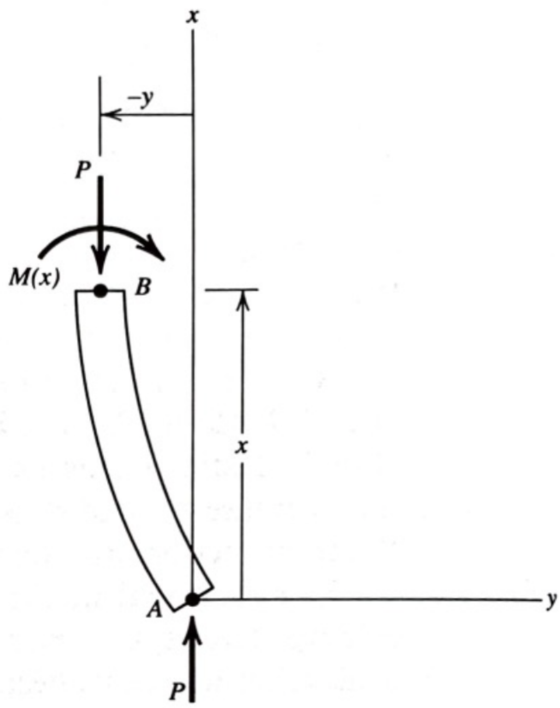

Figure 2: Free body diagram of the lower part of the ideal column. Reproduced from Advanced mechanics of materials

The moment $M(x)$ could be expressed as a function of the radius of curvature $R(x)$ of the centerline of the column in the displaced position as follows:

$$M(x) = \frac{EI}{R(x)} \tag{2}$$

The inverse of the curvature could be approximated as follows when the slope dy/dx of the displaced position is small,

$$\frac{1}{R} = \pm\frac{d^2y/dx^2}{[1+(dy/dx)^2]^{3/2}} = \pm\frac{d^2y}{dx^2} \tag{3}$$

Then substitute (Eq. 3) into (Eq. 2) we can rewrite the moment as follows:

$$M(x) = \pm EI\frac{d^2y}{dx^2} \tag{4}$$

Next, from (Eq.1) and (Eq. 4) the equilibrium equation can be expressed as follows (we can choose a positive $M(x)$ by choosing it in the clockwise senss):

$$\frac{d^2y}{d^2x} + k^2y = 0 \tag{5}$$

where

$$k^2 = \frac{P}{EI} \tag{6}$$

To solve the differential equation, we need to define the boundary conditions for the pinned ends column,

$$y = 0, \text{ for } x = 0 \text{ and } x = L \tag{7}$$

Nontrivial general solutions for (Eq. 5) is

$$y = A\sin kx + B\cos kx \tag{8}$$

where A and B are coefficients to be determined by the boundary conditions (Eq. 7).

$$A\sin kL = 0, \quad B = 0 \tag{9}$$

For $A \neq 0$, $\sin kL$ must be zero. Which means

$$k_n = \sqrt{\frac{P_n}{EI}} = \frac{n\pi}{L}, \quad n = 1, 2, 3, ... \tag{10}$$

Finally, substitute (Eq. 10) back to (Eq. 6) we then find the critical load for each mode (Fig. 3)

$$P_n = \frac{n^2\pi^2EI}{L^2}, \quad n = 1, 2, 3, ... \tag{11}$$

1.2 Elastic Buckling For Columns of General Constraints

For different constraints, the L in (Eq. 11) is different. For clamped free condition, the effective length L_e is $2L$, however, for clamped ends, the effective length L_e is $0.5L$. Therefore it is important to find the general solutions for general constraints.

Consider an axial compressive load P is applied to an ideal column with linearly elastic end restraints composed of rotational springs with elastic constants K_1, K_2 and extensional springs with elastic constants k_1, k_2 (Fig. 4). When P reaches P_{cr}, the potential energy of the system is

$$V = \frac{1}{2}K_2(\frac{dy_2}{dx})^2 + \frac{1}{2}k_2(y_2)^2 + \frac{1}{2}K_1(\frac{dy_1}{dx})^2 + \frac{1}{2}k_1(y_1)^2 + \frac{1}{2}EI\int_0^L(\frac{d^2y}{dx^2})^2dx - \frac{1}{2}P\int_0^L(\frac{dy}{dx})^2dx \tag{12}$$

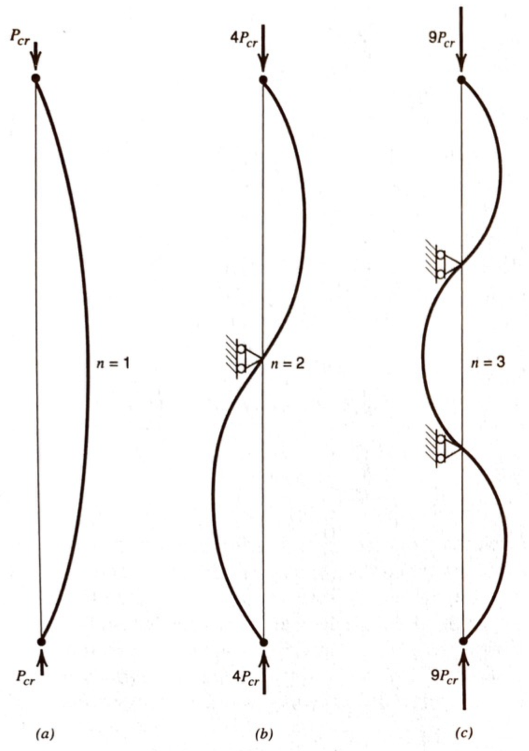

Figure 3: First three column buckling modes. Reproduced from Advanced mechanics of materials

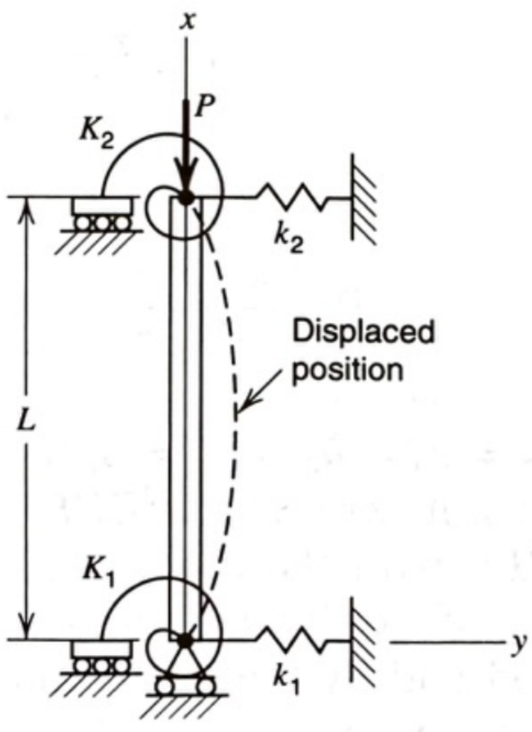

Figure 4: A ideal column with linearly elastic end restraints. Reproduced from Advanced mechanics of materials

where $y1$ represent the displacement at $x = 0$ and $y2$ represent the displacement at $x = L$. Following the principle of stationary potential energy, we set the first variation δV of V equal to zero.

$$\delta V = K_2(\frac{dy_2}{dx})(\delta\frac{dy_2}{dx}) + k_2 y_2 \delta y_2 + K_1(\frac{dy_1}{dx})(\delta\frac{dy_1}{dx}) + k_1 y_1 \delta y_1 + EI\int_0^L (\frac{d^2y}{dx^2})(\delta\frac{d^2y}{dx^2})dx - P\int_0^L (\frac{dy}{dx})(\delta\frac{dy}{dx})dx = 0 \tag{13}$$

The condition that $\delta V = 0$ are

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = 0 \tag{14}$$

and

$$(K_2(\frac{dy_2}{dx}) + EI(\frac{d^2y_2}{dx^2}))(\delta\frac{dy_2}{dx}) = 0 \tag{15}$$

$$(K_1(\frac{dy_1}{dx}) + EI(\frac{d^2y_1}{dx^2}))(\delta\frac{dy_1}{dx}) = 0 \tag{16}$$

$$(k_2 y_2 - EI(\frac{d^3y_2}{dx^3}) - P(\frac{dy_2}{dx}))(\delta y_2) = 0 \tag{17}$$

$$(k_1 y_1 - EI(\frac{d^3y_1}{dx^3}) - P(\frac{dy_1}{dx}))(\delta y_1) = 0 \tag{18}$$

(Eq. 14) is the Euler equation and (Eq. 15 - 18) are the boundary conditions. The general solutions for (Eq. 14) is

$$y = A\sin kx + B\cos kx + Cx + D \tag{19}$$

similar to the pinned ends case, A, B, C, D are constants and $k^2 = \frac{P}{EI}$. In the next section we plug in the boundary conditions of clamped-free columns and clamped-clamped columns.

1.3 Elastic Buckling For Clamped Free Column

For clamped free column, we set $x = 0$ to be clamped and $x = L$ to be free. At the fixed end, $x = 0$,

$$y(0) = y_1 = 0, \quad \frac{dy(0)}{dx} = \frac{dy_1}{dx} = 0 \tag{20}$$

Then $\delta y_1 = \delta\frac{dy_1}{dx} = 0$. Plug into (Eq. 19) we get

$$B + D = 0 \tag{21}$$

and

$$kA + C = 0 \tag{22}$$

For the free end, $K_2 = k_2 = 0$. Substitute the condition into (Eq. 15) and (Eq .17), we get

$$\frac{d^2y_2}{dx^2} = 0 \tag{23}$$

and

$$-EI(\frac{d^3y_2}{dx^3}) - P(\frac{dy_2}{dx}) = 0 \tag{24}$$

Substitute the free end boundary condition into (Eq. 19) we get

$$A + B\cos kL = 0 \tag{25}$$

and

$$A \cos kL = 0 \tag{26}$$

From (Eq. 21, 22, 25, 26) we get the solution

$$A = C = 0, \quad D = -B, \quad coskL = 0 \tag{27}$$

Finally, similar to the case in pinned ends. To find the critical load for different buckling modes, we focus on the condition that $\cos kL = 0$ such that

$$k_n L = (2n-1)\pi/2 \tag{28}$$

Compare the first mode critical load with the pinned case to find the effective length. For $n = 1, \quad k_1 = \frac{\pi}{2L}$. Substitue this back to (Eq. 6) we get

$$P_1 = \frac{\pi^2 EI}{4L^2} \tag{29}$$

$$(L_e)^2 = 4L^2, \quad L_e = 2L \tag{30}$$

1.4 Elastic Buckling For Clamped ends Column

At the fixed end, $x = 0 \quad and \quad x = L$,

$$y_1 = y_2 = 0, \quad \frac{dy_1}{dx} = \frac{dy_2}{dx} = 0 \tag{31}$$

Then $\delta y_1 = \delta \frac{dy_1}{dx} = \delta y_2 = \delta \frac{dy_2}{dx} = 0$. Plug the boundary condition at $x = 0$ into (Eq. 19) we get

$$B + D = 0 \tag{32}$$

and

$$kA + C = 0 \tag{33}$$

Plug the other boundary condition at $x = L$ into (Eq. 19) we get

$$A(\sin kL - kL) + B(\cos kL - 1) = 0 \tag{34}$$

and

$$A(\cos kL - 1) - B \sin kL = 0 \tag{35}$$

Similar to other examples, we focus on the terms with $\sin$ and $\cos$,

$$kL \sin kL = 2(1 - \cos kL), \quad k_n L = 2n\pi \tag{36}$$

Compare the first mode critical load with the pinned case to find the effective length. For $n = 1, \quad k_1 = \frac{2\pi}{L}$. Substitute this back to (Eq. 6) we get

$$P_1 = \frac{4\pi^2 EI}{L^2} \tag{37}$$

$$(L_e)^2 = L^2/4, \quad L_e = 0.5L \tag{38}$$

One last important note for elastic buckling is that the above derivation are limiting on a specific dimension. That is to say, columns are 3D objects, therefore they could buckle in both xz and yz planes and the actual order of mode is sorted by considering the difference in the moment of inertia of the cross section with respect to different axes.

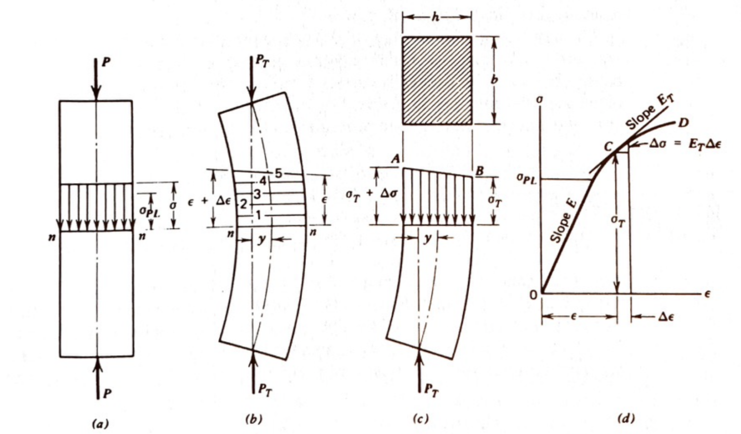

Figure 5: Inelastic buckling strain and stress relationship for tangent-modulus formula. Reproduced from Advanced mechanics of materials

2 Inelastic Buckling

An important geometric factor to be considered is the slenderness ratio L/r. The cross section A could be written as $r^2 = I/A$ where I is the moment of inertia of the cross sectional area and r is the least radius of gyration of the cross section area. For columns that has low slenderness ratio, the column could experience inelastic buckling before reaching the critical load that was derived from the Euler equation. To be more specific, the only term changed in the critical load calculation is the Young's modulus since at this point the slope of the stress strain curve is no longer the same (Fig. 5 (d)). For a small change in the strain, we could assume in that region the stress strain curve remains a straight line, and the Young's modulus used in inelastic buckling, the tangent modulus E_T is estimated by the tangent slope at the point C in (Fig. 5 (d)).

$$\Delta\sigma = E_T \Delta\epsilon \tag{39}$$

Similar to the derivation of the equilibrium method, the bending moment is balanced and written as follows:

$$P_T y = \frac{(\Delta\sigma/2)I}{h/2} \tag{40}$$

Substitute $\Delta\sigma$ with $E_T\Delta\epsilon$ by using (Eq. 39). Also, the strain ϵ can be related to the radius of curvature R of the column. $\Delta\epsilon = h/R$. Last, by using (Eq. 3) we get the following:

$$E_T I \frac{d^2y}{dz^2} = -P_T y \tag{41}$$

which is identical to (Eq. 5 and 6) in the equilibrium method. Therefore, for clamped free columns and clamped ends columns, the critical load in inelastic buckling are similar to the elastic buckling except the different of using E_T instead of E.