Final project - SI1336 Simulation and modeling

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1 Introduction

1.1 Background

A multi-staged rocket is a launch vehicle that consists of two or more stages, each containing its own propellant and engine. The rocket detaches burned-out stages and continues without them to lower the overall weight. Currently, there are two ways of configuring the multi-staged rocket: serial staging and parallel staging [1].

Rocket trajectories can be estimated analytically. However, important parameters must then be neglected or roughly approximated, resulting in inadequate results. To achieve much better estimations, a numerical simulation of a rocket launch can be implemented. In this project, a serial two-staged rocket will be simulated.

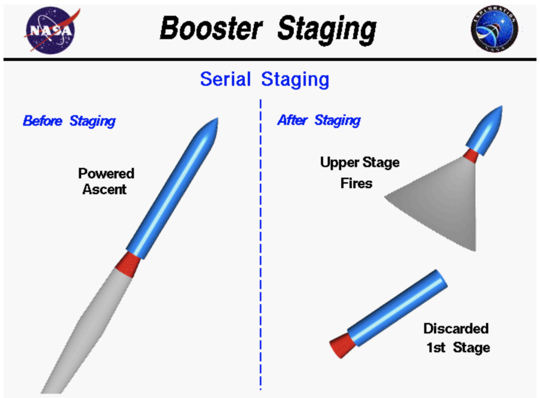


Figure 1: Serial staging.

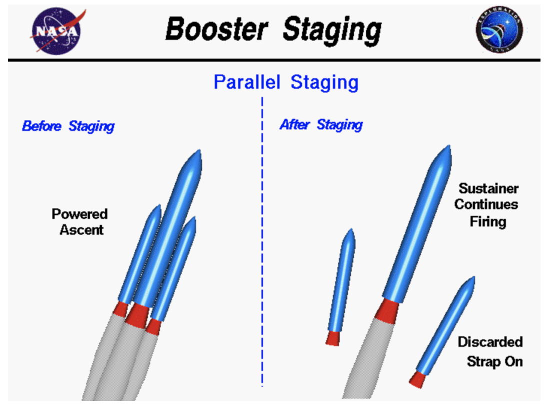


Figure 2: Parallel staging.

1.2 Physics model

1.2.1 Trajectory equations

For one dimensional rocket flight, the trajectory is obtained by Newton's second law of motion in the vertical direction of the rocket. The sum of the forces acting on the rocket can be written as:

$$thrust - weight - drag = mass \times acceleration$$

or in differential form:

$$T - m(t)g - \frac{1}{2}\rho(x)C_d A(\frac{dx(t)}{dt})^2 = m(t)\frac{d^2x(t)}{dt^2} \tag{1}$$

with initial conditions:

$$\frac{dx(t)}{dt}(0) = 0 \tag{2}$$

$$x(0) = 0 \tag{3}$$

1.2.2 Nomenclature

Symbol	Definition	Unit
t	Time	s
dt	Time-step	s
$x = x(t)$	Vertical displacement	m
$v = \frac{dx(t)}{dt}$	Vertical velocity	m/s
$a = \frac{dv(t)}{dt}$	Vertical acceleration	m/s^2
$\rho = \rho(x)$	Air density	kg/m^3
$m = m(t)$	Total mass	kg
m_s	Structural mass	kg
m_p	Propellant mass	kg
m_l	Payload mass	kg
g	Gravitational acceleration	m/s^2
T	Thrust	N
I_{sp}	Specific impulse	s
C_d	Drag coefficient	1
d	Diameter	m
A	Cross-sectional area	m^2

Table 1: Relevant parameters

1.2.3 Approximations

Firstly, this physics model assumes that the rocket is launched straight upwards to reduce complexity. Secondly, the flat Earth model is also assumed for simplicity. For air density, an exponential approximation that change with height can be done according to [2]:

$$\rho(x) \approx \rho_0 e^{\frac{-x}{H_n}} \tag{4}$$

where $\rho_0 = 1.225\ kg/m^3$ is the atmospheric air density at sea level and H_n is the height scale of the exponential fall for density, which is 10.4 km for Earth.

Furthermore, gravity, thrust and the aerodynamic drag coefficient C_d will be held constant. Lastly, horizontal winds (causing lift forces) and rocket spin (which take energy and consequently affects the final apogee altitude) will be neglected.

1.3 Aims and objectives

The aim of this project is to study how much rocket propellant is needed to reach the *Kármán line* - the altitude of 100 km where outer space begins [3]. This will be done by solving (1) in Python by both approximating it with Explicit Euler method and Fourth Order Runge-Kutta method (RK4). Furthermore, the report will examine the accuracy of the numerical solution by analyzing the influence of time-step dt for both numerical integrators.

The simulation procedure will be done according to:

1. Consider a two-stage model rocket. Stage 1 has parameters: $m_s = 2 \times 10^3, m_l = 6 \times 10^3, T = 200 \times 10^3, I_{sp} = 340, C_d = 0.75, d = 0.2$. Stage 2 has parameters: $m_s = 1 \times 10^3, m_l = 2 \times 10^3, T = 200 \times 10^3, I_{sp} = 380, C_d = 0.75, d = 0.2$. Let the propellant ratio between the two stages be 4:1. By implementing RK4, how much propellant will be needed for each stage for $dt = 0.01$?

2. Let the propellant ratio between the stages be 3:1, 1:1 and 1:4. How does the different ratios affect the results?

3. Is there a limit to how much propellant mass that can be used?

4. Using derived values of the propellant mass from 1), how does different values on dt affect the results for the Euler method and RK4 method respectively?

5. Using derived values of the propellant mass from 1), calculate the absolute error between the two methods for different dt to further demonstrate time-step influence.

2 Methods

2.1 Mass

Except from the rocket's altitude and velocity, the rocket's propellant mass is also a variable that changes with time. According to Tsiolkovsky rocket equation [4], the mass flow rate of the expended propellant can be calculated according to:

$$\frac{dm}{dt} = \frac{T}{I_{sp}g} \tag{5}$$

That implies that the current, total mass of the rocket at time t can be calculated as:

$$m(t) = m_0 - \frac{dm}{dt}(t) \tag{6}$$

where $m_0 = m_s + m_p + m_l$ is the initial mass including propellant, also known as wet mass.

2.2 Numerical integration

To solve the differential equation numerically, equation (1) must be separated into a system of two first order differential equations that must be solved simultaneously. These governing ODEs are:

$$\frac{dv}{dt} = \frac{1}{m}(T - mg - \rho C_d A v^2) = f(x, v, t) \tag{7}$$

$$\frac{dx}{dt} = v = g(v) \tag{8}$$

with initial conditions:

$$v(0) = 0 \tag{9}$$

$$x(0) = 0 \tag{10}$$

The calculus problem has now transformed into an algebraic problem, which can be solved by discretizing the solution domain and applying a numerical integrator.

2.3 Discretization

The continuous time interval must be discretized with equally spaced time-steps dt. This means that the variable t will be replaced with a time index, indicated by the subscript i, so that the time depending variables x, v and m are defined at $t_0, t_1, t_1, ..., t_i, ...$. Hence, we get:

$$t \to t_i$$

$$x(t) \to x_i$$

$$v(t) \to v_i$$

$$m(t) \to m_i$$

2.4 Explicit Euler method

The Explicit Euler method is the most basic numerical integrator, where the new solution is obtained from the previous solution in the time interval. It is easily implemented according to:

$$y_{i+1} = y_i + dtf_i$$

Applied on (7) and (8), we get:

$$v_{i+1} = v_i + dt \times f(x_i, v_i, t_i) \tag{11}$$

$$x_{i+1} = x_i + dt \times g(v_i) \tag{12}$$

2.5 Fourth Order Runge-Kutta method

The RK4 method is a commonly used higher order method, where the new solution is obtained by averaging increments over four points in the time interval. It is implemented according to:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where:

$$k_1 = dt \times f(t_i, y_i)$$

$$k_2 = dt \times f(t_i + \frac{dt}{2}, y_i + \frac{k_1}{2})$$

$$k_3 = dt \times f(t_i + \frac{dt}{2}, y_i + \frac{k_2}{2})$$

$$k_4 = dt \times f(t_i + dt, y_i + k_3)$$

Applied on (7) and (8), we get:

$$v_{i+1} = v_i + \frac{1}{6}(a_1 + 2a_2 + 2a_3 + a_4) \tag{13}$$

$$x_{i+1} = x_i + \frac{1}{6}(b_1 + 2b_2 + 2b_3 + b_4) \tag{14}$$

where:

$$a_1 = dt \times f(x_i, v_i, t_i)$$

$$a_2 = dt \times f(x_i + \frac{b_1}{2}, v_i + \frac{a_1}{2}, t_i + \frac{dt}{2})$$

$$a_3 = dt \times f(x_i + \frac{b_2}{2}, v_i + \frac{a_2}{2}, t_i + \frac{dt}{2})$$

$$a_2 = dt \times f(x_i + b_3, v_i + a_3, t_i + \frac{dt}{2})$$

$$b_1 = dt \times g(v_i)$$

$$b_2 = dt \times g(v_i + \frac{a_1}{2})$$

$$b_2 = dt \times g(v_i + \frac{a_2}{2})$$

$$b_4 = dt \times f(v_i + a_3)$$

3 Results and analysis

Following results answers the questions under subsection *"Aims and objectives"*.

1. A propellant mass ratio of 4:1 implicates that stage 1:s propellant mass m_1 make up 80% of the total and stage 2:s mass m_2 make up 20% of the total. When looping through a list of masses and defining above conditions to reach an apogee altitude of 100 km, following results were yielded:

Ratio	m_1 $[kg]$	m_2 $[kg]$	Total propellant mass $[kg]$
4:1	8352.0	2088.0	10440.0

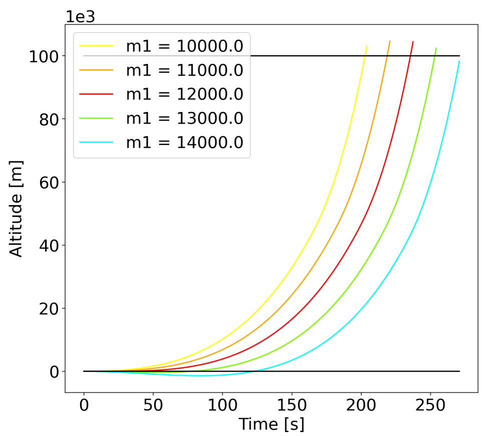
Table 2: Propellant mass distribution between stage 1 and stage 2.

2. When changing the propellant ratio and optimizing the mass distribution so that it would result in an apogee altitude of 100 km, the mass distributed between the stages according to *Table 3*. What can be observed is that less total propellant mass is needed when $m_2 > m_1$ and that this value seems to decrease as the ratio between m_2 and m_1 increases.

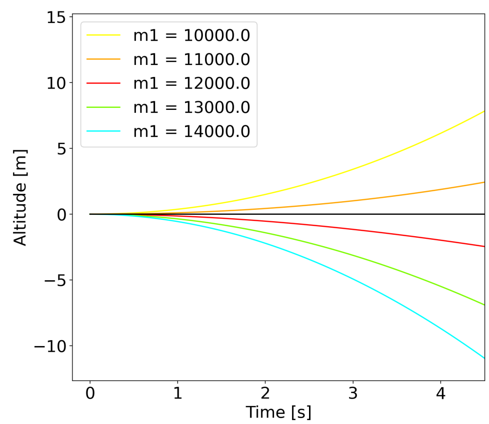
Ratio	$m_1[kg]$	$m_2[kg]$	Total propellant mass $[kg]$
3:1	6967.5	2322.5	9290.0
1:1	3330.0	3330.0	6660.0
1:4	1084.0	4336.0	5420.0

Table 3: Propellant mass distribution between stage 1 and stage 2 for different $m_1 : m_2$ ratios.

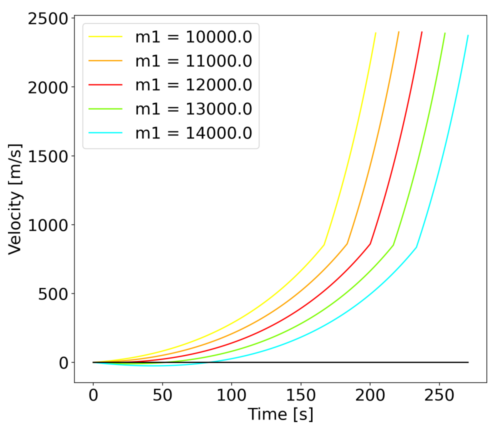
3. Observed from previous answer, the total propellant mass decreased for increasing mass ratio. To test the mass limit, m_2 where held constant of a value of 2088.0 kg, while m_1 was set to 10,000.0 kg and then increased with a mass-step of 1000.0 kg for every iteration. The results can be seen in figure $(a), (b), (c)$ and (d).



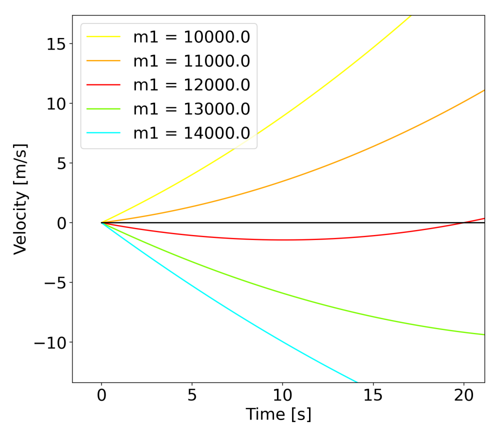
(a) Altitude dependence on time for different stage 1 masses, m_1.



(b) Zoom in of figure (a).



(c) Velocity dependence on time for different stage 1 masses, m_1.



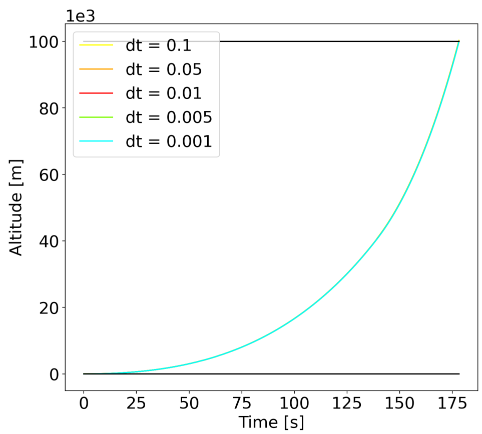
(d) Zoom in of figure (c).

What can be observed is that both the altitude and the velocity at the beginning of the launch becomes negative when m_1 becomes larger than 11,000.0 kg. This is physically impossible, yielding a total propellant mass limit of roughly $\approx$ 13,000.0 kg.

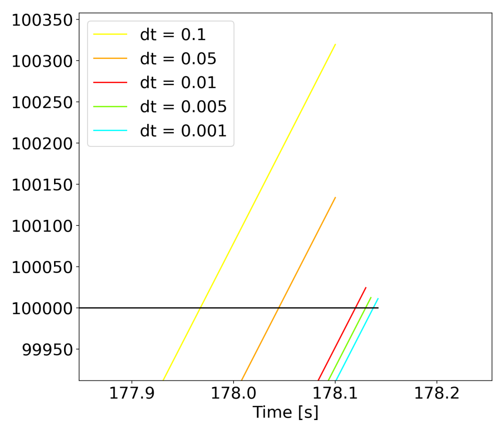
What also can be observed is that the final value of the curves shift in the x-axis as the propellant mass increases, which indicates that it takes longer time to reach the apogee altitude and velocity as heavier as the rocket gets. This aligns with what would happen in reality, as our physical intuition says that heavier objects are harder to move.

4. When implementing both Fourth Order Runge-Kutta and Explicit Euler for different dt, a clear influence of the time-step could be seen for both integrators. What can be observed from figure (f) and (j) is that the rocket's altitude came closer and closer to the desired apogee altitude of 100 km as dt decreased. It can also be observed that the rocket reached the desired altitude quicker for RK4 than for Explicit Euler for larger dt and converged to almost the same solution as dt got smaller.

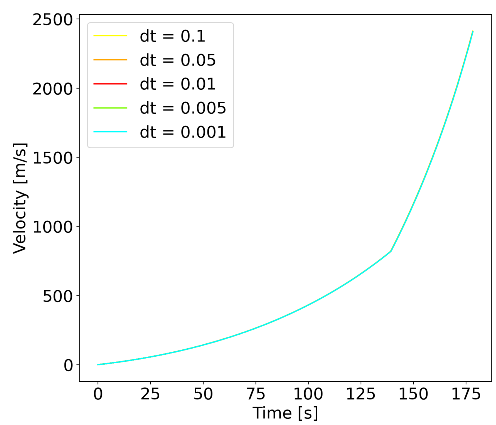
   Remark! In figure (f) and (j), the y-axis misses a label because the numbers became too large for a label to be included and for the figure to still remain the same sizes as the rest of the figures. However, as figure (f) and (j) is a zoom in of figure (e) and (i) respectively, it is very obvious that the their y-axis represents the rocket's altitude in meters as well.



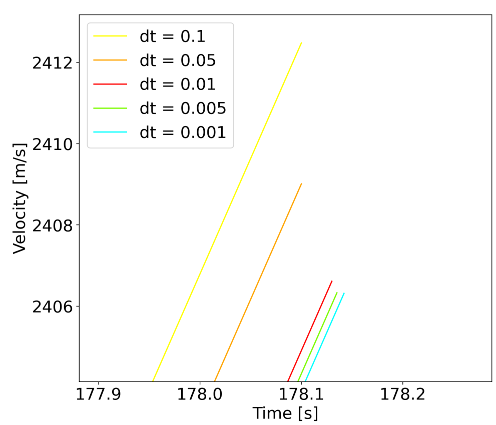
(e) Altitude dependence on time for different time-steps for RK4. The propellant mass distribution between the stages were optimized such that the rocket would reach an apogee altitude of 100 km, represented by the black line at the top of the plot.



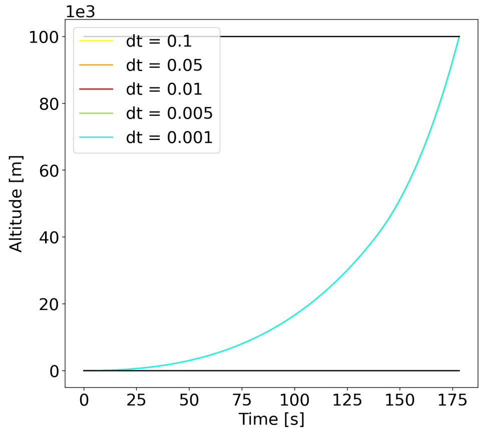
(f) Zoom in of figure (e).



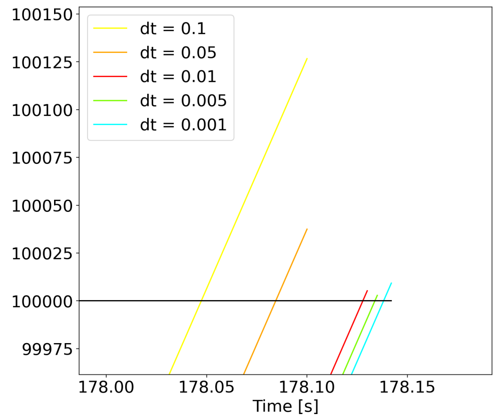
(g) Velocity dependence on time for different time-steps for RK4. As observed from the plot, a "breaking-point" can be seen at $t \approx 140$ s, which indicate the detachment of the two stages.



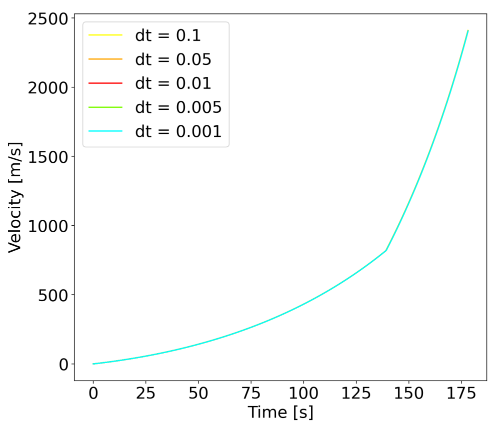
(h) Zoom in of figure (g).



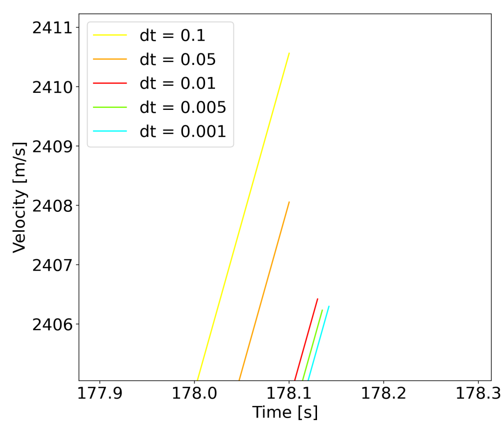
(i) Altitude dependence on time for different time-steps for Explicit Euler. The propellant mass distribution between the stages were optimized such that the rocket would reach an apogee altitude of 100 km, represented by the black line at the top of the plot



(j) Zoom in of figure (i).

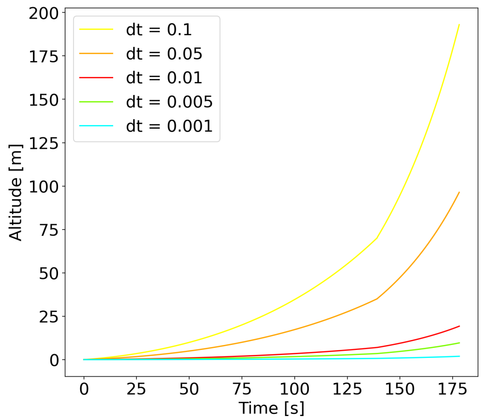


(k) Velocity dependence on time for different time-steps for Explicit Euler. As observed from the plot, a "breaking-point" can be seen at $t \approx 140$ s, which indicate the detachment of the two stages.

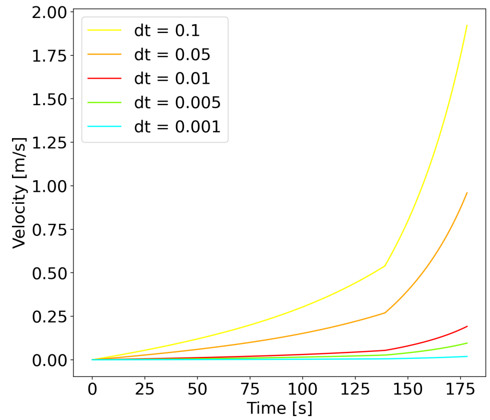


(l) Zoom in of figure (k).

5. To calculate the absolute error, each value of the altitude and velocity for RK4 and Explicit Euler respectively were subtracted from each other. Following results can be seen in figure (m) and (n). With these results, we can further demonstrate the time-step influence for the two different numerical integrators as it is shows that they differ for larger dt (with a total difference of 200 m in altitude when $dt = 0.1$!) and converge to roughly the same answer as dt decreases.



(m) Absolute altitude error.



(n) Absolute velocity error.

4 Discussion

Normally, less propellant will be needed in the second stage as this part of the rocket have less overall weight to carry into space and often have an engine which is built to generate more thrust in comparison to the first stage. However, there is not theoretically impossible to switch the ratios and let the propellant mass be less in the first stage, as it will only detach earlier and let the second stage travel a longer mileage. This turned out to be true when running the simulation, but what seemed quite odd is that the total propellant mass decreased as m_2 got larger and m_1 smaller. This can be the result of poor implementation and/or the fact that the parameters of the stages were made up, yielding inadequate results.

In regard to the propellant mass limit, it is only reasonable that the rocket cannot be too heavy as it adds weight and the rocket has to compensate it with more generated thrust or a lower apogee altitude and velocity. As the thrust were held at a constant value for this project's hypothetical rocket, the latter can be seen in figure (a) and (c).

Regarding the accuracy of the results, a first thought were to use data from a real two-staged rocket and use the relative error to determine how well the numerical integrators did. However, far too many approximations have been made for the relative error between the real and numerical value to be of any significance. That is why accuracy were assessed by studying how the numerical solution behave when taking shorter and shorter time-steps.

What can be concluded is that RK4 performs better than the Explicit Euler for larger time-steps dt. This result aligns with the fact that Explicit Euler is first order accurate, while RK4 is fourth order accurate. The disadvantage with RK4 is that it usually is the most computational demanding numerical integrator, as it requires four derivative function evaluation steps in comparison with Explicit Euler's single one function evaluation. This is often worth it for the high level of accuracy. However, it can be observed from figure (m) and (n) that the Explicit Euler method with a sufficiently small time-step yields a good numerical solution for the rocket trajectory equations.

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