Chapter 3 Learning in Two-Player Matrix Games

3.2 Nash Equilibria in Two-Player Matrix Games

For a two-player matrix game, we can set up a matrix with each element containing a reward for each joint action pair. Then the reward function $R_i$ for player $i(i = 1, 2)$ becomes a matrix.

A two-player matrix game is called a *zero-sum game* if the two player are fully competitive. In this way, we have $R_1 = -R_2$. A zero-sum game has a unique NE in the sense of the expected reward. This means that, although each player may have multiple NE strategies in a zero-sum game, the value of the expected reward $V_i$ under these NE strategies will be the same. A *general-sum matrix game* refers to all types of matrix games. In a general-sum matrix game, the NE is no longer unique and the game might have multiple NEs.

For a two-player matrix game, we define $\pi_i = (\pi_i(a_1), ..., \pi_i(a_{m_i}))$ as the set of all probability distributions over player $i$'s action set $A_i(i = 1, 2)$. Then $V_i$ becomes

$$V_i = \pi_1 R_i \pi_2^T \tag{1}$$

An NE for a two-player matrix game is the strategy pair $(\pi_1^*, \pi_2^*)$ for two players such that, for $i = 1, 2$,

$$V_i(\pi_i^*, \pi_{-i}^*) \geq V_i(\pi_i, \pi_{-i}^*), \quad \forall \pi_i \in PD(A_i) \tag{2}$$

where $-i$ denotes any other player than player $i$, and $PD(A_i)$ is the set of all probability distributions over player $i$'s action set $A_i$.

Given that each player has two actions in the game, we can define a two-player two-action general-sum game as

$$R_1 = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}, \quad \mathrm{R}_2 = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{3}$$

where $r_{lf}$ and $c_{lf}$ denote the reward to the row player (player 1) and the reward to the column player (player 2), respectively. The row player chooses action $l \in \{1, 2\}$ and the column player chooses action $f \in \{1, 2\}$. the pure strategies $l$ and $f$ are called a *strict NE in pure strategies* if

$$r_{lf} > r_{-lf}, c_{lf} > c_{l-f} \qquad for \quad l, f \in \{1, 2\} \tag{4}$$

where $-l$ and $-f$ denote any row other than row $l$ and any column other than column $f$ ,respectively.