Measurement Incompatibility, Quantum Steering, and High-dimensional Generalisations

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Two observables A and B are simulaneously measurable if they have a common eigenbasis.

This occurs if and only if they commute:

$$[A, B] = AB - BA = 0. \tag{2}$$

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- $M_a \geq 0 \qquad \forall a$
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- Observables correspond to M_a all being projectors.
- Doesn't say anything about post-measured state.

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Sets of POVMs

We denote a collection of POVMs as

$$\{M_{a|x}\}_{a,x} \equiv M_{a|x} \quad : \quad M_{a|x} \geq 0 \quad \forall a, x, \qquad \sum_a M_{a|x} = \mathbb{1} \quad \forall x$$

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Example

$$\begin{aligned} &Z && X \\ M_{+1|0} &= |0\rangle\langle 0| & M_{+1|1} &= |+\rangle\langle +| \quad (3) \\ M_{-1|0} &= |1\rangle\langle 1| & M_{-1|1} &= |-\rangle\langle -| \quad (4) \end{aligned}$$

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This is also equivalent to there existing Naimark dilations in which the projective measurements commute.

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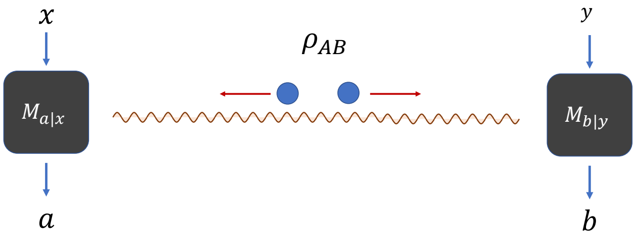
$\Longrightarrow$: Take $\lambda = \boldsymbol{a}$, $\quad p(a|x,\boldsymbol{a}) = \delta_{a,a_x}$

$\Longleftarrow$: Define $G_{a_1,\ldots,a_n} = \sum_\lambda \prod_x p(a_x|x,\lambda) G_\lambda$

Incompatible measurements in Quantum Information

Consider the Bell scenario:



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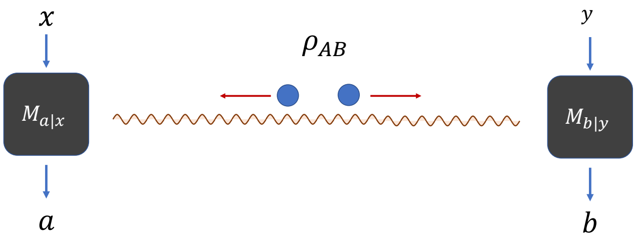

The resulting data is

$$p(a, b|x, y) = \mathrm{Tr}\left(M_{a|x} \otimes N_{b|y} \ \rho_{AB} \right)$$

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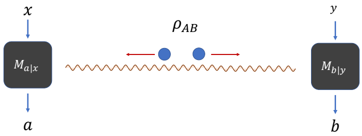

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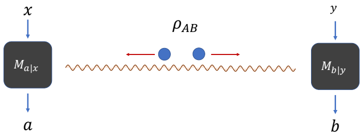

This data has a *local hidden variable* (LHV) model if it can be written as

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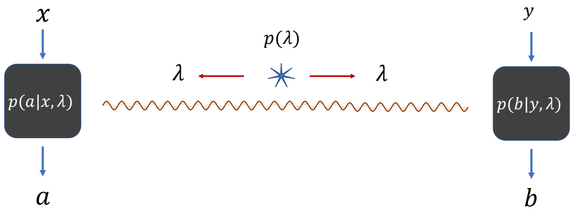

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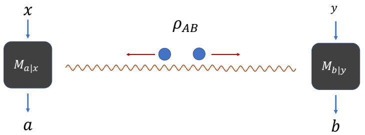

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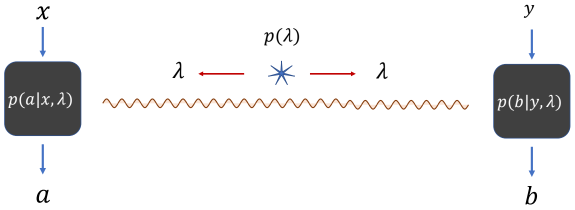

If there is no LHV model, the correlations are *nonlocal*.

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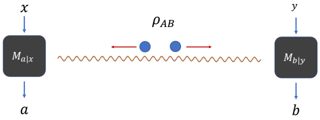

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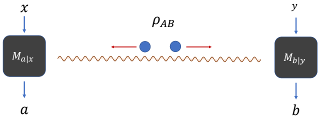

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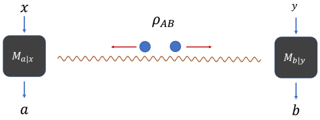

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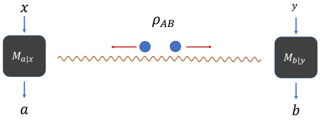

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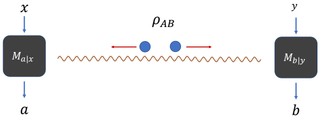

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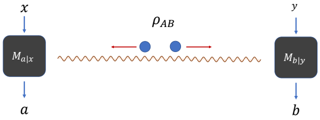

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Hence entanglement is necessary for nonlocal correlations!

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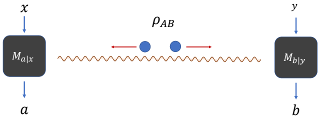

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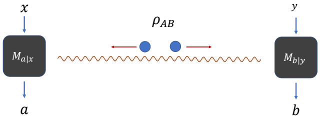

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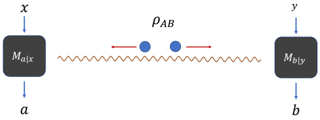

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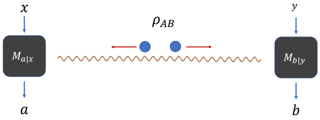

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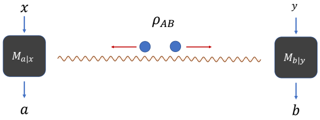

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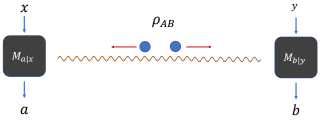
$$= \sum_\lambda p(\lambda)\ p(a|x,\lambda)\ p(b|y,\lambda) \tag{11}$$

Incompatible measurements in Quantum Information

$$Q: \quad p(a,b|x,y) = \mathrm{Tr}\Big(M_{a|x} \otimes N_{b|y}\ \rho_{AB}\Big)$$

$$LHV: p(a,b|x,y) = \textstyle\sum_\lambda p(\lambda)\ p(a|x,\lambda)\ p(b|y,\lambda)$$



If the one parties measurements are compatible : $M_{a|x} = \sum_\lambda p(a|x,\lambda) G_\lambda$.

$$\Longrightarrow p(a,b|x,y) = \sum_\lambda p(a|x,\lambda)\ \mathrm{Tr}\Big(G_\lambda \otimes N_{b|y}\ \rho_{AB}\Big) \tag{8}$$

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So incompatibility in both measurements is also necessary for nonlocality!

Summary of Measurement Incompatibility

- Incompatibility is an interesting, subtle concept for POVMs.

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- Measurements are compatible if $M_{a|x} = \sum_\lambda p(a|x,\lambda) G_\lambda$.
- Incompatibility is needed for nonlocality: can be viewed as a resource.
- One can think of incompatibility (of a set of measurements) as the 'dual' of entanglement (of a state).

– This can be made precise, more on this later!

Table of Contents

1. Measurement Incompatibility
2. Quantum Steering
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4. Our Contribution
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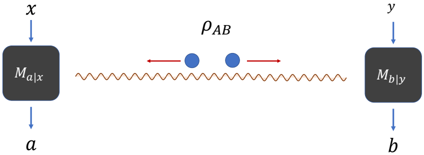
Nonlocality vs Entanglement

Again recall the Bell scenario



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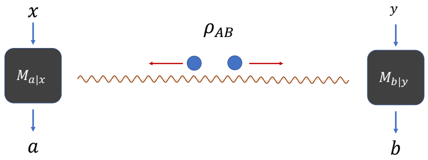
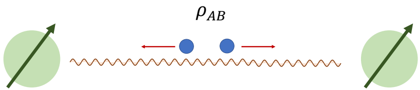


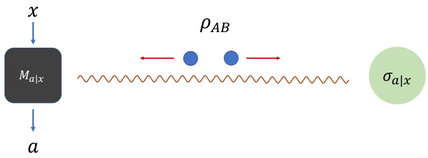
Here the data associated is a probability distribution $p(a, b|x, y)$ – neither party is *trusted.*

If both parties are trusted, the associated data is the full density matrix ρ_{AB}.



Quantum Steering

Quantum Steering is an intermediate case in which one part is trusted, and the other is untrusted.



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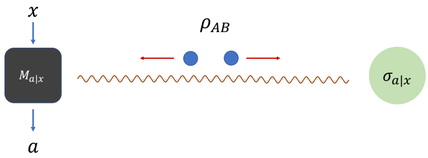

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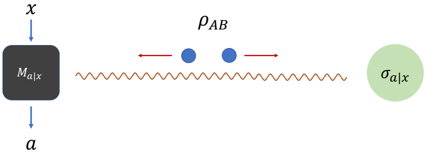

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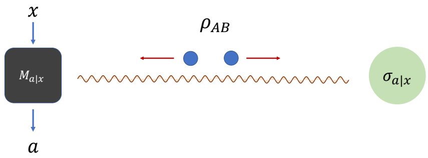

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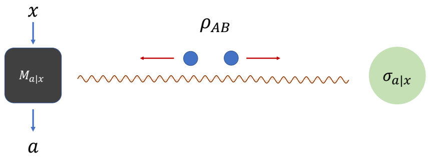

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- Could be relevant for communication protocols.
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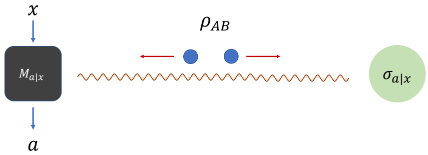

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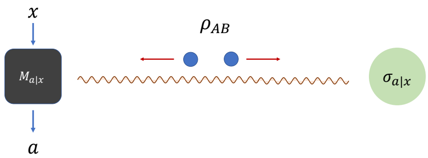

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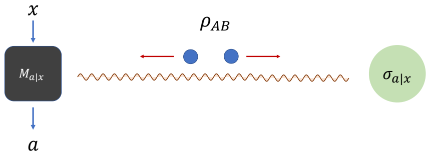

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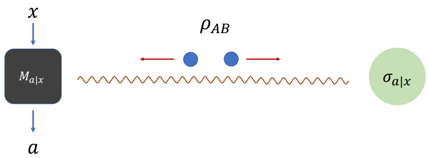

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- We have that $\sum_a \; \sigma_{a|x} = \mathrm{Tr}_A(\rho_{AB}) = \rho_B$ for all x.
- The data $\sigma_{a|x}$ is referred to as a (steering) **assemblage.**

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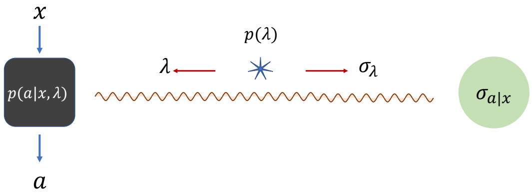

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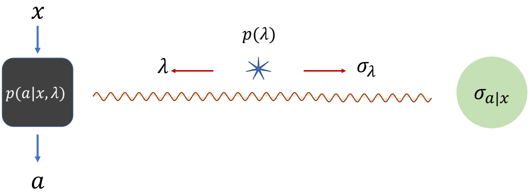

If there is no LHS model, we say the data demonstrates *steering*.

Entanglement and Incompatibility in Steering

The same arguments go through as before, we have for an assemblage

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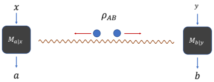
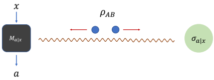
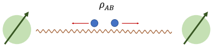
It is also true that any LHS assemblage can be prepared with a separable state and compatible measurements.

Nonlocality vs Steering vs Entanglement

	Nonlocality	**Steering**	**Entanglement**
Scenario:			
Data:	$p(a, b\|x, y)$	$\sigma_{a\|x}$	$\rho_{AB} = \begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}$
Local model:	$p(a, b\|x, y) = \sum_\lambda p(\lambda)\ p(a\|x, \lambda)\ p(b\|y, \lambda)$	$\sigma_{a\|x} = \sum_\lambda p(\lambda)\ p(a\|x, \lambda)\ \sigma_B^\lambda$	$\rho_{AB} = \sum_\lambda p(\lambda)\ \rho_A^\lambda \otimes \rho_B^\lambda$

Connections between Steering and Incompatibility

We already saw one connection between incompatibility and steering.

[1] Roope Uola et al. "One-to-one mapping between steering and joint measurability problems". In: *Physical review letters* 115.23 (2015), p. 230402.

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$$M_{a|x} \geq 0 \quad \forall a, x \qquad\qquad \sigma_{a|x} \geq 0 \quad \forall a, x \tag{13}$$

$$\sum_a M_{a|x} = \mathbb{1} \quad \forall a \qquad\qquad \sum_a \sigma_{a|x} = \rho_B \quad \forall a \tag{14}$$

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There is the following connection[1]:

Theorem

An assemblage $\sigma_{a|x}$ admits an LHS model if and only if the measurements $M_{a|x} := \rho_B^{-\frac{1}{2}} \sigma_{a|x} \rho_B^{-\frac{1}{2}}$ are compatible.

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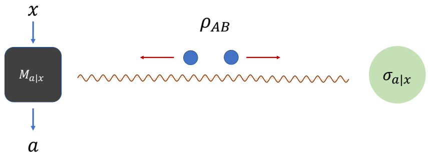
High-dimensional Quantum Steering

Recall the steering scenario



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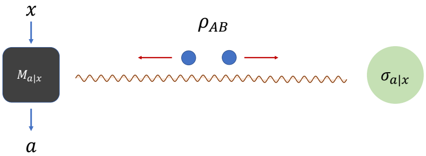

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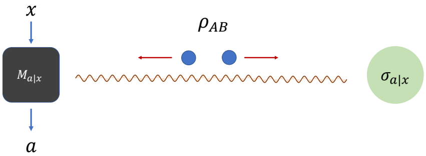

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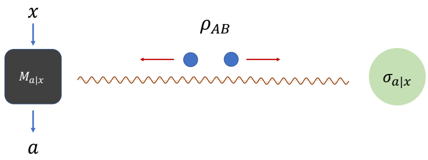

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– The **Schmidt number** is one way . . .

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Consider a bipartite pure state vector $|\psi\rangle_{AB} \in \mathbb{C}^d \otimes \mathbb{C}^d$.

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$$\mathrm{SR}(|\psi\rangle) := \min k \quad \text{s.t} \quad |\psi\rangle = \sum_{i=1}^{k} \lambda_i |a_i\rangle \otimes |b_i\rangle$$

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Note that

$$\mathrm{SR}(|\psi\rangle) = 1 \quad \Longleftrightarrow \quad |\psi\rangle = |\phi\rangle \otimes |\tau\rangle \quad \text{product state}$$

Schmidt number

The Schmidt rank is just for pure states. A general quantum state $\rho_{AB} = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ will have many decompositions.

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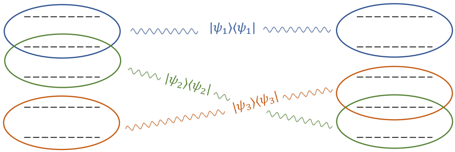

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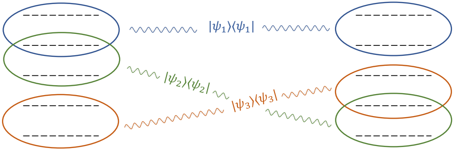
Again we have $\mathrm{SN}(\rho) \in \{1, \ldots, d\}$, and

$$\mathrm{SN}(\rho) = 1$$
$$\iff \quad \rho = \sum_k p_k \ |\phi_k\rangle\langle\phi_k| \otimes |\tau_k\rangle\langle\tau_k|$$

separable state.



High-dimensional Quantum Steering

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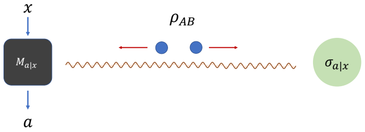

[2] Sébastien Designolle et al. "Genuine high-dimensional quantum steering". In: *Physical Review Letters* 126.20 (2021), p. 200404.

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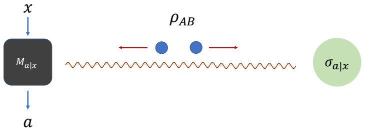

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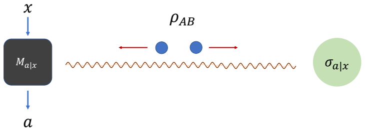

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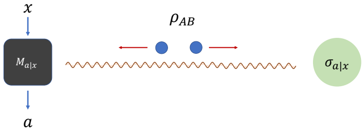

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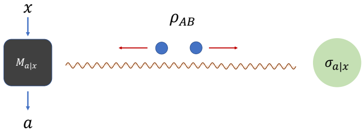

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- We know compatibility and LHS models are equivalent: does this generalise in terms of dimension?

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1. Measurement Incompatibility
2. Quantum Steering
3. Channel-state duality and the Heisenberg picture
4. Our Contribution
5. Conclusions and Future work

The Heisenberg picture

Under the Hilbert-Schmidt inner product $\langle A, B\rangle := \mathrm{Tr}(A^\dagger B)$, recall the definition of the dual map $\mathcal{F} \longleftrightarrow \mathcal{F}^*$:

$$\langle \mathcal{F}(A), B\rangle = \langle A, \mathcal{F}^*(B)\rangle \tag{16}$$

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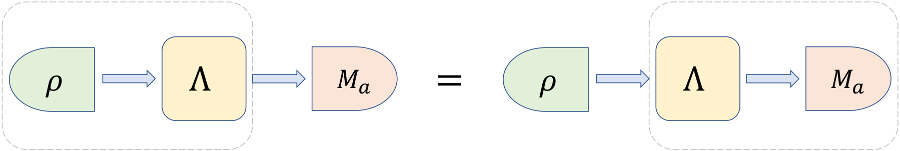

Channel state duality

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This is also referred to as the *Choi–Jamiołkowski* isomorphism, and ρ_Λ is often called the *Choi state* of the channel.

Generalised Channel State duality

States | Channels

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$$\Lambda^*_\rho(X) := \sigma^{-\frac{1}{2}} \mathrm{Tr}_A(X \otimes \mathbb{1} \rho)^T \sigma^{-\frac{1}{2}} \tag{23}$$

$$\iff \quad \Lambda_\rho(Y) = \mathrm{Tr}_B(\mathbb{1} \otimes (\sigma^{-\frac{1}{2}} Y \sigma^{-\frac{1}{2}})^T \rho) \tag{24}$$

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Recap

$$\mathrm{Tr}\big(M\,\Lambda(\rho)\big) = \mathrm{Tr}(\Lambda^*(M)\rho)$$

Dual Channels
$\Lambda^*: \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$

"map measurements to measurements"

Schrodinger vs Heisenberg picture

Quantum Channels
$\Lambda: \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$

"map states to states"

Channel-State Duality

Quantum states
$\rho \in \mathcal{L}(\mathcal{H}_A) \otimes \mathcal{L}(\mathcal{H}_B)$

"bipartite state"

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$$\Lambda_\rho(X) := \mathrm{Tr}_B(\mathbb{1} \otimes X^T\,\rho)$$

Comparing the pictures

It can be often helpful conceptually to use these connections:

Given some property of a channel, can also consider the associated dual and Choi state.

Λ	Λ^*	ρ_Λ
Trace-preserving	Unital	Unit trace
Completely-positive	Completely-positive	Positive semi-definite
Unitary	Unitary	Pure state
??	??	Separable
??	??	Schmidt number n

Entanglement breaking channels

Definition

A channel Λ is entanglement breaking if

$$\Lambda \otimes \mathbb{1}\ [\rho_{AB}] \quad \text{is separable} \quad \forall\ \rho_{AB}$$

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One can show that the Choi states of these channels are separable:

$$\Lambda \quad \text{entanglement breaking} \quad \Longleftrightarrow \quad \rho_\Lambda \quad \text{separable}$$

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This is also equivalent to there existing a Kraus decomposition

$$\Lambda(\cdot) = \sum_\lambda K_\lambda(\cdot)K_\lambda^\dagger \quad \text{such that:} \quad \mathrm{rank}(K_\lambda) \leq n \quad \forall \lambda$$

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So if the measurements can be written as $\Lambda^*(M_{a|x})$ for some n-PEB channel Λ, then the assemblage will also be n-preparable.

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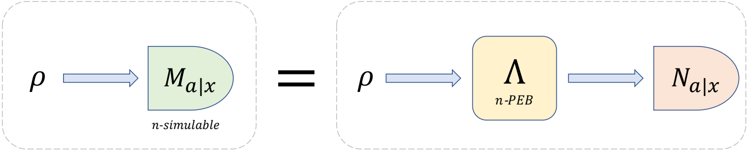
This can be viewed as a form of compression



Main results[6]

- If $M_{a|x}$ are n-simulable, the assemblage $\sigma_{a|x} = \mathrm{Tr}_A\big(M_{a|x} \otimes \mathbb{1}\ \rho\big)$ is n-preparable.

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- Analytical and numerical results on when noisy MUBs and the set of all projective measurements become n-simulable not discussed in this talk.

[6] Marie Ioannou et al. "Simulability of high-dimensional quantum measurements". In: *arXiv preprint arXiv:2202.12980* (2022), Benjamin D. M. Jones et al. "Equivalence between simulability of high-dimensional measurements and high-dimensional steering". In: *arXiv preprint arXiv:2207.04080* (2022).

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If $SN(\rho) \leq n$ or $M_{a|x}$ is n-simulable, then

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We also have a quantative version of this result using the convex weight.

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Consider a steering assemblage $\sigma_{a|x}$ and measurements $M_{a|x}$ such that $M_{a|x} = \rho_B^{-\frac{1}{2}} \sigma_{a|x} \rho_B^{-\frac{1}{2}}$, where $\rho_B := \sum_a \sigma_{a|x}$ is of full rank. Then $M_{a|x}$ is n-simulable if and only if $\sigma_{a|x}$ is n-preparable.

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Proof sketch: We have the following equivalences:

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A channel Λ is n-**partially incompatibility breaking** (n-PIB) if for any measurement assemblage $M_{a|x}$ the resulting measurement assemblage $\Lambda^*(M_{a|x})$ is n-simulable.

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i.e. for all $M_{a|x}$ there exists $N_{a|x}$ and a n-PEB channel Ω such that

$$\Lambda^*(M_{a|x}) = \Omega^*(N_{a|x}). \tag{32}$$

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Λ n-PIB if $\Lambda^*(M_{a|x})$ is n-simulable $\quad\forall\quad M_{a|x}$.

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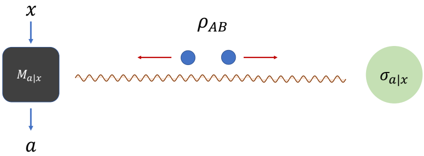

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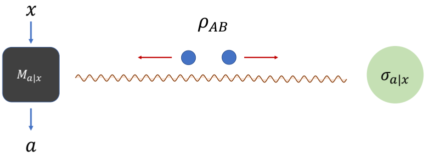

This is analogous to the existence of entangled yet unsteerable states: There exist states with Schmidt number n, but one can only verify a Schmidt number of $n' < n$.

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1. Measurement Incompatibility
2. Quantum Steering
3. Channel-state duality and the Heisenberg picture
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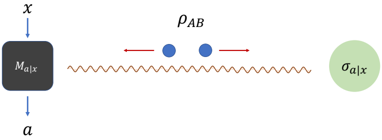
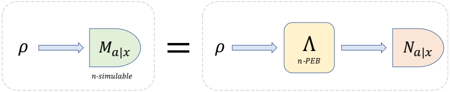
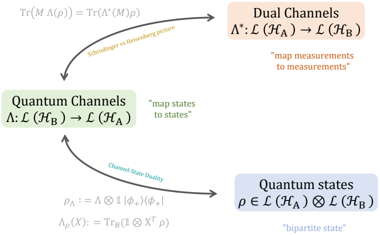
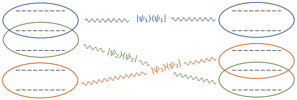
- Better bounds on how much noise before certain measurements/assemblages become n-simulable/n-preparable.







Thanks! :)



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