Multiclass Classification

Logistics

- MidTerm Solution & grade: Friday (April 3)
- Reading Materials will be uploaded the day before
- Slides will be uploaded the just before the class

This Lecture

- Multiclass fundamentals
- Feature extraction
- Multiclass logistic regression

Multiclass Fundamentals

Text Classification

A Cancer Conundrum: Too Many Drug Trials, Too Few Patients

Breakthroughs in immunotherapy and a rush to develop profitable new treatments have brought a crush of clinical trials scrambling for patients.

By GINA KOLATA

Yankees and Mets Are on Opposite Tracks This Subway Series

As they meet for a four-game series, the Yankees are playing for a postseason spot, and the most the Mets can hope for is to play spoiler.

By FILIP BONDY

Text Classification

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→ Health

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→ Sports

~20 classes

Image Classification

Dog

Car

Thousands of classes (ImageNet)

Entity Linking

Although he originally won the event , the United States Anti-Doping Agency announced in August 2012 that they had disqualified **Armstrong** from his seven consecutive Tour de France wins from 1999–2005.

Lance Edward Armstrong is an American former professional road cyclist

Armstrong County is a county in Pennsylvania…

Entity Linking

Although he originally won the event , the United States Anti-Doping Agency announced in August 2012 that they had disqualified **Armstrong** from his seven consecutive Tour de France wins from 1999–2005.

Lance Edward Armstrong is an American former professional road cyclist

Armstrong County is a county in Pennsylvania…

?

?

- 4,500,000 classes (all articles in Wikipedia)

Reading Comprehension

One day, James thought he would go into town and see what kind of trouble he could get into. He went to the grocery store and pulled all the pudding off the shelves and ate two jars. Then he walked to the fast food restaurant and ordered 15 bags of fries. He didn't pay, and instead headed home.

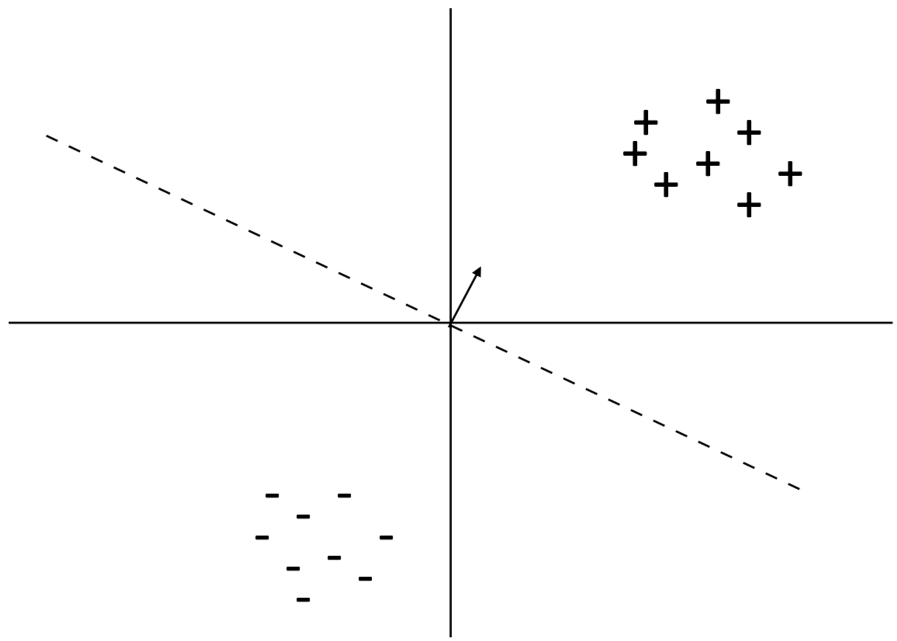
3) Where did James go after he went to the grocery store?
A) his deck
B) his freezer
C) a fast food restaurant
D) his room

- Multiple choice questions, 4 classes (but classes change per example)

Richardson (2013)

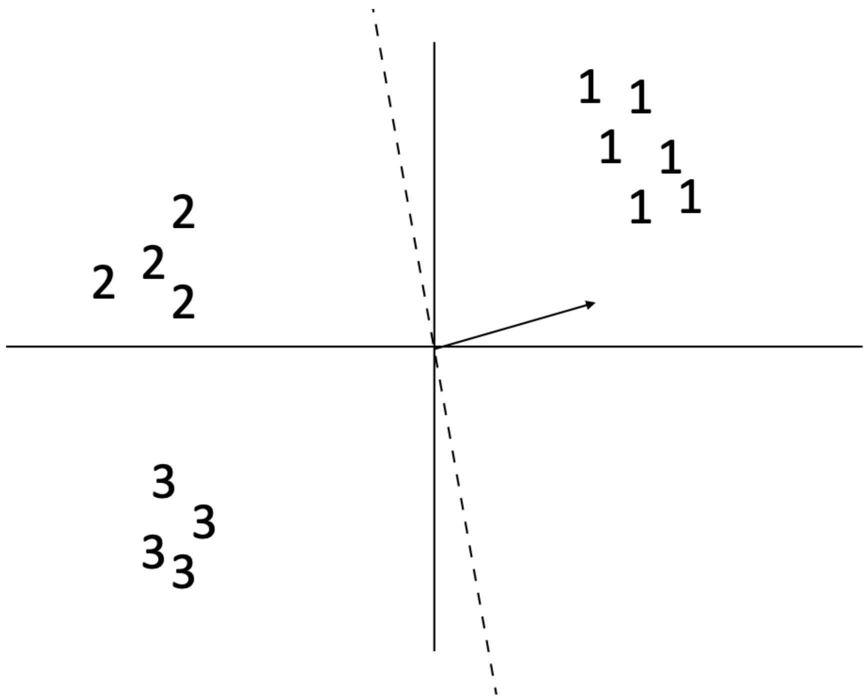
Binary Classification

- Binary classification: one weight vector defines positive and negative classes



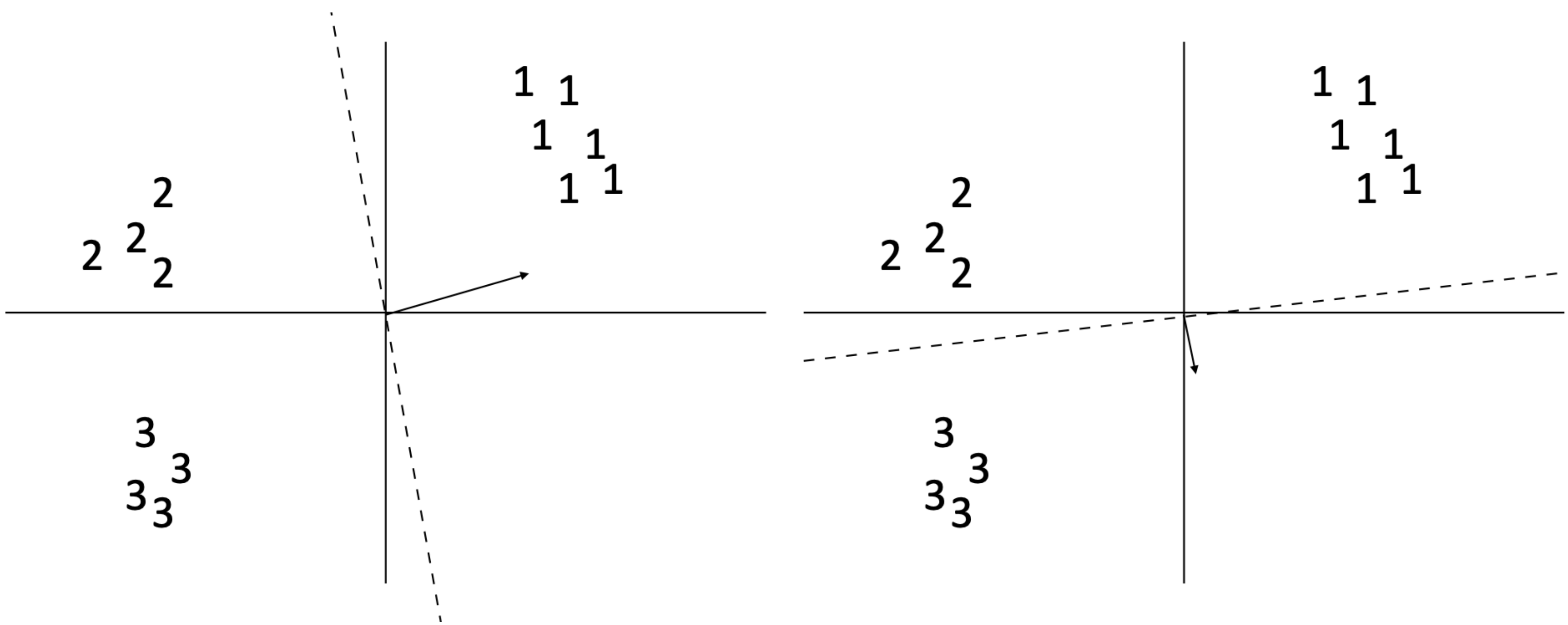
Multiclass Classification

- Can we just use binary classifiers here?



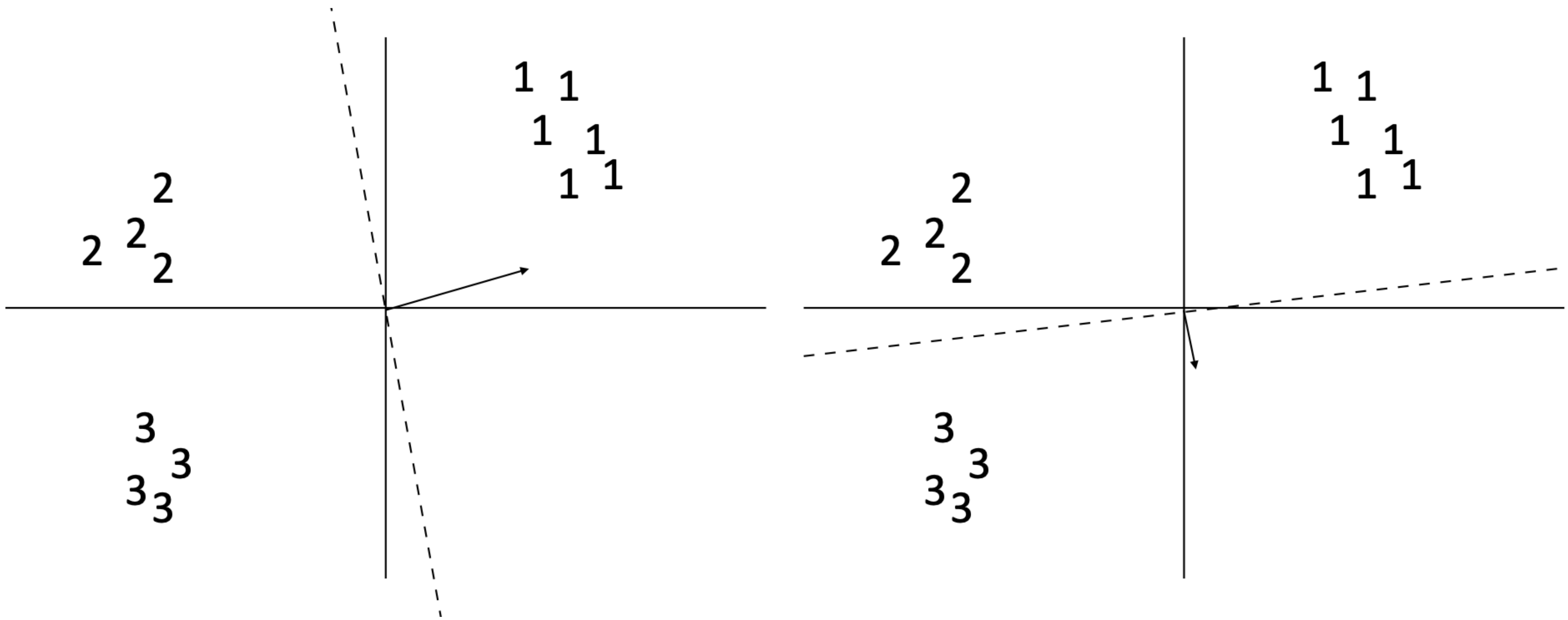
Multiclass Classification

- One-vs-all: train k classifiers, one to distinguish each class from all the rest



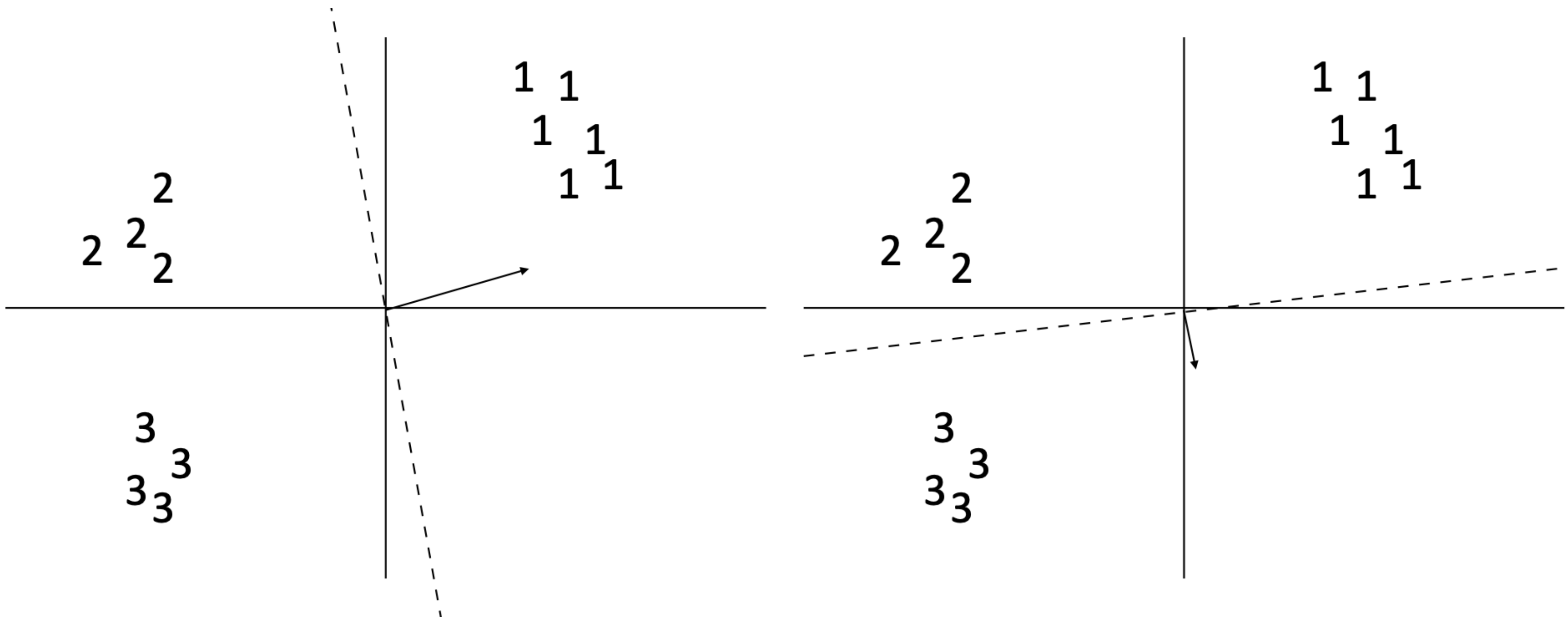
Multiclass Classification

- One-vs-all: train *k* classifiers, one to distinguish each class from all the rest
- How do we reconcile multiple positive predictions?



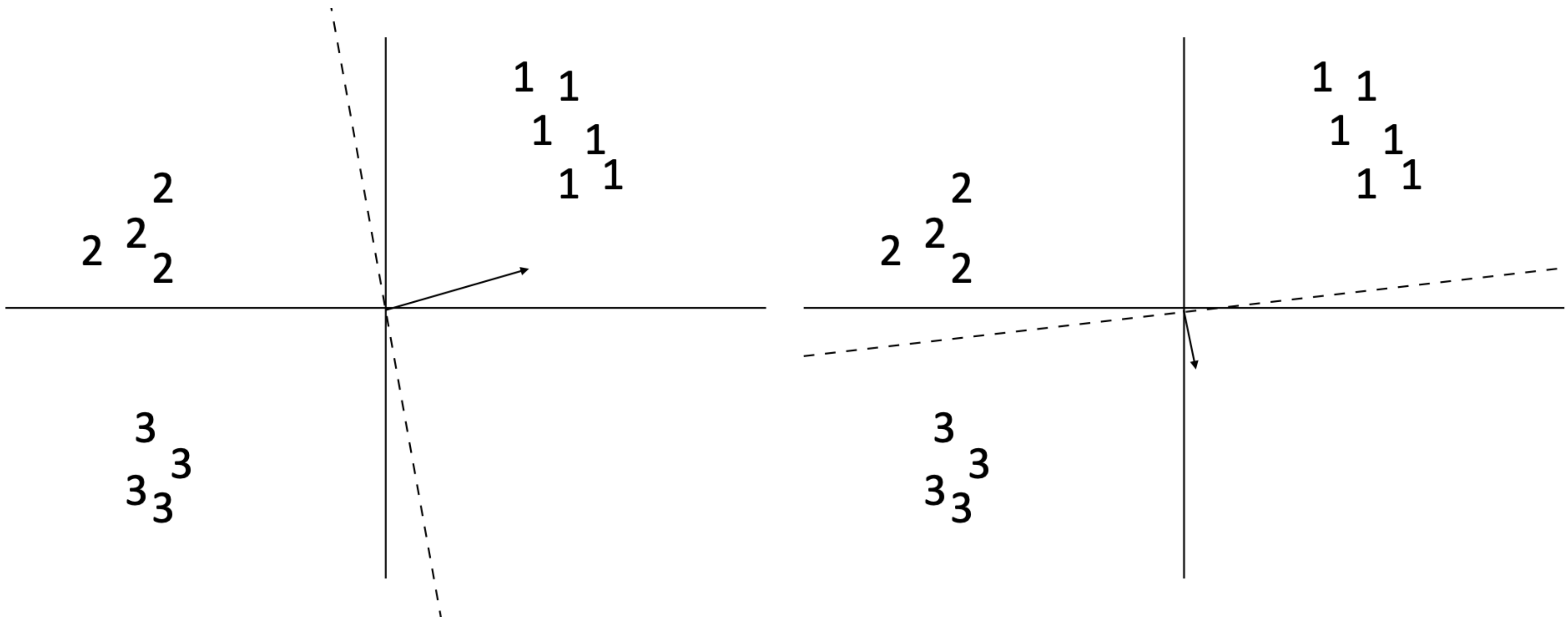
Multiclass Classification

- One-vs-all: train k classifiers, one to distinguish each class from all the rest
- How do we reconcile multiple positive predictions? **Highest score**



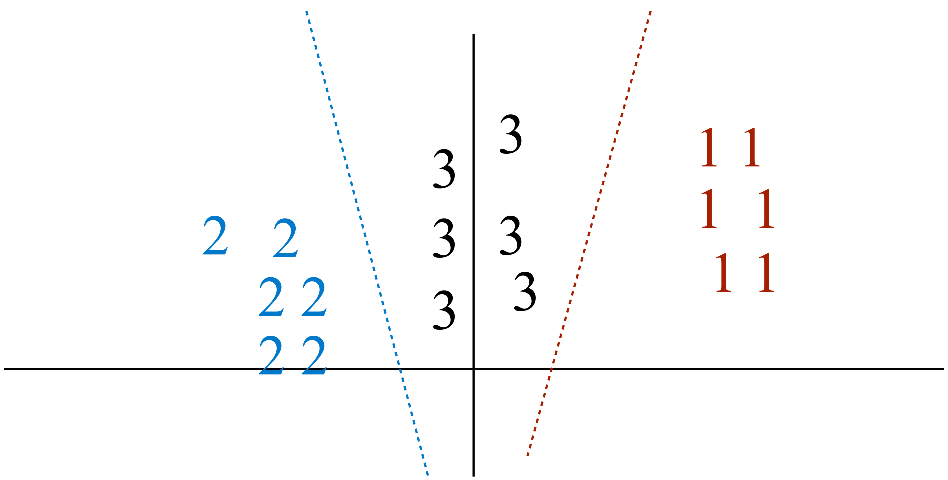
Multiclass Classification

- All-vs-all: train n(n-1)/2 classifiers to differentiate each pair of classes
- Again, how to reconcile?



Multiclass Classification

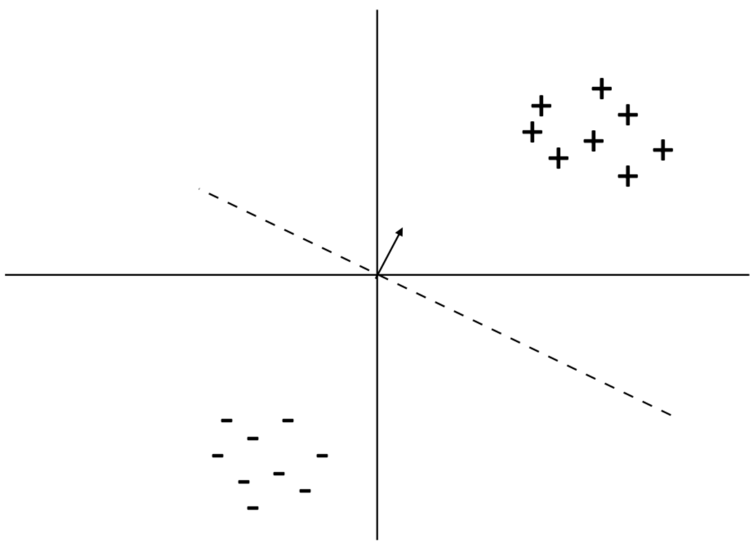
- Not all classes may even be separable using this approach



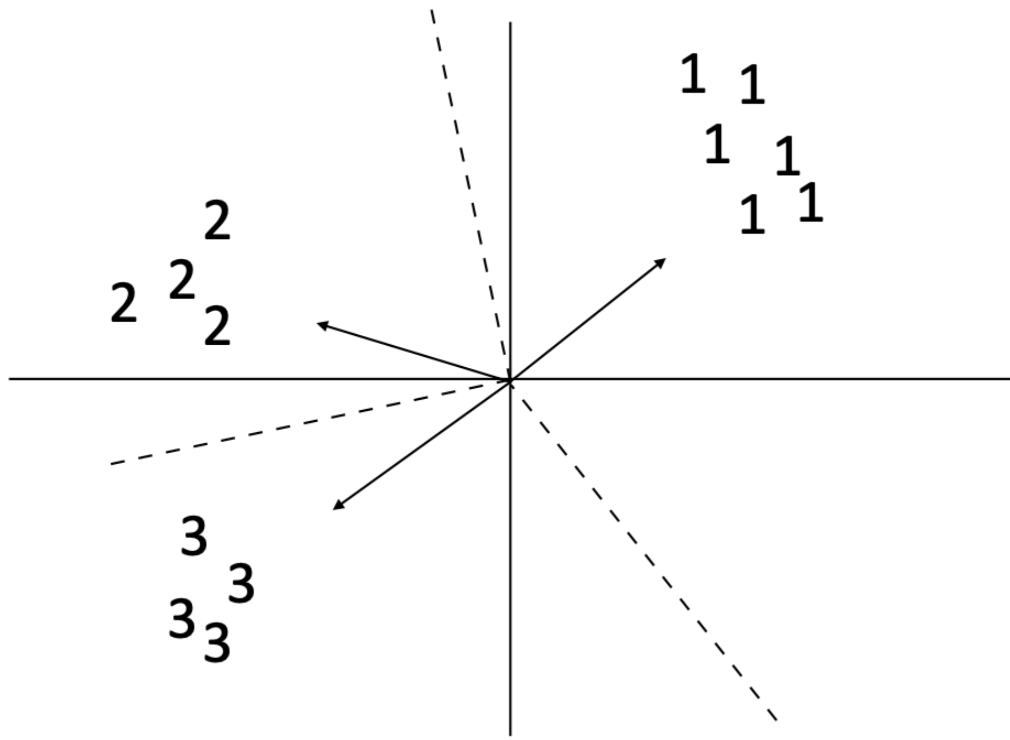
- Can separate 1 from 2+3 and 2 from 1+3 but not 3 from the others (with these features)

Multiclass Classification

- Binary classification: one weight vector defines both classes



- Multiclass classification: different weights and/or features per class



Multiclass Classification

- Formally: instead of two labels, we have an output space $\mathcal{Y}$ containing a number of possible classes

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  - Multiple feature vectors, one weight vector

Multiclass Classification

- Formally: instead of two labels, we have an output space $\mathcal{Y}$ containing a number of possible classes

- Decision rule: $\text{argmax}_{y \in \mathcal{Y}} w^\top f(x, y)$
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features depend on choice of label now! note: this isn't the gold label

Multiclass Classification

- Formally: instead of two labels, we have an output space $\mathcal{Y}$ containing a number of possible classes

- Decision rule: $\text{argmax}_{y \in \mathcal{Y}} w^\top f(x, y)$

features depend on choice of label now! note: this isn't the gold label

- Can also have one weight vector per class: $\text{argmax}_{y \in \mathcal{Y}} w_y^\top f(x)$

Feature Extraction

Block Feature Vectors

- Decision rule: $\text{argmax}_{y \in \mathcal{Y}} w^\top f(x, y)$

Block Feature Vectors

- Decision rule: $\mathrm{argmax}_{y \in \mathcal{Y}} w^\top f(x, y)$

too many drug trials, too few patients

Health

Sports

Science

Block Feature Vectors

- Decision rule: $\text{argmax}_{y \in \mathcal{Y}} w^\top f(x, y)$

too many drug trials, too few patients

Health

Sports

Science

- Base feature function:

$f(x) =$ I[contains *drug*], I[contains *patients*], I[contains *baseball*] = [1, 1, 0]

Block Feature Vectors

- Decision rule: $\text{argmax}_{y \in \mathcal{Y}} w^\top f(x, y)$

too many drug trials, too few patients → Health, Sports, Science

- Base feature function:

$f(x)$ = I[contains *drug*], I[contains *patients*], I[contains *baseball*] = [1, 1, 0]

feature vector blocks for each label

$f(x, y =$ Health $) = [\,1, 1, 0, 0, 0, 0, 0, 0, 0\,]$ I[contains *drug* & label = Health]

Block Feature Vectors

- Decision rule: $\text{argmax}_{y \in \mathcal{Y}} w^\top f(x, y)$

too many drug trials, too few patients

Health

Sports

Science

- Base feature function:

$f(x)$ = I[contains *drug*], I[contains *patients*], I[contains *baseball*] = [1, 1, 0]

feature vector blocks for each label

$f(x, y =$ Health $)$ = [1, 1, 0 , 0, 0, 0, 0, 0, 0] I[contains *drug* & label = Health]

$f(x, y =$ Sports $)$ = [0, 0, 0 , 1, 1, 0, 0, 0, 0] I[contains *drug* & label = Sports]

Block Feature Vectors

- Decision rule: $\text{argmax}_{y \in \mathcal{Y}} w^\top f(x, y)$

too many drug trials, too few patients → Health / Sports / Science

- Base feature function:

$f(x)$ = I[contains *drug*], I[contains *patients*], I[contains *baseball*] = [1, 1, 0]

feature vector blocks for each label

$f(x, y =$ Health $)$ = [1, 1, 0 , 0, 0, 0, 0, 0, 0] I[contains *drug* & label = Health]

$f(x, y =$ Sports $)$ = [0, 0, 0 , 1, 1, 0, 0, 0, 0] I[contains *drug* & label = Sports]

- Equivalent to having three weight vectors in this case

Making Decisions

too many drug trials, too few patients → Health

→ Sports

→ Science

Making Decisions

too many drug trials, too few patients

Health

Sports

Science

$f(x)$ = I[contains *drug*], I[contains *patients*], I[contains *baseball*]

$f(x, y =$ Health $) =$ [1, 1, 0, 0, 0, 0, 0, 0, 0]

$f(x, y =$ Sports $) =$ [0, 0, 0, 1, 1, 0, 0, 0, 0]

"word drug in Science article" = +1.1

Making Decisions

too many drug trials, too few patients → Health
→ Sports
→ Science

$f(x)$ = I[contains *drug*], I[contains *patients*], I[contains *baseball*]

$f(x, y =$ Health $) =$ [1, 1, 0, 0, 0, 0, 0, 0, 0]

$f(x, y =$ Sports $) =$ [0, 0, 0, 1, 1, 0, 0, 0, 0]

"word drug in Science article" = +1.1

$w =$ [+2.1, +2.3, -5, -2.1, -3.8, 0, +1.1, -1.7, -1.3]

Making Decisions

too many drug trials, too few patients

Health

Sports

Science

f (x) = I[contains *drug*], I[contains *patients*], I[contains *baseball*]

$f(x, y = \text{Health}) = [1, 1, 0, 0, 0, 0, 0, 0, 0]$

$f(x, y = \text{Sports}) = [0, 0, 0, 1, 1, 0, 0, 0, 0]$

"word drug in Science article" = +1.1

$w = [+2.1, +2.3, 0, -2.1, -3.8, 0, +1.1, -1.7, -1.3]$

$w^\top f(x, y)$ = Health: +4.4 Sports: -5.9 Science: -1.9

Another example: POS tagging

- Classify *blocks* as one of 36 POS tags

the router *blocks* *the packets*

NNS

VBZ

NN

DT

...

Another example: POS tagging

- Classify *blocks* as one of 36 POS tags
- Example *x* : sentence with a word (in this case, *blocks*) highlighted

the router *blocks* *the packets*

NNS
VBZ
NN
DT
…

Another example: POS tagging

- Classify *blocks* as one of 36 POS tags

- Example x : sentence with a word (in this case, *blocks*) highlighted

- Extract features with respect to this word:

the router *blocks* *the packets*

NNS
VBZ
NN
DT
…

$f(x, y = \text{VBZ})$ = I[curr_word= *blocks* & tag = VBZ],
I[prev_word=*router* & tag = VBZ]
I[next_word=*the* & tag = VBZ]
I[curr_suffix=*s* & tag = VBZ]

not saying that *the* is tagged as VBZ! saying that *the* follows the VBZ word

Multiclass Logistic Regression

Multiclass Logistic Regression

$$P_w(y|x) = \frac{\exp\left(w^\top f(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$$

sum over output space to normalize

Multiclass Logistic Regression

$$P_w(y|x) = \frac{\exp\left(w^\top f(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$$

sum over output space to normalize

- Compare to binary:

$$P(y=1|x) = \frac{\exp(w^\top f(x))}{1+\exp(w^\top f(x))}$$

negative class implicitly had *f*(*x*, *y*=0) = the zero vector

Multiclass Logistic Regression

$$P_w(y|x) = \frac{\exp\left(w^\top f(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$$

Multiclass Logistic Regression

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Softmax function

Multiclass Logistic Regression

Softmax function

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Why? Interpret raw classifier scores as **probabilities**

Multiclass Logistic Regression

$$P_w(y|x) = \frac{\exp\left(w^\top f(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$$

Softmax function

sum over output space to normalize

Why? Interpret raw classifier scores as **probabilities**

too many drug trials, too few patients

Health: +4.4

Sports: -5.9

Science: -1.9

$w^\top f(x,y)$

Multiclass Logistic Regression

Softmax function

$$P_w(y|x) = \frac{\exp\left(w^\top f(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$$

sum over output space to normalize

Why? Interpret raw classifier scores as **probabilities**

too many drug trials, too few patients

$w^\top f(x,y)$		exp
Health:	+4.4	81.45
Sports:	-5.9	0.002
Science:	-1.9	0.014

Multiclass Logistic Regression

$$P_w(y|x) = \frac{\exp\left(w^\top f(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$$

Softmax function

sum over output space to normalize

Why? Interpret raw classifier scores as **probabilities**

too many drug trials, too few patients

$w^\top f(x,y)$	exp		normalize (probabilities must sum to 1)
Health: +4.4		81.45	0.99
Sports: -5.9		0.002	$2.4*10^{-5}$
Science: -1.9		0.014	$1.7*10^{-4}$

probabilities

Multiclass Logistic Regression

$$P_w(y|x) = \frac{\exp\left(w^\top f(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$$

sum over output
space to normalize

Multiclass Logistic Regression

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sum over output
space to normalize

- Training: maximize $\mathcal{L}(x,y) = \sum_{j=1}^{n} \log P(y_j^*|x_j)$

Multiclass Logistic Regression

$$P_w(y|x) = \frac{\exp\left(w^\top f(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$$

sum over output space to normalize

- Training: maximize $\mathcal{L}(x,y) = \sum_{j=1}^{n} \log P(y_j^*|x_j)$

$$= \sum_{j=1}^{n} \left(w^\top f(x_j, y_j^*) - \log \sum_{y} \exp(w^\top f(x_j, y)) \right)$$

Multiclass Logistic Regression

$$P_w(y|x) = \frac{\boxed{\exp\left(w^\top f(x,y)\right)}}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$$

sum over output
space to normalize

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$$= \sum_{j=1}^{n} \left(w^\top f(x_j, y_j^*) - \boxed{\log \sum_{y} \exp(w^\top f(x_j, y))} \right)$$

Training

- Multiclass logistic regression $P_w(y|x) = \frac{\exp\left(w^\top f(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$
- Likelihood $\mathcal{L}(x_j, y_j^*) = w^\top f(x_j, y_j^*) - \log \sum_y \exp(w^\top f(x_j, y))$

$$\frac{\partial}{\partial w_i} \mathcal{L}(x_j, y_j^*) = f_i(x_j, y_j^*) - \frac{\sum_y f_i(x_j, y) \exp(w^\top f(x_j, y))}{\sum_y \exp(w^\top f(x_j, y))}$$

$$\frac{\partial}{\partial w_i} \mathcal{L}(x_j, y_j^*) = f_i(x_j, y_j^*) - \sum_y f_i(x_j, y) P_w(y|x_j)$$

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$$\frac{\partial}{\partial w_i}\mathcal{L}(x_j, y_j^*) = f_i(x_j, y_j^*) - \sum_y f_i(x_j, y) P_w(y|x_j)$$

$$\frac{\partial}{\partial w_i}\mathcal{L}(x_j, y_j^*) = f_i(x_j, y_j^*) - \mathbb{E}_y[f_i(x_j, y)]$$

model's expectation of feature value

gold feature value

Training

- Multiclass logistic regression $P_w(y|x) = \frac{\exp\left(w^\top f(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$
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gold feature value

model's expectation of feature value

Training

$$\frac{\partial}{\partial w_i}\mathcal{L}(x_j, y_j^*) = f_i(x_j, y_j^*) - \sum_y f_i(x_j, y) P_w(y|x_j)$$

Training

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too many drug trials, too few patients

$f(x, y =$ Health $) =$ [1, 1, 0, 0, 0, 0, 0, 0, 0]

$f(x, y =$ Sports $) =$ [0, 0, 0, 1, 1, 0, 0, 0, 0]

y^* = Health

$P_w(y|x)$ = [0.21, 0.77, 0.02]

Training

$$\frac{\partial}{\partial w_i}\mathcal{L}(x_j, y_j^*) = f_i(x_j, y_j^*) - \sum_y f_i(x_j, y) P_w(y|x_j)$$

too many drug trials, too few patients

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$f(x, y = \text{Health}) = [1, 1, 0, 0, 0, 0, 0, 0, 0]$

$f(x, y = \text{Sports}) = [0, 0, 0, 1, 1, 0, 0, 0, 0]$

$P_w(y|x) = [0.21, 0.77, 0.02]$

gradient: $[1, 1, 0, 0, 0, 0, 0, 0, 0] - 0.21\,[1, 1, 0, 0, 0, 0, 0, 0, 0]$
$- 0.77\,[0, 0, 0, 1, 1, 0, 0, 0, 0] - 0.02\,[0, 0, 0, 0, 0, 0, 1, 1, 0]$

$= [0.79, 0.79, 0, -0.77, -0.77, 0, -0.02, -0.02, 0]$

Training

$$\frac{\partial}{\partial w_i}\mathcal{L}(x_j, y_j^*) = f_i(x_j, y_j^*) - \sum_y f_i(x_j, y) P_w(y|x_j)$$

too many drug trials, too few patients

y^* = Health

$f(x, y = \text{ Health }) = [1, 1, 0, 0, 0, 0, 0, 0, 0]$

$P_w(y|x)$ = [0.21, 0.77, 0.02]

$f(x, y = \text{ Sports }) = [0, 0, 0, 1, 1, 0, 0, 0, 0]$

gradient: [1, 1, 0, 0, 0, 0, 0, 0, 0] — 0.21 [1, 1, 0, 0, 0, 0, 0, 0, 0]
— 0.77 [0, 0, 0, 1, 1, 0, 0, 0, 0] — 0.02 [0, 0, 0, 0, 0, 0, 1, 1, 0]

= [0.79, 0.79, 0, -0.77, -0.77, 0, -0.02, -0.02, 0]

update $w^\top$:

[1.3, 0.9, -5, 3.2, -0.1, 0, 1.1, -1.7, -1.3] + [0.79, 0.79, 0, -0.77, -0.77, 0, -0.02, -0.02, 0]
= [2.09, 1.69, 0, 2.43, -0.87, 0, 1.08, -1.72, 0]

new $P_w(y|x)$ = [0.89, 0.10, 0.01]

Training

$$\frac{\partial}{\partial w_i}\mathcal{L}(x_j, y_j^*) = f_i(x_j, y_j^*) - \sum_y f_i(x_j, y) P_w(y|x_j)$$

too many drug trials, too few patients

y^* = Health

$f(x, y =$ Health $) =$ [1, 1, 0, 0, 0, 0, 0, 0, 0]

$f(x, y =$ Sports $) =$ [0, 0, 0, 1, 1, 0, 0, 0, 0]

$P_w(y|x)$ = [0.21, 0.77, 0.02]

gradient: [1, 1, 0, 0, 0, 0, 0, 0, 0] — 0.21 [1, 1, 0, 0, 0, 0, 0, 0, 0]
— 0.77 [0, 0, 0, 1, 1, 0, 0, 0, 0] — 0.02 [0, 0, 0, 0, 0, 0, 1, 1, 0]

= [0.79, 0.79, 0, -0.77, -0.77, 0, -0.02, -0.02, 0]

update $w^\top$:
[1.3, 0.9, -5, 3.2, -0.1, 0, 1.1, -1.7, -1.3] + [0.79, 0.79, (
= [2.09, 1.69, 0, 2.43, -0.87, 0, 1.08, -1.72, 0]

Softmax function

$$P_w(y|x) = \frac{\exp\left(w^\top f(x, y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x, y')\right)}$$

new $P_w(y|x)$ = [0.89, 0.10, 0.01]

Logistic Regression: Summary

- Model: $P_w(y|x) = \frac{\exp\left(w^\top f(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(w^\top f(x,y')\right)}$
- Inference: $\text{argmax}_y P_w(y|x)$
- Learning: gradient ascent on the discriminative log-likelihood

$$f(x, y^*) - \mathbb{E}_y[f(x,y)] = f(x, y^*) - \sum_y [P_w(y|x) f(x,y)]$$

"towards gold feature value, away from expectation of feature value"