K-Nearest Neighbor

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- **Goal:** predict the output $\mathbf{y}$ for an **unseen** test example $\mathbf{x}$

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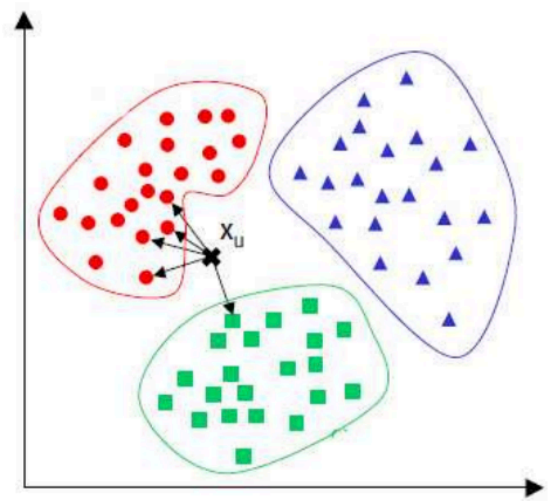

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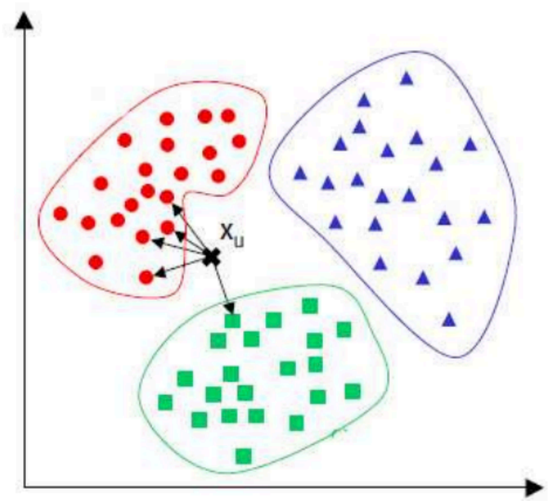

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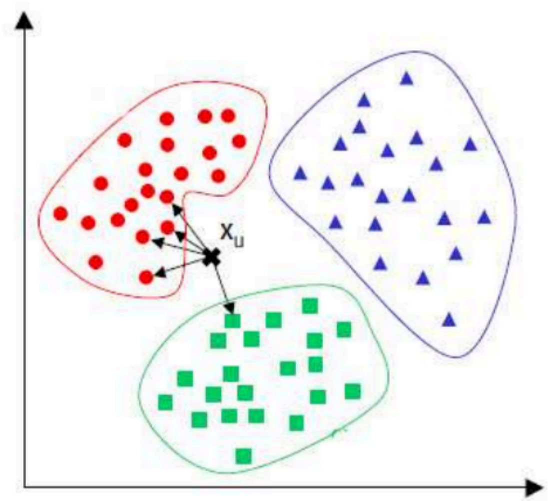

  - For classification: assign the majority class label (majority voting)
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- The algorithm requires:
  - Parameter K: number of nearest neighbors to look for
  - Distance function: To compute the similarities between examples

K-Nearest Neighbor Algorithm

- Compute the test point's distance from each training point
- Sort the distances in ascending (or descending) order
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- It simply uses the training data at the test time to make predictions

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$$d(\mathbf{x}_i, \mathbf{x}_j) = \sqrt{\sum_{m=1}^{D}(x_{im} - x_{jm})^2} = \sqrt{||\mathbf{x}_i||^2 + ||\mathbf{x}_j||^2 - 2\mathbf{x}_i^T \mathbf{x}_j}$$

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K-NN: Other Distance Measure

- Binary-valued features
  - Use Hamming distance: $d(x_i, x_j) = \sum_{m=1}^{D} \mathbb{I}(x_{im} \neq x_{jm})$
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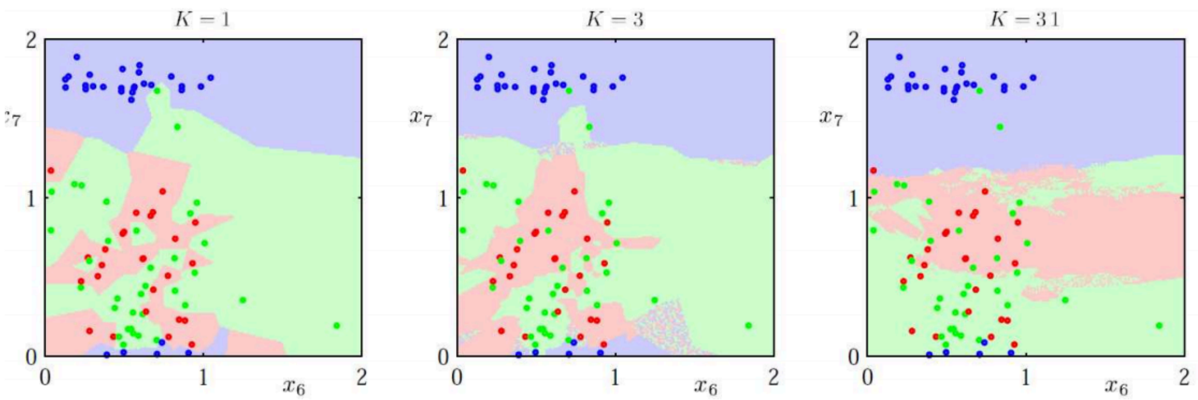
- Can also assign **weights** to features: $d(x_i, x_j) = \sum_{m=1}^{D} w_m d(x_{im}, x_{jm})$

K-NN: Choice of K



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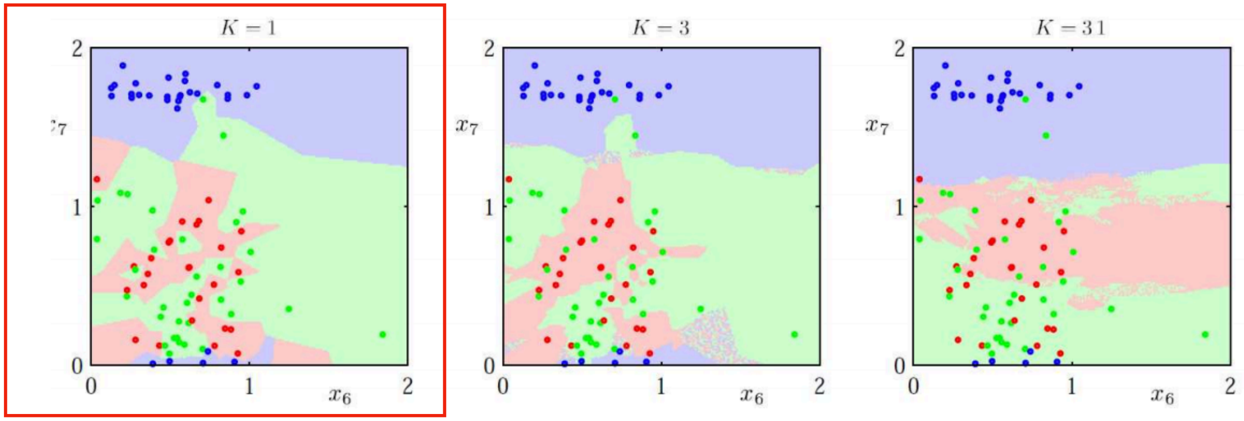

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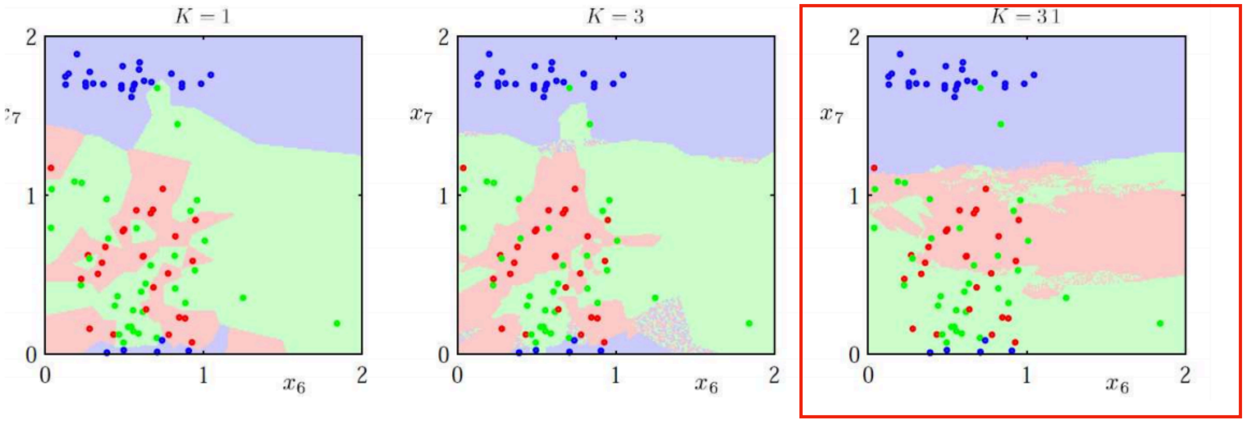

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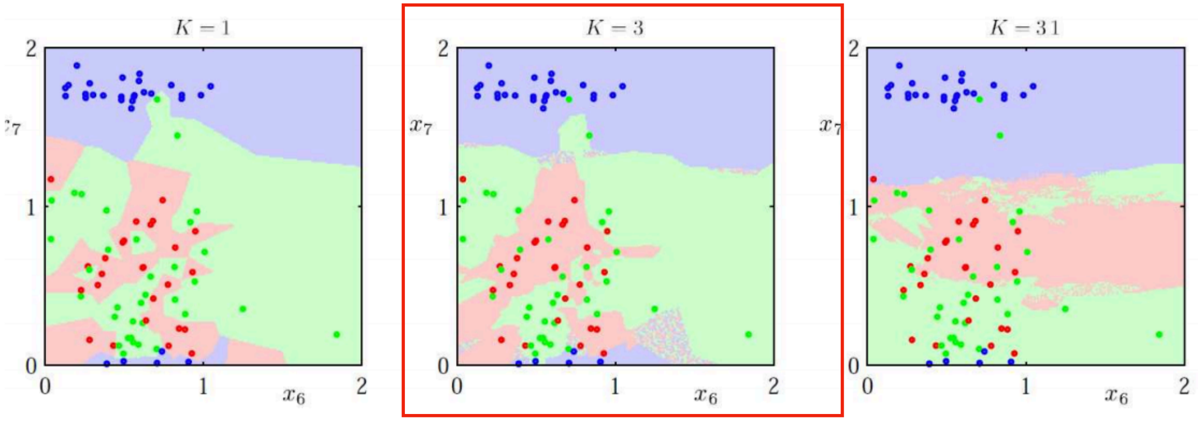

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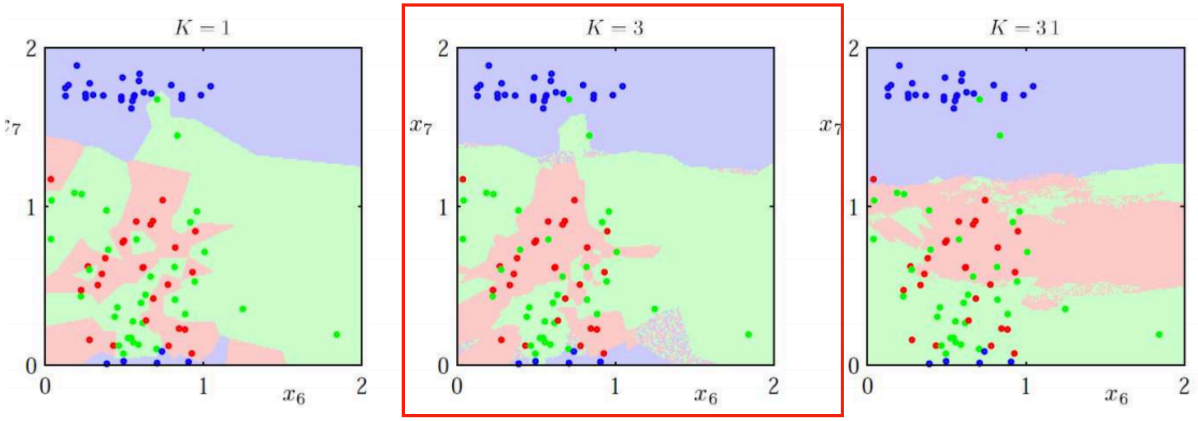

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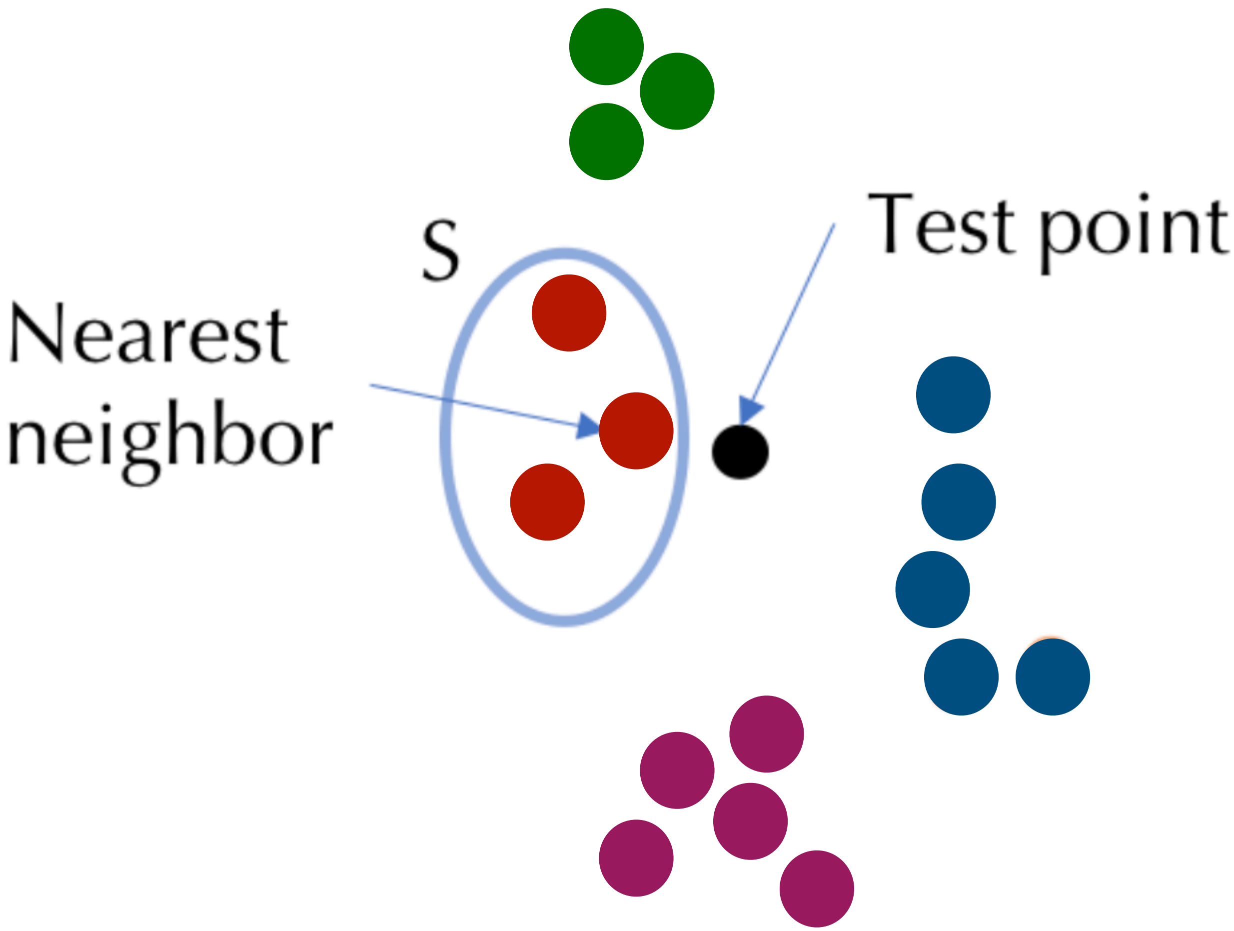
  - Often data dependent and heuristic based
  - Or using cross-validation (using some **held-out data**)
  - In general, a K too small or too big is bad!

K-NN: Properties

- What's nice
  - Simple and intuitive; easily implementable
  - Asymptotically consistent (a theoretical property)
    - With infinite training data and large enough K, K-NN approaches the best possible classifier (Bayes optimal)
- What's not so nice..
  - Store all the training data *in memory* even at test time
    - Can be memory intensive for large training datasets
    - An example of non-parametric, or **memory/instance-based** methods
    - Different from parametric, model-based learning models
  - Expensive at test time: $O(ND)$ computations for each test point
    - Have to search through all training data to find nearest neighbors
    - Distance computations with N training points (D features each)
  - Sensitive to noisy features
  - May perform badly in high dimensions (curse of dimensionality)
    - In high dimensions, distance notions can be counter-intuitive!

K-NN: Example



K-NN: Pseudocode

1. Calculate "$d(x, x_i)$" $i = 1, 2, \ldots, \mathbf{n}$; where **d** denotes the Euclidean distance between the points.
2. Arrange the calculated **n** Euclidean distances in non-decreasing order.
3. Let **k** be a +ve integer, take the first **k** distances from this sorted list.
4. Find those **k**-points corresponding to these **k**-distances.
5. Let $\mathbf{k}_i$ denotes the number of points belonging to the i^{th} class among **k** points i.e. $k \geq 0$
6. If $k_i > k_j \ \forall \ i \neq j$ then put x in class i.

Let (X_i, C_i) where $i = 1, 2, \ldots, n$ be data points. X_i denotes feature values & C_i denotes labels for X_i for each i.
Assuming the number of classes as 'c'
$c_i \in \{1, 2, 3, \ldots, c\}$ for all values of i

Let x be a point for which label is not known, and we would like to find the label class using k-nearest neighbor algorithms.