~~CS~~ 260 W'18 HW#1

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Last First Score: ______

CS 260 Winter 2018

Machine Learning Algorithms

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Homework #1 - Math Review

Due Date: Monday, January 15, 2018.

1. Assume that n is the number of people in a group where $n \geq 2$

(a) What is the probability that at least two people in this group share the same birthday?

Prob that two people have different b-days:

$$= \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right)$$

Prob that three people have different b-days:

$$= \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right)$$

Prob that three people DONT all have different b-days:

$$1 - \left(\frac{365}{365}\right) \times \left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right)$$

$$1 - \left(\frac{365 \times 364 \times 363 \ldots}{365 \times 365 \times 365 \ldots}\right)$$

$$= \boxed{1 - \frac{365!}{365^n \times (365-n)!}}$$

* Problem calculated excluding leap year.

(b) What is the minimum size of the group such that the probability is at least 0.5?

~~0.8247~~

$$0.5 \geq 1 - \left(\frac{365!}{365^n \times (365-n)!}\right)$$

Trial + Error → $\boxed{n=23}$, $\approx \cancel{80}$ 0.573

2. If your friend flipped a fair coin three times and told you that one of the tosses resulted in head. What is the probability that all three tosses resulted in heads?

Possible combinations: HHH

HTT

HHT

HT~~T~~H

THT

THH

TTH

=7 combinations

$\boxed{p = 1/7}$

3. A program selects a random integer X as follows: a random bit is first generated uniformly. If the bit is 0, X is drawn uniformly from $\{0, 1, \ldots, 7\}$; otherwise, X is drawn uniformly from $\{0, -1, -2, -3\}$. If we get an X from the program with $|X| = 1$, what is the probability that X is negative?

$$P(X \text{ is negative} \mid |X|=1) = \frac{P(|X|=1 \mid X \text{ is negative}) \times P(X \text{ is neg})}{P(|X|=1 \mid X \text{ is pos}) \times P(X \text{ is pos}) + P(|X|=1 \mid X \text{ is neg}) \times P(X \text{ is neg})}$$

$$= \frac{\frac{1}{4}(\frac{1}{2})}{\frac{1}{8}(\frac{1}{2}) + \frac{1}{4}(\frac{1}{2})} = \boxed{2/3}$$

4. The Bruin Walk Open, a tennis tournament takes place every years, has a Women's Doubles event with 16 seeded pairs. Unfortunately, it is unseasonably cold and wet, and many players are down with flu. In particular, 4 of the seeded players are sick so that they and their partner cannot play. What is the expected number of seeded pairs remaining in the tournament? Assume that any of the $\binom{32}{4}$ possible combinations of 4 seeded players is equally likely to be sick.

P(one player from 4 different teams gets sick) =
$(16/16 \times 15/16 \times 14/16 \times 13/16) \times 12$ seeded pairs remaining

$= \frac{43680}{65536} \times 12 = \frac{524,160}{65536}$

P(both players from 1 team gets sick and one player from 2 different teams gets sick) =
$(16/16 \times 15/16 \times 14/16 \times 1/16) \times 6$ combinations $\times 13$ seeded players remaining

$= \frac{3360}{65536} \times 6 \times 13 = \frac{262,080}{65536}$

P(both players from 2 teams gets sick) =
$(16/16 \times 15/16 \times 1/16 \times 1/16) \times 6$ combinations $\times 14$ seeded players remaining

$= \frac{240}{65536} \times 6 \times 14 = \frac{20,160}{65536}$

$$\frac{524{,}160}{65{,}536} + \frac{262{,}080}{65{,}536} + \frac{20{,}160}{65{,}536} = \frac{806{,}400}{65{,}536}$$

$$\approx \boxed{12.305}$$

5. Let X be a Gaussian vector with

$$E[X] = \begin{pmatrix} 10 \\ 5 \end{pmatrix}, \quad cov(X) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

What is the expression for the pdf of X that doesn't use matrix notation? i.e., if $X = [x_1, x_2]$ Write the joint pdf $P(x_1, x_2)$

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right)$$

$$= \frac{1}{2\pi(\sqrt{2})(\sqrt{1})\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left[\frac{(x-10)^2}{2} + \frac{(y-5)^2}{1} - \frac{2\rho(x-10)(y-5)}{(\sqrt{2})(\sqrt{1})}\right]\right)$$

$$= \boxed{\frac{1}{2\pi\sqrt{2}\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left[\frac{(x-10)^2}{2} + \frac{(y-5)^2}{1} - \frac{2\rho(x-10)(y-5)}{\sqrt{2}}\right]\right)}$$

6. What are the eigenvalues and eigenvectors of

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}?$$

$$A-\lambda I = \begin{pmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{pmatrix}$$

$\cancel{\det(A-\lambda I)} =$

$$\det(A-\lambda I) = (3-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ -1 & 1-\lambda \end{vmatrix}$$

$$+ 1\begin{vmatrix} 2 & 4-\lambda \\ -1 & -1 \end{vmatrix}$$

$$= (3-\lambda)((4-\lambda)(1-\lambda)-2(-1)) - 1(2(1-\lambda)-2(-1))$$

$$+ 1(2(-1)-(4-\lambda)(-1)) = 16-20\lambda+8\lambda^2-\lambda^3$$

$$= \lambda^3-8\lambda^2+20\lambda-16 = 0 \text{ when } \lambda = 2$$

$$(\lambda-2)(\lambda^2-6\lambda+8)$$

$\cancel{\lambda = 2-2-6}$

$$\lambda = \frac{-(-6) \pm \sqrt{(-6)^2-4(1)(8)}}{2(1)} = 2,4$$

$\boxed{\text{eigenvalues}: 2,4}$

$$\lambda = 2: A-2I =$$

$$\begin{pmatrix} 3-2 & 1 & 1 \\ 2 & 4-2 & 2 \\ -1 & -1 & 1-2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix}$$

$\rightarrow$ swap $R_1 + R_2$: $\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$

$\rightarrow R_2 = R_2 - \frac{1}{2}R_1 : \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$

$\rightarrow R_3 = R_3 + \frac{1}{2}R_1 : \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\rightarrow$ reduce $R_1 : \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x + y + z = 0$

$x = \cancel{0} - y - z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix}$$

$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ when $y = 1$ and $\cancel{y}\ z = 1$

$\lambda = 4 : A - 4I =$

$$\begin{pmatrix} 3-4 & 1 & 1 \\ 2 & 4-4 & 2 \\ -1 & -1 & 1-4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix}$$

$\rightarrow$ swap $R_1 + R_2 : \begin{pmatrix} 2 & 0 & 2 \\ -1 & 1 & 1 \\ -1 & -1 & -3 \end{pmatrix}$

$\rightarrow R_2 = R_2 + \frac{1}{2}R_1 : \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ -1 & -1 & -3 \end{pmatrix}$

$\rightarrow R_3 = R_3 + 1/2\, R_1$: $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix}$

$\rightarrow R_3 = R_3 + R_2$: $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$\rightarrow$ reduce R_1 : $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$x + z = 0$; $y + 2z = 0$

$x = -z$; $y = -2z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -2z \\ z \end{pmatrix}$$

$\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ when $z = 1$

$$\boxed{\begin{array}{l} \text{Eigenvectors for } \lambda = 2: \\ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ \text{Eigenvector for } \lambda = 4: \\ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \end{array}}$$

7. Consider $x \in \mathbb{R}^d$ and some $u \in \mathbb{R}^d$ with $\|u\| = 1$.

- (a) What is the maximum of $u^T x$?

max at a zero angle, $u^T x$ can be written as a ~~the~~ dot product

$\|x\| \|u\| \cos\theta = \|x\|(1)(1) = \boxed{\|x\|}$

(b) What u results in the maximum value?

When u and v subtend a zero angle (parallel) with the same orientation

(c) What is the minimum value of $u^T x$?

max at a 180° angle

$\|x\|(1)(-1) = \boxed{-\|x\|}$

(d) What u results in the minimum value?

when u and v are diametrically opposed (opposing directions)

(e) What is the minimum value of $|u^T x|$?

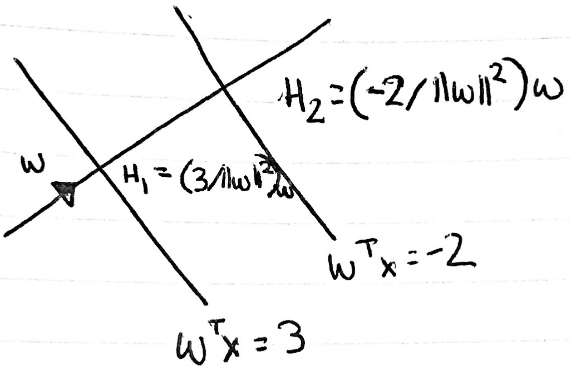
~~min~~ The minimum value is 0 at 90° since that is the smallest absolute value.

(F) What u results in the minimum value?

The u that results in the minimum value of $|u^T x|$ would be 0 (orthogonal) since $\|x\| \|0\| \cos\theta = 0$

8. Consider two parallel hyperplanes in R^d: $H1: w^T x = +3$, $H2: w^T x = -2$, where w is the norm vector. What is the distance between H_1 and H_2?

R^d:



$$d = \frac{|3-(-2)|}{\|w\|_2} = \boxed{\frac{5}{\|w\|_2}}$$

9. Let $f(x,y) = xy$, $x(u,v) = \cos(u+v)$, $y(u,v) = \sin(u-v)$. What is the $\partial f/\partial v$?

$f(u,v) = \cos(u+v)\sin(u-v)$

$$\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}(\cos(u+v))\sin(u-v) + \frac{\partial}{\partial v}(\sin(u-v))\cos(u+v)$$

$$= -\sin(u+v)(1)\sin(u-v) + (-\cos(u-v))(1)\cos(u+v)$$

$$= -\sin(u+v)\sin(u-v) - \cos(u-v)\cos(u+v)$$

$$= -(\sin(u+v)\sin(u-v) + \cos(u-v)\cos(u+v))$$

$$= -\cos(u+v-(u-v))$$

$$= \boxed{-\cos(2v)}$$

10. (a) Let $f(x) = \ln(1+e^{-2x})$. What is $\frac{df(x)}{dx}$?

$$= \frac{1}{1+e^{-2x}}(-2e^{-2x})$$

$$= \boxed{\frac{-2e^{-2x}}{1+e^{-2x}}}$$

(b) Let $g(x,y) = e^{x} + e^{2y} + e^{3xy^2}$. What is $\frac{\partial g(x,y)}{\partial y}$?

$$= 0 + 2e^{2y} + 6xye^{3xy^2}$$

$$= \boxed{2e^{2y} + 6xye^{3xy^2}}$$