Electromagnetic Fields (E2.3)

1. The Electromagnetic Spectrum
2. Maxwell's Equations
3. Waves and Transmission Lines
4. Transmission Line Devices
5. Electromagnetic Waves
6. Reflection and Refraction
7. Imaging
8. Radio and Radar
9. Microwaves
10. Guided Wave Optics

Links to the Future

- Preparation for the following courses in the third and fourth year

• Optoelectronics	(E3.12)
• Microwave Technology	(E3.18)
• Optical Communications	(E4.06)
• High Performance Analogue	(E4.17)
• Radio Frequency Electronics	(E4.18)

Recommended Reading

- J.D. Krauss “Electromagnetics” McGraw-Hill International
  Comprehensive, clear, lots of useful diagrams

- F.T. Ulaby “Electromagnetics for engineers” Pearson
  Simpler, quite easy to understand

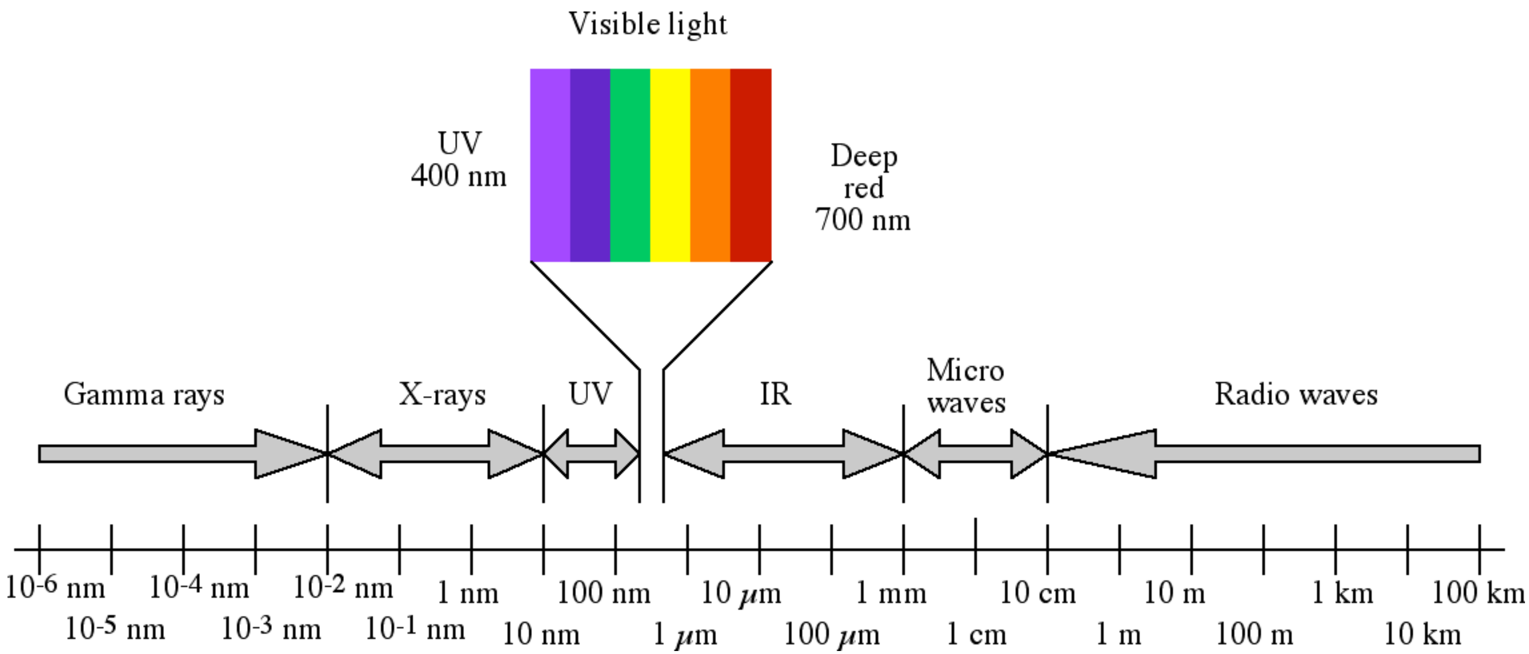
- L.Solymar “Lectures on electromagnetic theory” OUP
  Older, more elegant

- S. Ramo, J.R. Whinnery and T. Van Duzer “Fields and waves in communication electronics” John Wiley & Sons
  Very comprehensive, difficult in places

1: The Electromagnetic Spectrum

- Introduction
- The electromagnetic spectrum
- Key developments
- Applications for electromagnetic waves

The Electromagnetic Spectrum

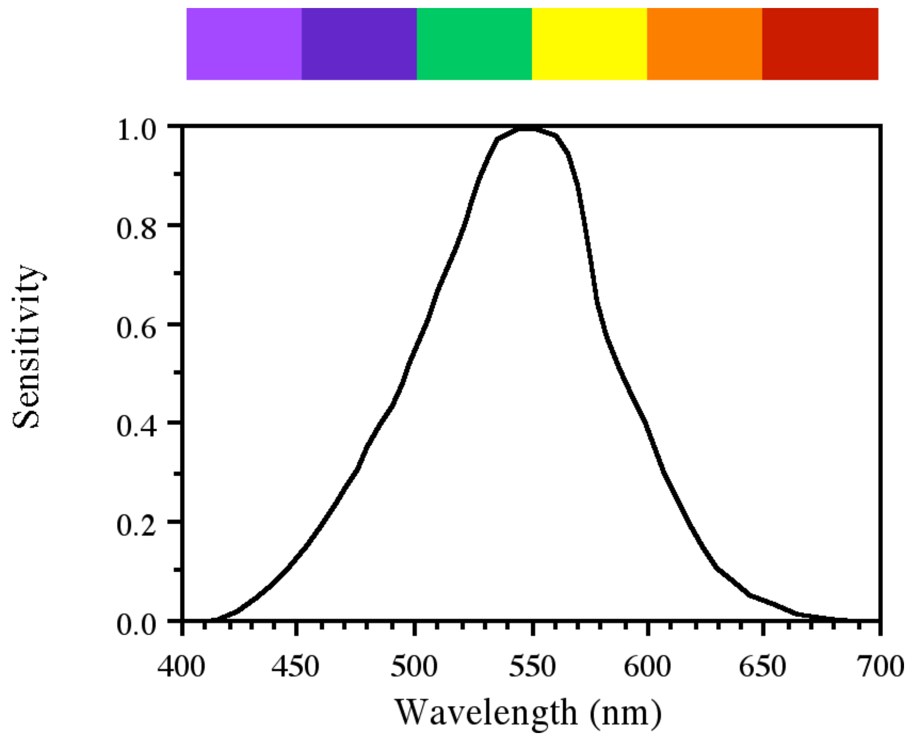


- The electromagnetic spectrum has enormous range; it includes:

  Gamma rays, x-rays, ultraviolet and visible light

  Infrared (heat waves), microwaves and radio waves

What Can We See?

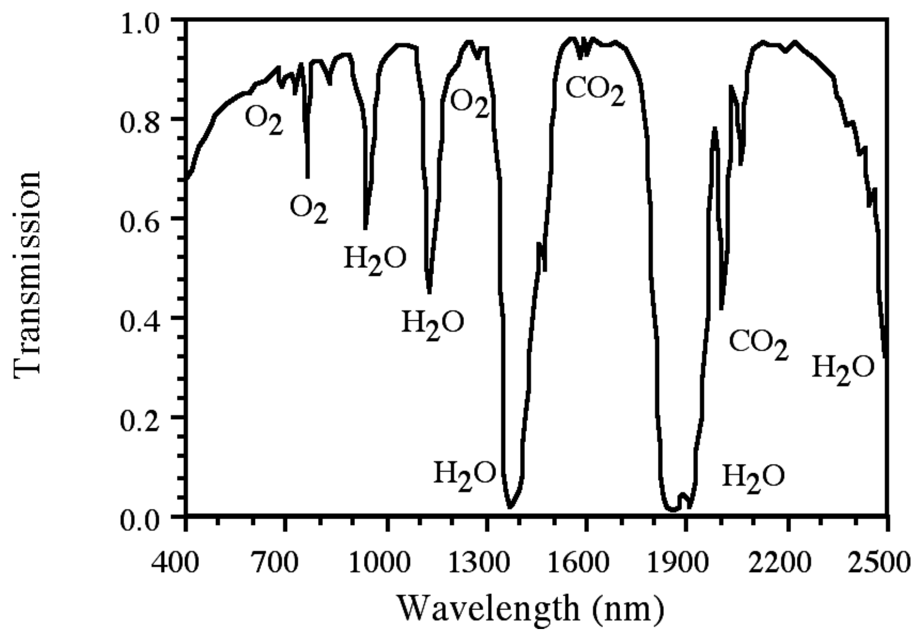


- Human eye sensitivity peaks at ca 550 nm - green light

  Cannot see shorter than blue (UV) - strong absorption in atmosphere

  Or longer than deep red (near IR) - too little energy for photochemistry

What Could We See?

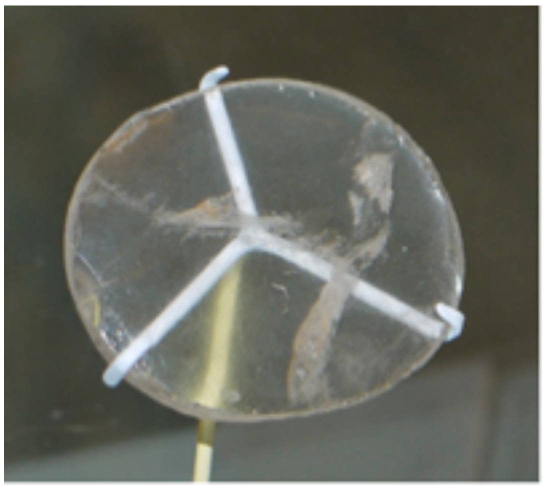


- Transmission very good for visible wavelengths
- Loss rises at short wavelengths (scattering and electronic absorption)

  UV propagation only possible in vacuum
- Losses high in infrared molecular 'absorption bands'

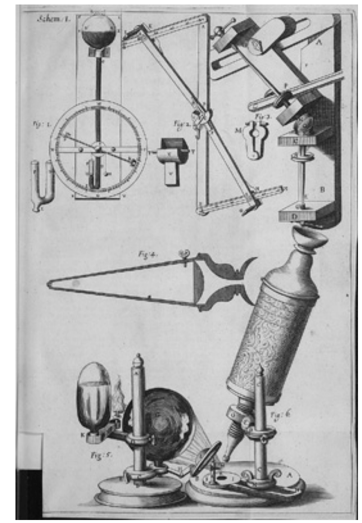
  8 - 12 μm waveband used for night vision

Optical Instruments



The Nimrud Lens,
750-710 BC

Willebrod Snell,
(1591-1626)



Robert Hooke's
microscope

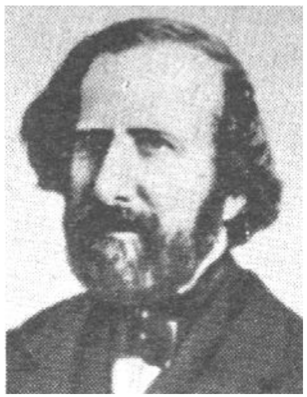
- Visible light most accessible part of the EM spectrum

  Lenses known since Ancient Assyrian times

  Refraction understood in 16th century

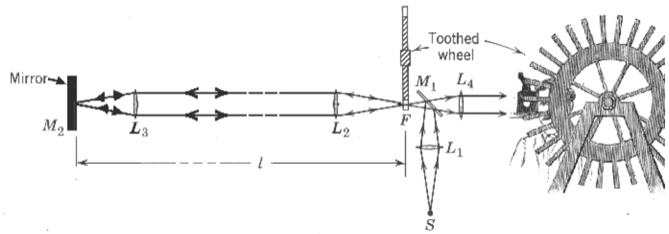
  Compound microscopes developed in 16th century

The Velocity of Light

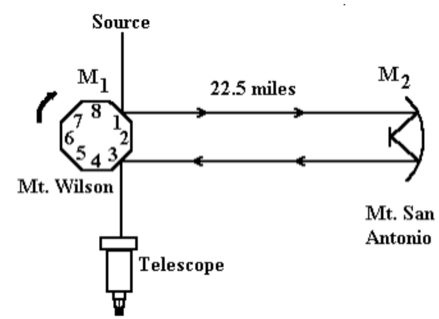


Armand Fizeau

(1819-1896)



Fizeau's apparatus



Michelson's apparatus

- Velocity measured by Armand Fizeau in 1849

  Used a beam of light reflected from a mirror 8 km away.

  Beam passed through the gaps between the teeth of a rapidly rotating wheel

  Speed increased until the returning light passed through the next gap

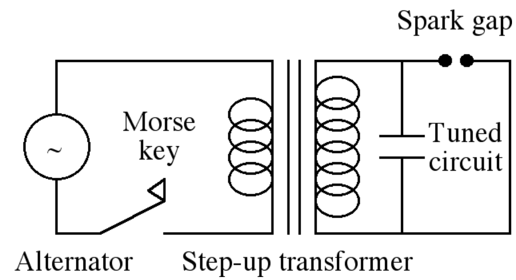
  Then c was calculated to be 315,000 km/s.

  Accuracy improved by Michelson in 1878 using longer path (22.5 miles)

Radio Waves

Heinrich Hertz
(1857-1894)

Hertz's transmitter
and receiver coil



Transmitter with
step-up transformer

- Discovered by Hertz in 1886

  Loop transmitter connected to an induction coil and a battery via a switch.

  When the switch was closed a spark appeared in the gap.

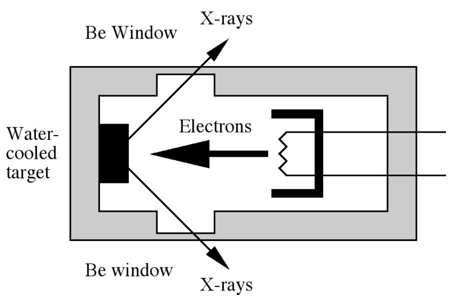
  The receiver was placed on a bench nearby.

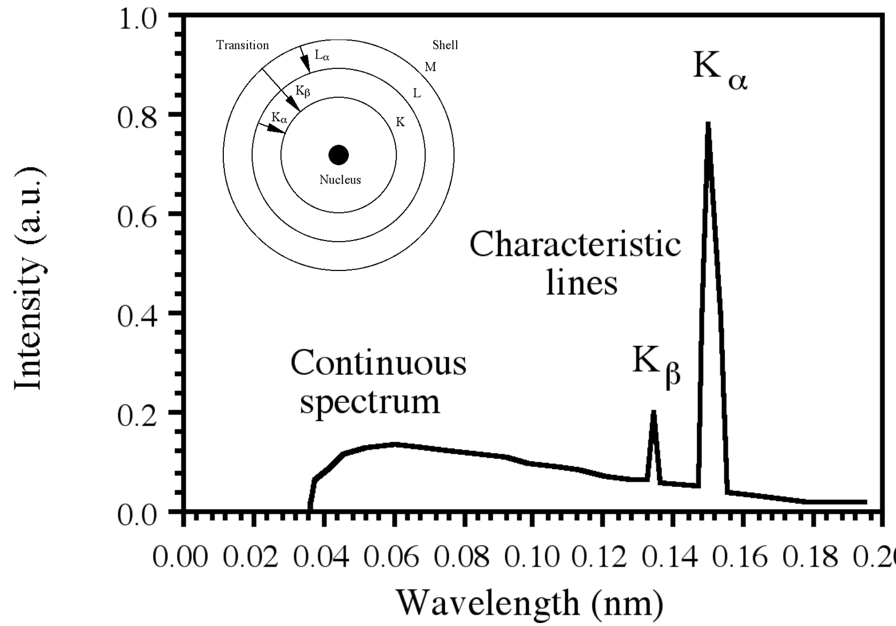
  When the induction coil was activated, a small spark also appeared in the receiver.

  Since there was no physical connection between the two this was something new.

X-Rays

Wilhelm Roentgen





- Discovered by Roentgen in 1895

  Produced when element is bombarded with electrons

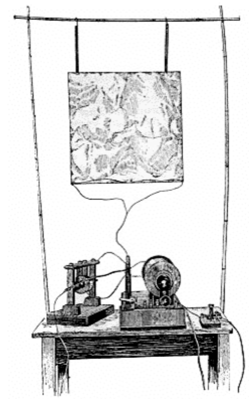
  K_α and K_β lines correspond to transitions back to the K-shell

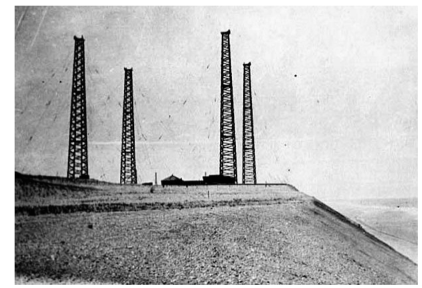
  Many materials transparent to X-rays. Very difficult to form X-ray lenses.

  Difference in absorption between bone, muscle and fat allows skeletal imaging.

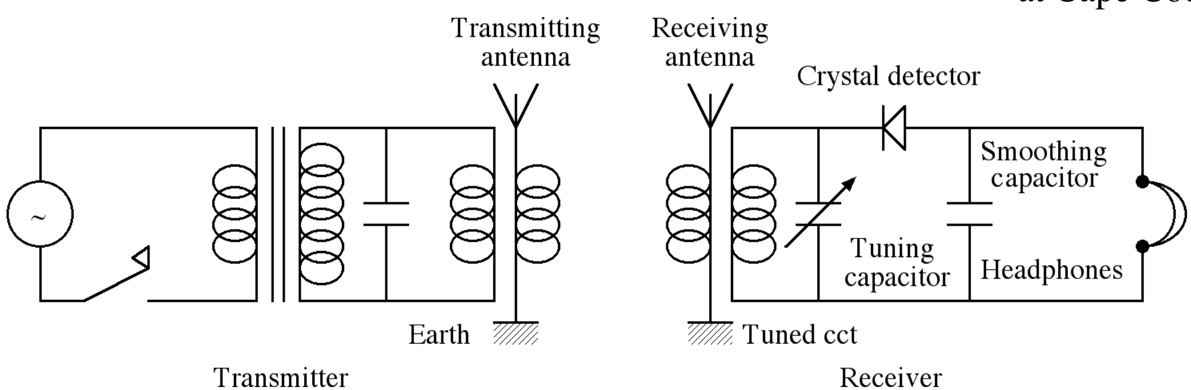
Radio Communications

Guglielmo Marconi
Nobel Prize 1909





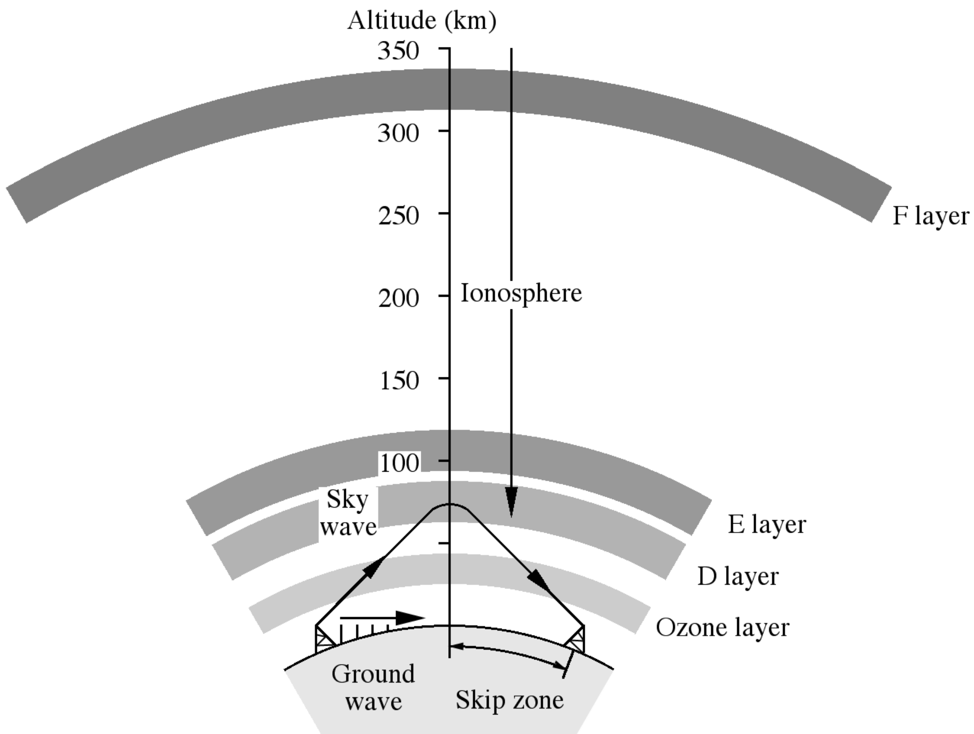
Marconi Wireless Station
at Cape Cod



- Trans-Atlantic radio transmission achieved by Marconi in 1901

  Used antenna instead of spark gap, and simple envelope demodulator

The Ionosphere



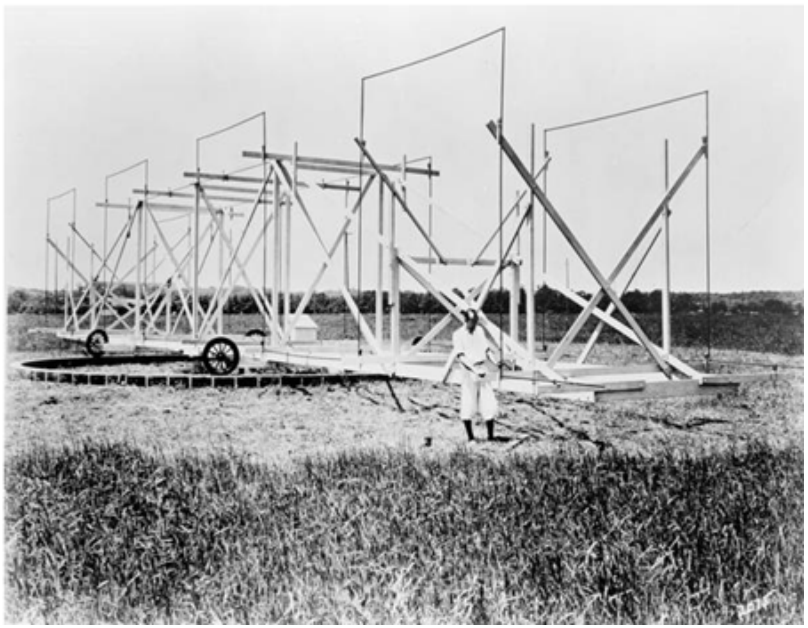
Edward Appleton

- Discovered by Edward Appleton in 1924

  Layers of ionized particles reflect low frequency radio waves

  Ionosphere and sea provide propagation path round curved surface of the earth

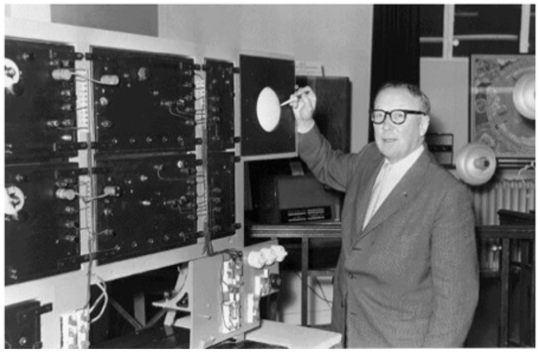
Radio Astronomy



Karl Jansky

- Discovered accidentally by Karl Jansky in 1931
  - Bell Labs Engineer, out testing equipment for interference
  - Instead discovered radio waves from stars

The Birth of Radar



Sir Robert Watson-Watt
Pioneer of air defence radar

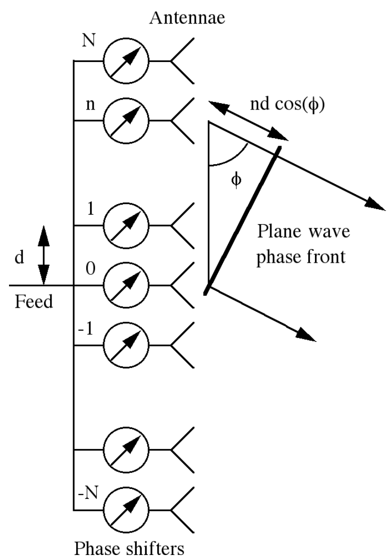
US SCR-584
first gunlaying radar, 1942

German Lichtenstein airborne radar in Ju 88 night fighter
Note large antenna due to lack of high-power high frequency magnetron source

- War-winning weapon, developed in late 1930s
  - Objects detected by observing reflection of radio wave
  - Return time of pulse gives range, Doppler shift gives velocity
  - Signal reduces as $1/r^4$ so high power transmitter needed for long range

Phased Array Radar

Soviet R-36 MIRV ICBM



Phased Array Radar

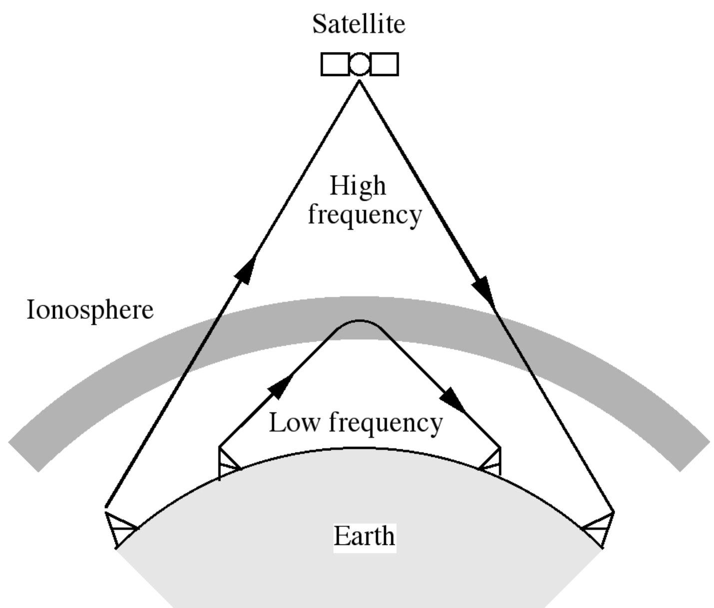
Fylingdales Solid State Phased Array Radar, Yorkshire

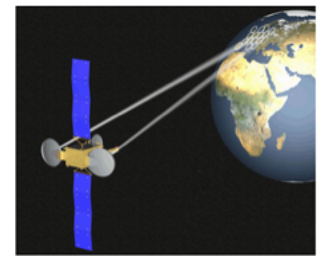
- Radar that can be electronically steered in any direction

  Uses array of antennae fed via phase shifters

- Early warning of nuclear strike during cold war

SatComms

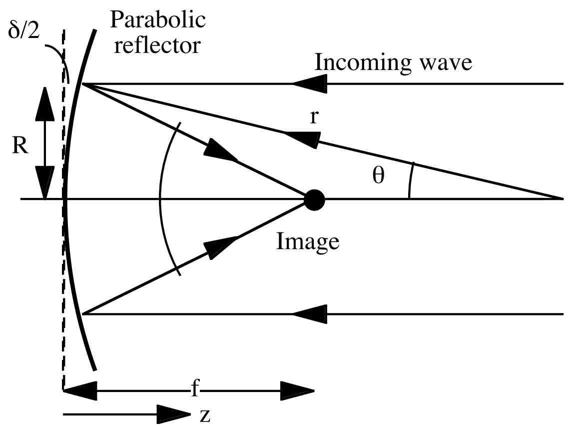




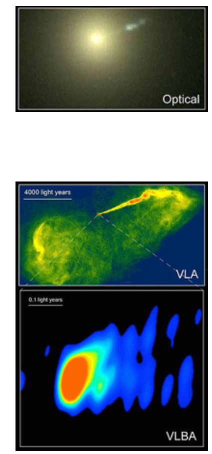
Arthur C. Clarke

- Proposed by Arthur C. Clarke in 1945
  - Ionosphere transparent to VHF waves; communication then via satellite
  - Made practical after WW2 due to development of rockets
  - Sputnik 1 launched 1957 by USSR; Echo 1 launched 1960 by USA

Radio Telescopes



Taffy Bowen with the Parkes radio telescope

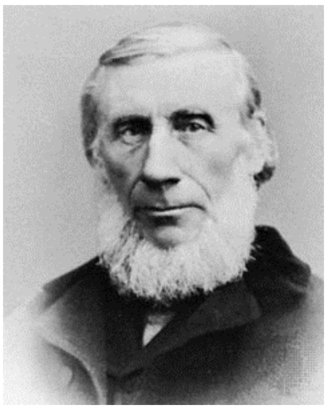


Optical and radio images of galaxy M87

- Wartime radar developers constructed first large radio telescopes using surplus equipment (e.g. the Parkes telescope, 1961)

  Radio astronomical images completely different from optical

Guiding of Light



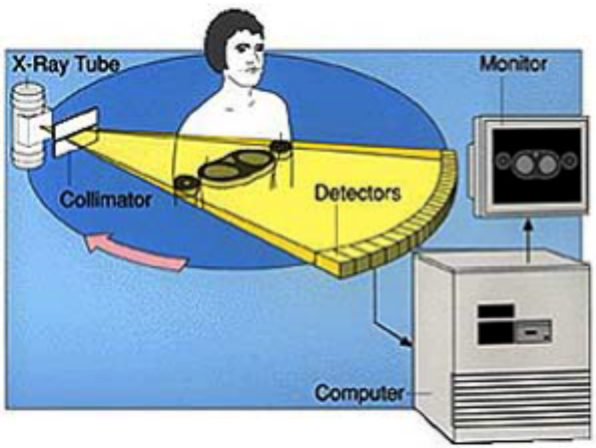
John Tyndall
(1820-1893)

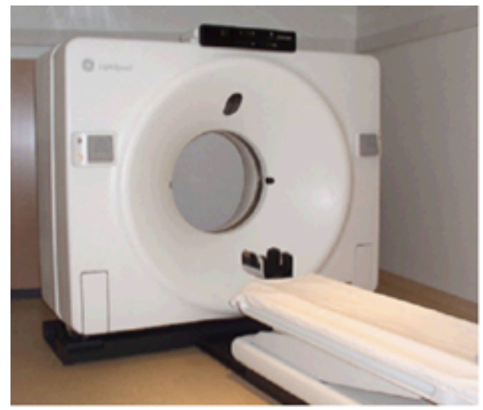
Guiding of light by water jet

Charles Kao
(1933-)

- Guiding in water jet first demonstrated by Tyndall in 1870
  Continuous guidance of light possible by total internal reflection
- Optical fibre communication proposed by Kao and Hockham in 1966
  Allows low loss over long distances

CAT Scanners

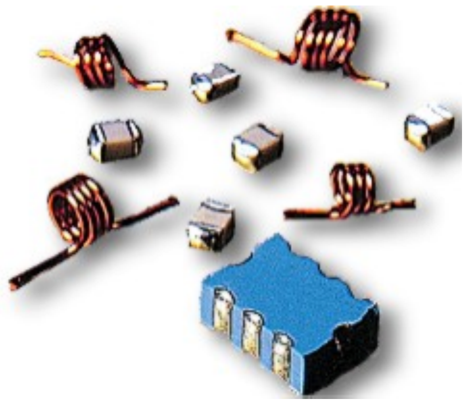


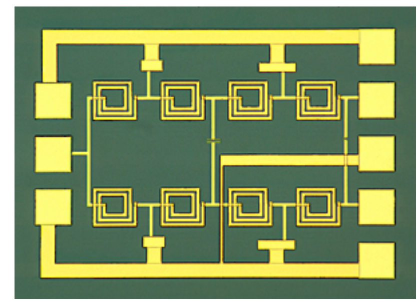


Godfrey Hounsfield
Nobel Prize 1979

- Invented at EMI in 1972
- Method of X-ray imaging that avoids the need for a lens
  - Operates by measuring absorption instead
  - Absorption measured over many different sight-lines
  - Simultaneous equations solve to determine absorber distribution

Applications (1)





Passive components

Transmission lines

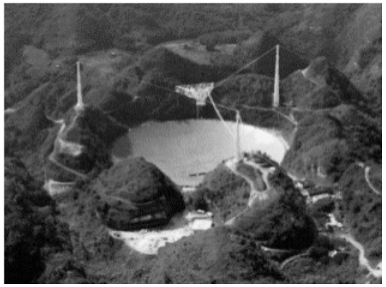
MMICs

- DC to RF

Applications (2)

Radio, mobile comms

TV



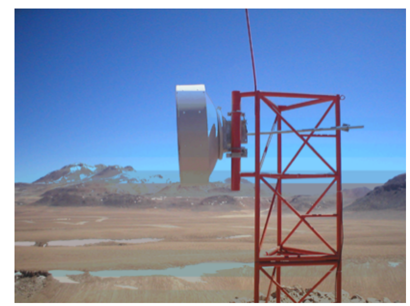
Radio astronomy

- Radio waves

Applications (3)

Radar

Microwave oven

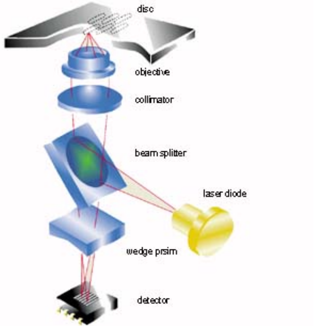


Microwave link

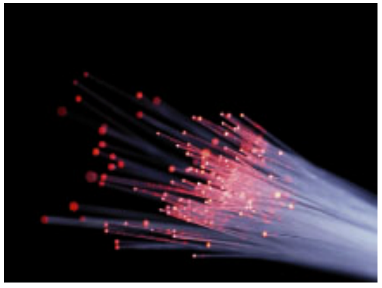
- Microwaves

Applications (4)

FLIR system



CD player, CD ROM

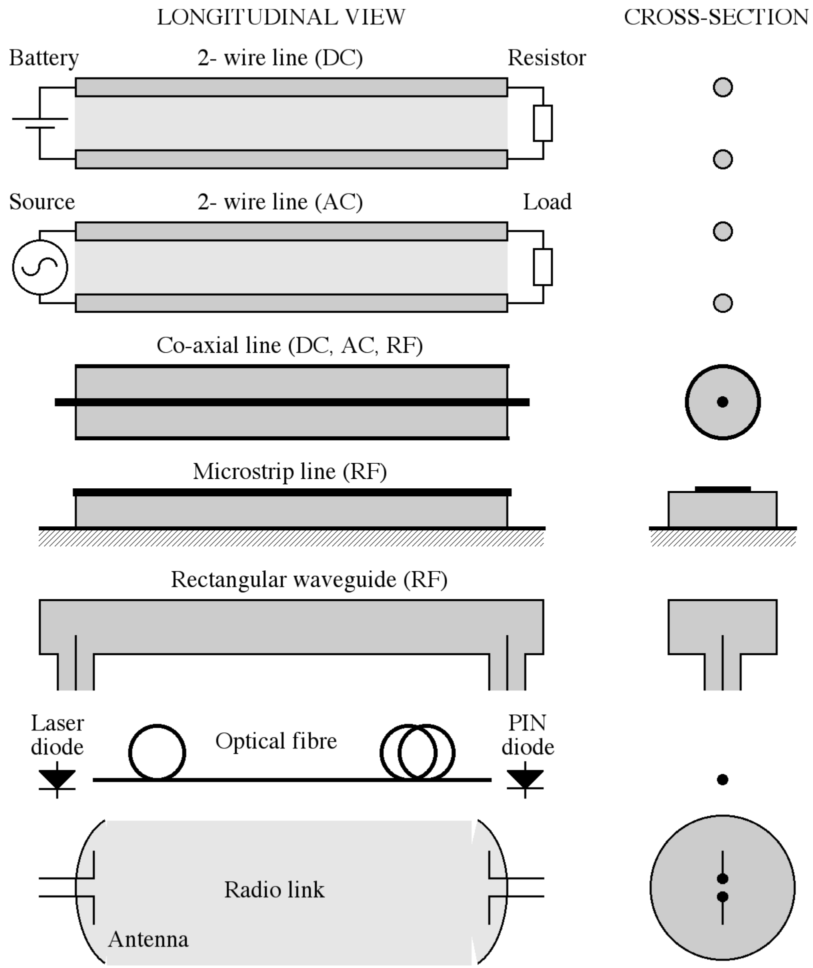


Optical fibre

X-ray CAT scanner

- Infra-red and visible optics; X-rays

Transmission Links



Limitations

- Line of sight comms limited by three mechanisms:
- Absorption

  Excitation of electronic transitions in the molecules of the earth's atmosphere at visible/UV wavelengths, and molecular vibrational transitions at infrared wavelengths. Losses are concentrated in bands.

- Scattering

  Rayleigh scattering, due to inhomogeneities (soot particles) and small-scale fluctuations in the molecular arrangement of the atmosphere. Scattering losses rise rapidly at short wavelengths (rising as $1/\lambda^4$)

- Diffraction

  Spreading of a bounded beam as it propagates. Diffraction increases rapidly as the beam dimensions approach that of the wavelength. Avoided using guided waves.

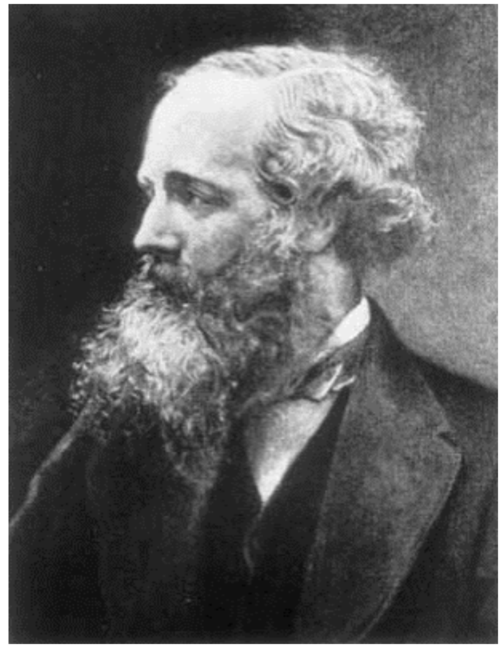
Other Links

- Radio system - one source communicates with many
- Radio telescope - radio star communicates with earth
- Radar system - communicates with itself via an aeroplane
- CD ROM - communicates with itself via a disk
- Microwave oven - communicates with food?

2: Maxwell’s Equations

- Maxwell’s Equations
- Gauss’ Law
- Capacitance of a co-ax cable
- Ampere’s Law
- Inductance of a coax cable

The Unifier of EM Theory



James Clerk Maxwell (1831 - 1879)
Added term to DC theory
and predicted the speed of light

EM Fields

- Electromagnetic fields are described by the quantities:

  $\underline{\mathbf{E}}$ = Electric field strength

  $\underline{\mathbf{D}}$ = Electric flux density

  $\underline{\mathbf{H}}$ = Magnetic field strength

  $\underline{\mathbf{B}}$ = Magnetic flux density

  $\underline{\mathbf{J}}$ = Current density

  Bold underlined = time- and space-varying vector

  In Cartesian co-ordinates

  $\underline{\mathbf{E}}(x, y, z, t) = [\ \mathbf{E}_x(x, y, z, t)\ ,\ \mathbf{E}_y(x, y, z, t)\ ,\ \mathbf{E}_z(x, y, z, t)\ ]$ etc.

- Often, time-variation is harmonic, so:

  $\underline{\mathbf{E}}(x, y, z, t) = \underline{E}(x, y, z) \exp(j\omega t)$

  Plain underlined = space-varying vector

  $\underline{E}(x, y, z) = [\ E_x(x, y, z)\ ,\ E_y(x, y, z)\ ,\ E_z(x, y, z)\ ]$

Maxwell's Equations

- Gauss' Law $\iint_A \underline{\mathbf{D}} \,.\, d\underline{a} = \iiint_V \rho \, dv$
  (ρ = charge density)

- Magnetic equivalent $\iint_A \underline{\mathbf{B}} \,.\, d\underline{a} = 0$
  (no magnetic monopoles)

- Ampere's Law $\int_L \underline{\mathbf{H}} \,.\, d\underline{L} = \iint_A [\, \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t \,] \,.\, d\underline{a}$
  (as modified by Maxwell)

- Faraday's Law $\int_L \underline{\mathbf{E}} \,.\, d\underline{L} = - \iint_A \partial\underline{\mathbf{B}}/\partial t \,.\, d\underline{a}$

Constitutive Equations

- $\underline{\mathbf{J}} = \sigma\, \underline{\mathbf{E}}$

  σ is the conductivity

- $\underline{\mathbf{D}} = \varepsilon\, \underline{\mathbf{E}}$

  ε is the permittivity, or $\varepsilon = \varepsilon_0\, \varepsilon_r$ where

  $\varepsilon_0 = (1/36\pi) \times 10^{-9} = 8.85 \times 10^{-12}$ F m^{-1} is the permittivity of free space

  ε_r is the relative permittivity

- $\underline{\mathbf{B}} = \mu\, \underline{\mathbf{H}}$

  μ is the permeability, or $\mu = \mu_0\, \mu_r$ where

  $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} is the permeability of free space

  μ_r is the relative permeability

Gauss

Johann Carl Friedrich Gauss (1777 – 1855)

Prime number theory; normal distribution,the Gaussian function, Electrostatics, magnetism, the theory of imaging

Gauss's Law (1) : Definition

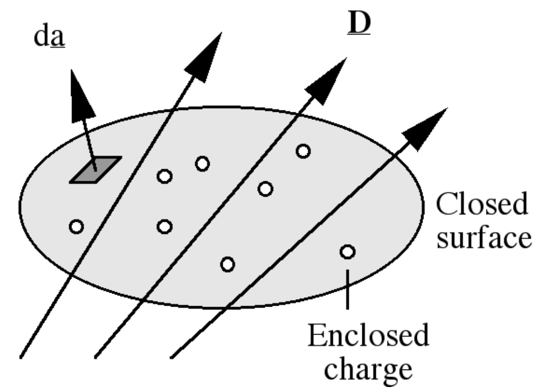
- In a few words:

  "Flux out = charge enclosed"

- In a few more words:

  "Integral over closed surface of normal component of flux =

  integral over enclosed volume of charge contained therein"

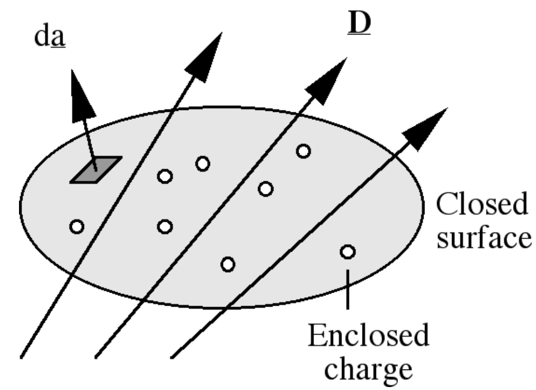


- In vectorial notation

  $\iint_A \underline{\mathbf{D}} \,.\, d\underline{a} = \iiint_V \rho \, dv$

Gauss' Law (2) : Revisited

- Consider flux out of surface enclosing charges



- Gauss' Law is $\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \rho \, dv$

  $d\underline{a}$ is a vector representing a small area

  The modulus of $d\underline{a}$ is the size of the area

  The direction of $d\underline{a}$ is the normal to the area

  $\underline{\mathbf{D}} \cdot d\underline{a}$ is the amount of flux normal to the small area

  $\iint_A \underline{\mathbf{D}} \cdot d\underline{a}$ is sum over the surface A, i.e. total flux out

  $\iiint_V \rho \, dv$ is the integral of the charge over the volume V

- Flux out = charge enclosed

Application: Co-ax Cable

- Assume line charge q per unit length

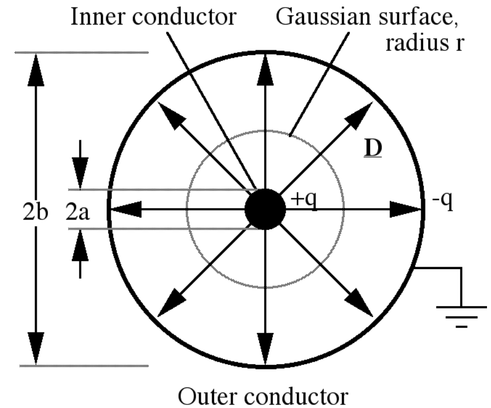
  Flux **D** is radial by symmetry

- Assume Gaussian surface at radius r

  NB Surface is cylinder of unit depth

  Gauss' law gives: $2\pi r \times 1 \times D_r = 2\pi r\varepsilon \times 1 \times E_r = q \times 1$

  Consequently $E_r = q/2\pi r\varepsilon$



Unit depth into page

- Potential V found from $E_r = -dV/dr$

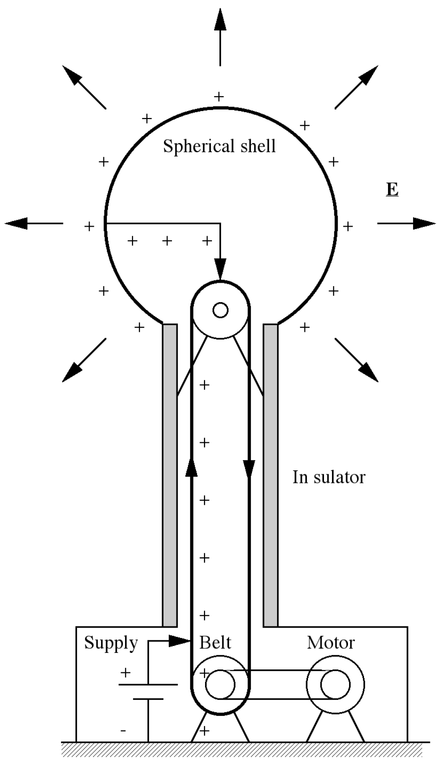
  Hence $V(b) - V(a) = -\int_a^b (q/2\pi r\varepsilon)\,dr = -(q/2\pi\varepsilon)\log_e(b/a)$

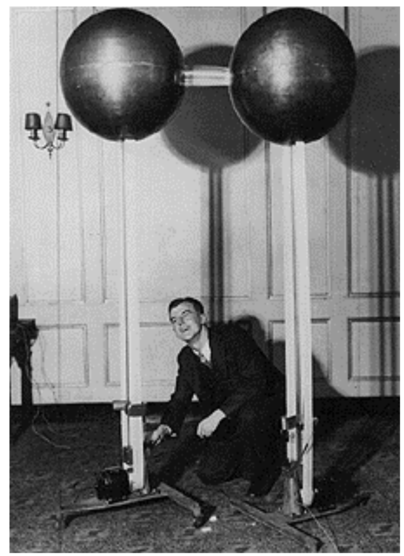
  If outer conductor is grounded, $V(a) = (q/2\pi\varepsilon)\log_e(b/a)$

- Capacitance per unit length (p.u.l.) is

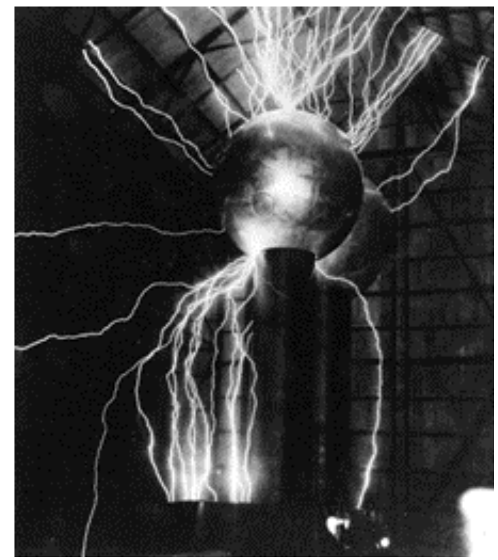
  $C_p = q/V = (2\pi\varepsilon) \div \log_e(b/a)$ F/m

Van de Graaf Generator





Robert J. Van de Graaff



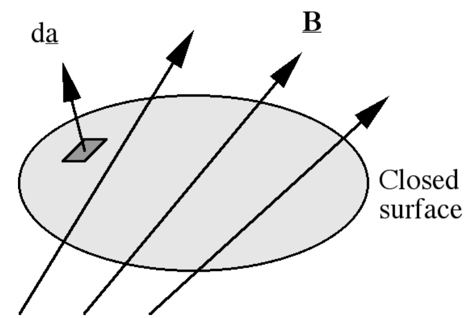
Let's break something

- Method of obtaining high voltage by pumping charge onto surface
  - Charges transferred onto and off moving belt by friction
  - High charge causes high electric field, which can create a discharge

Magnetic Version of Gauss' Law

- EM theory has many analogies between electric and magnetic behaviour; however, these are not exact

- There are isolated electric charges, but no magnetic monopoles.
- Cut a bar magnet in half, and you get two small magnets, not a North pole and a South pole



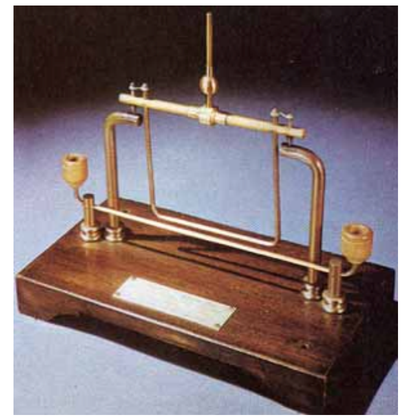
- Magnetic equivalent of Gauss' law is:

  $\iint_A \underline{\mathbf{B}} \,.\, d\underline{a} = 0$

- The RHS is zero because of the lack of monopoles

Ampere

André Marie Ampère (1775 - 1836)
Mathematics, Chemistry (discovered fluorine),
Magnetism (invented solenoid)



Ampere's current balance

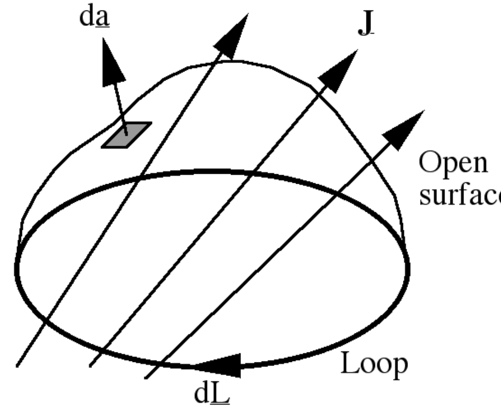
Ampere's Law (1): Definition

- In a few words:

  "Integral of magnetic field round path = current enclosed"

- In vectorial notation

  $\oint_L \underline{\mathbf{H}} \,.\, d\underline{L} = \iint_A \underline{\mathbf{J}} \,.\, d\underline{a}$



- Moving charges imply a circulating magnetic field

Ampere's Law (2): Revisited

- Ampere's law is $\oint_L \underline{\mathbf{H}} \cdot d\underline{L} = \iint_A \underline{\mathbf{J}} \cdot d\underline{a}$

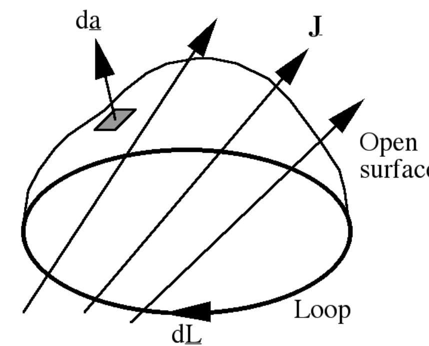
  $d\underline{L}$ is a short element of path

  Modulus of $d\underline{L}$ is size of element

  Direction of $d\underline{L}$ is direction of element

  $\underline{\mathbf{H}} \cdot d\underline{L}$ is component of field parallel to element

  $\oint_L \underline{\mathbf{H}} \cdot d\underline{L}$ is integral of this component round path



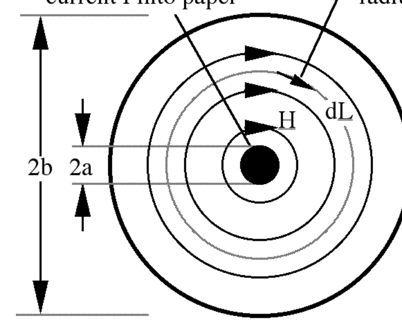
- LHS is integral round a path of component of magnetic field parallel to that path

- RHS is integral over surface of current density passing through, i.e. total current

Application: Co-ax cable

- Assume central conductor carries current I into paper

  Magnetic field <u>H</u> circumferential by symmetry



- Assume circular path, of radius r

  d<u>L</u> is always parallel to magnetic field

  Ampere's law gives $2\pi r H_\phi = I$,

  Hence $H_\phi = I/2\pi r$

- Flux linked between conductors per unit length is:

  $\Phi = {}_a\int^b B_\phi \, dr = {}_a\int^b \mu_0 H_\phi \, dr$

  $\Phi = {}_a\int^b (\mu_0 I/2\pi r) \, dr = (\mu_0 I/2\pi) \log_e(b/a)$

- Inductance per unit length is:

  $L_p = \Phi/I$

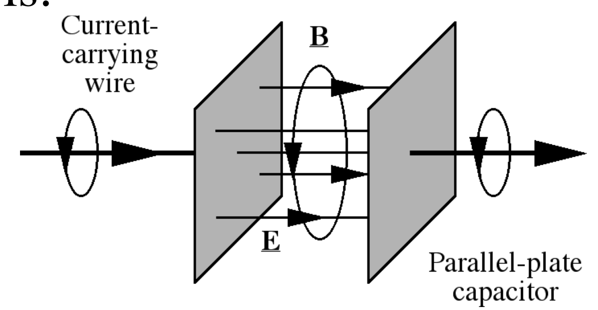
  $L_p = (\mu_0/2\pi) \log_e(b/a)$

Maxwell's Modification

- How is current continuous in circuit with capacitor?

- By Gauss' law, electric field between plates is:

  Flux p.u.a = charge p.u.a. so $\varepsilon E = Q/A$



- Maxwell noticed that

  $\varepsilon\, dE/dt = dD/dt = I/A$ is a current density

  He called it the displacement current, and added it to Ampere's law

  $\int_L \underline{\mathbf{H}} \cdot d\underline{L} = \iint_A [\, \underline{\mathbf{J}} + \partial \underline{\mathbf{D}}/\partial t \,] \cdot d\underline{a}$

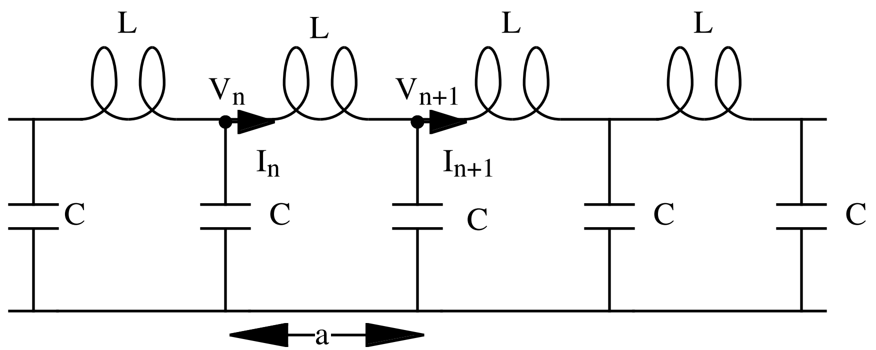
- At low frequency, $\underline{\mathbf{J}}$ dominates; at high frequency $\partial\underline{\mathbf{D}}/\partial t$ dominates.

  At a stroke, Maxwell unified the theory.

3: Waves & Transmission Lines

- Ladder networks
- Dispersion diagrams, phase and group velocity
- Transmission lines
- The Ionosphere

Ladder Networks (1)



- Periodic circuits used as the basis of many filters

  NOT the same as transmission line, but similar

- For n^{th} section of circuit above, Kirchhoff's laws give

  $V_{n+1} = V_n - j\omega L I_n$

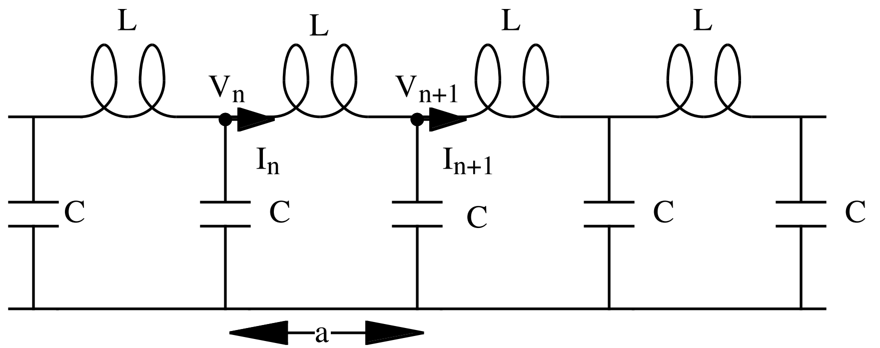
  $I_{n+1} = I_n - j\omega C V_{n+1}$

- Assume wave solutions $V_n = V_0 \exp(-jnka)$, $I_n = I_0 \exp(-jnka)$:

  $V_0 \exp\{-jk(n+1)a\} = V_0 \exp(-jkna) - j\omega L I_0 \exp(-jnka)$

  $I_0 \exp\{-jk(n+1)a\} = I_0 \exp(-jkna) - j\omega C\, V_0 \exp\{-jk(n+1)a\}$

Ladder Networks (2)



- Cancel the terms exp(-jnka) on both sides:

  $V_0 \exp(-jka) = V_0 - j\omega L\, I_0$

  $I_0 \exp(-jka) = I_0 - j\omega C\, V_0 \exp(-jka)$

- Re-arrange:

  $\{\exp(-jka) - 1\}V_0 + j\omega L I_0 = 0$

  $j\omega C \exp(-jka) V_0 + \{\exp(-jka) - 1\}I_0 = 0$

- Simultaneous equations for V_0 and I_0. Solution exists if:

  $\{\exp(-jka) - 1\}\{\exp(-jka) - 1\} + \omega^2 LC \exp(-jka) = 0$ NB independent of I_0, V_0!

Dispersion Characteristic (1)

- Need to solve:

  $\{\exp(-jka) -1\}\{\exp(-jka) -1\} + \omega^2 LC \exp(-jka) = 0$

- Multiply out:

  $\exp(-j2ka) - 2 \exp(-jka) + 1 + \omega^2 LC \exp(-jka) = 0$

- Cancel one exp(-jka) term:

  $\exp(-jka) - 2 + \exp(+jka) + \omega^2 LC = 0$

  $2 \cos(ka) - 2 + \omega^2 LC = 0$

- Defining $\omega_0^2 = 1/LC$, we get:

  $\omega^2/\omega_0^2 = 2 - 2 \cos(ka)$, or

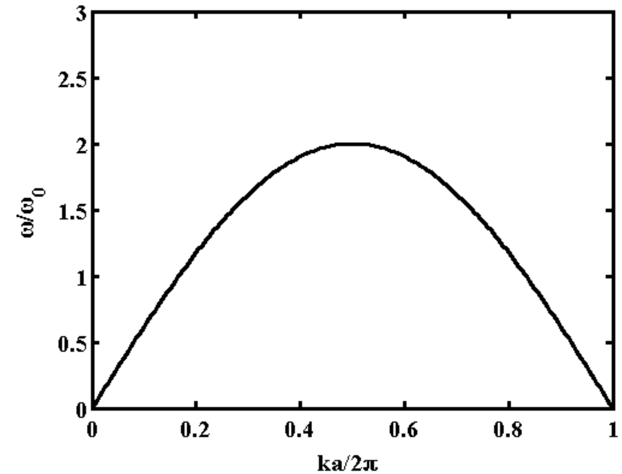
  $\omega/\omega_0 = 2 \sin(ka/2)$ This is the dispersion relation

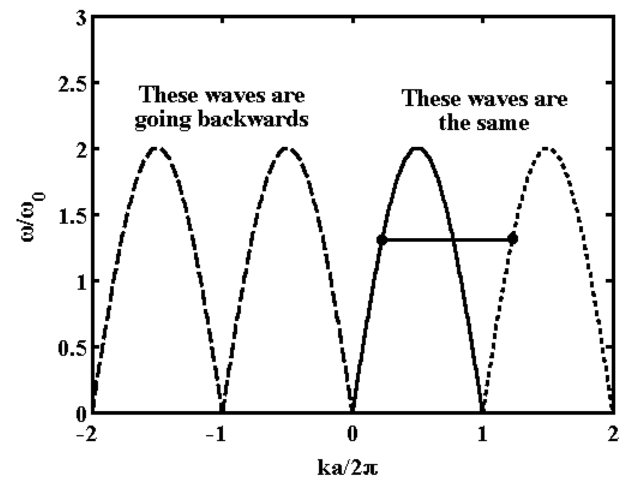
Dispersion Characteristic (2)

- Dispersion relation is $\omega/\omega_0 = 2 \sin(ka/2)$
  k is the propagation constant
  ka is the phase shift per section

- Plot of ω vs k is dispersion diagram
  Standard way of representing behaviour
  All waves have a dispersion characteristic

- Here, characteristic is a sinusoid
  For periodic structures,
  Only range $0 \leq ka \leq \pi$ significant



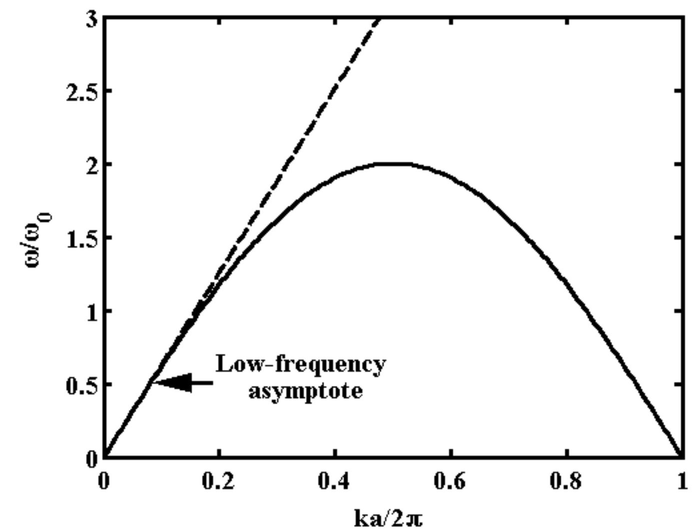


Phase Velocity

- Phase velocity is $v_{ph} = f\,\lambda = \omega/k$
  Velocity of single wave
  Cannot describe information or power transfer

- Generally, v_{ph} is not constant

- Here, $\omega/\omega_0 = 2\sin(ka/2)$
  At low frequency, $\omega/\omega_0 \approx 2 \text{ x } ka/2 \approx ka$
  Hence $\omega/k \approx a\omega_0$
  So $v_{ph} \approx (a^2/LC)^{1/2}$
  Dispersion plot approximates to straight line

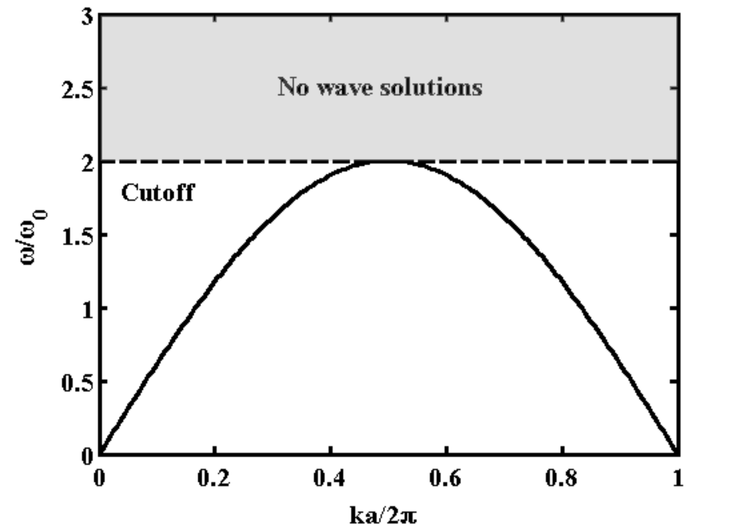


Cutoff

- Need to evaluate $ka = 2 \sin^{-1}(\omega/2\omega_0)$

  Real solutions only up to $\omega = 2\omega_0$

  This value is the 'cutoff' frequency



- If $y = \sin(x)$, and $y > 1$

  Put $x = \pi/2 - jz$

  $y = \sin(\pi/2)\cos(-jz) + \cos(\pi/2)\sin(-jz)$

  $y = \cos(-jz) = \cosh(z)$

  So $x = \pi/2 - j \cosh^{-1}(y)$

- So $ka = \pi - 2j \cosh^{-1}(\omega/2\omega_0)$ above cutoff

  k is complex, and $k = k' - jk''$

  Hence $V_0 \exp(-jnka) = V_0 \exp(-jnk'a) \exp(-nk''a)$

  Wave decays exponentially, and network is low-pass filter

Group Velocity (1)

- Consider signal with two components, at $\omega + d\omega$ and $\omega - d\omega$

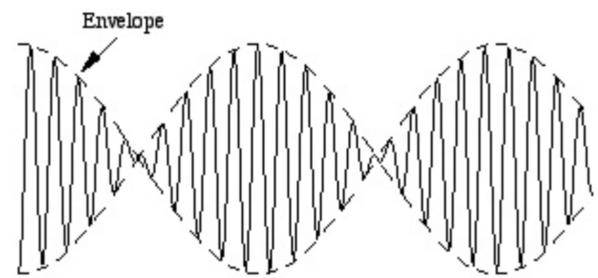
  Corresponding k-values are $k + dk$ and $k - dk$

- For equal amplitudes, voltage wave is:

  $V = V_0\,[\exp\{j((\omega + d\omega)t - (k + dk)z)\}$
  $+ \exp\{j((\omega - d\omega)t - (k - dk)z)\}]$

  Or $V = 2V_0 \exp\{j(\omega t - kz)\} \cos\{d\omega\, t - dk\, z\}$

  Hence, wave is amplitude-modulated carrier



- Velocity of envelope is group velocity

  $v_g = d\omega/dk$

  Envelope represents information-content (carrier has none!)

- For low signal distortion, v_g should be constant

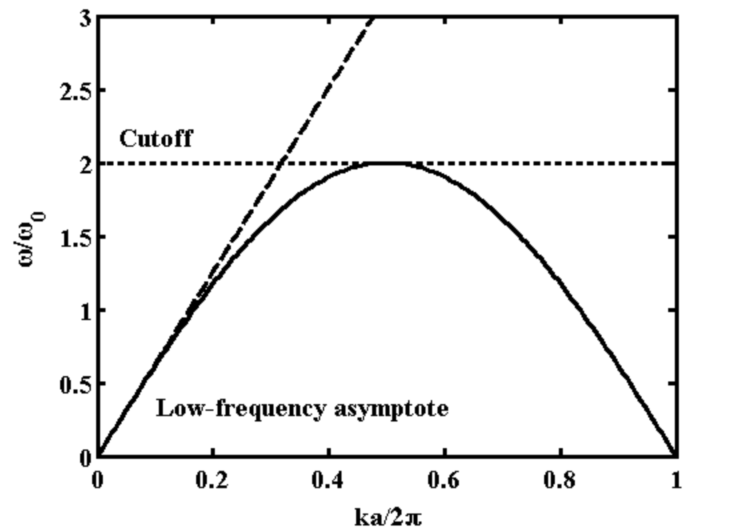
Group Velocity (2)

- v_g describes information and power flow
- Example: v_g for ladder network

  When $\omega \approx 0$

  $d\omega/dk \approx \omega/k$

  $v_g \approx v_{ph}$



- When $\omega \approx 2\omega_0$

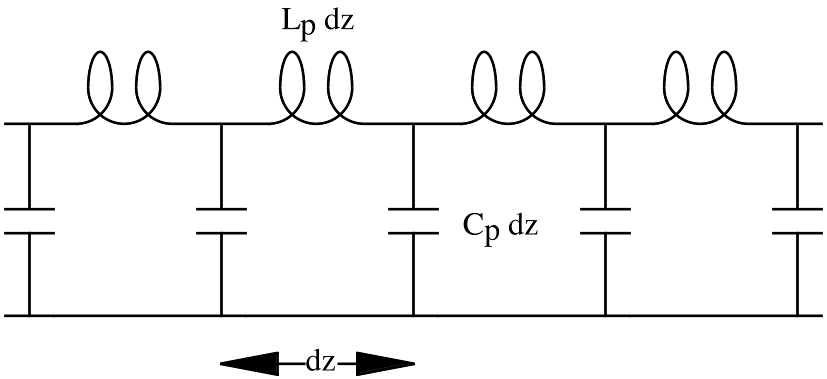
  $d\omega/dk \approx 0$

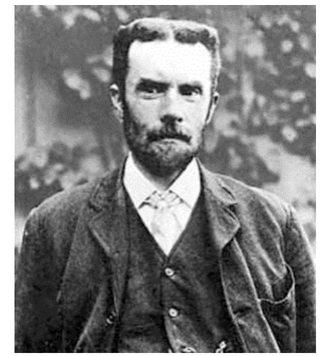
  $v_g \approx 0$

- Group velocity is zero at cutoff

  Information does not propagate; neither does power

Co-axial Transmission Line





Oliver Heaviside (1850-1925)
Transmission lines; Heaviside layer

- Ladder network model of continuous transmission line
  - Model due to Oliver Heaviside
  - Main difference from previous ladder is that sections are very small
- Co-ax cable has capacitance C_p and inductance L_p per unit length
  - Line is divided into sections of length dz
  - Each has inductance L_p dz and capacitance C_p dz

Transmission Line Equations (1)

- Assign nodal voltages V, V' and line currents I, I'

- At frequency ω, we get

  $V' = V - j\omega L_p dz\, I$

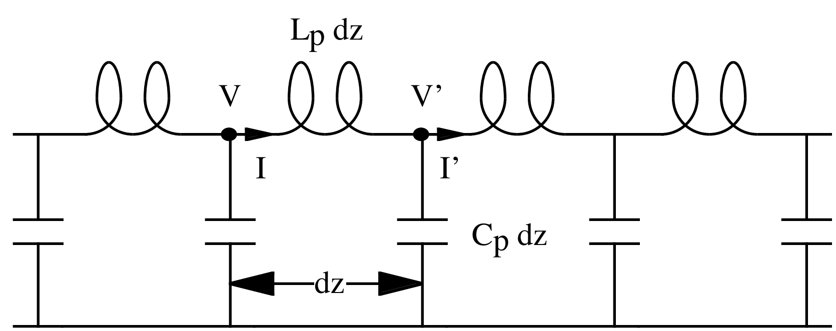
  $I' = I - j\omega C_p dz\, V' \approx I - j\omega C_p dz\, V$

NB: This approximation is good enough if V and V' are close, i.e. if V varies slowly

- Continuous system would have

  $V' = V + (dV/dz)\, dz$

  $I' = I + (dI/dz)\, dz$



- Comparison shows that:

  $dV/dz = -j\omega L_p I$

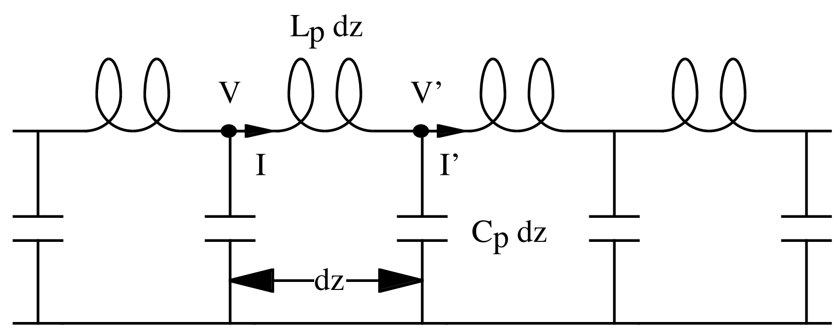
  $dI/dz = -j\omega C_p V$

Transmission Line Equations (2)

- Coupled equations are:

  $dV/dz = -j\omega L_p I$

  $dI/dz = -j\omega C_p V$



- Differentiating, we get:

  $d^2V/dz^2 = -\ j\omega L_p dI/dz$

- Substituting for dI/dz we get:

  $d^2V/dz^2 = -\omega^2 L_p C_p V$

- Similarly, we could have:

  $d^2I/dz^2 = -\omega^2 L_p C_p I$

- These are the transmission line equations

Wave Solutions

- Transmission line equation for voltage is:

  $d^2V/dz^2 = -\omega^2 L_p C_p V$

- Guess solution $V(z) = V_0 \exp(-jkz)$

  $dV/dz = -jkV$

  $d^2V/dz^2 = (-jk)^2 V = -k^2 V$

- Solution valid if

  $-k^2 V = -\omega^2 L_p C_p V$ or if

  $k = \omega(L_p C_p)^{1/2}$

- Full solution $\mathbf{V}(z, t) = V(z) \exp(j\omega t) = V_0 \exp\{j(\omega t - kz)\}$

  Result is travelling wave, with linked voltage and current

  $I(z) = I_0 \exp(-jkz)$ also solution of $d^2I/dz^2 = -\omega^2 L_p C_p I$

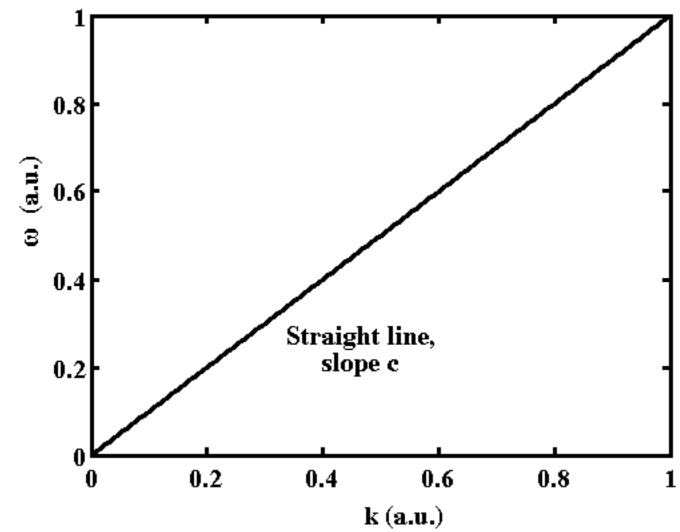
Velocity of an EM Wave

- Solutions $\mathbf{V} = V_0 \exp\{j(\omega t - kz)\}$, $\mathbf{I} = I_0 \exp\{j(\omega t - kz)\}$

  Here $\omega = 2\pi f = 2\pi/T$, where T is time period

  And $k = 2\pi/\lambda$, where λ is spatial wavelength

  k is the propagation constant



- Phase velocity is:

  $v = f\,\lambda = \omega/k = 1/(L_p C_p)^{1/2}$

- For co-ax cable:

  $L_p = (\mu_0/2\pi)\log_e(b/a)$

  $C_p = (2\pi\varepsilon) \div \log_e(b/a)$

  Hence $L_p C_p = \mu_0\varepsilon$ and $v = 1/(\mu_0\varepsilon)^{1/2}$

- This is speed of light in medium of permeability μ_0 and permittivity ε

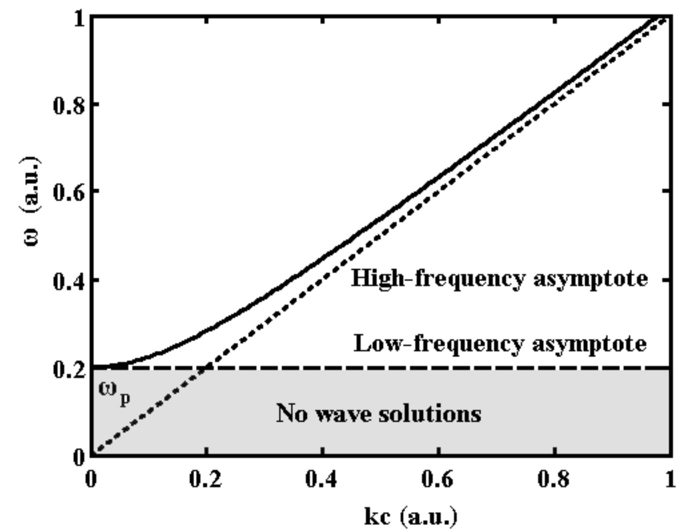
The Ionosphere (1)

- Ionosphere contains charged particles

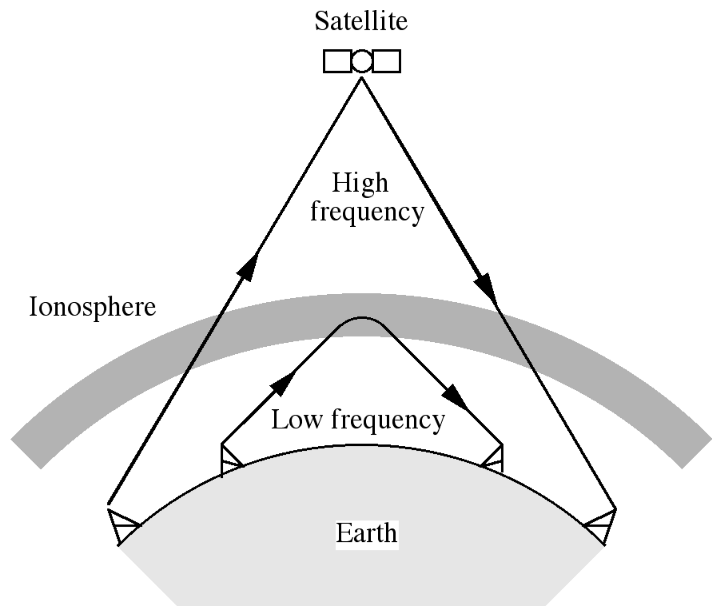
- Dispersion relation is
  $\omega^2 = \omega_p{}^2 + c^2k^2$
  Where ω_p is the plasma frequency

- When $k \approx 0$, $\omega \approx \omega_p$
- When $k >> 0$, $\omega \approx ck$

- $k = (1/c)\ (\omega^2 - \omega_p{}^2)^{1/2}$
  Imaginary solutions for $\omega < \omega_p$
  Ionosphere is high-pass filter



The Ionosphere (2)



- Atmosphere broadly transparent to radio waves
- Ionosphere reflects at low frequency - allows over-the horizon communication
- Ionosphere transparent at high frequency - allows communication via satellite

4: Transmission Line Devices

- Characteristic impedance
- Lossy lines
- Reflection and transmission at junctions
- Input impedance
- Splitters and matching sections

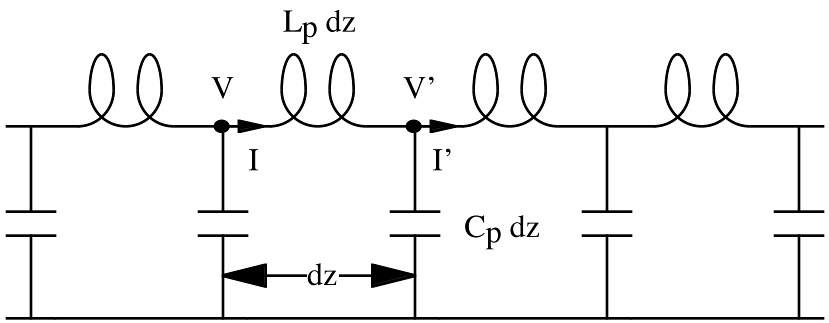
Transmission Line Equations

- Consider ladder model of transmission line again

- At frequency ω, we get

  $V' = V - j\omega L_p dz\ I$

  $I' = I - j\omega C_p dz\ V$



- Continuous system would have

  $V' = V + (dV/dz)\ dz$

  $I' = I + (dI/dz)\ dz$

- Comparison shows that:

  $dV/dz = -j\omega L_p I$

  $dI/dz = -j\omega C_p V$

Characteristic Impedance

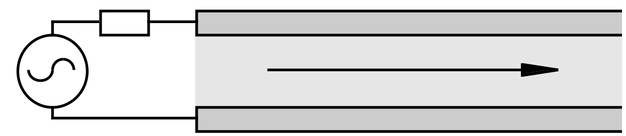
- Wave solutions are $V = V_0 \exp(-jkz)$, $I = I_0 \exp(-jkz)$

  But $dV/dz = -j\omega L_p I$

  So $-jkV = -j\omega L_p I$ and $-jkV_0 = -j\omega L_p I_0$

- Hence $V_0/I_0 = \omega L_p/k = (L_p/C_p)^{1/2}$

  Result has dimensions of Ohms



Line,
Characteristic impedance Z_0

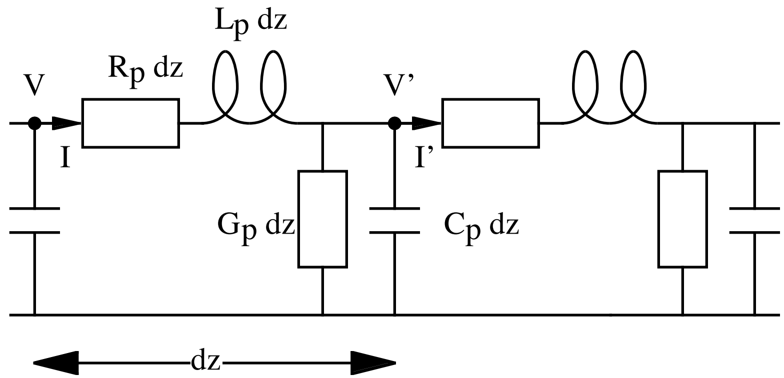
- Ratio is called the characteristic impedance Z_0

- Why is Z_0 real?

  Signal injected into infinite line never emerges, and is effectively lost

  NB Impedance of finite line is not the same!

Lossy lines (1)



- Model loss in inductor by series resistance R_p
- Model loss in capacitor by shunt conductance G_p

- Transmission line equations become:

$dV/dz = -(R_p + j\omega L_p)I$

$dI/dz = -(G_p + j\omega C_p)V$

$d^2V/dz^2 = (R_p + j\omega L_p)(G_p + j\omega C_p)\ V = -k^2V$

Lossy lines (2)

- If $-k^2 = (R_p + j\omega L_p)(G_p + j\omega C_p)$

  Then $k^2 = \omega^2 L_p C_p - j\omega C_p R_p - j\omega L_p G_p + R_p G_p$

- If losses are small, neglect $R_p G_p$ term and rewrite as:

  $k^2 \approx \omega^2 L_p C_p \text{ x } \{1 - j(\omega C_p R_p + \omega L_p G_p)/\omega^2 L_p C_p\}$

- Hence:

  $k \approx \omega(L_p C_p)^{1/2} \text{ x } \{1 - j(C_p R_p + L_p G_p)/\omega L_p C_p\}^{1/2}$

- Now use binomial approximation $(1 + x)^{1/2} \approx 1 + x/2$:

  $k \approx \omega(L_p C_p)^{1/2} \text{ x } \{1 - j(C_p R_p + L_p G_p)/2\omega L_p C_p\}$

- So $k = k' - jk''$ with $k' = \omega(L_p C_p)^{1/2}$ and $k'' = (C_p R_p + L_p G_p)/2 (L_p C_p)^{1/2}$

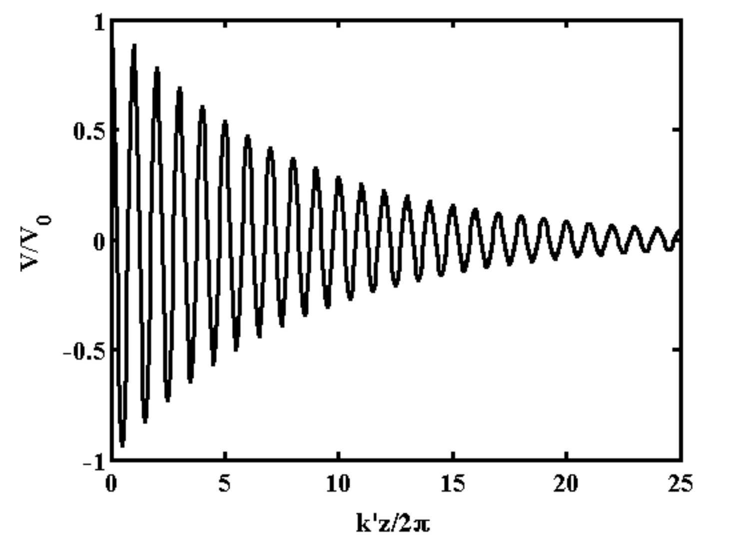
Lossy lines (3)

- Effect of complex k found as follows:

- Generally $V = V_0 \exp(-jkz)$

- If $k = k' - jk''$ then

  $V = V_0 \exp(-jk'z) \exp(-k''z)$



- Wave attenuates as it propagates

  For good performance, need $k'' \ll k'$

- k''/k' known as 'tan δ' or loss tangent

  More expensive cables have lower tan δ

Backward-going Waves

- By convention, forward-going waves are written:

  $V(z) = V_0 \exp(-jkz)$ and $I(z) = I_0 \exp(-jkz)$ i.e. negative sign in exponential

- For backward-going voltage wave, assume:

  $V(z) = V_0 \exp(+jkz)$ and $I(z) = I_0 \exp(+jkz)$ i.e. positive sign

- If $dV/dz = -j\omega L_p I$, then

  $I = (-1/j\omega L_p)\ dV/dz$ or

  $I = (-1/j\omega L_p) \times (+jk)\ V_0 \exp(+jkz)$, or

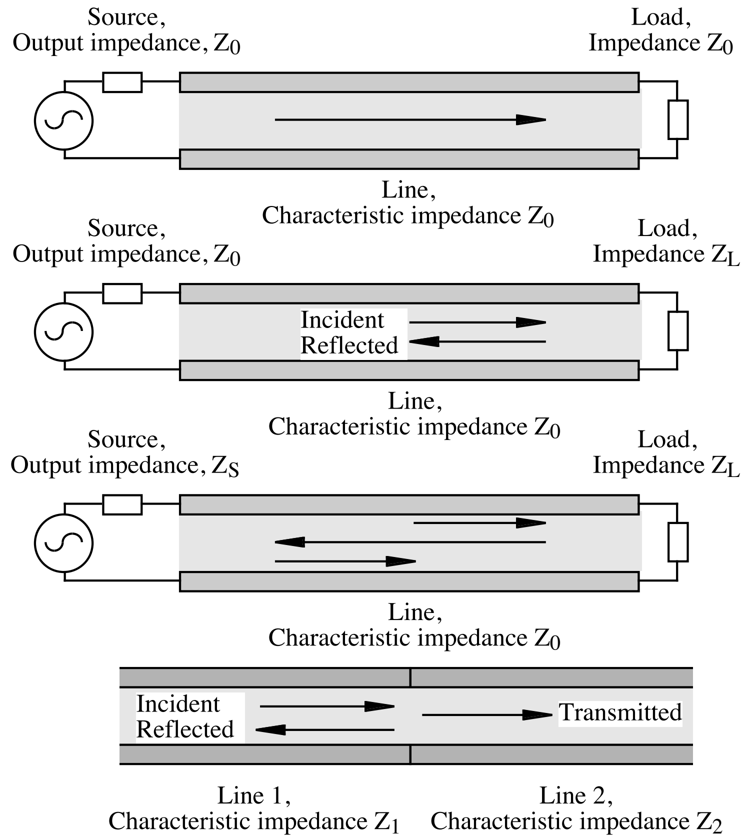
  $I = -(k/\omega L_p) \exp(+jkz)$

- This means that

  $I(z) = I_0 \exp(+jkz)$ where $I_0 = -V_0/Z_0$

  NB: Note minus sign in definition of I_0!

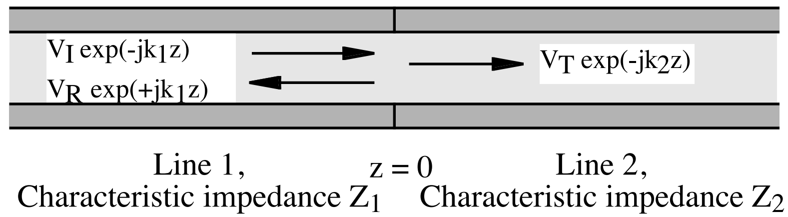
Reflected Waves



- Any transmission line discontinuity will cause reflections

  Ideally, impedances should all be matched

Reflection and Transmission (1)



- Consider discontinuity between two lines with (Z_1, k_1) and (Z_2, k_2)
- Assume incident and reflected waves on LHS:

  $V_1 = V_I \exp(-jk_1z) + V_R \exp(+jk_1z)$

  $I_1 = (V_I/Z_1) \exp(-jk_1z) - (V_R/Z_1) \exp(+jk_1z)$

- Assume transmitted wave on RHS:

  $V_2 = V_T \exp(-jk_2z)$

  $I_2 = (V_T/Z_2) \exp(-jk_2z)$

- If junction is at z = 0, boundary conditions are easy to apply

Reflection and Transmission (2)

- Match voltages and currents on boundary (z = 0) to get

  $V_I + V_R = V_T$

  $(V_I - V_R)/Z_1 = V_T/Z_2$

- Substitute upper equation in lower:

  $(V_I - V_R)/Z_1 = (V_I + V_R)/Z_2$

- Re-arrange:

  $(V_I - V_R)Z_2 = (V_I + V_R)Z_1$

  $V_I(Z_2 - Z_1) = V_R(Z_2 + Z_1)$

- Express reflected and transmitted voltages as ratios:

  $V_R/V_I = (Z_2 - Z_1)/(Z_2 + Z_1)$ Reflection coefficient, R_V

  $V_T/V_I = 2Z_2/(Z_2 + Z_1) = 1 + R_V$ Transmission coefficient, T_V

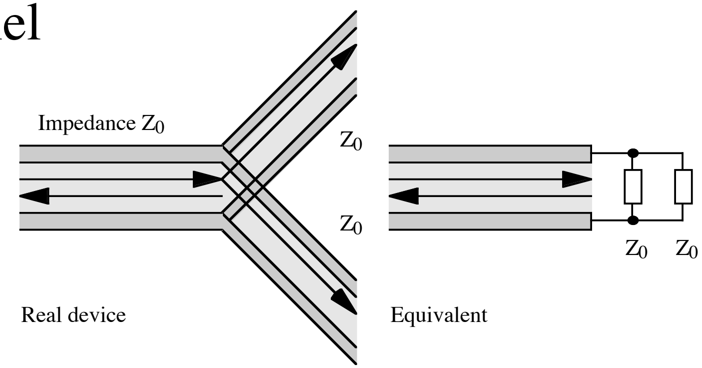
Matched Splitter

- Simple splitter has two outputs in parallel

  So $Z_1 = Z_0$ and $Z_2 = Z_0/2$

  So $R_V = (Z_0/2 - Z_0)/(Z_0/2 + Z_0) = -1/3$

  Hence reflection always occurs



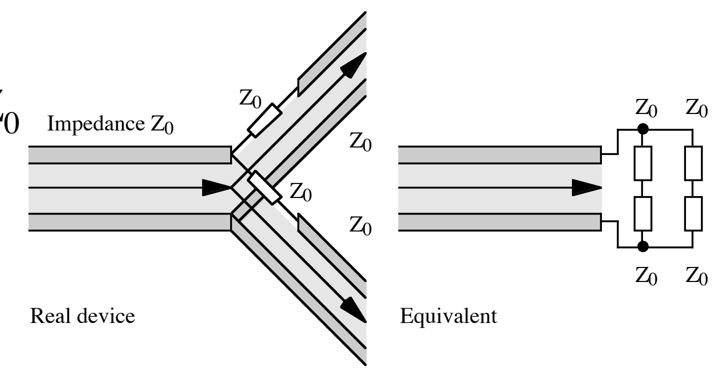
- How to form a reflection-less splitter?

  Insert series resistors equal to Z_0

  Output lines now each have impedance $2Z_0$

  Since these are in parallel, we get $Z_2 = Z_0$

  Splitter is matched at all frequencies



- Device is not perfect

  Because of resistors, device consumes power

Input Impedance (1)

- Need to find impedance of line with terminating load Z_L

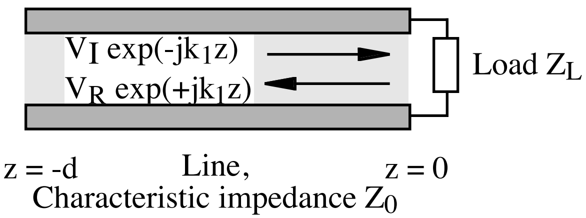
  Line has length d impedance Z_0 and propagation constant k

  There must be a reflection at mismatched load

- At z = -d, voltage and current are:

  $V = V_I \exp(+jkd) + V_R \exp(-jkd)$

  $I = (V_I/Z_0) \exp(+jkd) - (V_R/Z_0) \exp(-jkd)$



- Now, the input impedance is $Z_{in} = V(-d)/I(-d)$

- We know $V_R/V_I = R_V = (Z_L - Z_0)/(Z_L + Z_0)$, so

  $V(-d) = V_I \{\exp(+jkd) + \exp(-jkd) (Z_L - Z_0)/(Z_L + Z_0)\}$

  $I(-d) = (V_I/Z_0) \{\exp(+jkd) - \exp(-jkd) (Z_L - Z_0)/(Z_L + Z_0)\}$

Input Impedance (2)

- If $V(-d) = V_I \{\exp(+jkd) + \exp(-jkd)\ (Z_L - Z_0)/(Z_L + Z_0)\}$ then

$$V(-d)\ (Z_L + Z_0) = V_I \{(Z_L + Z_0) \exp(+jkd) + (Z_L - Z_0) \exp(-jkd)\}$$
$$= V_I \{Z_L[\exp(+jkd) + \exp(-jkd)] + Z_0[\exp(+jkd) - \exp(-jkd)]\}$$
$$= 2V_I \{Z_L \cos(kd) + jZ_0 \sin(kd)\}$$

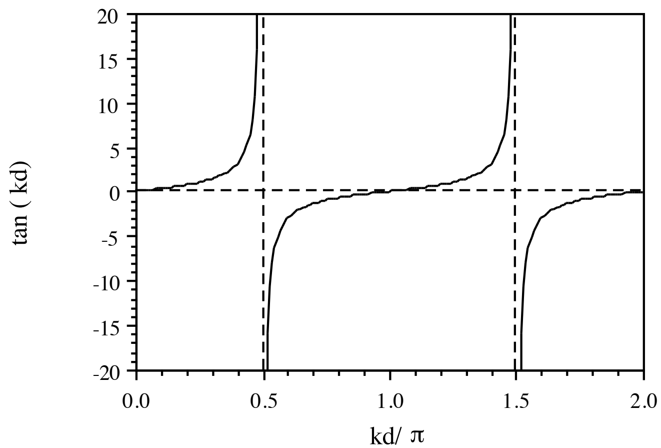
- If $I(-d) = (V_I/Z_0) \{\exp(+jkd) - \exp(-jkd)\ (Z_L - Z_0)/(Z_L + Z_0)\}$ then

$$I(-d)\ (Z_L + Z_0) = (V_I/Z_0) \{(Z_L + Z_0) \exp(+jkd) - (Z_L - Z_0) \exp(-jkd)$$
$$= (V_I/Z_0) \{Z_L[\exp(+jkd) - \exp(-jkd)] + Z_0[\exp(+jkd) + \exp(-jkd)]\}$$
$$= 2(V_I/Z_0) \{jZ_L \sin(kd) + Z_0 \cos(kd)\}$$

- Input impedance $Z_{in} = V(-d)/I(-d)$ then found by division as:

$$Z_{in} = Z_0 \{Z_L \cos(kd) + jZ_0 \sin(kd)\} / \{Z_0 \cos(kd) + jZ_L \sin(kd)\}$$

Special Cases



- General input impedance is:

  $Z_{in} = Z_0 \{Z_L \cos(kd) + jZ_0 \sin(kd)\} / \{Z_0 \cos(kd) + jZ_L \sin(kd)\}$, or

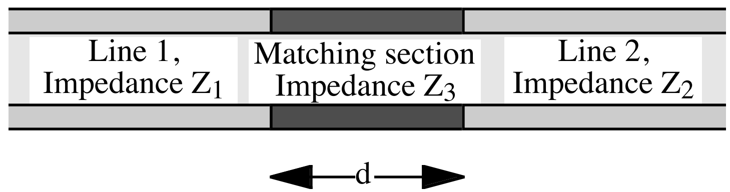
  $Z_{in} = Z_0 \{Z_L + jZ_0 \tan(kd)\} / \{Z_0 + jZ_L \tan(kd)\}$

- If line is short circuited ($Z_L = 0$)

  $Z_{in} = jZ_0 \tan(kd)$ so Z_{in} can look like capacitor or inductor, depending on d

  Method of making inductors when wound components no longer work

Quarter-Wave Transformer



- Need to match together two lines of impedance Z_1 and Z_2

  Insert short section of line of length $d = \lambda/4$ so $k_3 d = \pi/2$

- Input impedance of Line 3 and Line 2 together is:

  $Z_{in} = Z_3 \{Z_2 + jZ_3 \tan(k_3 d)\} / \{Z_3 + jZ_2 \tan(k_3 d)\}$

  $Z_{in} = Z_3 \{Z_2 + jZ_3 \tan(\pi/2)\} / \{Z_3 + jZ_2 \tan(\pi/2)\}$

- Since tan function goes to infinity, $Z_{in} = Z_3^2/Z_2$

- Z_{in} presents matched load to Line 1 if $Z_1 = Z_3^2/Z_2$

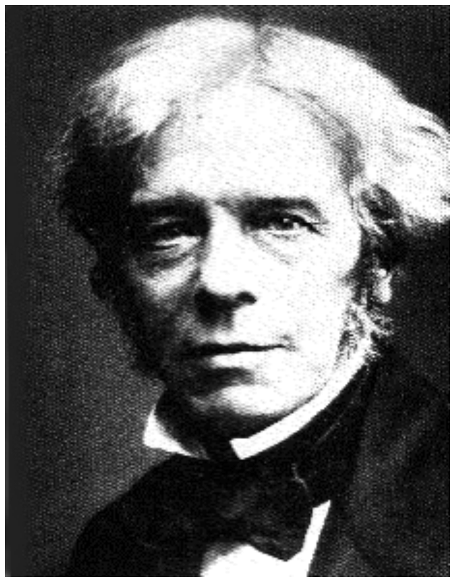
  Hence Z_3 should be geometric mean of Z_1 and Z_2

  Simple device, but only works well at one frequency

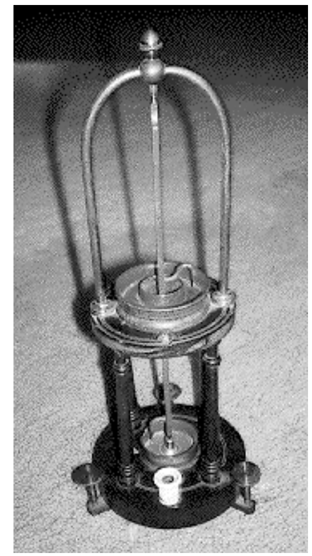
5: Electromagnetic Waves

- Faraday’s Law
- The wave equation
- Electromagnetic waves

Faraday



Michael Faraday (1791 - 1867)
Electro-magnetic induction,
the magneto-optical effect,
diamagnetism, field theory



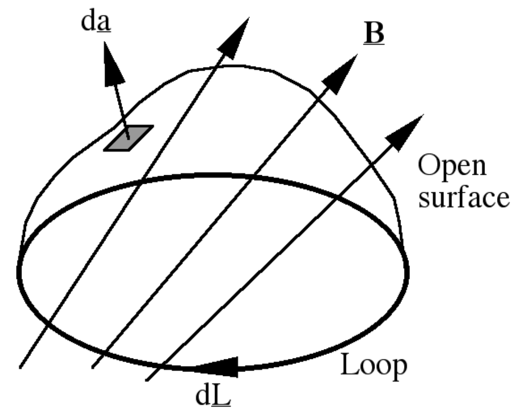
An early Faraday motor

Faraday's Law

- Consider time-varying magnetic flux $\underline{\mathbf{B}}$ passing through closed loop L, defining rim of open surface

  Flux of magnetic induction ψ_B through surface is:

  $\psi_B = \iint_A \underline{\mathbf{B}} \cdot d\underline{a}$



- EMF E induced round loop is:

  $\mathsf{E} = -\partial\psi_B/\partial t$

  Induced voltage related to field by $\mathsf{E} = \oint_L \underline{\mathbf{E}} \cdot d\underline{L}$

- Combining the above we obtain Faraday's law:

  $\oint_L \underline{\mathbf{E}} \cdot d\underline{L} = - \iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

- Time-varying magnetic field has associated electric field

Problems with Maxwell

- So far, Maxwell's equations are integral equations
  We don't know how to solve integral equations
  We do know how to solve differential equations

- We can transform integral equations into differential ones
  Need extra maths: Gauss' Theorem and Stokes Theorem

- Can show how this works for Gauss' and Faraday's Law

Gauss Theorem, Gauss' Law

- Remember Gauss' Law:

  $\iint_A \underline{\mathbf{D}} \,.\, d\underline{a} \;=\; \iiint_V \rho \, dv$

- Gauss' Theorem states that:

  $\iint_A \underline{\mathbf{F}} \,.\, d\underline{a} \;=\; \iiint_V \mathrm{div}(\underline{\mathbf{F}}) \, dv$

  NB Don't muddle up Gauss' Law and Gauss' Theorem!

- Apply to Gauss' Law:

  $\iint_A \underline{\mathbf{D}} \,.\, d\underline{a} \;=\; \iiint_V \mathrm{div}(\underline{\mathbf{D}}) \, dv \;=\; \iiint_V \rho \, dv$

- Since integration region V is undefined, we must have:

  $\mathrm{div}(\underline{\mathbf{D}}) = \rho$ (This is the the desired differential form)

Stokes' Theorem, Faraday's Law

- Remember Faraday's Law:

  $\int_L \underline{\mathbf{E}} \cdot d\underline{L} = - \iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

- Stokes' Theorem states that:

  $\int_L \underline{\mathbf{F}} \cdot d\underline{L} = \iint_A \mathrm{curl}(\underline{\mathbf{F}}) \cdot d\underline{a}$

- Apply to Faraday's Law:

  $\int_L \underline{\mathbf{E}} \cdot d\underline{L} = \iint_A \mathrm{curl}(\underline{\mathbf{E}}) \cdot d\underline{a} = - \iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

- Since integration region A is undefined, we must have:

  $\mathrm{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$ (This is the desired differential form)

All the Equations Together

- Using Gauss' Theorem and Stokes Theorem, we can transform the magnetic equivalent of Gauss' Law and Ampere's Law into differential forms.

- The equations together then look like this:

  $\mathrm{div}(\underline{\mathbf{D}}) = \rho$

  $\mathrm{div}(\underline{\mathbf{B}}) = 0$

  $\mathrm{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$

  $\mathrm{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$

- And again we need the constitutive equations

  $\underline{\mathbf{J}} = \sigma\,\underline{\mathbf{E}}$; $\underline{\mathbf{D}} = \varepsilon\,\underline{\mathbf{E}}$; $\underline{\mathbf{B}} = \mu\,\underline{\mathbf{H}}$

The Wave Equation (1)

- We now create a wave equation for electromagnetic waves

  Assume non-magnetic materials (so $\mu = \mu_0$ and $\underline{\mathbf{B}} = \mu_0\, \underline{\mathbf{H}}$)

  Assume dielectric materials (so $\underline{\mathbf{D}} = \varepsilon\, \underline{\mathbf{E}}$)

  Assume no charges (so $\rho = 0$)

  Assume no conductivity (so $\sigma = 0$ and $\underline{\mathbf{J}} = 0$)

- Equations to be solved are:

  $\mathrm{div}(\underline{\mathbf{D}}) = 0$ (1) NB: If ε is constant, $\mathrm{div}(\underline{\mathbf{E}}) = 0$

  $\mathrm{div}(\underline{\mathbf{B}}) = 0$ (2)

  $\mathrm{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$ (3)

  $\mathrm{curl}(\underline{\mathbf{H}}) = \partial\underline{\mathbf{D}}/\partial t$ (4)

- Nice and symmetric!

The Wave Equation (2)

- Take curl of Equation 3:

  curl [curl($\underline{\mathbf{E}}$)] = - curl [$\partial\underline{\mathbf{B}}/\partial t$]

- Substitute for $\underline{\mathbf{B}}$:

  curl [curl($\underline{\mathbf{E}}$)] = - μ_0 curl [$\partial\underline{\mathbf{H}}/\partial t$]

- Re-arrange:

  curl [curl($\underline{\mathbf{E}}$)] = - μ_0 ∂[curl($\underline{\mathbf{H}}$)/∂t]

- Substitute for $\underline{\mathbf{D}}$ in Equation 4:

  curl($\underline{\mathbf{H}}$) = $\varepsilon\ \partial\underline{\mathbf{E}}/\partial t$

- Use the result to eliminate $\underline{\mathbf{H}}$:

  curl [curl($\underline{\mathbf{E}}$)] = - $\mu_0\varepsilon\ \partial^2\underline{\mathbf{E}}/\partial t^2$

Actually this is already a wave equation
But we don't know what curl curl means ...

The Wave Equation (3)

- Start with

  $\text{curl} [\text{curl}(\underline{\mathbf{E}})] = - \mu_0\varepsilon \, \partial^2\underline{\mathbf{E}}/\partial t^2$

- Use standard vector identity

  $\text{curl} [\text{curl}(\underline{\mathbf{F}})] = \text{grad} [\text{div}(\underline{\mathbf{F}})] - \nabla^2\underline{\mathbf{F}}$

- Hence

  $\text{grad} [\text{div}(\underline{\mathbf{E}})] - \nabla^2\underline{\mathbf{E}} = - \mu_0 \, \partial^2(\varepsilon\underline{\mathbf{E}})/\partial t^2$

- With no charges $\text{div}(\underline{\mathbf{D}}) = 0$ so

  $\text{div}(\underline{\mathbf{E}}) = 0$ if ε is constant

- Hence

  $\nabla^2\underline{\mathbf{E}} = \mu_0\varepsilon \, \partial^2\underline{\mathbf{E}}/\partial t^2$

  This is the vector wave equation for light!

Time-Independent Wave Eqn

- Vector wave equation is $\nabla^2\underline{\mathbf{E}} = \mu_0\varepsilon\ \partial^2\underline{\mathbf{E}}/\partial t^2$

  Assume fields vary harmonically as $\underline{\mathbf{E}}(x, y, z, t) = \underline{E}(x, y, z) \exp(j\omega t)$

- So

  $\partial\underline{\mathbf{E}}/\partial t = j\omega\ \underline{\mathbf{E}}$

  $\partial^2\underline{\mathbf{E}}/\partial t^2 = -\omega^2\ \underline{\mathbf{E}}$

- The vector wave equation reduces to

  $\nabla^2\underline{E} = -\omega^2\mu_0\varepsilon_0\varepsilon_r\ \underline{E}$

- This is a time-independent vector wave equation

Scalar Wave Equation

- The time-independent vector wave equation is:

  $\nabla^2\underline{E} = -\omega^2\mu_0\varepsilon_0\varepsilon_r\,\underline{E}$

- Can write separate equation for each co-ordinate

- For $E_{\mathbf{x}}$, we get:

  $\nabla^2 E_x = -\omega^2\mu_0\varepsilon\, E_x$

  This is the scalar wave equation

- In long-hand:

  $\partial^2 E_x/\partial x^2 + \partial^2 E_x/\partial y^2 + \partial^2 E_x/\partial z^2 = -\omega^2\mu_0\varepsilon\, E_x$

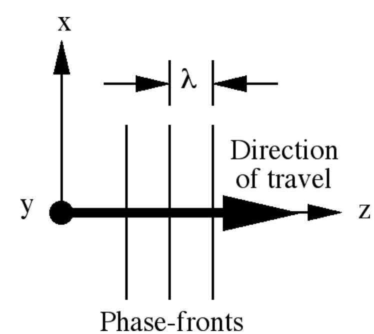
Plane Wave Solutions

- E-field indicates direction of polarization
- Wave with a single component of $\underline{E}$ is linearly polarized

  If $\underline{E} = E_x\,\underline{i}$, wave is x-polarized

- Plane waves have uniform amplitude and planes of constant phase perpendicular to travel

  For a z-going wave, $\partial\underline{E}/\partial x = \partial\underline{E}/\partial y = \underline{0}$



- Time-independent scalar wave equation is:

  $d^2E_x/dz^2 = -\omega^2\mu_0\varepsilon\, E_x$

- Guess solution $E_x = E_0 \exp(-jkz)$

  $dE_x/dz = -jkE_x$; $d^2E_x/dz^2 = -k^2E_x$

  Solution viable if $k^2 = \omega^2\mu_0\varepsilon$

Velocity of Light

- Time independent solution:

  $\underline{E}(z) = E_0 \exp(-jkz)\ \underline{i}$

- Whole solution is travelling wave:

  $\underline{\mathbf{E}}(z, t) = \underline{E} \exp(j\omega t) = E_0 \exp\{j(\omega t - kz)\}\ \underline{i}$

  $\omega = 2\pi/T = 2\pi f$ where f is temporal frequency

  $k = 2\pi/\lambda$ where λ is wavelength

- Phase velocity is:

  $v_{ph} = f\lambda = \omega/k = 1/\sqrt{(\mu_0\varepsilon)}$

- In vacuum:

  $\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}\ s^2C^2/m^3kg$; $\mu_0 = 4\pi \times 10^{-7}\ m\ kg/C^2$

  Hence $v = 2.99 \times 10^8$ m/s!

Refractive Index

- In the presence of matter, $\varepsilon_r \neq 1$

  Velocity is then $v_{ph} = 1/\sqrt{(\mu_0\varepsilon_0\varepsilon_r)}$

  This can be written $v_{ph} = c/\sqrt{\varepsilon_r}$

  Here, $n = \sqrt{\varepsilon_r}$ is the refractive index

- Waves generally travel slower in matter

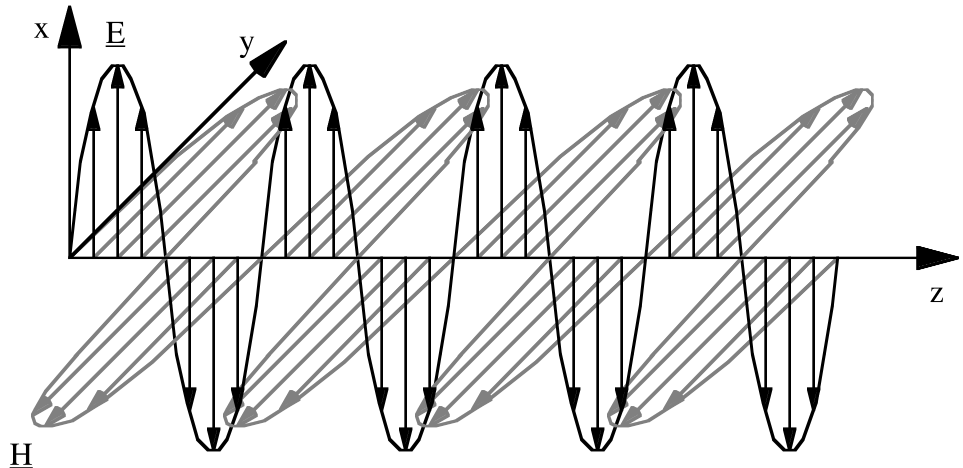
- Other quantities scale with n:

  Wavelength: $\lambda = \lambda_0/n$

  Propagation constant: $k = nk_0$

  Here the subscript zero means "vacuum value"

The Nature of EM Waves



- We know that $\underline{E} = E_0 \exp(-jkz)\ \underline{i}$: What is $\underline{H}$?

- For time-independent fields, curl($\underline{E}$) = $-j\omega\mu_0\ \underline{H}$

  Apply to solution; curl($\underline{E}$) = $-jk\ E_0 \exp(-jkz)\ \underline{j}$

  Hence $\underline{H}$ must have the form $\underline{H} = H_0 \exp(-jkz)\ \underline{j}$

  With a field amplitude of $H_0 = (k/\omega\mu_0)\ E_0$

  Total field is transverse electric and magnetic (TEM) wave

6: Reflection and Refraction

- Impedance and power
- Boundary matching
- Reflection and refraction
- Total internal reflection

Wave Impedance

- For z-going wave, assume:

  $\underline{E} = E_0 \exp(-jkz)\ \underline{i}$ and $\underline{H} = H_0 \exp(-jkz)\ \underline{j}$

  Maxwell curl equation implies that $H_0 = (k/\omega\mu_0) E_0$

  Since $k = \omega(\mu_0\varepsilon)^{1/2}$ we must have $E_0/H_0 = (\mu_0/\varepsilon)^{1/2}$

  Hence, the two amplitudes are linked

- The ratio E_0/H_0 has dimensions of Ohms

  It is called the wave impedance, Z_0

- For free space:

  $Z_0 = \{4\pi \times 10^{-7} / (1/36\pi) \times 10^{-9}\}^{1/2} = 120\pi = 377\ \Omega$

  To launch a wave into space, we must match to this value

  For medium of refractive index n, $Z = Z_0/n$

Power Flow (1)

- Poynting vector is $\underline{\mathbf{S}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}}$ - gives instantaneous power

  $\underline{\mathbf{S}}$ is difficult to measure, since it contains components at 2ω

  More useful is the irradiance $\underline{S} = (1/T) \int_T \underline{\mathbf{S}}\, dt$

- Because only real parts are important, we can write:

  $\underline{S} = (1/T) \int_T \mathrm{Re}\{\ \underline{E} \exp(j\omega t)\ \} \times \mathrm{Re}\{\ \underline{H} \exp(j\omega t)\ \}\, dt$

  Now $\mathrm{Re}(z) = (z + z^*)/2$, so

  $\underline{S} = (1/4T) \int_T \{\underline{E} \exp(j\omega t) + \underline{E}^* \exp(-j\omega t)\} \times \{\underline{H} \exp(j\omega t) + \underline{H}^* \exp(-j\omega t)\}\, dt$

- Product of first and third terms varies as $\exp(j2\omega t)$
- Product of second and forth terms varies as $\exp(-j2\omega t)$

- Both average to zero. Hence:

  $\underline{S} = (1/4T) \int_T \{\underline{E} \times \underline{H}^* + \underline{E}^* \times \underline{H}\}\, dt$

  $\underline{S} = \{\ \underline{E} \times \underline{H}^* + \underline{E}^* \times \underline{H}\ \}/4$ or $\underline{S} = 1/2\ \mathrm{Re}\ \{\underline{E} \times \underline{H}^*\}$

Power Flow (2)

- Time-averaged power flow is:

  $\underline{S} = 1/2 \; \mathrm{Re}\,\{\underline{E} \times \underline{H}^*\}$

- Example: z-going plane wave

  Assume:

  $\underline{E} = E_0 \exp(-jkz)\, \underline{i}$

  $\underline{H} = (k/\omega\mu_0)\, E_0 \exp(-jkz)\, \underline{j}$

- In this case:

  $\underline{S} = (k/2\omega\mu_0)\, E_0^2\, \underline{k}$

  $\underline{S} = (1/2Z_0)\, E_0^2\, \underline{k}$

- Power flows in direction of travel
- Power density is proportional to electric field squared

Waves in Lossy Dielectrics

- Wave equation for x-polarization is $d^2E_x/dz^2 = -\omega^2\mu_0\varepsilon\ E_x$

  In a lossy dielectric, ε_r is complex-valued, so $\varepsilon_r = \varepsilon_r' - j\varepsilon_r''$

- Wave equation now $d^2E_x/dz^2 + \omega^2\mu_0\varepsilon_0\ (\varepsilon_r' - j\varepsilon_r'')\ E_x = 0$

  Assume travelling wave solution $E_x = E_0 \exp(-jkz)$

  Hence $k = \omega\ \sqrt{}(\mu_0\varepsilon_0\varepsilon_r')\ \sqrt{}(1 - j\varepsilon_r''/\varepsilon_r')$

- If loss is small, so $\varepsilon_r'' \ll \varepsilon_r'$, we can use binomial approximation:

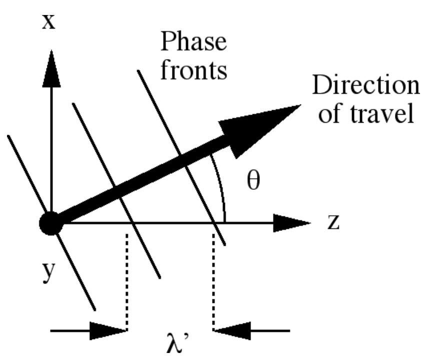
  $k = \omega\ \sqrt{}(\mu_0\varepsilon_0\varepsilon_r')\ (1 - j\varepsilon_r''/2\varepsilon_r')$, or:

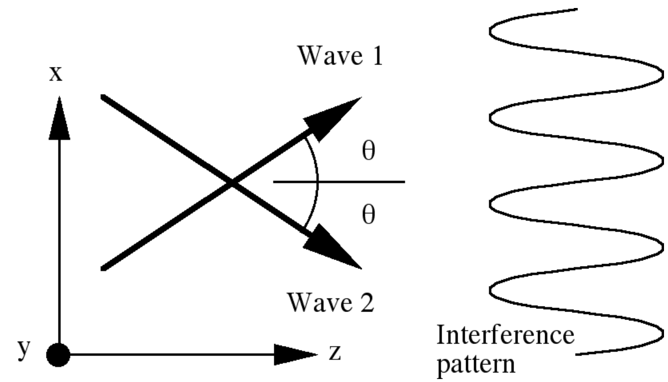
  $k = k' - jk''$, where $k' = \omega\ \sqrt{}(\mu_0\varepsilon_0\varepsilon_r')$, and $k'' = k'\varepsilon_r''/2\varepsilon_r'$

  Wave therefore propagates as $E_x = E_0 \exp(-k''z) \exp(-jk'z)$

- This is a plane wave with exponential decay

Oblique Waves





- For y-polarized, oblique wave in the x-z plane:

  $\underline{E} = \underline{j}\, E_y \exp\{-j(kx \sin(\theta) + kz \cos(\theta))\}$

- If two oblique waves cross, sum their fields as:

  $\underline{E} = \underline{j}\, E_y \{\exp[-jk_0(z \cos(\theta) + x \sin(\theta))] + \exp[-jk_0(z \cos(\theta) - x \sin(\theta))]\}$

- We can write this alternatively as

  $\underline{E} = \underline{j}\, 2E_y \exp[-jk_0 z \cos(\theta)] \cos[k_0 x \sin(\theta)]$

  This is a sinusoidal interference pattern

Boundary Matching (1)

- Boundary conditions must be satisfied at junctions between media

- Apply Maxwell's equations at junction between media.

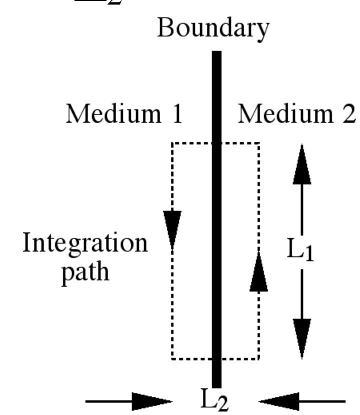
  Medium 1 has time-dependent field $\underline{\mathbf{E}}_1$, and Medium 2 has field $\underline{\mathbf{E}}_2$

- Faraday's law says $\int_L \underline{\mathbf{E}} \cdot d\underline{L} = - \iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$

  Perform line integration round rectangular path

  If L_2 tends to zero, RHS must tend to zero.

  $\int_L \underline{\mathbf{E}} \cdot d\underline{L} = (\mathbf{E}_{t1} - \mathbf{E}_{t2}) L_1 = 0$ so $\mathbf{E}_{t1} - \mathbf{E}_{t2} = 0$



- Tangential components of $\underline{\mathbf{E}}$ must match across boundary

Boundary Matching (2)

- Similar calculation can be done for all field quantities

- Full set of boundary conditions is then:

  Tangential components of $\underline{\mathbf{E}}$ and $\underline{\mathbf{H}}$ must match on a boundary

  Normal components of $\underline{\mathbf{D}}$ and $\underline{\mathbf{B}}$ must match on a boundary*

  *Apart from surface effects

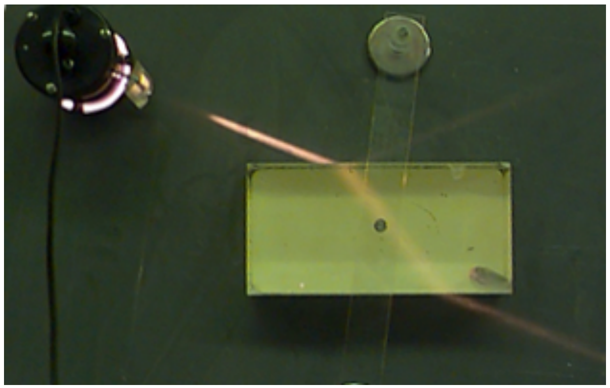
- Or in vector form:

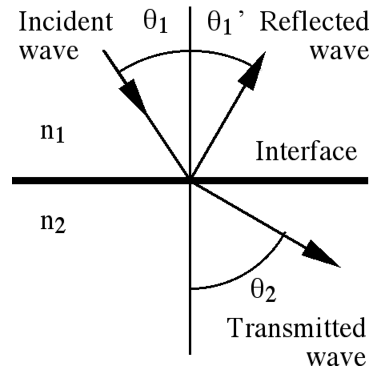
  $\underline{n} \times (\underline{\mathbf{E}}_2 - \underline{\mathbf{E}}_1) = 0$ $\qquad$ $n \times (\underline{\mathbf{H}}_2 - \underline{\mathbf{H}}_1) = \underline{\mathbf{K}}$

  $\underline{n} \cdot (\underline{\mathbf{D}}_2 - \underline{\mathbf{D}}_1) = \rho_s$ $\qquad$ $\underline{n} \cdot (\underline{\mathbf{B}}_2 - \underline{\mathbf{B}}_1) = 0$

  Here $\underline{\mathbf{K}}$ = surface current, ρ_s = surface charge (often zero)

Reflection and Refraction





Snell, 1591-1626

- A plane light wave strikes a dielectric interface.

  For near-normal incidence (small θ_1) there will be a plane reflected wave in Medium 1, and a transmitted beam in Medium 2.

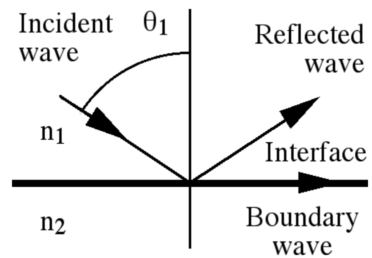
- Reflected wave angle is equal and opposite to angle of incidence.

  This is Alhazen's Law, $\theta_1' = \theta_1$

- Transmitted wave angle is defined by Snell's law:

  $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

Total Internal Reflection



- Snell's law says $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

  For $\theta_1 = \theta_C = \sin^{-1}(n_2/n_1)$, we get $\theta_2 = \pi/2$

  Transmitted wave now parallel to the interface

  For $\theta_1 > \theta_c$, no real solution for θ_2; total internal reflection occurs

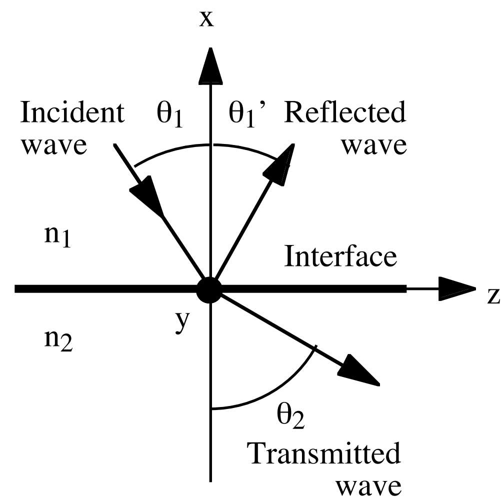
- This angle of incidence is the critical angle

  Example: For glass/air interface ($n_1 = 1.5$ and $n_2 = 1.0$), $\theta_c = 41.8^o$.

- For total internal reflection, we must have $n_2 < n_1$

  Incidence must be from the high index side

EM Solution: Matching $\underline{E}$



- Assume $\underline{E}$ is y-polarised:

$$E_{y1} = E_I \exp[-jk_0 n_1(z \sin(\theta_1) - x \cos(\theta_1))] + E_R \exp[-jk_0 n_1(z \sin(\theta_1') + x \cos(\theta_1'))]$$

$$E_{y2} = E_T \exp[-jk_0 n_2(z \sin(\theta_2) - x \cos(\theta_2))]$$

- Tangential components of $\underline{E}$ must match

Since $\underline{E}$ is tangential, $E_{y1} = E_{y2}$ on $x = 0$:

$$E_I \exp[-jk_0 n_1 z \sin(\theta_1)] + E_R \exp[-jk_0 n_1 z \sin(\theta_1')] = E_T \exp[-jk_0 n_2 z \sin(\theta_2)]$$

- Equation only satisfied for all z if:

$n_1 \sin(\theta_1) = n_1 \sin(\theta_1') = n_2 \sin(\theta_2)$

Thus, $\theta_1 = \theta_1$' (Alhazen's law)

And $n_1 \sin(\theta_1) = n_1 \sin(\theta_1') = n_2 \sin(\theta_2)$ (Snell's law)

And $E_I + E_R = E_T$

EM Solution: Matching $\underline{H}$

- Tangential components of $\underline{\mathbf{H}}$ must also match

- Find $\underline{\mathbf{H}}$ from curl relation

  Since $E_x = E_z = 0$, $H_x = (-j/\omega\mu_0)\ \partial E_y/\partial z$; $H_y = 0$; $H_z = (j/\omega\mu_0)\ \partial E_y/\partial x$

  The only tangential component is H_z, so $\partial E_{y1}/\partial x = \partial E_{y2}/\partial x$ on $x = 0$

- Differentiating and putting x = 0 we get:

  $jk_0n_1 \cos(\theta_1)\ E_I \exp[-jk_0n_1z \sin(\theta_1)]$

  $- jk_0n_1 \cos(\theta_1)\ E_R \exp[-jk_0n_1z \sin(\theta_1)$

  $= jk_0n_2 \cos(\theta_2)\ E_T \exp[-jk_0n_2z \sin(\theta_2)]$

- Cancel the exponential terms as before to get:

  $n_1 \cos(\theta_1)\ E_I - n_1 \cos(\theta_1)\ E_R = n_2 \cos(\theta_2)\ E_T$

Fresnel Coefficients

- We now have 2 simultaneous equations:

  $E_I + E_R = E_T$

  $n_1 \cos(\theta_1) E_I - n_1 \cos(\theta_1) E_R = n_2 \cos(\theta_2) E_T$

- Substitute upper equation into lower:

  $n_1 \cos(\theta_1) E_I - n_1 \cos(\theta_1) E_R = n_2 \cos(\theta_2) (E_I + E_R)$

  $[n_1 \cos(\theta_1) - n_2 \cos(\theta_2)] E_I = [n_2 \cos(\theta_2) + n_1 \cos(\theta_1)] E_R$

- Write results as $\Gamma_E = E_R/E_I$ and $T_E = E_T/E_I$:

  $\Gamma_E = [n_1 \cos(\theta_1) - n_2 \cos(\theta_2)] / [n_1 \cos(\theta_1) + n_2 \cos(\theta_2)]$

  $T_E = E_T/E_I = 2n_1 \cos(\theta_1)/ [n_1 \cos(\theta_1) + n_2 \cos(\theta_2)]$

- These are called Fresnel coefficients

  At normal incidence, $\Gamma_E = (n_1 - n_2) / (n_1 + n_2)$

Total Internal Reflection

- Snell's law says $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

  For $\theta_1 = \theta_C = \sin^{-1}(n_2/n_1)$, we get $\theta_2 = \pi/2$

- What happens when $\theta_1 > \theta_C$?

  $\sin(\theta_2) = (n_1/n_2) \sin(\theta_1)$ (now greater than one)

  $\cos(\theta_2) = \{1 - (n_1^2/n_2^2) \sin^2(\theta_1)\}^{1/2}$

  $\cos(\theta_2) = \pm j\{(n_1^2/n_2^2) \sin^2(\theta_1) - 1\}^{1/2}$

  Hence we may put $\cos(\theta_2) = \pm j\alpha$

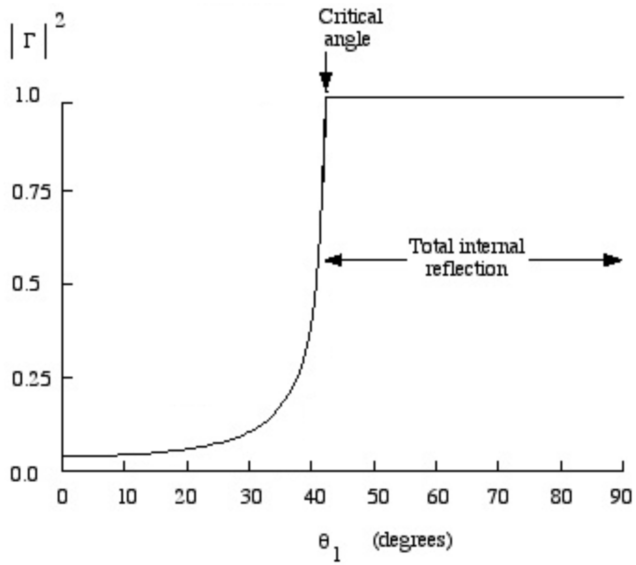
- Fresnel reflection coefficient is:

  $\Gamma_E = \{n_1 \cos(\theta_1) - n_2 \cos(\theta_2)\} / \{n_1 \cos(\theta_1) + n_2 \cos(\theta_2)\}$

  For $\theta_1 > \theta_C$, $\Gamma_E = \{n_1 \cos(\theta_1) - j\alpha\, n_2\} / \{n_1 \cos(\theta_1) + j\alpha\, n_2\}$

  Power reflectivity is $\Gamma_E \Gamma_E^* = 1$ so all power is now reflected

Angular Variation of Reflectivity



Augustin Jean Fresnel
1788 - 1827

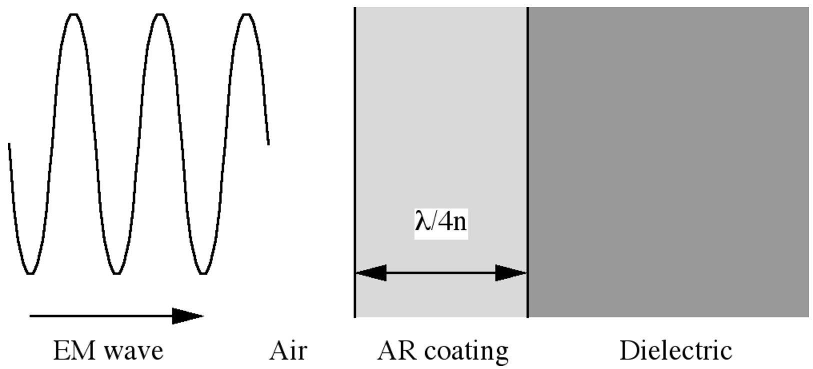
- Variation of reflection coefficient for glass/air interface

  Incidence is from glass side

  At normal incidence, $\Gamma_E = (1.5 - 1) / (1.5 + 1) = 0.2$; $|\Gamma_E|^2 = 0.04$

  Reflectivity initially low, then rises to 100% at critical angle

Anti-Reflection Coating



- Used to suppress reflection at air/dielectric interface
  - Operation identical to quarter wave transformer
  - Quarter-wave thickness of material used as intermediate coating
  - Chose $\lambda \sim 0.5\ \mu m$ - peak eye sensitivity
- Impedance of coating is geometric mean of air and dielectric values
  - Impedance of medium with index n is Z_0/n, hence $n_{coating} = (n_{air}n_{diel})^{1/2}$
  - For plastics ($n_{diel} \sim 1.6$), need $n_{coat} \sim 1.3$ (MgF suitable)
  - Only works properly at normal incidence - strong reflection otherwise

7: Imaging

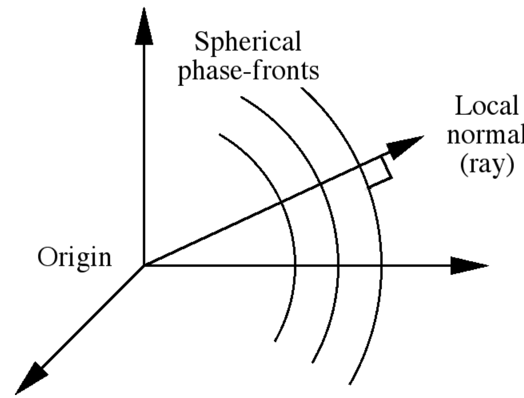
- Spherical waves
- Paraxial waves
- Lenses and mirrors
- Image formation
- Optical imaging systems

Spherical Waves (1)

- Scalar wave equation is $\nabla^2 E + \omega^2 \mu_0 \varepsilon_0 E = 0$

  Symmetric solutions have $E(x, y, z) = E(r)$ only

  Need to find $\nabla^2 E$ in radial co-ordinates



- Now the chain rule gives

  $\partial E/\partial x = dE/dr \; \partial r/\partial x$ and

  $\partial^2 E/\partial x^2 = d^2E/dr^2 \; (\partial r/\partial x)^2 + dE/dr \; \partial^2 r/\partial x^2$

- Since $r^2 = x^2 + y^2 + z^2$,

  $\partial r/\partial x = x/r$ and

  $\partial^2 r/\partial x^2 = (1/r) + x \; \partial(1/r)/\partial x = (1/r)(1 - x^2/r^2)$ and

  $\partial^2 E/\partial x^2 = (x^2/r^2) \; d^2E/dr^2 + (1/r)(1 - x^2/r^2) \; dE/dr$

Spherical Waves (2)

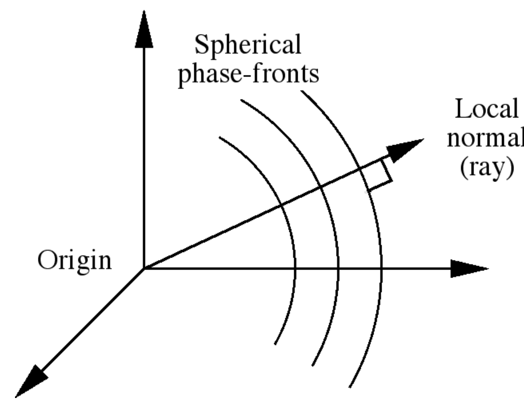
- If:

  $\partial^2E/\partial x^2 = (x^2/r^2)\ d^2E/dr^2 + (1/r)\ (1 - x^2/r^2)\ dE/dr$

- Then by inspection:

  $\partial^2E/\partial y^2 = (y^2/r^2)\ d^2E/dr^2 + (1/r)\ (1 - y^2/r^2)\ dE/dr$

  $\partial^2E/\partial z^2 = (z^2/r^2)\ d^2E/dr^2 + (1/r)\ (1 - z^2/r^2)\ dE/dr$



- Now:

  $\nabla^2E = \partial^2E/\partial x^2 + \partial^2E/\partial y^2 + \partial^2E/\partial z^2$

- Adding all three terms together:

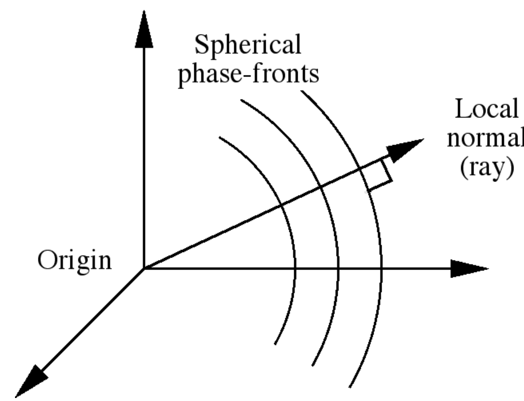
  $\nabla^2E = d^2E/dr^2 + (1/r)\ \{3 - (x^2 + y^2 + z^2)/r^2\}\ dE/dr$ or

  $\nabla^2E = d^2E/dr^2 + (2/r)\ dE/dr$

Spherical Waves (3)

- Scalar wave equation reduces to:

  $d^2E/dr^2 + (2/r)\, dE/dr + \omega^2\mu_0\varepsilon_0\, E = 0$



- Define variable $F(r) = r\, E(r)$

  $dF/dr = r\, dE/dr + E$ and

  $d^2F/dr^2 = r\, d^2E/dr^2 + 2\, dE/dr$

- Scalar wave equation now

  $d^2F/dr^2 + \omega^2\mu_0\varepsilon_0\, F = 0$

  So $F(r) = E_0 \exp(-jk_0r)$ by inspection

  And $E(r) = (E_0/r) \exp(-jk_0r)$ - spherical wave

- Need an approximation for small distance from optical axis

Paraxial Waves

- Spherical wave is $E(r) = E_0/r \exp(-jk_0 r)$

  At distance z, $r^2 = z^2 + R^2$ where $R^2 = x^2 + y^2$

  For small R, $r = z (1 + R^2/z^2)^{1/2} \approx z + R^2/2z$

- Approximate phase by:

  $\exp(-jk_0 r) \approx \exp(-jk_0 z) . \exp(-jk_0 R^2/2z)$

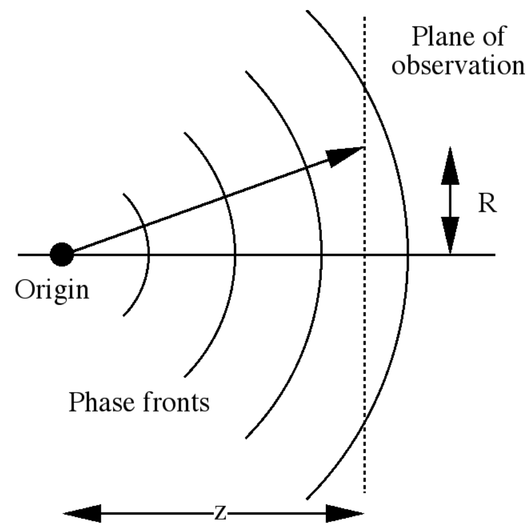
- Approximate amplitude by $E_0/r \approx E_0/z$

- Paraxial wave is:

  $E = E_0/z \exp(-jk_0 R^2/2z) . \exp(-jk_0 z)$

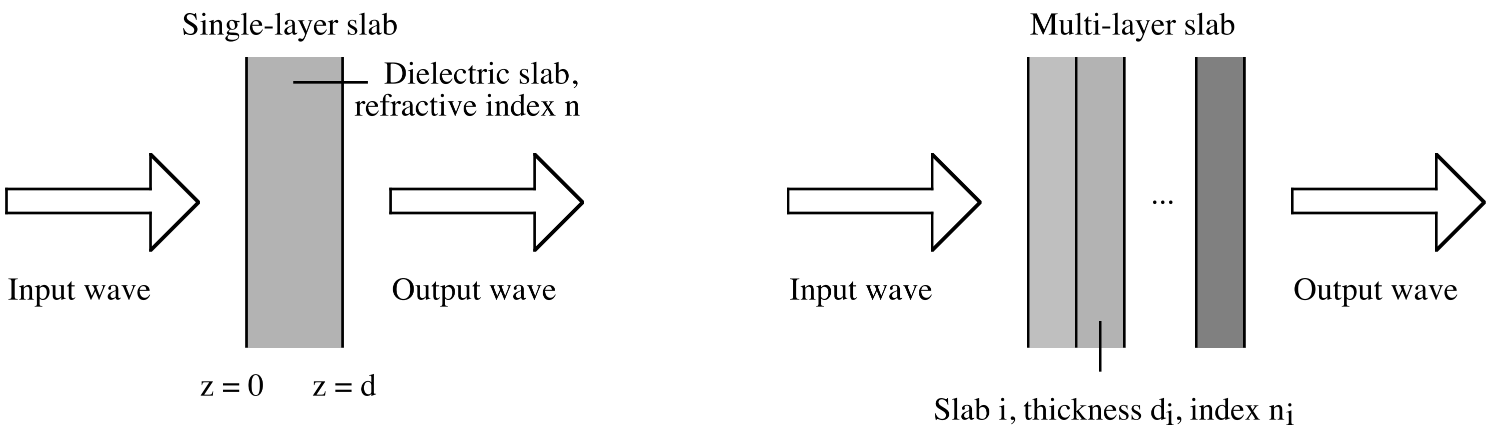
  $E \approx A(z) \exp(-jk_0 R^2/2z)$

  Wave diverges in z-direction, with a parabolic phase front



- Converging wave is $E'(R, z) \approx A(z)' \exp(+jk_0 R^2/2z)$

Optical Transfer Function



- For parallel-sided slab of thickness d:

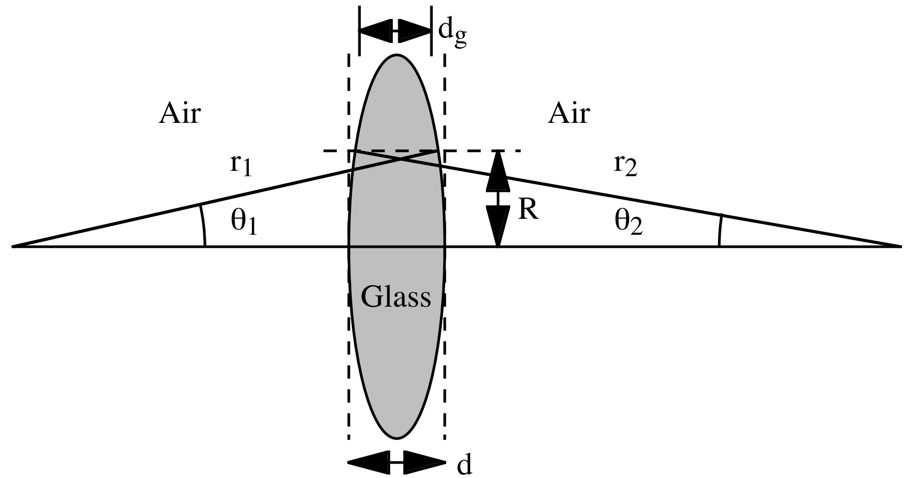
  $E_{out} = E_{in} \exp(-jk_0nd)$

- If we write $E_{out} = E_{in} \tau_S$, transfer function is:

  $\tau_S = \exp(-jk_0nd)$

- Optical thickness is $\delta = nd$
- For multilayer slab, $\delta = \Sigma\, n_i d_i$

Lenses (1)



- Lens has spherical surfaces with radii r_1 and r_2

  Need to find transfer function $\tau_S = \exp(-jk_0 n\delta)$ with $\delta = \Sigma\, n_i d_i$

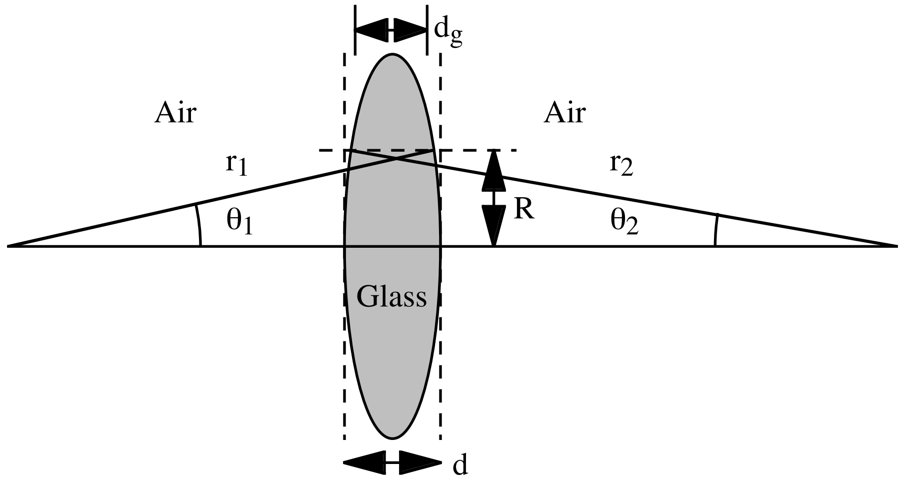
- Thickness of glass at radial distance R is:

  $d_g = d - \{r_1 [1 - \cos(\theta_1)] + r_2 [1 - \cos(\theta_2)]\}$

  Since $\cos(\theta) \approx 1 - \theta^2/2$ $\quad\quad$ $d_g \approx d - \{r_1\theta_1^2/2 + r_2\, \theta_2^2/2\}$

  Since $\theta_1 \approx R/r_1$ & $\theta_2 \approx R/r_2$ $\quad\quad$ $d_g \approx d - (R^2/2)\,(1/r_1 + 1/r_2)$

Lenses (2)



- Optical thickness found by adding paths in glass and air (with n = 1):

  $\delta = n \times d_g + 1 \times (d - d_g)$

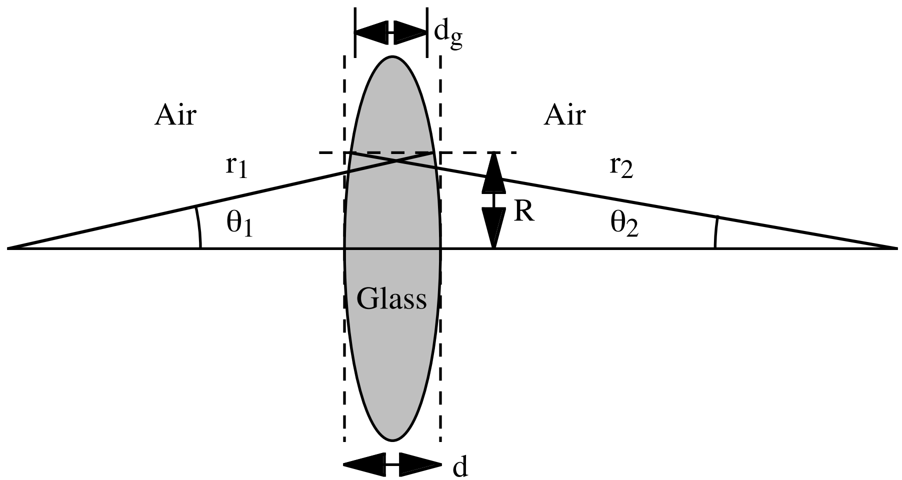
- Thickness of glass at radial distance R is:

  $d_g \approx d - (R^2/2)(1/r_1 + 1/r_2)$

- Hence, optical thickness is:

  $\delta \approx nd - (n - 1)(R^2/2)(1/r_1 + 1/r_2)$

Lenses (3)



- Hence, transfer function is:

  $\tau_L = \exp(-jk_0\delta)$ or

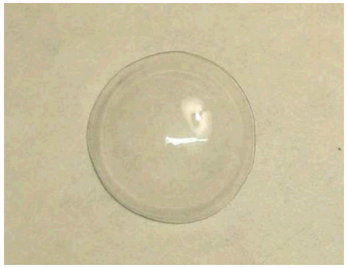
  $\tau_L = \exp(-jk_0 nd) \exp\{+jk_0(n - 1) (R^2/2) (1/r_1 + 1/r_2)\}$

- Write this as:

  $\tau_L = \tau_S \exp(+jk_0R^2/2f)$ where f is the focal length

  where $1/f = (n - 1) (1/r_1 + 1/r_2)$ the lens-maker's formula

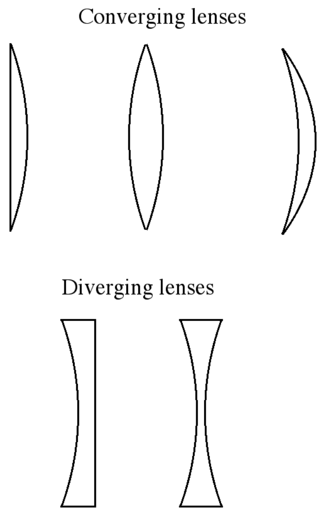
Lens Types



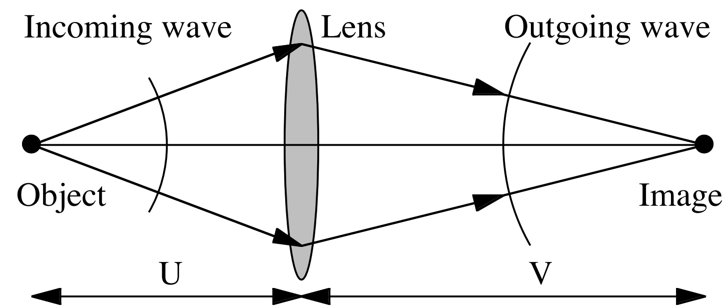
- Lens maker's formula
  Focal length is $1/f = (n - 1) (1/r_1 + 1/r_2)$

- Value of f is governed by r_1 and r_2
  Convex surface has +ve radius
  Concave surface has -ve radius

- Focal length can be positive or negative
  Positive f gives converging lens
  Negative f gives diverging lens



Imaging



- How do lenses form images?

- Assume paraxial wave input:

  $E_{in} = A(u) \exp(-jk_0R^2/2u)$

- Output found by multiplying by τ_L:

  $E_{out} = \tau_L\, E_{in} = A(u)\, \tau_S \exp(+jk_0R^2/2f) \exp(-jk_0R^2/2u)$

  Since this contains R^2, it is also paraxial wave

- Combining the exponentials, the output can be written as:

  $E_{out} = A'(u) \exp(+jk_0R^2/2v)$ where $A'(u)= A(u)\, \tau_S$

  And v satisfies $1/u + 1/v = 1/f$ - the imaging equation

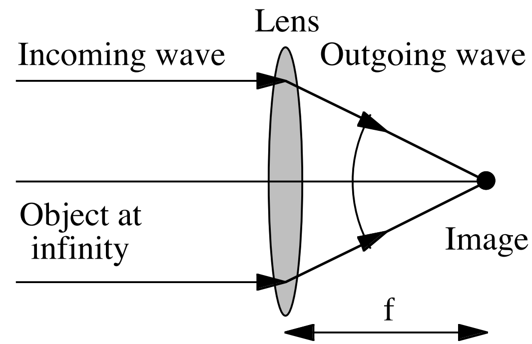
- Knowing object position u and focal length f,

  we can find image position v. The magnification is v/u.

Imaging Examples

- Imaging equation is:

  $1/u + 1/v = 1/f$



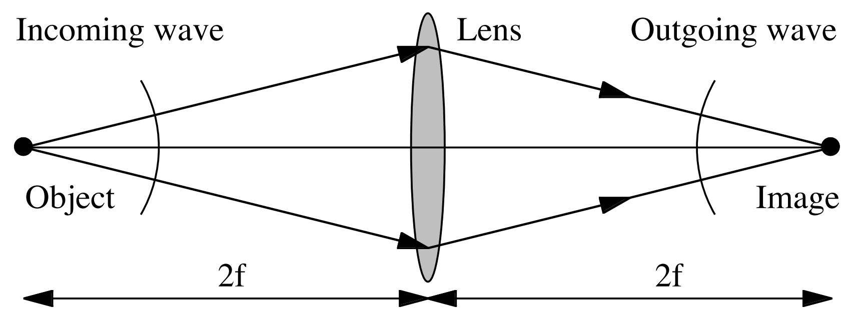
- If u is infinite (parallel beam incident):

  $1/v = 1/f$ so beam is focussed f from lens

- If $u = 2f$ then $v = 2f$

  Image is formed 2f from lens

  Magnification is unity



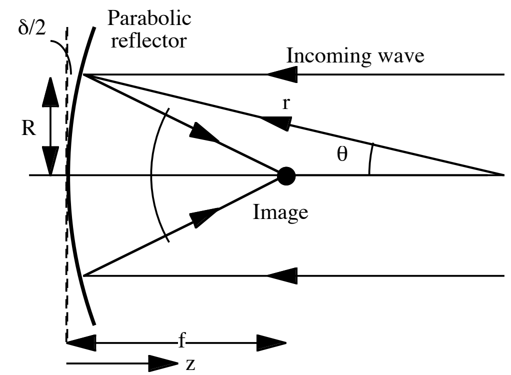
Parabolic Reflectors

- Reflective equivalent of lens

  Used when no transparent material available

  Non-dispersive, so no chromatic aberrations

  Lightweight - important for large telescopes



- Variation of phase shift across aperture:

  $\delta/2 = r\{1 - \cos(\theta)\}$ where r is radius of curvature

  $\delta/2 \sim r\theta^2/2$ for small angles so $\delta \sim R^2/r$

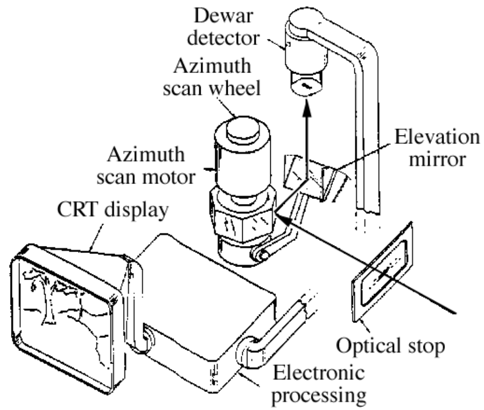
- If input wave is $E_0 \exp(+jk_0z)$, then output is:

  $E_0' \exp(-jk_0z) \exp(jk_0\delta) = E_0' \exp(-jk_0z) \exp(+jk_0R^2/r)$

  Wave converges to a focus r/2 away

- Focal length is half radius of curvature

Imaging by Scanning



- Sometimes it is impossible to detect a parallel image

  Construction of full focal plane detector array may be too difficult

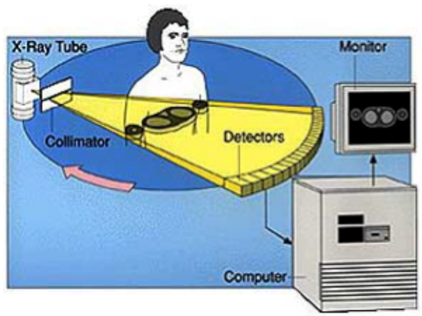
- Example is forward looking infrared (FLIR) system

  Detects photons with long wavelength and low energy (night vision)

  Uses low bandgap CdHgTe detector - requires cooling to reduce noise

  Image created by scanning image over single detector or linear array

Imaging by Reconstruction (1)

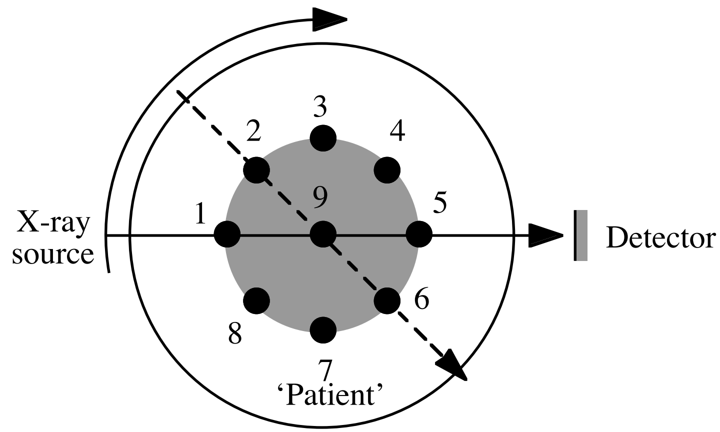


Godfrey Hounsfield
Nobel Prize 1979

- Sometimes a true optical image cannot be formed
  - Example is X-ray imaging
  - All materials have n ≈ 1 at X-ray wavelengths - can't build a lens!

- Computer aided tomography (CAT) operates by measuring absorption
  - Many different absorbing paths are measured
  - Each path contains contributions from many absorbing points
  - Image formed by solving simultaneous equations to find absorber values

Imaging by Reconstruction (2)



- Consider the system above, where the patient is modelled as 9 absorbers.

  Absorption A_1 along the full line is $A_1 = a_1 + a_5 + a_9$

  Absorption A_2 along the dashed line is $A_2 = a_2 + a_6 + a_9$
- Rotating the source and detector together we can form eight equations, one fewer than needed to find the nine unknowns a_1, a_2 … a_9.
- However additional equations can be provided using additional detectors.
- A matrix equation is then set up and solved to find the unknown absorber values and build up a picture of the patient.

8: Radio and Radar

- Antenna types
- Antenna properties
- Radio and the Friis transmission equation
- Radar and the radar equation

Radio System

- Signals carried from transmitter to receiver by EM wave

- Consider simple amplitude modulated signal:

  Signal amplitude A_S, signal and carrier at ω_S and ω_C

  $A(t) = \cos(\omega_C t)\{1 + mA_S \cos(\omega_S t)\}$

  $A(t) = \cos(\omega_C t) + mA_S \{\cos[(\omega_C + \omega_S)t] + \cos[(\omega_C - \omega_S)t] /2$

  Signal up-shifted in frequency: ω_S has become $\omega_C + \omega_S$

- Consequences:

  Wavelength reduced so wave can be launched from small antenna

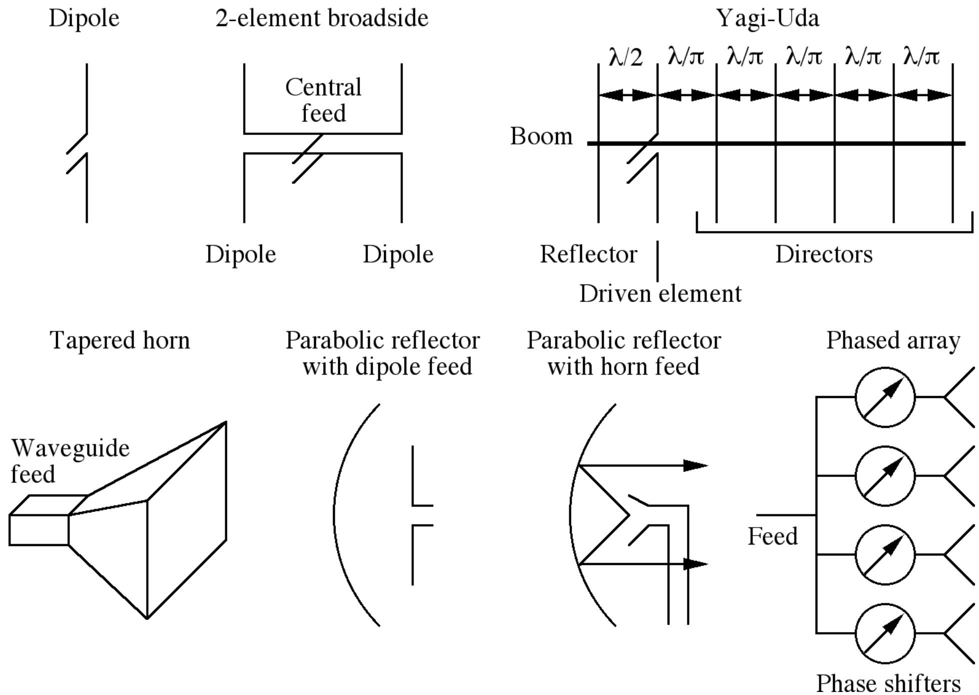
  Frequency variation reduced, so antenna can be narrow-band

  Different carriers can be used so system can be frequency multiplexed

- Signal falls off as $1/r^2$ so is weak at large distance

  Main function of antenna is to overcome this problem

Antenna Types



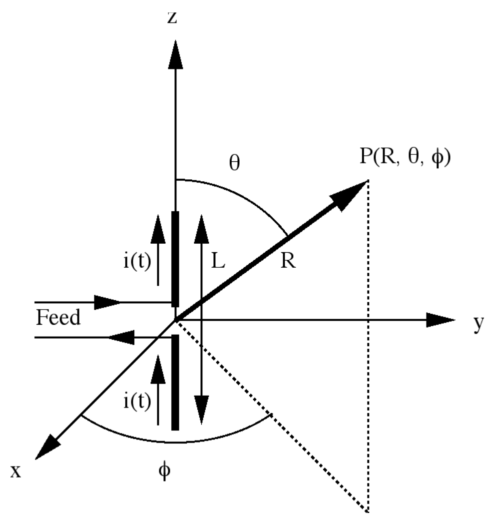
- Antenna improves directivity (signal gain in a particular direction)

  Directivity is the same in transmission and reception

  High directivity is easy if many elements are driven, but expensive

  Yagi-Uda array allows directivity with single connection - very cheap!

Short Dipole Antenna



- Elementary antenna to excite EM waves by electrical feed
  Assume dipole of length L and uniform current $I(t) = I_0 \cos(\omega t)$
- In far field ($R >> \lambda$), both field components decay as 1/R:
  $E_\theta = j\,(I_0 L k Z_0 \sin(\theta)/4\pi R)\exp(-jkR)$
  $H_\phi = E_\theta/Z_0$
  Hence antenna radiates approximately spherical wave

Radiation Pattern

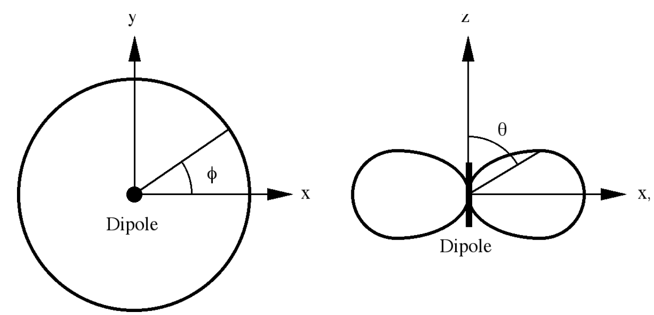
- Polar diagram of power density is called radiation pattern

- Power density is $\underline{S} = 1/2 \,\mathrm{Re}\,(\underline{E} \times \underline{H}^*)$

- Example: for short dipole

  $E_\theta = j\,(I_0 L k Z_0 \sin(\theta)/4\pi R)\,\exp(-jkR)$

  $H_\phi = E_\theta/Z_0$



Radiation pattern

- In this case we get

  $\underline{S} = S(R, \theta)\,\underline{r} = (I_0^2 L^2 k^2 Z_0/32\pi^2 R^2)\,\sin^2(\theta)\,\underline{r}$

- Radiation pattern isotropic in (x, y) plane, doughnut shaped otherwise

Antenna Directivity (1)

- Normalized radiation pattern is $F(\theta, \phi) = S(R, \theta, \phi) / S_{max}$
  Where S_{max} = maximum S at given R

- Directivity D is ratio of maximum of F to average of F
  Maximum of F is unity

- Average of F is $1/4\pi \int_0^{2\pi} \int_0^{\pi} F(\theta, \phi) \sin(\theta)\, d\theta\, d\phi$
  Hence $D = 1/ \{1/4\pi \int_0^{2\pi} \int_0^{\pi} F(\theta, \phi) \sin(\theta)\, d\theta\, d\phi\}$
  Generally hard to evaluate integral, but can be done

Antenna Directivity (2)

- Directivity is $D = 1/ \{1/4\pi \int_0^{2\pi} \int_0^{\pi} F(\theta, \phi) \sin(\theta)\, d\theta\, d\phi\}$
- Example: for short dipole

  $S(R, \theta) = (I_0^2 L^2 k^2 Z_0 / 32\pi^2 R^2) \sin^2(\theta)$

  Hence $F = S/S_{max} = \sin^2(\theta)$

- Hence $D = 1/ \{1/4\pi \int_0^{2\pi} \int_0^{\pi} \sin^3(\theta)\, d\theta\, d\phi\}$

  Or $D = 1/ \{1/2 \int_0^{\pi} \sin^3(\theta)\, d\theta\}$

  Or $D = 1/ \{1/2 [-\cos(\theta) + \cos^3(\theta)/3]_0^{\pi}\}$

  Or $D = (1/2 \,.\, 4/3)^{-1} = 1.5$

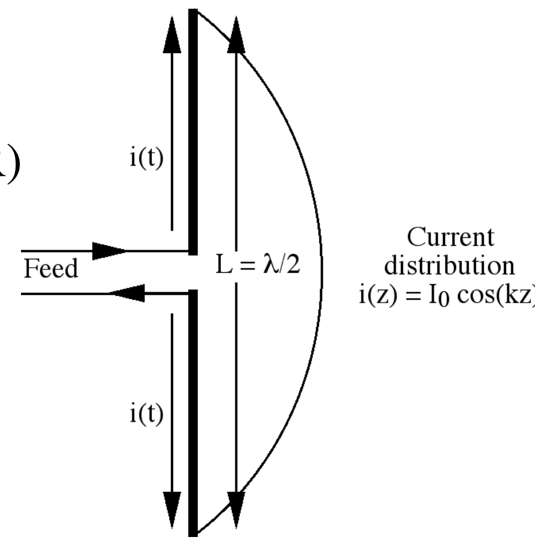
- Short dipole almost same as isotropic antenna (D = 1): not much use!

Half-wave Dipole Antenna

- Practical antenna to excite EM waves by electrical feed

  Resonant system, so current is much larger

  Non-uniform current $I(t) = I_0 \cos(\omega t) \cos(kz)$ in length $L = \lambda/2$



- In far field, only two significant components:

  $E_\theta = j\, 60 I_0 \{\cos[(\pi/2)\cos(\theta)] / R \sin(\theta)\} \exp(-jkR)$

  $H_\phi = E_\theta / Z_0$

- Hence

  $F(\theta, \phi) = \{\cos[(\pi/2)\cos(\theta)] / \sin(\theta)\}^2$

- More of the radiation now confined near $\theta = 0$

  Antenna more directive (D = 1.64), but still isotropic in (x, y) plane

Antenna Arrays

- Dipoles are not directive
  - Good for broadcasting signals
  - Bad for receiving signals
- Sensitivity requires directivity
  - But antenna must be pointed in right direction
- How to improve directivity?
  - Use arrays of half-wave dipoles
- Array elements can be driven or undriven
  - Common variant is Yagi-Uda array (only 1 driven)
  - Used for many domestic TV receivers

Hidetsugu Yagi

TV aerial

2-Element Broadside Array

- Simplest array, constructed from two dipoles

- Two signals received in far field:

  Relative phase delay is $\psi = kd\cos(\phi)$

- If individual antenna field is $E(\phi)$:

  Array gives $E'(\phi) = E(\phi)\{1 + \exp(-j\psi)\}$

  Or $E'(\phi) = E(\phi)\exp(-j\psi/2)\{\exp(+j\psi/2) + \exp(-j\psi/2)\}$

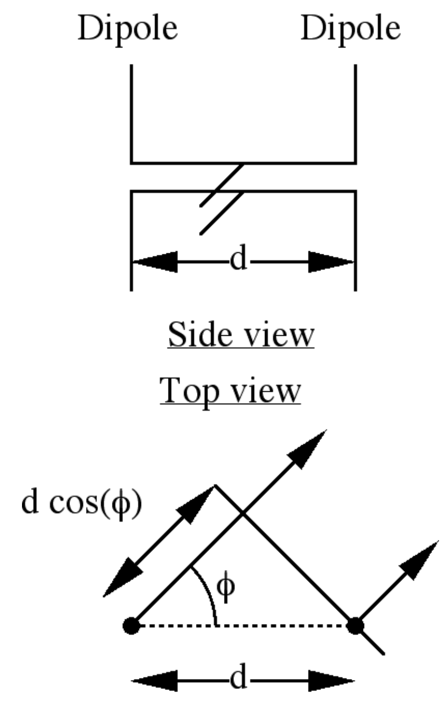
  Or $E'(\phi) = 2E(\phi)\exp(-j\psi/2)\cos(\psi/2)$

- Since $E(\phi) = 1$:

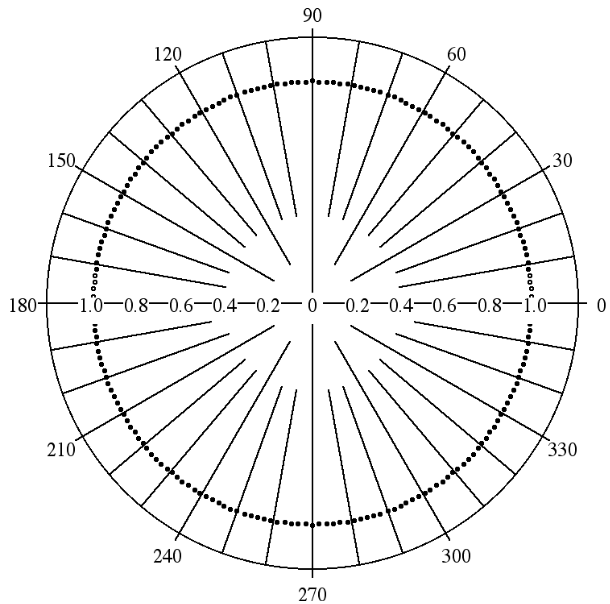
  $S'(\phi) = 4\cos^2(\psi/2)$

  $F'(\phi) = \cos^2(\psi/2) = \cos^2\{kd\cos(\phi)/2\}$

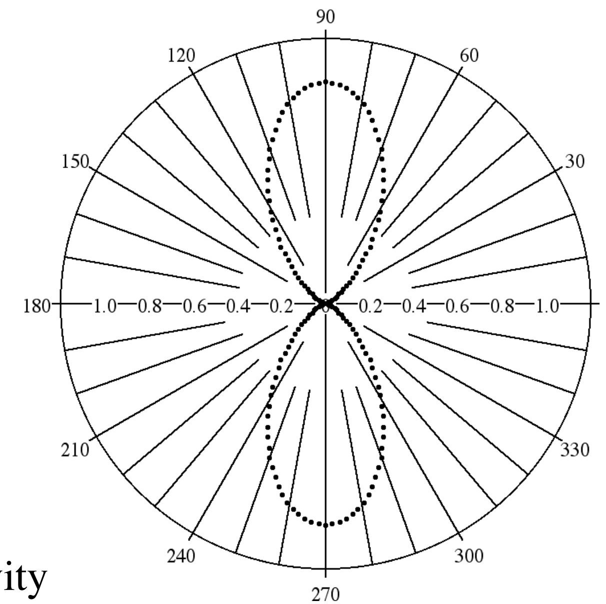
  Now potentially directive since F' depends on ϕ



Array Radiation Pattern



d << λ;
no directivity



d = λ/2;
directivity

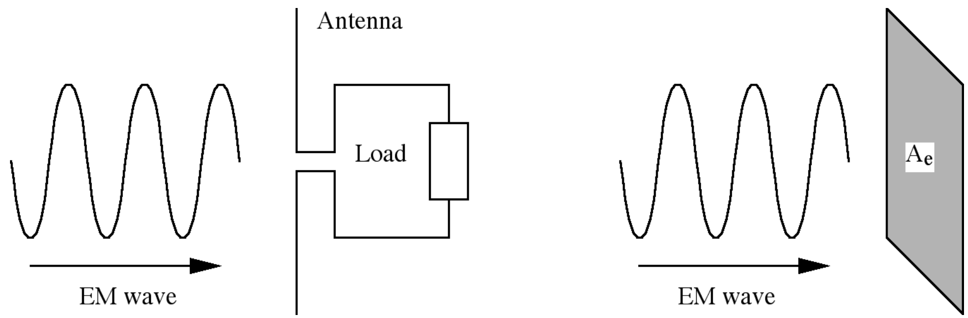
- $F'(\phi) = \cos^2(\psi/2) = \cos^2\{kd \cos(\phi)/2\}$

  $F'(\phi) = 1$ when $\phi = \pi/2$

  If $d = \lambda/2$, $F'(\phi) = 0$ when $\phi = 0$

- Radiation pattern has broadside gain (think of old battleships)

Antenna Gain and Effective Area



- Real antenna are not 100% efficient

  If the radiation efficiency is η, the antenna gain is $G = \eta D$

- The effective area A_e is the equivalent area from which antenna can gather power, and deliver it to matched load

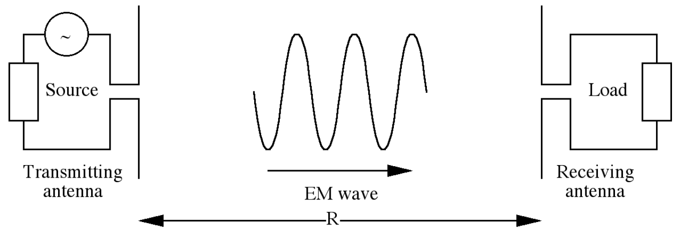
- Effective area is related to directivity and gain by

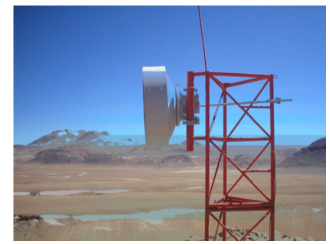
  $A_e = \lambda^2 D/4\pi = \lambda^2 G/4\pi\eta$

- Example: For short dipole,

  $D = 3/2$ so $A_e = 3\lambda^2/8\pi$

Friis Transmission Equation





- Gives received power P_R, assuming transmit power P_T
- Four-step calculation

1) Assume transmitting antenna is isotropic and loss-less:

Ideal power density at receive antenna is $S_{ISO} = P_T/4\pi R^2$

2) Now note that real antennae are neither isotropic nor loss-less:

Real power density is $S_{REAL} = \eta_T D_T S_{ISO}$ or

$S_{REAL} = \eta_T\ (4\pi A_T/\lambda^2)\ S_{ISO}$ so

$S_{REAL} = P_T\ (\eta_T A_T/R^2\lambda^2)$

Friis Transmission Equation

- Now use effective area to find intercepted power

  3) First assume intercepted by receive antenna is ideal

  $P_{INT} = S_{REAL}\ A_R$

  Or $P_T\,(\eta_T A_T A_R / R^2 \lambda^2)$

- Now assume receive antenna is not ideal either

  4) Power at receiver is then:

  $P_R = \eta_R P_{INT}$, or

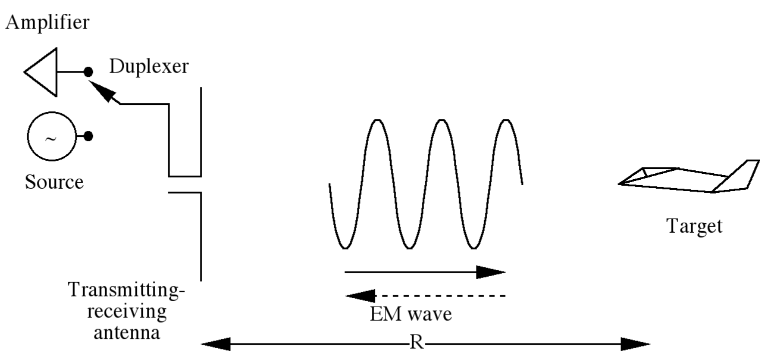
  $P_R = P_T\,(\eta_T \eta_R A_T A_R / R^2 \lambda^2)$

- Finally, note that result can be written alternatively as

  $P_R = P_T\ G_T G_R (\lambda / 4\pi R)^2$

- Receive power falls off as $1/r^2$; high antenna gains needed

Radar Range Equation



- Radar equation gives the received power in a radar system
  - Same antenna used for transmission and reception
  - Switching between transmitter and receiver carried out by duplexer
  - Care needed to avoid blowing up receiver

- Power calculation involves variant of Friis equation
  - Two-step process: radar-to-target, then target-to-radar
  - Received power determines effective range of radar

Radar Range Equation

- From Friis formula (3), intercepted power at range R is

  $P_{INT} = P_T (\eta_T A_T A_R / R^2 \lambda^2)$

- In radar system, A_R is called σ, the target's radar cross-section, so

  $P_{INT} = P_T (\eta_T A_T \sigma / \lambda^2 R^2)$

- Assume intercepted power is scattered as a spherical wave

  Perform second Friis-type calculation

- Power received back at transmit antenna is:

  $P_R = P_{INT} \times (\eta_T A_T / 4\pi R^2)$, or

  $P_R = P_T (\eta_T^2 A_T^2 \sigma / 4\pi R^4 \lambda^2)$

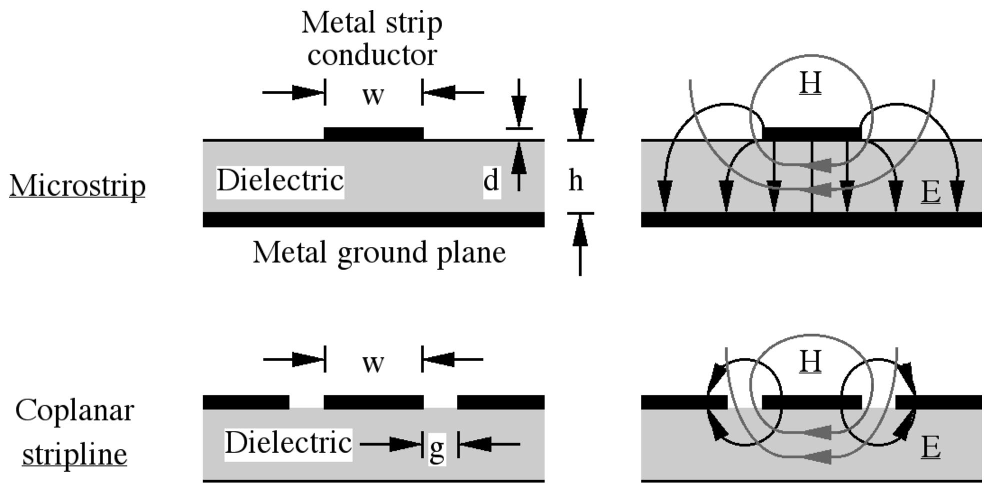
- For given sensitivity P_R, range R is proportional to $P_T^{1/4}$

  High power transmitter and high value of A_T essential

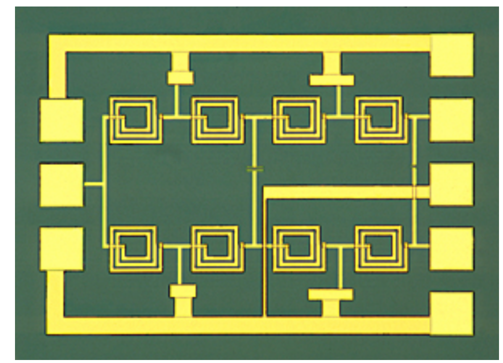
9: Microwaves

- Microstrip waveguides
- Waves in metals
- Waveguides and guided modes
- Resonators

Waveguides (1)



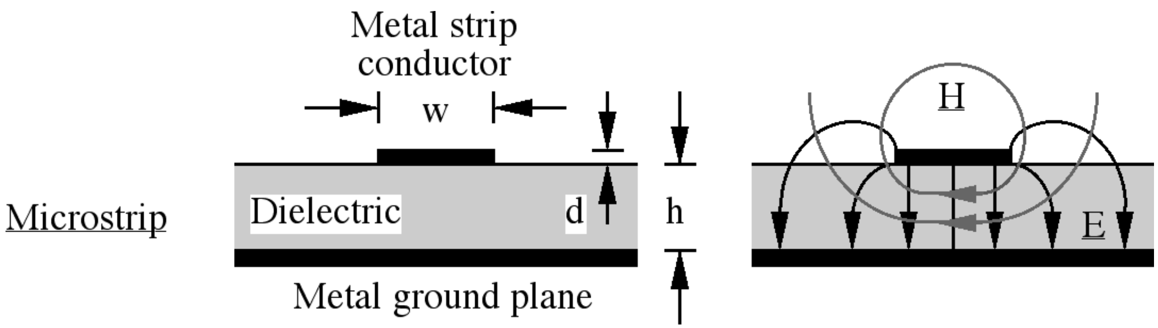
Geometry   Fields



Monolithic microwave
Integrated circuit (MMIC)

- At mm wavelengths, metal strips used to form circuits
- Two common forms, microstrip and coplanar strip
- Microstrip requires rear ground plane

Waveguides (2)



- Characteristic impedance of microstrip found as follows:

  Parallel plate capacitor between strip and ground: $C_P \approx \varepsilon w/h$

  Including fringing field (spreads h on either side): $C_P \approx \varepsilon(w + 2h)/h$

- By Ampere's law, magnetic field is $Hw \approx I$, so $B = \mu_0 I/w$

  Linked flux is $\Phi_P = Bh = \mu_0 Ih/w$

  Inductance is $L_P = \Phi_{pul}/I = \mu_0 h/w$

- Hence $Z_0 = (L_P/C_P)^{1/2} \approx \{\mu_0 h/\varepsilon w(2 + w/h)\}^{1/2}$ - adjustable to value

Waves in Metals (1)

- In metals, σ and $\underline{\mathbf{J}}$ are both non-zero

  In Ampere's law, need $\mathrm{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$, not just $\partial\underline{\mathbf{D}}/\partial t$

- Derive wave equation with

  $\mathrm{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$

  $\mathrm{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$

  From material equations

  $\mathrm{curl}(\underline{\mathbf{E}}) = -\mu_0\partial\underline{\mathbf{H}}/\partial t$

  $\mathrm{curl}(\underline{\mathbf{H}}) = \sigma\underline{\mathbf{E}} + \varepsilon\partial\underline{\mathbf{E}}/\partial t$

- Assuming $\underline{\mathbf{E}} = \underline{\mathrm{E}} \exp(j\omega t)$, $\underline{\mathbf{H}} = \underline{\mathrm{H}} \exp(j\omega t)$

  $\mathrm{curl}(\underline{\mathrm{E}}) = -j\omega\mu_0\underline{\mathrm{H}}$

  $\mathrm{curl}(\underline{\mathrm{H}}) = (\sigma + j\omega\varepsilon)\underline{\mathrm{E}} = j\omega\varepsilon(1 + \sigma/j\omega\varepsilon)\underline{\mathrm{E}}$

- Hence can replace ε with $\varepsilon(1 + \sigma/j\omega\varepsilon)$ or $\sigma/j\omega$ if σ is very large

Waves in Metals (2)

- For z-going plane waves, time-independent scalar wave equation is:

  $d^2E_x/dz^2 = -\omega^2\mu_0\varepsilon\, E_x$

- Propagation constant is

  $k = \omega(\mu_0\varepsilon)^{1/2}$

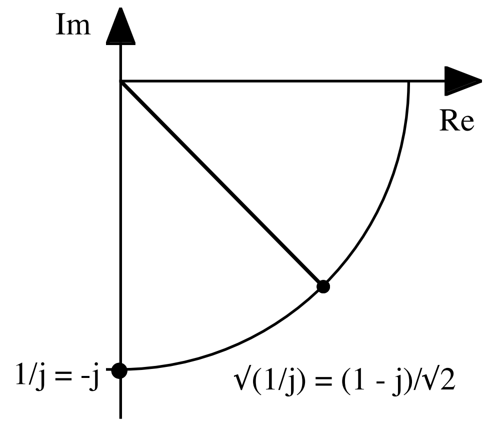
- If $\varepsilon = \sigma/j\omega$

  $k = \omega(\mu_0\sigma/j\omega)^{1/2}$

  Hence $k = (1 - j)\,(\omega\mu_0\sigma/2)^{1/2}$

  So $k = k' - jk''$

Need to find $(1/j)^{1/2}$



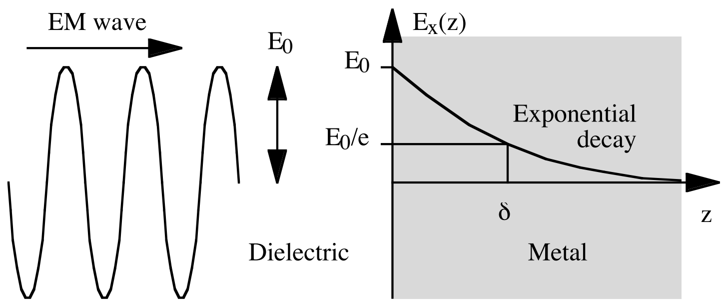
- Since $E = E_0 \exp(-jkz) = E_0 \exp(-jk'z) \exp(-k''z)$

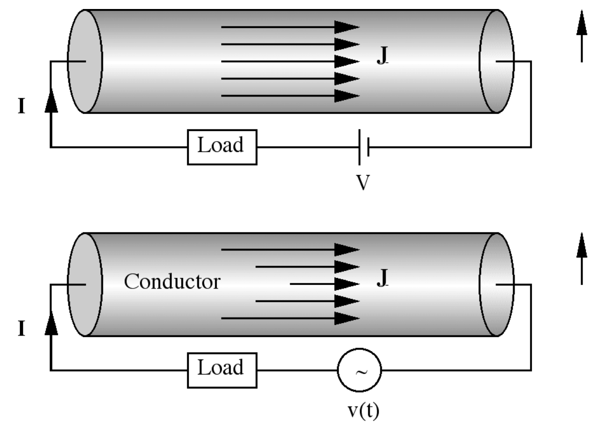
  Once again, wave decays as it propagates

  Decay to 1/e of original amplitude when $z = 1/k'' = (1/\pi f\mu_0\sigma)^{1/2}$

  This distance known as the skin depth δ

Consequences of Skin Effect





- EM wave decays as it travels into metal

  Current confined near conductor surface at RF frequency

- For cylinder; need to solve wave equation in cylindrical polar co-ords

  In wire, current contained in annular strip of area $2\pi r\delta$

  Wire resistance increases, from $R_P = 1/\sigma\pi r^2$ to $R_P = 1/\sigma 2\pi r\delta$

  Inductors don't work at high frequency due to high loss

Reflectivity of Metals

- At high frequencies (μwave to optical) σ no longer real

  Can be shown (not here!) that $\sigma \sim Ne^2/jm^*\omega$

  Here N is electron density and m* effective mass

- Propagation constant now $k^2 \sim -j\omega\mu_0\sigma \sim -\mu_0 Ne^2/m^*$

  Generally, $k^2 = \varepsilon_r k_0^2$ so ε_r must be large and negative

  Refractive index is $n = \varepsilon_r^{1/2}$ so n must be imaginary (say, $n = jn'$)

- Reflection coefficient for dielectric interface is

  $\Gamma_E = (n_1 - n_2) / (n_1 + n_2)$

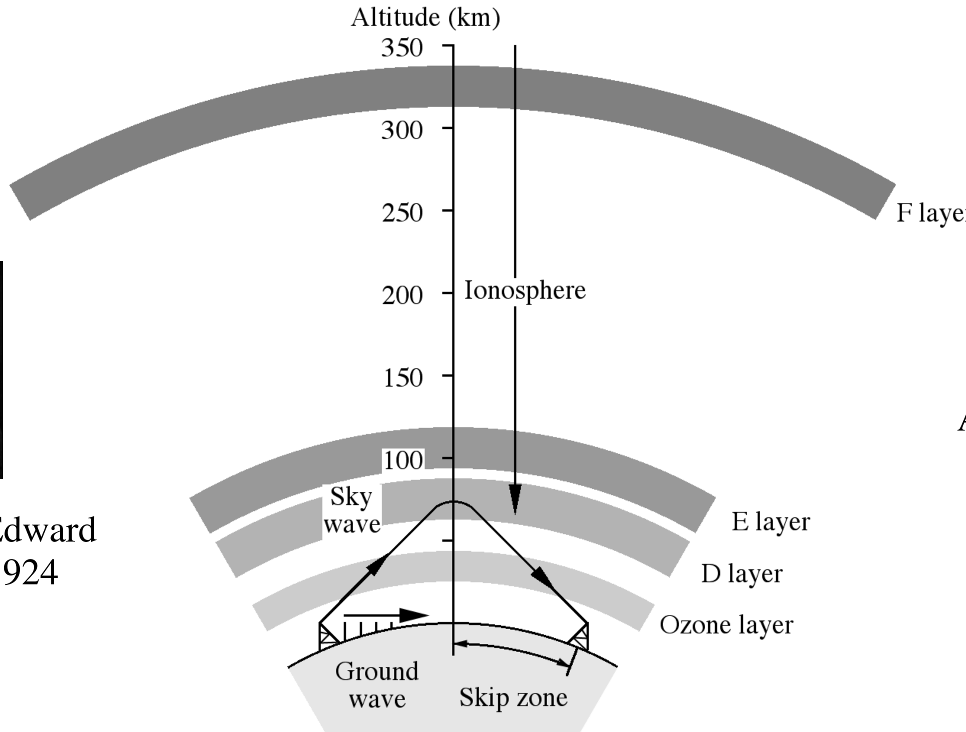
- For dielectric/metal interface we then get:

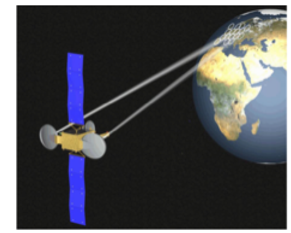
  $\Gamma_E = (n_1 - jn') / (n_1 + jn')$

  As n' becomes large, Γ_E tends to - 1, so electric field is zero in metal

The Ionosphere

Discovered by Edward Appleton in 1924





Satcomms proposed by Arthur C. Clarke in 1945

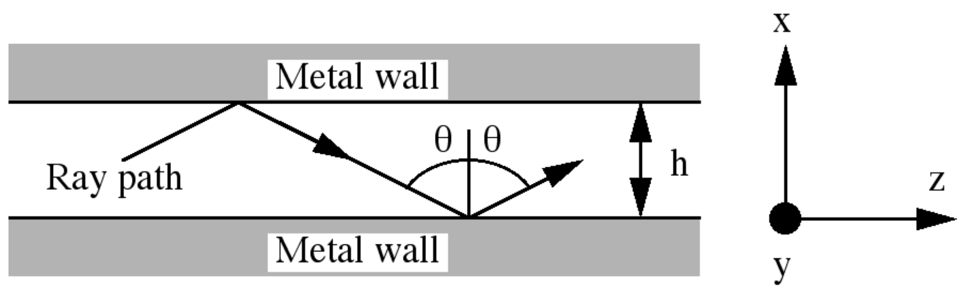
- Energetic particles create ions in upper atmosphere

  Layers of ionized particles reflect low frequency radio waves

  Provides propagation path round curved surface of the earth

  Ionosphere transparent to VHF waves; communication then via satellite

Metal-walled Waveguides



- Useful at microwave frequencies

  Assume field is sum of upward- and downward-going waves:

  $E_y = E_+ \exp\{-jk_0[z \sin(\theta) + x \cos(\theta)]\} + E_- \exp\{-jk_0[z \sin(\theta) - x \cos(\theta)]\}$

- Boundary condition 1:

  $E_y = 0$ on $x = 0$ satisfied if $E_- = - E_+$ so

  $E_y = E_+ (\exp\{-jk_0[z \sin(\theta) + x \cos(\theta)]\} - \exp\{-jk_0[z \sin(\theta) - x \cos(\theta)]\})$

  $E_y = E \sin\{k_0 x \cos(\theta)\} \exp\{-jk_0 z \sin(\theta)\}$

The Eigenvalue Equation

- Assume $E_y = E \sin\{k_0 x \cos(\theta)\} \exp\{-jk_0 z \sin(\theta)\}$
- Boundary condition 2

  $E_y = 0$ on $x = h$

  Satisfied if $\sin\{k_0 h \cos(\theta)\} = 0$ so

  $k_0 h \cos(\theta) = \nu\pi$ (where $\nu = 1, 2$ .... etc.)

- This is the eigenvalue equation.

  Each solution corresponds to a mode, defined by the mode index ν

- Field can be written as $E_y = E(x) \exp(-j\beta z)$

  Here $E(x) = E \sin\{k_0 x \cos(\theta)\}$ is the transverse field

  And $\beta = k_0 \sin(\theta)$ is the propagation constant

  Since $k_0 h \cos(\theta) = \nu\pi$, $\beta = k_0\{1 - (\nu\pi/k_0 h)^2\}^{1/2}$

Guided Modes

- Since $\beta = k_0\{1 - (\nu\pi/k_0h)^2\}^{1/2}$

- Only fixed number of modes can propagate

  If $\nu\pi/k_0h > 1$, there is no solution

  Mode of order ν only exists if $h > \nu\pi/k_0$

- If guide width h is too small, no modes are supported.

  This occurs when $h < \pi/k_0$ or $h < \lambda_0/2$

  If h slightly bigger, one mode is supported (single-mode guide)

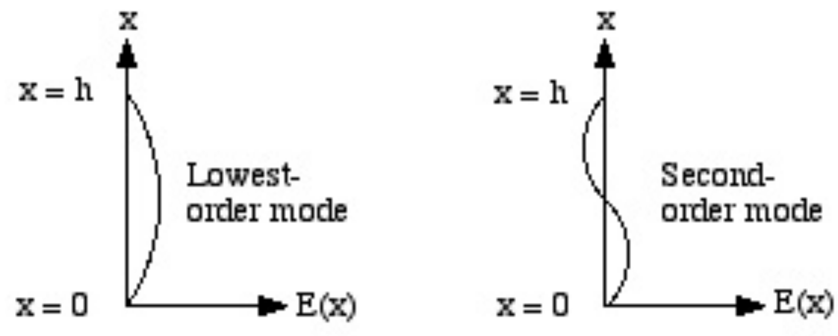
  If h bigger still, more modes are supported (multi-mode guide)

  Any mode that cannot propagate is cut off

- The higher the mode number, the smaller the value of θ.

  At cut-off, the ray angle is zero, so the rays bounce up and down between the walls, making no forward progress.

Modal Transverse Fields



- Transverse variation of field is $E(x) = E \sin\{k_0 x \cos(\theta)\}$

  Or $E_\nu = E \sin(\nu\pi x/h)$

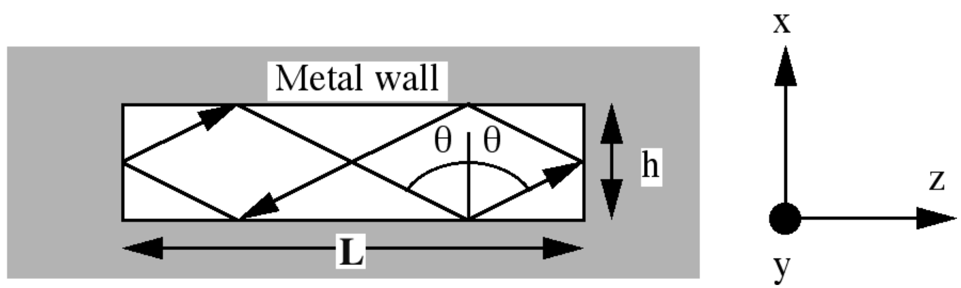
- Modal fields are sinusoidal standing wave patterns

  The higher the mode number the higher the periodicity

- Different modes have different phase and group velocities

  Only single-mode guides are good for transmission

Cavity Resonators (1)



- 2D resonator constructed using extra metal walls at end of guide

  These act as mirrors, reflecting the guided mode

- Field now sum of forward- and backward-going modes

  $E_y = E \sin(\nu\pi x/h) \{A \exp(-j\beta z) + B \exp(+j\beta z)\}$

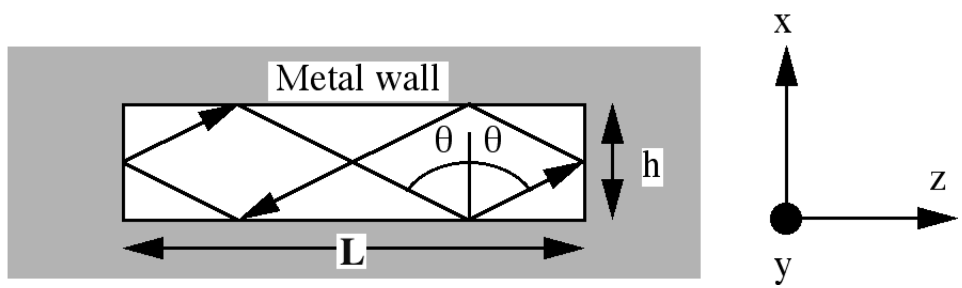
- Boundary condition 1

  E_y must be zero at z = 0 so B = -A, and

  $E_y = E \sin(\nu\pi x/h) \{\exp(-j\beta z) - \exp(+j\beta z)\}$

  $E_y = E' \sin(\nu\pi x/h) \sin(\beta z)$

Cavity Resonators (2)



- Assume $E_y = E' \sin(\nu\pi x/h) \sin(\beta z)$
- Boundary condition 2

  E_y must be zero at $z = L$ so $\sin(\beta L) = 0$ and $\beta = \mu\pi/L$

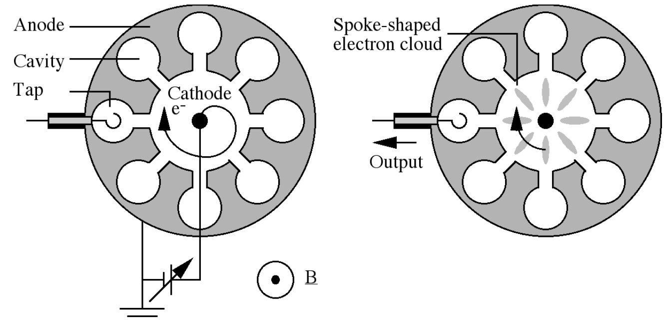
  Hence $E_{\nu,\mu} = E' \sin(\nu\pi x/h) \sin(\mu\pi z/L)$
- From waveguide analysis, $\beta = k_0\{1 - (\nu\pi/k_0h)^2\}^{1/2}$

  Hence $(\nu\pi/k_0h)^2 + (\mu\pi/k_0L)^2 = 1$

  Only satisfied for one $k_0 = 2\pi/\lambda$ so mode exists only at one wavelength

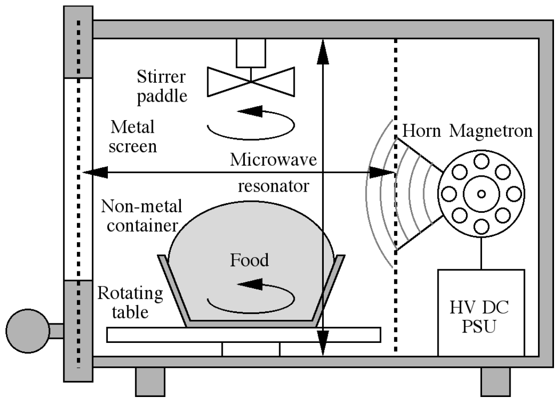
  Cavity resonator is frequency-selective filter

The Cavity Magnetron



- High power oscillator at microwave frequencies
- Allowed long-range radar and compact antennae
  - Machined copper anode containing cathode in evacuated enclosure
  - High voltage extracts electrons from cathode
  - Magnetic field makes electrons spiral round cathode
  - Electrons bunch into spoke shaped clouds
  - Rotating cloud excites coupled cavities into resonance
  - Power extracted by tap in one cavity; water cooling allows high power
- Secret weapon of WW2

Microwave Oven



- Cooks (wet) food by absorption of microwave energy
- Magnetron used to generate high power (700 W) at 2.45 GHz

  Energy coupled into cooking chamber by tapered waveguide horn

  Standing wave pattern set up inside chamber

  Pattern ‘stirred’ using metal paddle to avoid cold spots

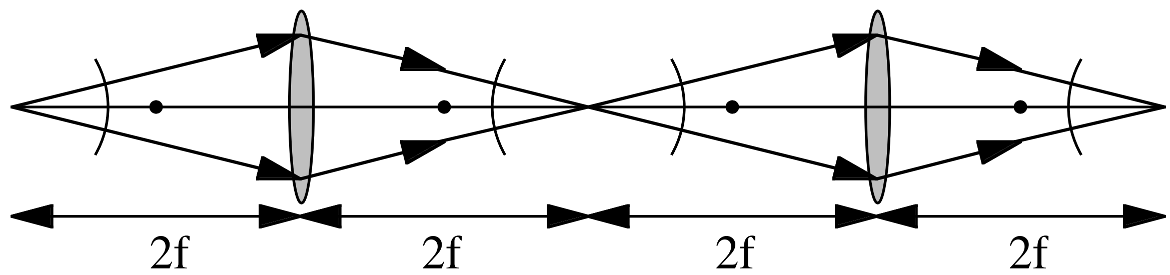
  Food placed in non-metal container and rotated on table

  Energy absorbed by water molecules realigning dipoles

10: Guided Wave Optics

- Lens waveguides
- Optical fibres and fibre fabrication
- Step-index fibre
- Parabolic index fibre
- The laser

Lens Waveguide



- Imaging equation is

  1/u + 1/v = 1/f

  If u = v, 2/u = 1/f so u = 2f

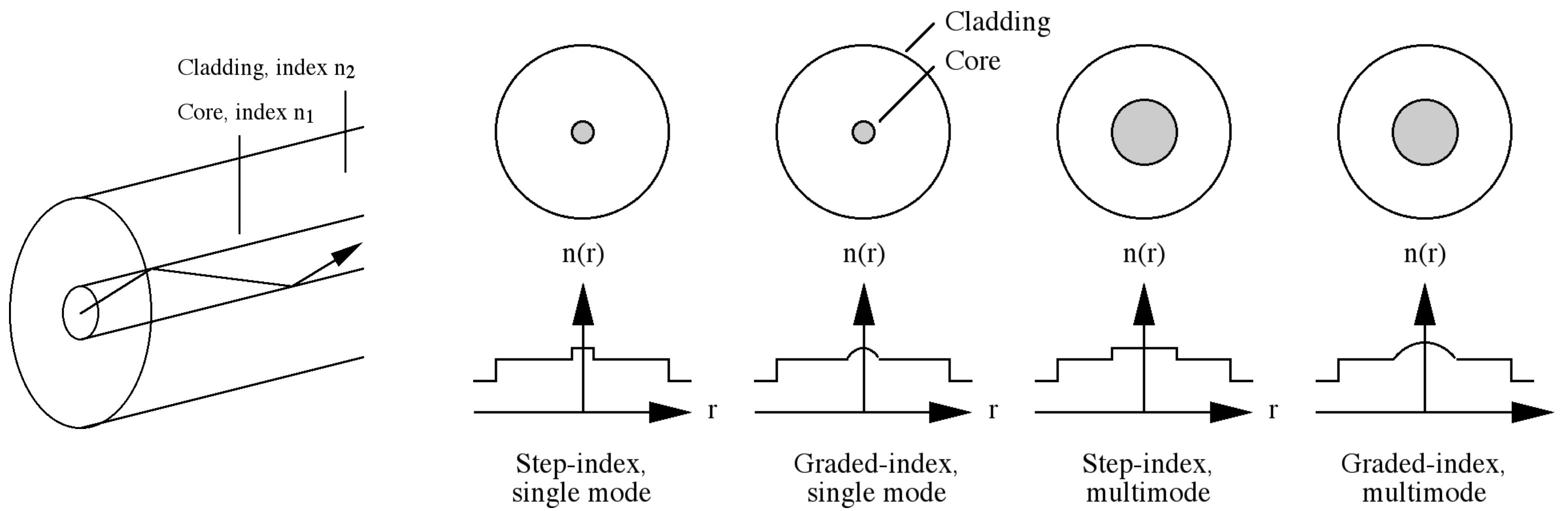
- Chain of lenses 4f apart forms guiding structure

  Early optical waveguide developed at Bell Labs

  Propagation length limited by misalignment and scattering

  Cumbersome; not all chains form stable guide

Optical Fibre Types



- Cylindrical core inside cylindrical cladding

  Stable guiding structure; classified by core size and shape

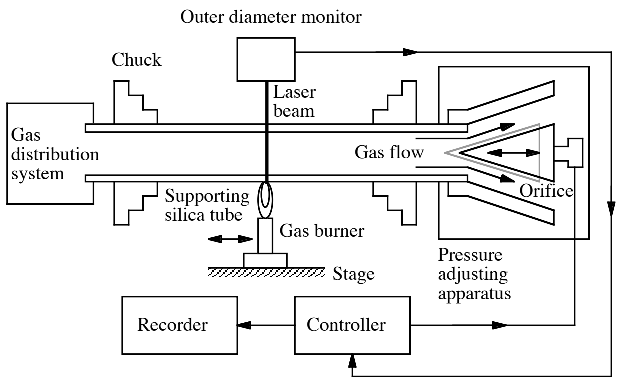
  Small core -single mode, large core - multi-mode

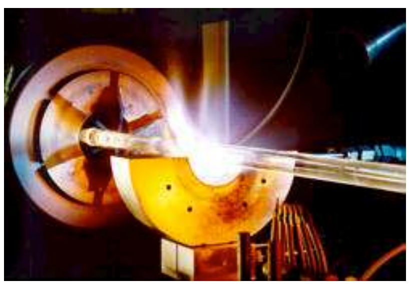
  Graded-index core offers lowest dispersion

- Materials

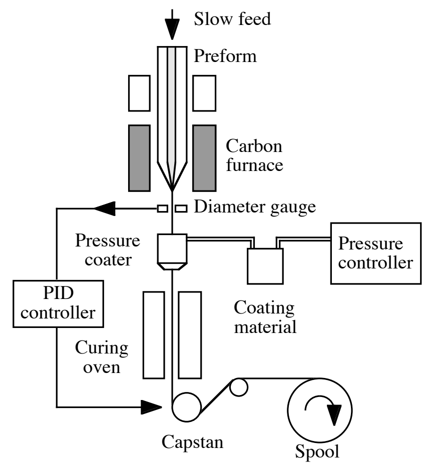
  Glasses (low loss; SiO_2 - GeO_2 best) and plastics (higher loss)

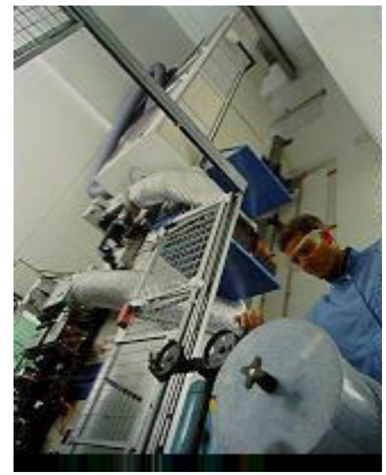
Optical Fibre Fabrication





Preform fabrication





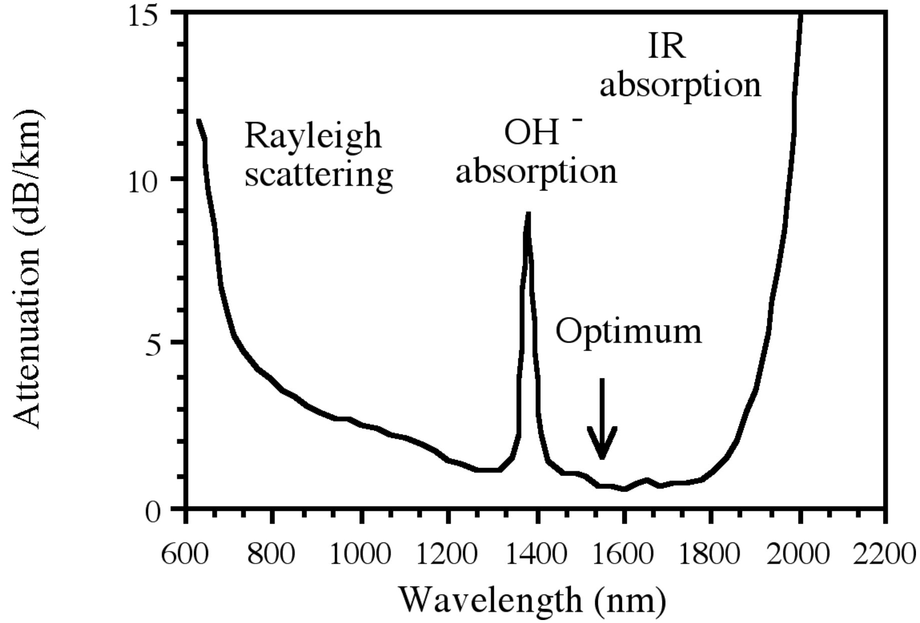
Fibre pulling

- Two-step process

  Preform with refractive index variation formed by gaseous deposition

  Fibre formed by pulling strand from molten end of preform

Optical Fibre Attenuation



- Attenuation in silica fibre

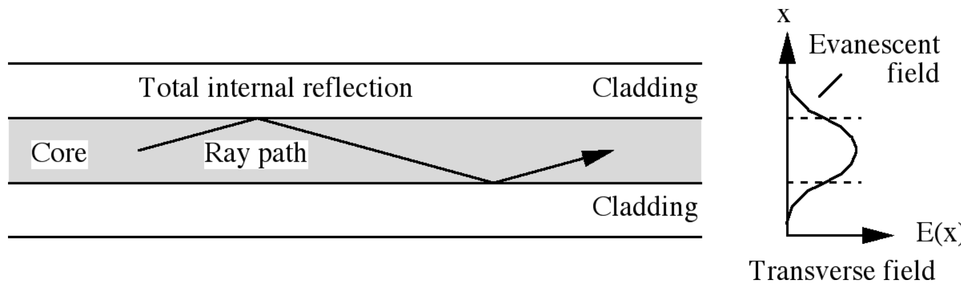
  Increases at short wavelength due to Rayleigh scattering (rises as $1/\lambda^4$)

  Increases at long wavelength because of mid-IR absorption bands

  Generally has peak near $\lambda = 1.4$ μm because of OH^- absorption

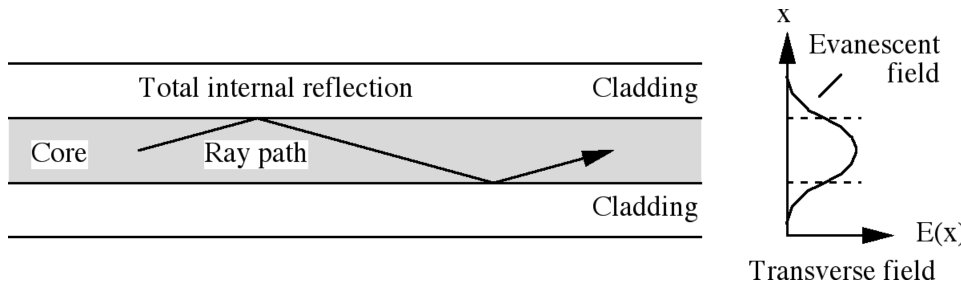
  Is lowest near $\lambda = 1.55$ μm (0.2 dB/km)

Step-index Fibre (1)



- In step-index fibre, propagation is by repeated TIR
- Guided mode contains evanescent field outside the core
  - Well confined modes have effective index n_{core}
  - Modes near cutoff have effective index n_{clad}
- For multimode fibre, mode index ranges from n_{core} to n_{clad}
  - Different modes therefore have different speeds
  - Phase velocities range from $v_{core} = c/n_{core}$ to $v_{clad} = c/n_{clad}$

Step-index Fibre (2)



- Phase velocities range from c/n_{core} to c/n_{clad}

  In distance L, arrival times range from L/v_{core} to L/v_{clad}

  Time spread of a pulse is $\Delta t = (L/c)\,(n_{core} - n_{clad})$

- Adjacent pulses of width T overlap when $T = \Delta t$

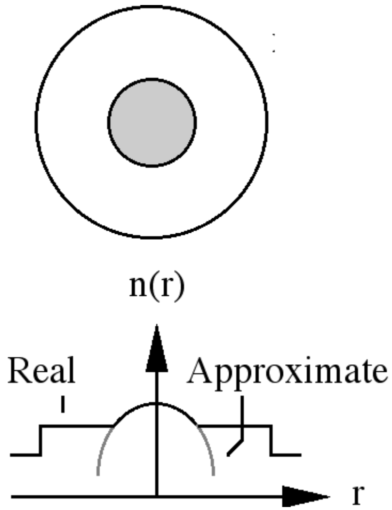
  Maximum bit rate is therefore $B = 1/(2\Delta t)$

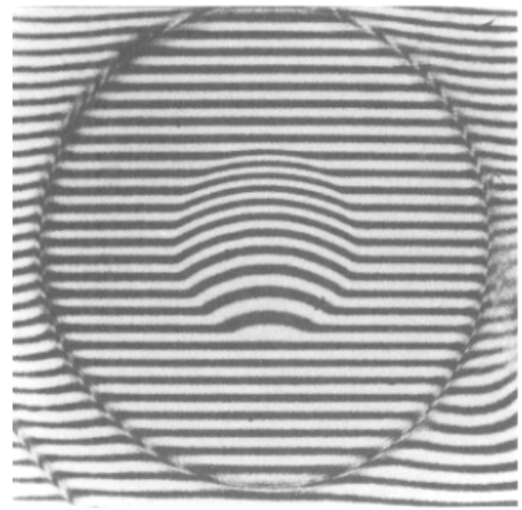
- Bit-rate: length product is $BL = c/\{2(n_{core} - n_{clad})\}$

  If $n_{core} - n_{clad} = 0.01$, and L = 1 km, B = 15 MB/s

  If $n_{core} - n_{clad} = 0.01$, and L = 10 km, B = 1.5 MB/s

Parabolic Index Fibre (1)





- Parabolic index fibre guides by continuous refraction

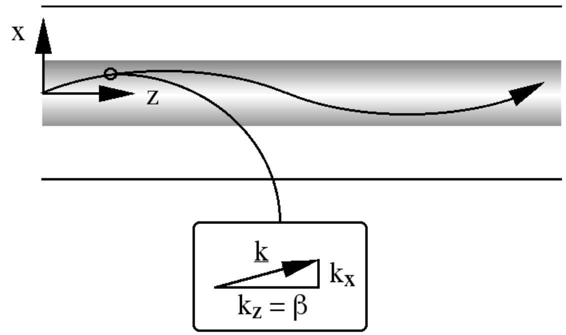
  Refractive index profile is $n(r) = n_0 \sqrt{1 - (r/r_0)^2}$

  Ignore fact that index can become imaginary - r usually small

- In two dimensions

  $n(x) = n_0 \sqrt{1 - (x/x_0)^2}$

Ray Trajectory Equation (1)



- Local ray direction given by $dx/dz = k_x/k_z$

  Where $k_z = \beta$, and β is the modal propagation constant.

  Squaring and rearranging, we get $k_z^2 (dx/dz)^2 = k_x^2$
- Since $k_x^2 + k_z^2 = |\underline{k}|^2$ and $|\underline{k}| = k_0 n$ we can write:

  $k_x^2 = k_0^2 n^2 - \beta^2$, or

  $k_x^2 = (k_0^2 n_0^2 - \beta^2) - k_0^2 n_0^2 (x/x_0)^2$
- Hence, equation for ray trajectory is:

  $\beta^2 (dx/dz)^2 = (k_0^2 n_0^2 - \beta^2) - k_0^2 n_0^2 (x/x_0)^2$

Ray Trajectory Equation (2)

- Trajectory equation is:

  $\beta^2 (dx/dz)^2 = (k_0{}^2 n_0{}^2 - \beta^2) - k_0{}^2 n_0{}^2 (x/x_0)^2$

  Nonlinear differential equation - impossible to solve?

- Guess solution $x = A \sin(Bz + \psi)$

  Trajectory is a sinusoid

- Differentiate:

  $dx/dz = AB \cos(Bz + \psi)$

- Substitute:

  $\beta^2 A^2 B^2 \cos^2(Bz + \psi) = (k_0{}^2 n_0{}^2 - \beta^2) - (k_0{}^2 n_0{}^2 A^2/x_0{}^2) \sin^2(Bz + \psi)$

- Re-arrange to get

  $(k_0{}^2 n_0{}^2 - \beta^2) - (k_0{}^2 n_0{}^2 A^2/x_0{}^2) \sin^2(Bz + \psi) - \beta^2 A^2 B^2 \cos^2(Bz + \psi) = 0$

Ray Trajectory Equation (3)

- Trajectory equation is:

  $(k_0^2 n_0^2 - \beta^2) - (k_0^2 n_0^2 A^2/x_0^2) \sin^2(Bz + \psi) - \beta^2 A^2 B^2 \cos^2(Bz + \psi) = 0$

- Solution possible if $\sin^2$ and $\cos^2$ terms add to give a constant

  Hence $(k_0^2 n_0^2 A^2/x_0^2) = \beta^2 A^2 B^2$

  So $B = k_0 n_0/\beta x_0$

- Remaining equation is

  $(k_0^2 n_0^2 - \beta^2) - (k_0^2 n_0^2 A^2/x_0^2) = 0$

  Hence $A = x_0 \sqrt{(1 - \beta^2/k_0^2 n_0^2)}$

- Final solution is

  $x = x_0 \sqrt{(1 - \beta^2/k_0^2 n_0^2)} \sin\{(k_0 n_0/\beta x_0)z + \psi\}$

- For weak guides (when $\beta \approx k_0 n_0$), $x \approx A \sin(2\pi z/P + \psi)$

  where the trajectory period $P = 2\pi x_0$ is independent of β

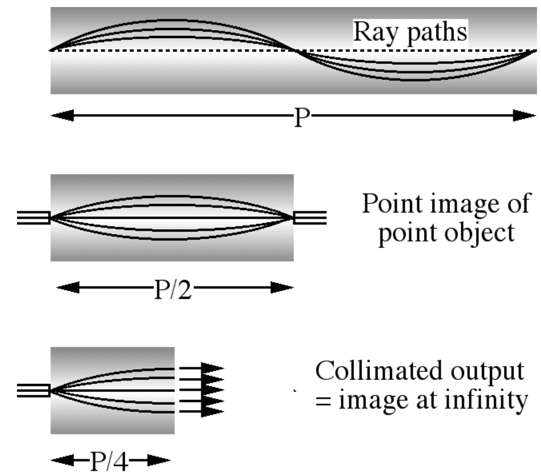
Parabolic Index Fibre (2)

- For weak guides, trajectory is

  $x \approx A \sin(2\pi z/P + \psi)$

  where $A = x_0 \sqrt{(1 - \beta^2/k_0^2 n_0^2)}$

  and $P = 2\pi x_0$



- Pitch P independent of amplitude A

  Rays starting from a point end on a point

- Fibre can carry out imaging operations

  Short length known as Gradient Index Rod (GRIN) lens

- Quarter-pitch GRIN lens can collimate a point source

Parabolic Index Fibre (3)

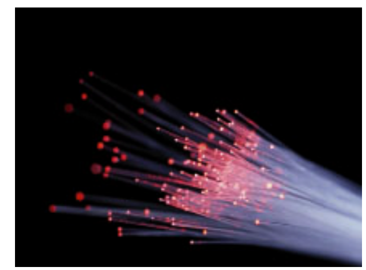
- Can show all modes have similar group velocity

  Intermodal dispersion then almost zero in MM fibre

  Parabolic fibre bit rate: length product much higher than step index

- Improved performance using SM fibre

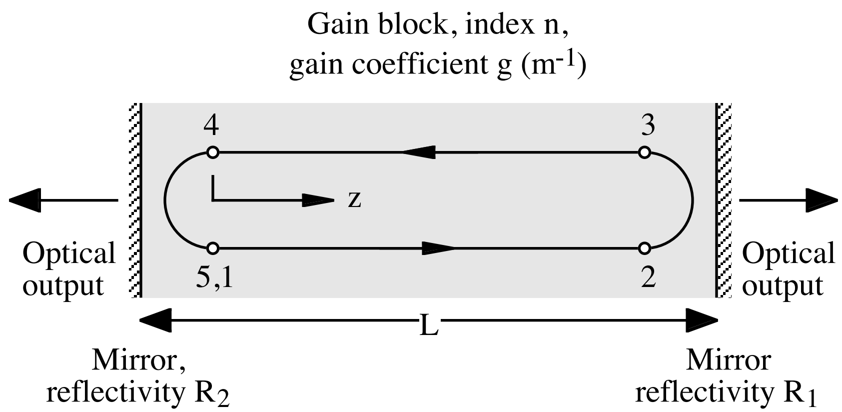
  Intermodal dispersion then eliminated



- Shape of graded core used to cancel material dispersion

  Very high bit-rate : length product (10 GB/s x 100 km) then possible

Lasers (1)



- Optical waveguide with gain g and end mirrors R_1, R_2

  Assume field at 1 is $E_1 = E_0$

  Field at 2 is $E_2 = E_0 \exp(-j\beta L) \exp(gL)$

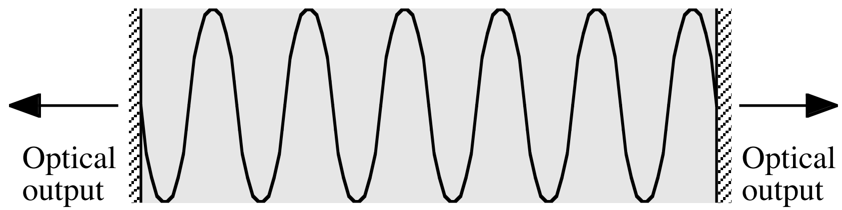
  Field at 3 is $E_3 = E_0 \exp(-j\beta L) \exp(gL)\, R_1$

  Field at 4 is $E_4 = E_0 \exp(-j2\beta L) \exp(2gL)\, R_1$

  Field at 5 is $E_5 = E_0 \exp(-j2\beta L) \exp(2gL)\, R_1R_2$

  Laser oscillates if $E_5 = E_1$

Lasers (2)



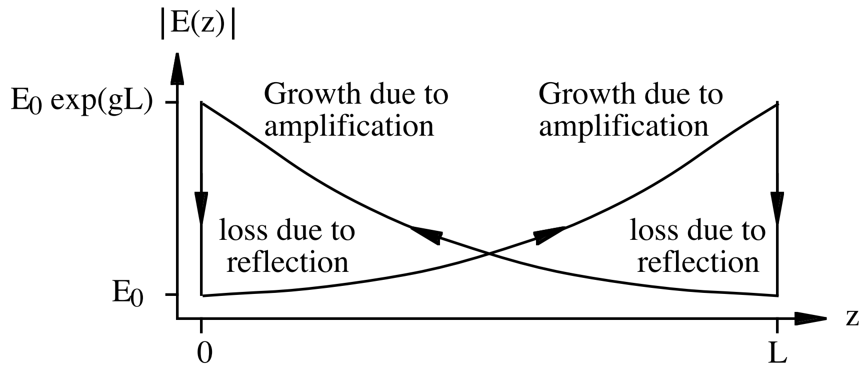
- If $\exp(-j2\beta L) \exp(2gL) R_1R_2 = 1$

  LHS must be real since RHS is real

- Hence

$\exp(-j2\beta L) = 1$	Phase condition
$2\beta L = 2\nu\pi$	Where ν is an integer
$2\ (2\pi n/\lambda)\ L = 2\nu\pi$	
$\lambda = (2nL/\nu)$	Only particular wavelengths oscillate

Lasers (3)



- If $\exp(-j2\beta L)\ \exp(2gL)\ R_1R_2 = 1$

  LHS must be unity since RHS is unity

- Hence

  $\exp(2gL)\ R_1R_2 = 1$ Gain condition

  $g = (1/2L)\ \log_e(1/R_1R_2)$

  Gain must exceed mirror losses for laser to oscillate

  Need a high-gain optical medium