Representation Theory of Noetherian Categories

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Chapter 1

Introduction

In this paper, we examine the main results made by Sam and Snowden in their paper titled "A Gröbner Approach to Combinatorial Categories" by studying the category of functors from a small category $\mathcal{C}$ to the category of Modules over a ring, $\mathbf{k}$, $\mathrm{Mod}_{\mathbf{k}}$. The functor category from $\mathcal{C}$ to an arbitrary category $\mathcal{D}$ is the category whose objects are all functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are all natural transformations between such functors. In particular, when we let $\mathcal{D}$ be the category $\mathrm{Mod}_{\mathbf{k}}$, the functor category from $\mathcal{C}$ to $\mathrm{Mod}_{\mathbf{k}}$ is called the **representation category** of $\mathcal{C}$, denoted $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$.

If we consider any group G, it is possible to view G as a category $\mathcal{C}_G$ where the only object of $\mathcal{C}_G$ is G, and the morphisms of $\mathcal{C}_G$ is the underlying set of G, where composition of morphisms is given by the binary operation on G. In this way, we can see that categories are wonderful ways of generalizing abstract structures such as sets, rings, and groups. A **representation** of a group G is a homomorphism ϕ from G to the group of automorphisms of a $\mathbf{k}$-module (or vector space, if $\mathbf{k}$ is a field), M, typically denoted (ϕ, M). Representation theory studies how groups behave by *representing* their elements as linear transformations, thus making an abstract algebraic object more concrete by describing its elements in terms of structures that are easy to study and manipulate. In particular, when we think of G as a category, we see that the functor category of $\mathcal{C}_G$ is equivalent to the representations of G. Thus, $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is a generalization of representation theory, and we therefore call such functors $\mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$ **representations** of the $\mathcal{C}$.

One category of particular interest to us is **FI**, the category of finite sets and injections. Representations of **FI** come up in the study of cohomology of configuration spaces in the work of Benson Farb, Thomas Church, Jordan S. Ellenberg, and Rohit Nagpal to encode sequences of representations of symmetric groups.[1]

In the study of categories of representations, a basic question one may ask is whether or not a category is **Noetherian**. We say a category of modules is Noetherian if a submodule of any finitely generated module is finitely generated. The purpose of this paper, in coherence with the work of Sam and Snowden, is to develop general criteria for proving a category is Noetherian. For example, the category **FI** is Noetherian.

A key point in proving Noetherianity comes from the theory of Gröbner bases. In this paper, we explore the methods Sam and Snowden use to develop an analogous study of Gröbner theory on modules and apply it to representations of categories. In particular, classic Gröbner bases can be used to prove that polynomial algebras are Noetherian. Our main result (and proof) are analogous to this result.

All substantial results in this paper, particularly from Chapters 4,5, and 6, come from Sam and Snowden's paper, and thus will not be referenced every time. Please refer to reference [2] for further reading and sources.

Chapter 2

Posets and Noetherianity

2.1 Poset Theory

Definition 2.1.1. A **poset** is a set X together with a binary relation $\leq$ satisfying the axioms of reflexivity, antisymmetry, and transitivity. That is, for all $x, y, z \in X$, $\leq$ must satisfy:

- $x \leq x$
- If $x \leq y$ and $y \leq x$, then $x = y$
- If $x \leq y$ and $y \leq z$, then $x \leq z$.

Elements x, y of a poset X are **comparable** if $x \leq y$ or $y \leq x$. A **chain** in X is a subset $\{x_1, x_2, ...\}$ of X in which any pair of element x_i, x_j are comparable. An **anti-chain** in X is a subset $\{x_1, x_2, ...\}$ of X in which any two distinct elements are incomparable, i.e. $x_i \not\leq x_j$ for all $i \neq j$.

Example 2.1.1. For any poset $(X, \leq)$, the sets $\varnothing$, the empty set, and $\{x\}$, any singleton set, are both chains and anti-chains.

Example 2.1.2. Any subset of $\mathbb{R}$ is a chain, since all elements of $\mathbb{R}$ are comparable.

Example 2.1.3. Let $X = \{1, 2\}$. The power set of X is the set of all subsets of X, $\mathcal{P}(X) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. The poset $(\mathcal{P}(X), \leq)$ where $\leq$ is given by inclusion, is demonstrated in Figure 2.1. We see that the set $\{\{1\}, \{2\}\}$ is an anti chain in $(\mathcal{P}(X), \leq)$ and the set $\{\varnothing, \{1\}, \{1, 2\}\}$ is a chain.

$$\begin{tikzcd}
 & \{1,2\} & \\
\{1\} \arrow[ur] & & \{2\} \arrow[ul] \\
 & \varnothing \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 2.1: $P(x)$ poset diagram for $X = \{1,2\}$

2.2 Noetherian Posets

We say that X satisfies the **ascending chain condition** (ACC) if every ascending chain in X stabilizes. That is, given a chain $x_1 \leq x_2 \leq \ldots$ in X, we have $x_i = x_{i+1}$ for all $i \geq j$, where j is some index greater than zero. The **descending chain condition** (DCC) for a poset X is the dual notion of the ACC on X, which requires that all decreasing sequences in X stabilize.
An **ideal** in X is a subset $I \subseteq X$ such that if $x \in I$ and $x \leq y$ then $y \in I$. There is a natural way to define a poset on subsets of any set by inclusion. That is, given a set X with subsets $J, K \subseteq X$, we say that $J \leq K$ if $J \subseteq K$. We use the notation $\mathcal{J}(X)$ to denote the poset of ideals of X ordered by inclusion. The **principal ideal** generated by $x \in X$ is $\{y|y \geq x\}$. We say that an ideal is **finitely generated** if it is a finite union of principal ideals.

Example 2.2.1. In the poset $(\mathbb{R}, \leq)$, intervals of the form (a, ∞) and $[a, \infty)$ are ideals. A subset of an ideal is not necessarily an ideal. The subset $U = [0,8) \cup (8, \infty) \subseteq [0, \infty)$ is not an ideal since $8 \geq 0$ but $8 \notin U$. In $\mathbb{R}$, ideals of the form (a, ∞) are not finitely generated, because they cannot be written as a finite union of principal ideals.

Definition 2.2.1. A poset X is **Noetherian** if every ideal of X is finitely generated. There are many equivalent ways of saying a poset is Noetherian, which are given by the following proposition.

Proposition 2.2.1. Given a poset X, the following conditions are equivalent:

a. X is Noetherian
b. The poset $\mathcal{J}(X)$ satisfies ACC.
c. Given a sequence $x_1, x_2, \ldots$ in X, there exists $i < j$ such that $x_i \leq x_j$.
d. X satisfies DCC and has no infinite anti-chains.

The proof is left to the reader.

Proposition 2.2.2. Let X be a Noetherian poset and let $x_1, x_2, \ldots$ be a sequence in X. Then there exists an infinite sequence of indices $i_1 < i_2 < \ldots$ such that $x_{i_1} \leq x_{i_2} \leq \ldots$.

Proof. Let I be the set of indices such that $i \in I$ and $j > i$ implies that $x_i \not\leq x_j$. I cannot be infinite, otherwise there would exist some $i < i'$ with $i, i' \in I$ such that $x_i \leq x_{i'}$ by Noetherianity, contradicting the definition of I. Thus I is finite, and letting i_1 be any number larger than all elements of I gives us the sequence $x_{i_1} \leq x_{i_2} \leq \ldots$.

□

Proposition 2.2.3. Let X and Y be posets. Let $\mathcal{F} = \mathcal{F}(X, Y)$ be the set of all order preserving functions $f : X \to Y$ partially ordered by $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$. Then, we have the following:

a. If X is Noetherian and Y satisfies ACC, then $\mathcal{F}$ satisfies ACC.
b. If $\mathcal{F}$ satisfies ACC and X is nonempty, then Y satisfies ACC.
c. If $\mathcal{F}$ satisfies ACC and Y has two distinct comparable elements, then X is Noetherian.

Proof. (a) Assume X is Noetherian and for a contradiction, suppose $\mathcal{F}$ does not satisfy ACC. For $f_i \in \mathcal{F}$, let $f_1 < f_2 < \ldots$ be an ascending chain. Then for each i, we may choose $x_i \in X$ such that $f_i(x_i) < f_{i+1}(x_i)$, and passing to a subsequence gives us that $x_1 \leq x_2 \leq \ldots$ by Proposition 2.2.2. Let $y_i = f_i(x_i)$. Then, since $f_i(x_i) < f_{i+1}(x_i) \leq f_{i+1}(x_{i+1})$, this gives us that $y_1 < y_2 < \ldots$ is an ascending sequence in Y, and thus Y does not satisfy ACC, a contradiction.
(b) Taking the assumptions from (b), we see that Y embeds into $\mathcal{F}$ as the set of constant functions, giving us that Y satisfies ACC since $\mathcal{F}$ does.
(c) Suppose that $\mathcal{F}$ satisfies ACC and that Y contains distinct elements $y_1 < y_2$. Let I be an ideal of X¡ and define the function $\chi_I \in \mathcal{F}$ by

$$\chi_I(x) = \begin{cases} y_2, & x \in I \\ y_1, & x \notin I. \end{cases}$$

This construction gives rise to the function $I \mapsto \chi_I$ which embeds $\mathcal{J}(X)$ into $\mathcal{F}$, and thus $\mathcal{J}(X)$ satisfies ACC. Then by Proposition 2.2.1, X is Noetherian. □

Chapter 3

Category Theory

3.1 Basic Definitions of Category Theory

Definition 3.1.1. A **category** $\mathcal{C}$ consists of collections $\text{Ob}(\mathcal{C})$, the objects of $\mathcal{C}$ and $\text{Mor}(\mathcal{C})$, the morphisms, or arrows, between the objects of $\mathcal{C}$. Each arrow f has a source object x and a target object y where x and y are in $\text{Ob}(\mathcal{C})$, where we write $f : x \to y$ to denote f being an arrow from x to y. The objects and arrows of $\mathcal{C}$ must satisfy the following axioms:

i. for any arrows $f : x \to y, g : y \to z \in \text{Mor}(\mathcal{C})$ there is an arrow

$$g \circ f : x \to z$$

called the composite of f and g,

ii. For each object $x \in \text{Ob}(\mathcal{C})$, the arrow

$$1_x : x \to x$$

exists, and is called the identity arrow of x.

iii. Composition of arrows is associative, that is

$$h \circ (g \circ f) = (h \circ g) \circ f$$

for all $f : x \to y, g : y \to z, h : z \to w$, and

iv. $f \circ 1_x = f = 1_y \circ f$ for all $f : x \to y$.

Thus, in category theory it is not the objects themselves that are of the main interest, but rather the morphisms between the objects that we are most concerned with. A morphism between categories is called a **functor**, and it has the following properties.

Definition 3.1.2. A **functor**

$$F : \mathcal{C} \to \mathcal{D}$$

between categories $\mathcal{C}$ and $\mathcal{D}$ is a mapping of objects to objects and arrows to arrows, such that for x and y in $\mathrm{Ob}(\mathcal{C})$ and arrows f and g in $\mathrm{Mor}(\mathcal{C})$

i. $F(f : x \to y) = F(f) : F(x) \to F(y)$,
ii. $F(1_x) = 1_{F(x)}$,
iii. $F(g \circ f) = F(g) \circ F(f)$.

Definition 3.1.3. If F and G are functors between categories $\mathcal{C}$ and $\mathcal{D}$, then a **natural transformation** $\eta : F \to G$ is a family of morphisms that satisfies the requirements:

i. η associates to every object $x \in \mathrm{Ob}(\mathcal{C})$ a morphism $\eta_x : F(X) \to G(X)$ between objects of $\mathcal{D}$, where η_x is called the component of η at x.
ii. Components of η must satisfy the following commutative diagram for every morphism $f : x \to y \in \mathrm{Mor}(\mathcal{C})$:

$$\begin{array}{ccc} F(x) & \xrightarrow{\eta_x} & G(x) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(y) & \xrightarrow{\eta_y} & G(y) \end{array}$$

That is, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

Definition 3.1.4. A functor $F : \mathcal{C} \to \mathcal{D}$ is **essentially surjective** if each object y in $\mathcal{D}$ is isomorphic to an object $F(x)$ for an object x in $\mathcal{C}$, where two objects x and y in a category $\mathcal{C}$ are **isomorphic** if there exists a morphism $f : x \to y$ which admits a two-sided inverse. In other words, there exists a morphism $g : y \to x$ in $\mathcal{C}$ such that $g \circ f = 1_x$ and $f \circ g = 1_y$.

Definition 3.1.5. A functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence** of categories if F has the following properties:

i. F is full: for any two objects x and y in $\mathrm{Ob}(\mathcal{C})$, the map $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ induced by F is surjective.
ii. F is faithful: for any two objects x and y in $\mathrm{Ob}(\mathcal{C})$, the map $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ induced by F is injective, and
iii. F is essentially surjective.

When there exists such a functor F between categories $\mathcal{C}$ and $\mathcal{D}$, we say that $\mathcal{C}$ is **equivalent** to $\mathcal{D}$.

Example 3.1.1. A very important functor in category theory is the **Forgetful Functor**, $\phi : \mathcal{C} \to \textbf{Set}$. For example, let the category $\mathcal{C}$ be the category of all ordered sets $(S, \leq)$ (where $\leq$ is a given total order on S), with order preserving injections as morphisms. Then ϕ sends every ordered set $(S, \leq)$ to its underlying set S in **Set**, "forgetting" the ordering, and sends morphisms to themselves. Clearly, the forgetful functor $\phi : \textbf{OI} \to \textbf{FI}$, where **OI** is the category of finite ordered sets with order preserving injections as morphisms, and **FI** is the category of finite sets with injections as morphisms, is an essentially surjective functor.

Definition 3.1.6. A category $\mathcal{C}$ is **small** if both the collection of objects and the collection of morphisms of $\mathcal{C}$ are sets. If not, $\mathcal{C}$ is **large**. A category $\mathcal{C}$ is called **locally small** if for all objects, $X, Y \in \mathcal{C}$, the collection $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \{f \in Ob(\mathcal{C}) | f : X \to Y\}$ is a set.

Example 3.1.2. Any finite category is a small category. The category $\textbf{Set}_{\textbf{fin}}$ of finite sets and functions is a *essentially* small category, meaning that it is equivalent to a small category. A category being essentially small allows us to apply many of the same results to it that we could a small category. The category **Set** of all sets is not small, since by Russel's paradox we know the collection of all sets cannot be a set. However, **Set** is locally small, since $\mathrm{Hom}_{\textbf{Set}}(X, Y) = Y^X$ is the *set* of all functions from X to Y.

Definition 3.1.7. Given a category $\mathcal{C}$ and an object x of $\mathcal{C}$, we can define the **slice category** $\mathcal{C}_x$ over object $x \in \mathcal{C}$ as the category with

i. objects as arrows $f : y \to x$ for $y \in Ob(\mathcal{C})$
ii. morphisms as arrows $g : y \to y'$ from $f : Y \to X$ to $f' : y' \to x$ such that $f' \circ g = f$.

$$\begin{tikzcd}
& x & \\
y \arrow[ur, "f"] \arrow[rr, "g"] & & y' \arrow[ul, "f'"']
\end{tikzcd}$$

Figure 3.1: Commutative diagram as an arrow in C_x

We say $\mathcal{C}$ is **directed** if every self map in $\mathcal{C}$ is the identity. Let $|\mathcal{C}|$ denote the set of isomorphism classes in $\mathcal{C}$. If $\mathcal{C}$ is essentially small and directed, then $|\mathcal{C}|$ may be considered as a poset with binary relation denoted $x \leq y$ if there exists morphism $x \to y \in \mathrm{Mor}(\mathcal{C})$. We say $\mathcal{C}$ is **Hom-finite** if $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is finite for all $x, y \in \mathrm{Ob}(\mathcal{C})$.

3.1.1 Yoneda Lemma

Let $\mathcal{C}$ be a locally small category, and let x_0 be an object of $\mathcal{C}$. Then x_0 gives rise to the functor

$$\mathrm{Hom}_{\mathcal{C}}(x_0, -)$$

from $\mathcal{C} \to$ **Set**, which sends an object y of $\mathcal{C}$ to the set of morphism $\mathrm{Hom}_{\mathcal{C}}(x_0, y)$ and sends a morphism $f : x \to y$ to the morphism $f \circ -$ that sends a morphism g in $\mathrm{Hom}_{\mathcal{C}}(x_0, x)$ to the morphism $f \circ g$ in $\mathrm{Hom}_{\mathcal{C}}(x_0, y)$. We call $\mathrm{Hom}_{\mathcal{C}}(x_0, -)$ the functor "represented by x_0". Thus, a functor is **representable** if it is isomorphic to a functor of this form.
Since **Set** is a category we have a good understanding of, instead of trying to understand the category $\mathcal{C}$, perhaps it would be easier to study the categories of all functors of $\mathcal{C}$ $\to$**Set**. This idea leads us to the following important Lemma:

Lemma 3.1.1. (Yoneda) Let F be an arbitrary functors from $\mathcal{C}$ $\to$**Set**. For each object x_0 of $\mathcal{C}$, the set of natural transformations from $\mathrm{Hom}_{\mathcal{C}}(x_0, -)$ to F is isomorphic to $F(x_0)$. In other words,

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(x_0, -), F) \cong F(x_0). \tag{3.1}$$

Proof. Let ϕ be a natural transformation from $\mathrm{Hom}_{\mathcal{C}}(x_0, -) \to F$. To prove the isomorphism of equation 3.1, we will demonstrate that ϕ is completely determined by where its component ϕ_{x_0} sends the identity morphism id_{x_0}. Consider the commutative diagram

$$\begin{tikzcd}
id_{x_0} \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(x_0, x_0) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(x_0, f)"] \arrow[d, "\phi_{x_0}"'] & \mathrm{Hom}_{\mathcal{C}}(x_0, x) \arrow[d, "\phi_x"] \\
\phi_{x_0}(id_{x_0}) = u & F(x_0) \arrow[r, "Ff"] & F(x)
\end{tikzcd}$$

For each morphism $f : x_0 \to x$, we can see from above that $\phi_x(f) = (Ff)(u)$. Since every element $u \in F(x_0)$ will define such a natural transformation, this establishes the isomorphism. [3] $\square$

Studying a category this way is similar to, and in fact generalizes, studying a ring by investigating the modules one can form over that ring. We can think of the ring as analogous to the category $\mathcal{C}$, and the category of modules over the ring as the category of functors $\mathcal{C} \to$ **Set** (we will discuss this concept in more detail in the proceeding chapter).

3.2 Categorical Properties

Definition 3.2.1. An **initial** object of $\mathcal{C}$ is an object $I \in \mathcal{C}$ such that for every object $C \in \mathcal{C}$, there exists a unique morphism $I \to C$. Dually, a **terminal** object is an object $T \in \mathcal{C}$ such that for every object $C \in \mathcal{C}$, there exists a unique morphism $C \to T$.

If an object is both initial and terminal, then it is called a **zero object**. We call a category with a zero object a pointed category.

Example 3.2.1. In **Set**, the only initial object is the empty set, since the empty morphism $\varnothing \to C$ is the unique morphism between the empty set and any other set, $C \in$ **Set**. The terminal objects in **Set** are the singleton sets, sets which contain only one element. Every set C may be mapped uniquely to a singleton $\{*\}$ by mapping every element $c \in C$ to $* \in \{*\}$. Since the initial object is not equal to the terminal objects, **Set** has no zero object.

Example 3.2.2. In the category of groups, **Grp**, every trivial group E_i is a zero object. Indeed every group can be mapped uniquely into E_i uniquely be sending every element to the identity element e_i of E_i, and E_i can be mapped uniquely into any group G by sending e_i to the identity element of G.

Remark. Many properties in category theory have a corresponding **dual** property. Thus, given a statement for a category $\mathcal{C}$, when we switch places of the source and target for each morphism and switch the order of the composition of two morphisms, the resulting statement is the dual statement for the dual category, $\mathcal{C}^{op}$. In the last definition, it is easy to see that being an initial object in a category $\mathcal{C}$ is dual to being a terminal object in the category $\mathcal{C}^{op}$.

Definition 3.2.2. For a category $\mathcal{C}$, the **product** of two objects $A, B \in \mathcal{C}$ is an object $A \times B$ together with the morphisms $\pi_a : A \times B \to A, \pi_b : A \times B \to B$ that satisfy the universal property: for every object X and pair of morphisms $f_a : X \to A, f_b : X \to B$, there exists a unique morphism $f : X \to A \times B$ such that $\pi_b \circ f = f_b$ and $\pi_a \circ f = f_a$.. The dual of a product is a **coproduct**.

Categorical products are generalizations of direct products of groups and Cartesian products of sets, while coproducts are generalizations of disjoint unions of sets and direct sums of abelian groups, modules, and vector spaces. When a category has a **zero object** (an object that is both initial and final), there is a canonical morphism from the coproduct of a set of objects to the product of the same set of objects. In an additive category (e.g., the category of abelian groups) this is the morphism from direct sum to direct product. This morphism is an

$$
\begin{tikzcd}
 & X \arrow[dl, "f_a"'] \arrow[d, dashed, "f"] \arrow[dr, "f_b"] & \\
A & A \times B \arrow[l, "\pi_a"] \arrow[r, "\pi_b"'] & B
\end{tikzcd}
$$

Figure 3.2: Universal mapping property of the product of A and B.

isomorphism when the number of factors is finite. Thus in an additive category, finite direct sums and finite products can be identified. One uses the term **biproduct** as a joint name for both constructions.

Definition 3.2.3. Let $\mathcal{C}$ be a category and $f : x \to y$ a morphism in $\mathcal{C}$. The morphism f is called a **left zero morphism** (dually **right zero morphism**) if for any object w in $\mathcal{C}$ and any morphisms $g, h : w \to x$, $fg = fh$.

$$
\begin{tikzcd}
w \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & x \arrow[r, "f"] & y
\end{tikzcd}
$$

Figure 3.3: Left Zero Morphism

We say a category has **zero morphisms** when, for every two objects a and b in $\mathcal{C}$, there is a fixed morphisms

$$0_{ab} : a \to b$$

such that for all objects x, y, z in $\mathcal{C}$ and all morphisms $f : y \to z, g : x \to y$, we have that $f0_{xy} = 0_{xz}, 0_{yz}g = 0_{xz}$, and $f0_{xy} = 0_{yz}g$. In other words, the following diagram commutes.

$$
\begin{tikzcd}
x \arrow[r, "0_{xy}"] \arrow[d, "g"'] \arrow[dr, "0_{xz}"] & y \arrow[d, "f"] \\
y \arrow[r, "0_{yz}"'] & z
\end{tikzcd}
$$

Figure 3.4: Commutative Zero Morphism Diagram

In order to define the following property, we must consider a category $\mathcal{C}$ which has zero morphisms.

Definition 3.2.4. Let $f : x \to y$ be a morphism in $\mathcal{C}$. We define the **kernel** of f (dually **cokernel**) to be the *equalizer* of f and the zero morphism 0_{xy}, i.e. the kernel of f is the object $\mathrm{Ker}(f)$ along with morphism $k : \mathrm{Ker}(f) \to x$ such that $fk = 0_{\mathrm{Ker}(f)y}$ with the following universal property: Given any morphism $\alpha : k' \to x$ such that $f\alpha = 0_{k'y}$, there exists a unique morphism $u : k' \to \mathrm{Ker}(f)$ such that $ku = \alpha$.

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "k"] & x \arrow[r, shift left, "f"] \arrow[r, shift right, "0_{xy}"'] & y \end{tikzcd}$$

Figure 3.5: Kernel of f

The following definition generalizes the idea of injective maps of sets (dually: surjective maps).

Definition 3.2.5. A morphism $f : x \to y$ in a category $\mathcal{C}$ is said to be a **monomorphism** (dual: **epimorphism**) if for every object z and every pair of morphisms $g, h : z \to x$, we have

$$(fg = fh) \implies (g = h).$$

Remark. The monomorphisms in **Set** are precisely the injective functions and epimorphisms are precisely surjective functions, as stated above. Therefore every isomorphism is both a monomorphism and an epimorphism. However, in a general category, a morphism that is both a monomorphism and epimorphism is not necessarily an isomorphism. Take, for example, the category of rings and ring homomorphisms, **Rng**. The inclusion

$$\mathbb{Z} \overset{\iota}{\hookrightarrow} \mathbb{Q}$$

is a monomorphism and an epimorphism, but is clearly not an isomorphism.

We say a monomorphism $f : a \to b$ is **normal** if it is the kernel of some morphism $g : b \to c$. This concept generalizes the concept of *normal subgroup inclusions*. We say a category is normal if every monomorphism is normal, and **binormal** if all monomorphisms are normal and epimorphisms are conormal.

3.2.1 Abelian Categories

Definition 3.2.6. We say a category $\mathcal{C}$ is **abelian** if

i. it has a zero object,
ii. it has all binary biproducts,

iii. it has all kernels and cokernels, and
iv. it is binormal.

This definition is equivalent to the following definition, which we build up piecewise:

- A category is **preadditive** if all Hom-sets are abelian groups and composition of morphisms is bilinear.
- A preadditive category is **additive** if every finite set of objects has a biproduct.
- An additive category is **preabelian** if every morphism has both a kernel and cokernel.
- Finally, we say a preabilian category is **abelian** if every monomorphism and every epimorphism is normal.

The enriched structure that the second definition gives us is a consequence of the first three axioms of the first definition.

Abelian categories are of particular interest to us because they are very *stable* categories and have many nice features that we will explore in this section. The most common example of an abelian category is **Ab**, the category of all abelian groups, as well as $\mathbf{Ab}_{\text{fin}}$, the category of all finite abelian groups. However, the following examples are most relevant to the content of this paper.

Example 3.2.3. For a ring $\mathbf{k}$, the category of all left (or right) modules over $\mathbf{k}$, denoted $\text{Mod}_{\mathbf{k}}$, is an abelian category. We check the axioms of an abelian category:

- The zero object in $\text{Mod}_{\mathbf{k}}$ is the **zero module**, O, since for any module M there are unique ring homomorphisms $\iota_x : O \to M$ and $0_x : M \to O$.
- For any two modules M, N in $\text{Mod}_{\mathbf{k}}$, the biproduct generalizes the notion of the **direct sum** $M \oplus N$ which is defined for all non empty modules.
- For any module homomorphism $f : M \to N$, the kernel of f is the set of all elements of M which get mapped to zero. It is a submodule of M and thus an object of $\text{Mod}_{\mathbf{k}}$. The cokernel of f is defined by the module $N/\operatorname{Im}(f)$.
- In $\text{Mod}_{\mathbf{k}}$, monomorphisms are precisely injective module homomorphisms and epimorphisms are precisely surjective module homomorphisms. In addition, every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel. From this, binormalcy of $\text{Mod}_{\mathbf{k}}$ follows.

Example 3.2.4. For a left-Noetherian ring $\mathbf{k}$, the category of finitely generated left modules over $\mathbf{k}$ is an abelian category.

Lemma 3.2.1. If $\mathcal{C}$ is a small category and $\mathcal{A}$ is an abelian category, then the category of all functors from $\mathcal{C}$ to $\mathcal{A}$ is an abelian category.

In particular, the category of representations from a small category $\mathcal{C}$ to $\mathrm{Mod}_{\mathbf{k}}$ is abelian. These categories will become our focus for the remainder of the paper.

Chapter 4

Representations of Categories

Just as groups are seen as an abstract study of symmetries and permutations, categories are thought of as the study of functions, or transformations. A representation of a group is a way of thinking of permutations as linear transformations, and thus a representation of a category allows us to view abstract transformations as concrete linear transformations.

4.1 Module Theory

4.1.1 Basic Definitions and Theory

The focus of this thesis is the study of representations of a category, which, as we will see in the next section, are modules. Thus, it will behoove us to examine some important properties of modules before proceeding. For a module M over a ring R, we will refer to M as an R-module.

Definition 4.1.1. Let M be an R-module. We say that M is a **free module** if it has a basis, i.e. a generating set of linearly independent elements of M.

Given a free module, the cardinality of its basis is called the **rank** of the module.

Example 4.1.1. For a ring R, R is a free module of rank 1 over itself, with any unit element of R as the basis.

Definition 4.1.2. An R-module, M, is **finitely generated** if it has a finite generating set. That is, M is finitely generated if there exists a finite set of elements $A = \{a_1, ..., a_n\}$ in M such that for any x in M, there exist $r_1, ..., r_n$ in R with $x = r_1a_1 + ... + r_na_n$. The set A is called the **generating set** of M.

If a module is generated by one element, it is called a **cyclic module**.

Lemma 4.1.1 (Structure Theorem for Finitely Generated Modules over a PID)**.** Every finitely generated module over a principal ideal domain is the direct sum of cyclic submodules (submodules generated by a single element).

It is important to note that the generating set A of a finitely generated R-module M is not necessarily a basis for M, as its elements are not necessarily linearly independent over R. To see an example of this, consider $\mathbb{Z}_n$ as a module over Z for any $n \geq 1$. $\mathbb{Z}_n$ is finitely generated; in fact, it is generated by a single element, $\{1\}$. However M is not free, since for any $g \in \mathbb{Z}_n$, $ng = 0$.
This observation motivates the following proposition.

Proposition 4.1.1. An $R-$module M is finitely generated if and only if it is a quotient of a finitely generated free module.

Proof. Assume that M is a quotient of the finitely generated free module F. Let $q : F \twoheadrightarrow M$ be the quotient homomorphism, and let $\{x_1, ..., x_n\}$ be a basis of F. Then $\{q(x_1), q(x_2), ..., q(x_n)\}$ generates M, making M finitely generated. Conversely, suppose M is generated by $\{a_1, ..., a_n\}$. Then M is the quotient of the free module $\oplus_{i=1}^{n} R$. □

4.1.2 Properties of Noetherian Modules

In this section, we will address some useful properties of Noetherian modules that will carry over to Noetherian representations in future sections.

Definition 4.1.3. A module M is a **Noetherian module** if it satisfies the ascending chain condition on its submodules, where the submodules are partially ordered by inclusion.

Proposition 4.1.2. Let M be an R-module. The following are equivalent:

a. M satisfied the ascending chain condition on its submodules
b. Every submodule of M is finitely generated.

Proof. First, assume M satisfied ACC on its submodules, and assume that N is a submodule of M which is not finitely generated. Thus by our assumption $N \neq 0$, and for $n_1 \in N$, $\langle n_1 \rangle \subsetneq N$. So there exists an element $n_2 \in N - \langle n_1 \rangle$, and we may thus construct inductively a sequence of element $n_1, \ldots, n_{\mathbf{k}} \in N$ such that

$$\langle n_1 \rangle \subsetneq \langle n_1, n_2 \rangle \subsetneq \ldots \subsetneq \langle n_1, \ldots, n_{\mathbf{k}} \rangle.$$

Since N is not finitely generated, $\langle n_1, \ldots, n_{\mathbf{k}} \rangle \subsetneq N$, so there exists $n_{k+1} \in N - \langle n_1, \ldots, n_{\mathbf{k}} \rangle$. Thus there exists an infinite ascending chain of submodules in M which never stabilizes, which contradicts our assumption.

Conversely, assume that every submodule of M is finitely generated and let $N_1 \subseteq N_2 \subseteq \ldots$ be an ascending chain of submodules. Let $N = \cup N_i$, which, as a submodule of M, is finitely generated. Thus $N = \langle n_1, n_2, \ldots, n_{\mathbf{k}} \rangle$. For each i, let m_i be such that $n_i \in N_{m_i}$, and let $m = \max\{m_1, \ldots, m_{\mathbf{k}}\}$. Thus, we have that $N \subseteq N_m \subseteq N_{m+k} \subseteq \cup N_i \subseteq N$, so $N = N_m = N_{m+k}$ for all $\mathbf{k}$. Thus, the chain $N_1 \subseteq N_2 \subseteq \ldots$ stabilizes after finitely many steps. □

The following theorems are common results of Noetherian modules that we will want to keep in mind as we further explore Noetherianity in further sections. Since the results are standard, some proofs are left to the reader.

Theorem 4.1.2. If M is a Noetherian R-module, then every submodule S of M is Noetherian.

Proof. By definition 4.1.3, since every submodule of S is also a submodule of M, all submodules of S are finitely generated. □

Theorem 4.1.3. If M is a Noetherian $R - module$ then every quotient module M/N is Noetherian.

Proof. Every submodule of M/N will be of the form L/N where $N \subset L \subset M$. Then since M is Noetherian, L is finitely generated and the quotient homomorphism will map its generators to generators of L/N. □

Theorem 4.1.4. Let M be an R-module and N be a submodule. Then M is Noetherian if and only if N and M/N are Noetherian.

Proof. If M is Noetherian then N is Noetherian and by Theorem 4.1.3 M/N is Noetherian. Conversely, suppose N and M/N are Noetherian and let L be a submodule of M. The image of L in M/N, denoted I, and $L \cap N$ are both finitely generated. Let $x_1, ..., x_m \in L$ generate I and $y_1, \ldots, y_n$ generate $L \cap N$. Then for any $x \in L$ we have $x \equiv r_1 x_1 + \ldots + r_{\mathbf{k}} x_{\mathbf{k}} \mathrm{mod} N$ for some $r_i \in R$, and thus $x - \sum r_i x_i \in L \cap N$. So $x - \sum r_i x_i = \sum s_j y_j$ for $s_j \in R$, and $x = \sum r_i x_i + \sum s_j y_j$. Therefore L is spanned by $x_1, \ldots, x_m, y_1, \ldots, y_n$. □

Theorem 4.1.5. If M and N are Noetherian R-modules then their direct sum $M \oplus N$ is a Noetherian R-module.

Proof. Apply Theorem 4.1.4 to the module $M \oplus N$ and the submodule $M \oplus 0 \cong M$, where $(M \oplus N)/(M \oplus 0) \cong N$. □

Theorem 4.1.6. If $M_1, ..., M_{\mathbf{k}}$ are Noetherian R-modules, then $M_1 \oplus ... \oplus M_{\mathbf{k}}$ is a Noetherian R-module.

Proof. This theorem is a direct result of applying induction on $\mathbf{k}$ and applying the results from Theorem 4.1.5 □

4.2 Noetherian Categories

4.2.1 Finitely Generated Representation

We now have the tools needed to define a representation of a category as follows.

Definition 4.2.1. A **representation** of $\mathcal{C}$, also known as a $\mathcal{C}$-**module** over a ring $\mathbf{k}$, is a functor $\mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$. A functor $H : \mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$ is a **subrepresentation** of a representation $F : \mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$ if H is a subfunctor of F. That is, it must satisfy the following conditions:

i. For all objects $x \in \mathcal{C}$, $H(x) \subseteq F(x)$, and
ii. For all arrows $f : x \to y \in \mathcal{C}$, $H(f)$ is the restriction of $F(f)$ to $H(x)$.

A map between representations is a natural transformation. When we say an **element** of a representation M, we mean an element of $M(x)$ for some object $x \in \mathrm{Ob}(\mathcal{C})$.
Consider a representation $M : \mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}(\mathcal{C})$, and a set $S = \{s_1, ..., s_{\mathbf{k}}\}$ of elements of M. The subrepresentation of M **generated by the set** S, denoted $\langle S \rangle$, is the smallest representation of M that contains S. The elements of $\langle S \rangle$ are all the elements of M that can be obtained from elements of S by applying morphisms of $\mathcal{C}$ and taking $\mathbf{k}$-linear combinations. If there exists some finite set S' for which $\langle S' \rangle = M$, then we say that M is **finitely generated**. In other words, M is finitely generated if there exists a set $B = \{b_1, b_2, \ldots, b_{\mathbf{k}}\}$ of elements of M, where $b_i \in M(x_i)$ with x_i objects of $\mathcal{C}$, such that for every object y in $\mathcal{C}$ and every element m in $M(y)$, there exist scalars $\alpha_1, \ldots, \alpha_{\mathbf{k}}$ in $\mathbf{k}$ and morphisms $f_1, \ldots, f_{\mathbf{k}} : x_i \to y$ such that

$$m = \sum_{i=1}^{k} \alpha_i M(f_i)(b_i),$$

where $f_i : x_i \to y$ are morphisms in $\mathcal{C}$.
To understand this concept more clearly, let us examine the following example.

Example 4.2.1. Consider the category $\mathcal{C}$ with one object x and whose endomorphisms form a monoid A, i.e.

$$\mathrm{End}(x) = A,$$

where A is a set with an identity elements and is closed under an associative binary operation (in this case, composition of endomorphisms). A representation of $\mathcal{C}$ is a $\mathbf{k}$-module $M = M(x)$ where A acts on $M(x)$ such that for each $a \in A$, $M(a) : M \to M$ is a homomorphism. If $S \subseteq M$ is a subset of the representation, then we can consider the **subrepresentation generated by** S, denoted $\langle S \rangle$. This is the set of all elements of M which can be obtained by applying the module operations and the A-action for each element $a \in A$, and is equivalent to the definition given in the previous paragraph of a subrepresentation generated by a set.

4.2.2 $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$

The *category of representations* of $\mathcal{C}$, denoted $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$, is the category whose objects are representations, i.e. functors $\mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$, and morphisms are natural transformations between such functors.

Functors can be classified as Noetherian in an analogous way to how we classified modules as Noetherian. That is, a functor F in $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is **Noetherian** if every subrepresentation is finitely generated. Equivalently, we can say that a representation is Noetherian if every ascending chain of subrepresentations of F stabilizes.

Definition 4.2.2. We say that $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is Noetherian if every finitely generated object (representation) in it is Noetherian.

Example 4.2.2. (Noetherian Representation) Let $\mathcal{C}$ be the category which has as elements the elements of $\mathbb{N} = \{\bar{1}, \bar{2}, \bar{3}, ..\}$ and the single arrow between two elements n and m if and only if $n \leq m$:

$$\bar{1} \to \bar{2} \to \bar{3} \to \cdots .$$

Consider the representation F, given by the diagram

$$\mathbf{k} \to \mathbf{k} \to \mathbf{k} \to \cdots ,$$

where all morphisms in the diagram are the identity on the ring, $\mathbf{k}$. The only possible subrepresentations of F will be of the form

$$0 \to 0 \to \cdots \to 0 \to \mathbf{k} \to \mathbf{k} \to \cdots ,$$

where there are zeros in the first i positions, and then $\mathbf{k}$ in every subsequent position. To convince ourselves of this, we must show that for an arbitrary sub functor M of F,

i. The only possible subobjects of $M(x)$ are 0 and $\mathbf{k}$, and
ii. $\mathbf{k}$ will always be followed by $\mathbf{k}$.

Recall that M is a subfunctor of F if M is a functor, and there is a natural transformation $i : M \to F$ where each component of i is monic. To see why (i) is true, just notice that there are no nontrivial subobjects of $\mathbf{k}$ in $\mathrm{Mod}_{\mathbf{k}}$, and if $M(x)$ was a module over $\mathbf{k}$ not equal to $\mathbf{k}$, then the component $i_x : M(x) \to F(x)$ would not be monic. Thus the only possible subobjects of $M(x)$ are 0 and $\mathbf{k}$. To prove (ii), consider the following diagram

$$\begin{tikzcd} k \arrow[r] \arrow[d, "i"] & 0 \arrow[d, "i"] \\ k \arrow[r] & k \end{tikzcd}$$

This diagram does not commute, since any non trivial element of $\mathbf{k}$ will be mapped to zero in one direction and to itself in the other direction.

Now, since every subrepresentation is of this form, we may conclude that F is a Noetherian representation since every ascending chain of subrepresentations stabilizes.

4.3 Principal Projectives

For a representation F in $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$, an element of F is the element $F(x)$ in $\mathrm{Mod}_{\mathbf{k}}$ for some $x \in \mathcal{C}$. Given a morphism $f : x \to y$ in $\mathcal{C}$, we will denote f_* as the map of $\mathbf{k}$-modules $F(x) \to F(y)$.

Remark. One might wonder if a representation $R \in \mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is finitely generated, does that imply that it is Noetherian? This is not necessarily the case. Here is an example of a category whose category of representations is not Noetherian (even when $\mathbf{k}$ is a field):

Example 4.3.1. (Finitely generated Non-Noetherian Representation) Let $\mathcal{C}$ be the "star shaped" category, with objects $x_0, x_1, x_2, ...$ and morphisms: for all $n > 0$ there exists a unique morphism $x_0 \to x_n$, and all other morphisms are the identity.

Let M be the representation which maps every object to $\mathbf{k}$ and every morphism to the identity (note the subtle difference between this representation and the

$$
\begin{tikzcd}
 & x_1 \\
 & x_2 \\
x_0 \arrow[uur] \arrow[ur] \arrow[dr] & \vdots \\
 & x_n \\
 & \vdots
\end{tikzcd}
$$

Figure 4.1: Star Shaped Category

one in the previous example). M is finitely generated, in fact it is generated by the identity element of $M(x_0) =\mathbf{k}$. Consider the subrepresentation M_i of M which maps x_1 through x_i to $\mathbf{k}$ and all other elements to 0 (this is a subrepresentation since the only morphisms are the identity arrows). Then $M_1, M_2, M_3, ...$ is an increasing chain of subrepresentations which never stabilizes, so M is not Noetherian. We can also see non-Noetherianity of M by noting the following subrepresentation of M is not finitely generated: Let N be the representation which maps x_0 to 0 and every other object to $\mathbf{k}$.

$$
\begin{tikzcd}
 & \mathbf{k} \\
 & \mathbf{k} \\
0 \arrow[uur] \arrow[ur] \arrow[dr] & \vdots \\
 & \mathbf{k} \\
 & \vdots
\end{tikzcd}
$$

Figure 4.2: Non-finitely generated subrepresentation, N

In order to be finitely generated, there must exist a finite set S of elements of N such that for any object $x_i \in \mathcal{C}$ we must be able to arrive at any object $N(x_i)$ by a series of available operations in $\mathrm{Mod}_{\mathbf{k}}$. Consider the element $1 \in \mathbf{k}$ and let $N(x_i)$ be an arbitrary element of N. 1 gets mapped to 0 in $N(x_0)$, and then again to 0 in $N(x_i)$, but 0 does not generate $\mathbf{k}$, so N cannot be finitely generated.

For an object x of $\mathcal{C}$, we define the **principal projective** representation P_x of $\mathcal{C}$ by defining

$$P_x(y) = \mathbf{k}[\mathrm{Hom}_{\mathcal{C}}(x,y)],$$

where $\mathrm{Hom}_{\mathcal{C}}(x,y) = \{\alpha : x \to y \mid \alpha \text{ is a } \mathcal{C} \text{ homomorphism.}\}$ $P_x(y)$ is the free left $\mathbf{k}$-module generated by $\mathrm{Hom}_{\mathcal{C}}(x,y)$, where we write e_α as the corresponding element in $P_x(y)$ to the morphism $\alpha : x \to y$ in $\mathcal{C}$. Thus, an element of $P_x(y)$ will be of the form

$$\sum_{\alpha \in \mathrm{Hom}_{\mathcal{C}}(x,y)} \lambda_\alpha e_\alpha,$$

where all but finitely many of the coefficients λ_α are zero.
Let M be another representation. By the Yoneda Lemma, we see that

$$\mathrm{Hom}_{\mathcal{C}}(P_x, M) = M(x).$$

This means that $\mathrm{Hom}_{\mathcal{C}}(P_x, -)$ is an exact functor, and thus P_x is a projective object of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ (thus, the name principal projective).
Every representation M in $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ can be written as the quotient

$$M = \oplus_{i \in I} P_{x_i} / \oplus_{j \in J} P_{x'_j}, \tag{4.1}$$

for indexing sets I, J and objects $x_i, x'_j \in \mathcal{C}$.
To see how to one can arrive at equation 4.1 from a given representation M, let x be a fixed object of $\mathcal{C}$. Then $M(x)$ is a $\mathbf{k}$-module, and by the Yoneda Lemma, there is a natural canonical natural transformation (in other words, a morphism of representations)

$$M(x) \otimes P_x(-) \to M(-) \tag{4.2}$$

that is surjective onto $M(x)$. Furthermore, $M(x)$ can be presented as a quotient of a free module. Let's say that we have a surjection

$$\bigoplus_{i \in I} \mathbf{k} \twoheadrightarrow M(x). \tag{4.3}$$

Tensoring with $P_x(-)$ and combining with the previous morphism we obtain a morphism of representations

$$(\oplus_i k) \otimes P_x(-) \to M(-) \tag{4.4}$$

Note that this is a morphism from a sum of principal representations to M, and it is surjective onto M(x). Repeating the procedure for every object of C, and taking a direct sum we obtain a surjective morphism from a direct sum of principal projectives to M. This is equivalent to saying that M can be written as a quotient of a direct sum of principal projectives.

A representation is **finitely generated** if the set I is finite. The following proposition states this more clearly.

Proposition 4.3.1. A representation is finitely generated if and only if it is a quotient of a finite direct sum of principal projective objects.

Proposition 4.3.1 is proved in essentially the same way as the analogous statement about modules over a ring (Proposition 4.1.1).
Equivalently, we can say that M is finitely generated if there is a surjection

$$\oplus_{i=1}^{n} P_{x_i} \to M$$

for a finite set of elements $\{x_i\}$ in $\mathcal{C}$. We say a map of representations is a surjection if it evaluates to a surjection in $\mathrm{Mod}_{\mathbf{k}}(\mathcal{C})$ when evaluated at each object in $\mathcal{C}$.

Proposition 4.3.2. The category $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is Noetherian if and only if every principal projective is Noetherian.

Proof. If $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is Noetherian then every representation of $\mathcal{C}$ is Noetherian, in particular every principal projective is Noetherian. Conversely, assume that every principal projective is Noetherian and let M be a finitely generated object. By Proposition 4.3.1, M is a quotient of a finite direct sum of principal projectives. Since Noetherianity descends through quotients and across direct sums, M is Noetherian, which completes the proof. $\square$

4.4 Finiteness Property of a Functor

The main results in this paper revolve around showing certain categories have Noetherian representation categories. However, if we know that a category $\mathcal{C}$ has Noetherian $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$, and we can construct a certain functor $\mathcal{C}$ to another category $\mathcal{C}'$, then it is also possible to tell whether $\mathcal{C}'$ has a Noetherian representation category. The goal of this section is to explore such functors more thoroughly.
Let $\phi : \mathcal{C} \to \mathcal{C}'$ be a functor.

Definition 4.4.1. A functor ϕ is an $\mathcal{F}$**- functor** if the following condition holds: given any object x' of $\mathcal{C}'$ there exist finitely many objects $y_1, ..., y_n$ of $\mathcal{C}$ and morphisms $f_i : x' \to \phi(y_i)$ in $\mathcal{C}'$ such that for any object y of $\mathcal{C}$ and any morphism $f : x' \to \phi(y)$ in $\mathcal{C}'$, there exists a morphism $g : y_i \to y$ such that $f = \phi(g) \circ f_i$.

For a functor $\phi : \mathcal{C} \to \mathcal{C}'$, we can define a pullback functor on representations

$$\phi^* : \mathrm{Rep}_{\mathbf{k}}(\mathcal{C}') \to \mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$$

such that, for y an object of $\mathcal{C}$ and P_x a principal projective of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C}')$, we have that

$$\phi^*(P_{x'})(y) = k[\mathrm{Hom}_{\mathcal{C}'}(x', \phi(y)]. \tag{4.5}$$

While this looks like a principal projective functor, it is not necessarily. In general, when we have a functor $\phi : \mathcal{C} \to \mathcal{C}'$, the pullback functor ϕ^* will induce a representation from $\mathcal{C} \to \mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$, but it will not be a principal projective.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\phi"] \arrow[d] & \mathcal{C}' \arrow[d] \\
\mathrm{Rep}_{\mathbf{k}}(\mathcal{C}) & \mathrm{Rep}_{\mathbf{k}}(\mathcal{C}') \arrow[l, "\phi^*"]
\end{tikzcd}$$

Figure 4.3: Covariant Functor ϕ^*

We can categorize whether or not a functor ϕ is an $\mathcal{F}$-functor using ϕ^* with the following proposition:

Proposition 4.4.1. A functor $\phi : \mathcal{C} \to \mathcal{C}'$ is an $\mathcal{F}$-functor if and only if ϕ^* takes finitely generated objects of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C}')$ to finitely generated objects of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$.

Proof. Assume that ϕ is an $\mathcal{F}$ functor. By Proposition 4.3.1, it is enough to show that ϕ^* takes principal projectives to finitely generated representations. Let $P_{x'}$ be the principal projective of $\mathcal{C}'$ at object $x' \in \mathcal{C}'$. From equation 4.5, we see that $\phi^*(P_{x'})(y)$ has basis elements e_f for $f \in \mathrm{Hom}_{\mathcal{C}'}(x', \phi(y))$. Let $f_i : x' \to \phi(y_i)$ be as in the Definition 4.4.1, in particular that there are finitely many such f_i. For each $f : x' \to \phi(y)$ in $\mathcal{C}'$ there exists a morphism $g : y_i \to y$ in $\mathcal{C}$ such that $f = \phi(g) \circ f_i$. Thus, e_{f_i} generates $\phi^*(P_{x'})$, and thus it is finitely generated. The converse is left to the reader, as it is not used in this paper. □

Proposition 4.4.2. Suppose that $\phi : \mathcal{C} \to \mathcal{C}'$ is an essentially surjective functor. Let M' be an object of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C}')$. Let $M = \phi^*(M')$. If M is finitely generated (resp. Noetherian) then M' is finitely generated (resp. Noetherian).

Proof. Let $S = \{s_1, \ldots, s_n\}$ be a generating set of M. This means that there exist objects $x_1, \ldots, x_n$ of $\mathcal{C}$ such that $s_i \in M(x_i) = M'(\phi_(x_i))$ for $i = 1, \ldots, n$. Thus we may think of elements of S as elements of M' as well as of M. Our goal is to show that S generates M'. Let y' be an object of $\mathcal{C}'$, and s' an element of $M'(y')$. We need to show that s' can be written as a linear combination of images of elements of S under some morphisms $\phi(x_i) \to y'$ in M'. Since ϕ is essentially surjective, y' is isomorphic to an object in the image of ϕ. This means that we may assume that y' is in the image of ϕ. Let us write $y' = \phi(y)$, and $M(y) = M'(y')$. It follows that we may consider s' to be an element of $M(y)$ as well as of $M'(y')$. Since S generates M, there exist morphisms $\alpha_i : x_i \to y$ in $\mathcal{C}$, and elements $r_1, \ldots, r_n$ of $\mathbf{k}$ such that $s' = \Sigma_i r_i M(\alpha_i)(s_i)$. But then $s' = \Sigma_i r_i M'(\phi(\alpha_i))(s_i)$. This means that s' is in the subrepresentation of M' generated by S, where s' was any element of M'. It follows that S generates M'.

Now suppose $M = \phi^*(M')$ is a Noetherian representation of $\mathcal{C}$. We need to show that M' is a Noetherian representation of $\mathcal{C}'$. Let N' be a subrepresentation of M'. We need to show that N' is finitely generated. Clearly, $\phi^*(N')$ is a subrepresentation of M. Since M is Noetherian, $\phi^*(N')$ is finitely generated. By the first half of the proof, N' is finitely generated.

□

Now we can prove the main result of this section

Corollary 4.4.1. Let $\phi : \mathcal{C} \to \mathcal{C}'$ be an essentially surjective $\mathcal{F}$-functor, and suppose $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is Noetherian. Then $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C}')$ is Noetherian.

Proof. Let M be a finitely generated representation of $\mathcal{C}'$. To show $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C}')$ is Noetherian, it is enough to show that M is Noetherian. Since M is finitely generated, $\phi^*(M)$ is finitely generated by 4.4.1. Thus $\phi^*(M)$ is Noetherian since $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is Noetherian, and so M is Noetherian by Proposition 4.4.2. □

Chapter 5

Gröbner Approach to the Noetherian Property

In this chapter, we address Sam and Snowden's approach to proving the Noetherian property of categories using Gröbner theory. In 5.1, we will see how the authors make a connection between monomial and principal sub representations. In Section 5.2, we will demonstrate how they define an ordering on the set of monomial subrepresentations, and in Section 5.3, we will use these ideas to define a Gröbner basis for a representation.

5.1 Monomial Representations

Let $\mathcal{C}$ be an essentially small category. Define $S : \mathcal{C} \rightarrow$**Set** to be a functor taking an element $x \in \mathcal{C}$ to the set $S(x)$ and an arrow $\alpha : x \to y \in \mathcal{C}$ to the arrow $\alpha_* : S(x) \to S(y)$. From S, we can define a representation $P : \mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$ by

$$P(x) = \mathbf{k}[S(x)], \tag{5.1}$$

that is, the free $\mathbf{k}$-module generated by the set $S(x)$. For $f \in S(x)$, we write e_f to denote the corresponding element in $P(x)$.

Definition 5.1.1. A subfunctor of S is **principal** if it is generated by a single element.

For example, suppose $\mathcal{C}$ is a category with two objects x, y, and n morphisms $\alpha_1, \ldots, \alpha_n : x \to y$. Let $S : \mathcal{C} \to$ **Set** be the functor defined by $S(x) = S(y) = \mathbb{N}$ (the set of natural numbers), and $\alpha_{i*}(n) = n + i$, so that α_{i*} is adding i. The principal subfunctor of S generated by the element $1 \in S(x)$ is a functor that sends x to $\{1\}$ and y to $\{2, 3, \ldots, n+1\}$. The principal subfunctor of S generated by $1 \in S(y)$ is the functor that sends x to the empty set and y to $\{1\}$.

Definition 5.1.2. A **monomial** element of P is of the form λe_f for some $\lambda \in \mathbf{k}$ and $f \in S(x)$. A subrepresentation M of P is **monomial** if $M(x)$ is spanned by all of the monomials it contains for each $x \in \mathcal{C}$.

The aim of this section is to give a sufficient condition for the representation P to be Noetherian by examining its monomial subrepresentations. First we give an ordering to the set of principal subrepresentations, $|S|$, ordered by reverse inclusion, for a small category $\mathcal{C}$ as follows.

Let $\tilde{S} = \cup_{x \in \mathcal{C}} S(x)$. Define a relation on $\tilde{S}$ by saying that for $f \in S(x)$ and $g \in S(y)$, $f \leq g$ if there exists $\alpha : x \to y$ such that $\alpha_*(f) = g$. Then we have the equivalence relation $f \sim g$ if $f \leq g$ and $g \leq f$. $|S|$ can then be defined by the quotient $\tilde{S}/\sim$ with the induced partial order on representatives.
Given a subrepresentation M of P, and given $f \in \tilde{S}$, consider the set

$$I_M(f) = \{\lambda \in k \mid \lambda e_f \in M\}.$$

Remark. $I_M(f)$ is an ideal of $\mathbf{k}$.

Since M is a subrepresentation, $I_M(f)$ is closed under addition and scalar multiplication. That is for any $a \in k$, if λ_1 and $\lambda_2 \in I_M(f)$, then $a(\lambda_1 + \lambda_2) \in I_M(f)$. Further, if $f \leq g$, then $I_M(f) \subseteq I_M(g)$. Indeed, consider $\lambda \in I_M(f)$, and morphism $\alpha : x \to y$ such that $\alpha_*(f) = g$. Then the induced morphism $\mathbf{k}[\alpha_*] : M(x) \to M(y)$ sends λe_f to λe_g, and thus $\lambda \in I_M(g)$. This gives us that if $f \sim g$, then $I_M(f) = I_M(g)$.

Define $\mathcal{J}(\mathbf{k})$ to be the **poset of left ideals** in $\mathbf{k}$ and $\mathcal{M}(P)$ to be the **poset of monomial subrepresentations**, both ordered by inclusion, and let $\mathcal{F} = \mathcal{F}(|S|, \mathcal{J}(\mathbf{k}))$ be the set of all order preserving functions $I : |S| \to \mathcal{J}(\mathbf{k})$, partially ordered by $I \leq I'$ if $I(f) \leq I'(f)$ for all $f \in |S|$. Given $M \in \mathcal{M}(P)$, we have an order preserving function

$$I_M : |S| \to \mathcal{J}(\mathbf{k}),$$

where $I_M \in \mathcal{F}(|S|, \mathcal{J}(\mathbf{k}))$. From this construction, the obtain the following proposition.

Proposition 5.1.1. The map $I : \mathcal{M}(P) \to \mathcal{F}(|S|, \mathcal{J}(\mathbf{k}))$ is an isomorphism of posets.

Proof : We must show that I is an order preserving bijection. To see that I is order preserving, take $M \subseteq M' \in \mathcal{M}(P)$. I maps these monomial representations to I_M and $I_{M'}$, respectively. Then for each $f \in |S|$, it is clear that $I_M(f) \subseteq I_{M'}(f)$. To construct the inverse of I, let $L : |S| \to \mathcal{J}(\mathbf{k})$ be an order preserving function. For $f \in |S|$, $L(f)$ is a left ideal of $\mathbf{k}$, and whenever $f \leq g$, $L(f) \subseteq L(g)$. Now we define $I'(L)$ to be the monomial subrepresentation

$$M(x) = \sum_{f \in S(x)} L(f)e_f.$$

To check that I' is order preserving, take two maps $L, L' : |S| \to \mathcal{J}(\mathbf{k})$ such that $L \leq L'$, i.e. $L(f) \subseteq L'(f)$ for all $f \in |S|$. Then,

$$L \mapsto M : M(x) = \sum_{f \in S(x)} L(f)e_f,$$
$$L' \mapsto M' : M'(x) = \sum_{f \in S(x)} L'(f)e_f.$$

Thus $M(x) \subseteq M'(x)$ and so I is order preserving. Lastly, we show that I and I' are each others inverses. If M is any subrepresentation of P (not necessarily monomial), it is easy to see that $I'(I(M))$ is the maximal monomial subrepresentation of M. Thus if M is monomial, $I'(I(M)) = M$. The proof that $I(I'(L)) = L$ for all L is equally easy.

We now can state the following equivalence statement.

Theorem 5.1.1. Let $P : \mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$ be a representation as defined in equation 5.1 . The following are equivalent, assuming P is non-zero.

a. Every monomial subrepresentation of P is finitely generated.
b. The poset $\mathcal{M}(P)$ satisfies ACC.
c. The poset $|S|$ is Noetherian and $\mathbf{k}$ is left-Noetherian.

Proof. That (a) and (b) are equivalent is standard and has already been addressed in analogous cases in this paper. (b) being equivalent to (c) follows using the map from the previous proposition and Proposition 2.2.3 (c), where $|S|$ is nonempty if $P \neq 0$ and $\mathcal{J}(\mathbf{k})$ contains two distinct comparable elements: zero and unit ideals. □

5.2 init(M)

Now that we have connected principal subrepresentations with monomial subrepresentations of P, we will do the same for arbitrary subrepresentations of P.

First, though, we must build a theory of *monomial orders*. Let **WO** be the category of well ordered sets and strict order preserving functions, and let S be a functor from $\mathcal{C}$ to **set**. There is a forgetful functor F :**WO**$\to$**Set** sending a well ordered set $(R, \preceq)$ to its underlying set and "forgetting" the ordering on it. We can define an **ordering** on S as a lifting of S to **WO**, sending S to a well ordered set $(S, \preceq)$ such that for every morphism $x \to y \in \mathcal{C}$, the induced map $S(x) \to S(y)$ is strictly order preserving. We say S is **orderable** if it admits an ordering.

$$\begin{tikzcd}
& \mathbf{WO} \arrow[d, "F"] & (S, \preceq) \\
\mathcal{C} \arrow[r, "S"] & \mathbf{Set} & S \arrow[u, dashed, "\text{ordering}"']
\end{tikzcd}$$

Recall the representation $P : \mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}(\mathcal{C})$ defined by $P(x) = k[S(x)]$.

Definition 5.2.1. Given an ordering $\preceq$ on S and a non-zero element

$$\alpha = \sum_{f \in S(x)} \lambda_f e_f$$

in $P(x)$, an **initial term** of α, denoted $\mathrm{init}(\alpha)$, is $\lambda_g e_g$ where $g = \max_{\preceq}\{f \mid \lambda_f \neq 0\}$. The **initial variable** of α, denoted $\mathrm{init}_0(\alpha)$, is g.

Example 5.2.1. If $\alpha \in P(x)$ is equal to $\lambda_1 e_1 + \lambda_5 e_5 + \lambda_7 e_7$, where $e_i < e_j$ for $i < j$, then $\mathrm{init}_0(\alpha) = 7$ and $\mathrm{init}(\alpha) = \lambda_7 e_7$, where $7 \in S(x)$ and e_7 is the corresponding element of $P(x)$.

Definition 5.2.2. For a subrepresentation M of P, the **initial representation** of M, denoted $\mathrm{init}(M)$, is defined by setting $\mathrm{init}(M)(x)$ to be the **k**-span of the elements $\mathrm{init}(\alpha)$ for non-zero $\alpha \in M(x)$.

Proposition 5.2.1. $\mathrm{init}(M)$ is a monomial subrepesentation of P.

Proof. That $\mathrm{init}(M)$ is monomial follows directly from its definition, since it is the span of the elements of $\mathrm{init}(\alpha)$, all of which are monomials. Thus, we just need to prove that $\mathrm{init}(M)$ is indeed a subrepresentation of P. This equates to showing that a morphism $g : x \to y$ in $\mathcal{C}$ induces a map $g_* : S(x) \to S(y)$ that maps $\mathrm{init}(M)(x)$ into $\mathrm{init}(M)(y)$. Define an element of $M(x)$, $\alpha = \sum_{i=1}^{n} \lambda_i e_{f_i}$, with each λ_i non-zero and ordered so that $f_i \prec f_1$ for all $i > 1$. Thus, $g_*(\alpha) = \sum_{i=1}^{n} \lambda_i e_{g_*(f_i)}$ and, since g_* is order preserving, $g_*(f_i) \prec g_*(f_1)$. This gives us that $\mathrm{init}(g_*(\alpha)) = \lambda_1 e_{g_* f_1} = g_*(\mathrm{init}(\alpha))$, proving our claim. $\square$

Proposition 5.2.2. If $N \subseteq M$ are subrepresentations of P and $\text{init}(N) = \text{init}(M)$, then $M = N$.

Proof. $M = N$ means that $M(x) = N(x)$ for all $x \in \mathcal{C}$. So assume that $M(x) \neq N(x)$ for some $x \in \mathcal{C}$, and let $K \subseteq S(x)$ be the set of all elements which are the initial variable of some element of $M(x) \setminus N(x)$. By our assumption, $K \neq \varnothing$, and since $S(x)$ is well ordered, there is some minimal element f with respect to $\preceq$. Now, pick an element $\alpha \in M(x) \setminus N(x)$ such that $\text{init}(\alpha) = f$ (by the definition of K, we know f is some initial variable). By assumption, there exists β in $N(x)$ with $\text{init}(\alpha) = \text{init}(\beta)$. This means, by definition of α, that $\alpha - \beta \in M(x) \setminus N(x)$, giving us that $\text{init}_0(\alpha - \beta) \prec \text{init}_0(\alpha)$, which contradicts f being minimal. Thus, $M = N$. □

5.3 Gröbner Basis

Definition 5.3.1. Let M be a subrepresentation of P. A set $\mathcal{G}$ of elements of M is a **Gröbner Basis** of M if $\{\text{init}(\alpha) \mid \alpha \in \mathcal{G}\}$ generates $\text{init}(M)$. M has a **finite Gröbner basis** if and only if $\text{init}(M)$ is finitely generated.

The following proposition follows directly from Definition 5.3.1 and Proposition 5.2.2

Proposition 5.3.1. Let $\mathcal{G}$ be a Gröbner basis of M. Then $\mathcal{G}$ generates M.

The following theorem is the first main result of our paper, as it allows us to classify when a principal projective is Noetherian.

Theorem 5.3.1. Let $\mathbf{k}$ be left-Noetherian, S orderable and $|S|$ Noetherian. Then every subrepresentation of P has a finite Gröbner basis. In particular, P is Noetherian in $\text{Rep}_{\mathbf{k}}(\mathcal{C})$.

Proof. Let M be a subrepresentation of P. To show that M has a finite Grobner basis $\mathcal{G}$ is equivalent to showing that $\text{init}(M)$ is finitely generated. The conditions that $\mathbf{k}$ be left Noetherian, S orderable and $|S|$ Noetherian give us that every monomial subrepresentation of P is finitely generated by Theorem 5.1.1. By Proposition 5.2.1, $\text{init}(M)$ is a monomial subrepresentation, and thus it is finitely generated. The theorem follows. By Proposition 5.3.1 since $\mathcal{G}$ is finite, M is finitely generated. Thus P is Noetherian. □

Let $\mathcal{C}$ be an essentially small category. Then for an object x of $\mathcal{C}$, we can define the functor $S_x : \mathcal{C} \to \mathbf{Set}$ by setting

$$S_x(y) = \text{Hom}_{\mathcal{C}}(x, y), \tag{5.2}$$

where $P_x = \mathbf{k}[S_x]$.

Definition 5.3.2. A category $\mathcal{C}$ is **Gröbner** if, for all objects x in $\mathcal{C}$, the functor S_x is orderable and the poset $|S_x|$ is Noetherian. $\mathcal{C}$ is **quasi-Gröbner** if there exists a Gröbner category $\mathcal{C}'$ and an essentially surjective $\mathcal{F}$-functor $\mathcal{C}' \to \mathcal{C}$.[2]

Theorem 5.3.2. Let $\mathcal{C}$ be a quasi-Gröbner category. Then for any left-Noetherian ring $\mathbf{k}$, the category $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is Noetherian.

Proof. If $\mathcal{C}$ is Gröbner, then every principal projective of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is Noetherian by the previous theorem, so $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is Noetherian by Proposition 4.3.2. If $\mathcal{C}$ is quasi-Gröbner, then $\phi : \mathcal{C}' \to \mathcal{C}$ is an essentially surjective $\mathcal{F}$-functor with $\mathcal{C}'$ a Gröbner category. Thus, $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C}')$ is Noetherian, and thus by Corollary 4.4.1 $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is Noetherian. □

Chapter 6

Categories of Injections

The goal of the final chapter is to give a concrete example of a Noetherian representation category.

6.1 Formal Languages and Lingual Categories

Before we can prove the main result of the chapter, we must set up the framework of languages. The reason for this is that we will be constructing a Noetherian language and showing that it is isomorphic to a certain poset in order to prove our category of interest is Gröbner. By Higman's Lemma (6.1), it is much easier to prove Noetherianity of our language than of the poset we are working with. The theory of formal language is an interesting and complex topic which will only be touched on in this paper.

Let Σ be a fixed finite set of symbols, which we call an **alphabet**. Symbols in an alphabet may be assigned a special meaning, and the formation rules, which are typically defined recursively, specify which strings of symbols count as well-formed. We let Σ^* denote the set of all finite words in Σ, where a word is well-formed string of symbols of the alphabet. A **language** $\mathcal{L}$ on Σ is a subset of Σ^*.

Example 6.1.1. (Language of Propositional Calculus)
The language $\mathcal{P}$ of propositional calculus could be defined in the following way:

a. The alphabet Σ of $\mathcal{P}$ consists of English letters with optional indexes and the following special symbols: $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\Rightarrow$ (implies), and () (grouping).

b. The formation rules are: every English letter and every letter with an index is a formula, and if A and B are formulas, then so are ¬A, A ∨ B, A ∧ B, A⇒ B, and (A). [4]

Example 6.1.2. (Language of Natural Numbers)
Consider the alphabet

$$\Sigma = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 0, +, =\}.$$

We can express the language $\mathcal{L}$, the natural numbers, with well formed additions and well-formed addition equalities by imposing the following rules:

i. Every nonempty word that does not contain "+" or "=" and does not start with "0" is in $\mathcal{L}$.
ii. The word "0" is in $\mathcal{L}$.
iii. A word containing "=" is in $\mathcal{L}$ if and only if there is exactly one "=" and it separates two valid words of $\mathcal{L}$.
iv. A word containing "+" but not "=" is in $\mathcal{L}$ if and only if every "+" in the word separates two valid words of $\mathcal{L}$.
v. No word is in $\mathcal{L}$ other than those implied by the previous rules.

From these rules, we see that the word "1+5=1000" is in $\mathcal{L}$ however the word "+=434" is not. In other words, the words formed by these rules are not necessarily true, but we understand how to interpret them.

Example 6.1.3. Let Σ be an alphabet, and suppose that Σ is equipped with a partial order. It induces a partial order on Σ^* as follows: Let $v = a_1 \dots a_m$ and $w = b_1 \dots b_n$ be two words. We say that $v \leq w$ if $m \leq n$, and there exists indices $1 \leq i_1 \leq i_2 \dots \leq n$ such that the following inequalities hold in Σ:

$$a_1 \leq b_{i_1}, \dots, a_m \leq b_{i_m}$$

An important special case is when Σ has the trivial partial order, in which no distinct elements are comparable. In this case the induced order on Σ^* is called the **subsequence order**.

6.2 The categories $\mathbf{OI}_d$ and $\mathbf{FI}_d$

For a positive integer d, we define the category $\mathbf{FI}_d$ to be the category whose objects are finite sets and whose morphisms are pairs (f, g) where $f : S \to T$ is an injection and $g : T \setminus f(S) \to \{1, \dots, d\}$ is a function. Composition of morphisms is defined as follows: for injections $f : S \to T, f' : T \to U$ and functions $g : T \setminus f(S) \to \{1, \dots, d\}, g' : U \setminus f'(T) \to \{1, \dots, d\}$, the composition

$f' \circ f : S \to U$ is defined by composition of functions, and is injective since the composition of injections is injective. The composition of g' and g is equal to the disjoint union of g with g'.

$$\begin{tikzcd}
S \arrow[r, "f"] & T \arrow[rr, "f'"] & & U \\
& T \setminus f(S) \arrow[dr, "g"'] & U \setminus f'(f(S)) \coprod U \setminus f'(T \setminus f(S)) \arrow[d, "g \coprod g'"] & U \setminus f'(T) \arrow[dl, "g'"] \\
& & \{1, \ldots, d\} &
\end{tikzcd}$$

Figure 6.1: composition of morphisms in $\mathbf{FI}_d$

The category $\mathbf{OI}_d$ is defined analogously with objects now being totally ordered finite sets and morphisms (f, g) with f an order preserving injection and no additional conditions placed on g.

Let $\Sigma = \{0, \ldots, d\}$ with the trivial partial order. Since Σ is a finite subset of $\mathbb{N}$, it is Noetherian. Let $\mathcal{L} \subseteq \Sigma^*$ be the language on Σ with words of the form $w_1 \cdots w_r$ where exactly n of the w_i are equal to zero (e.g. in the case where $d = 5, n = 1$: $w = 153032$ is a word in $\mathcal{L}$ but $w' = 10101$ is not.) We will equip $\mathcal{L}$ with the **subsequence order** described in remark 6.1.3, i.e.: if $s : [i] \to \Sigma$ and $t : [j] \to \Sigma$ are words, then $s \leq t$ if there exists $I \subseteq [j]$ such that $s = t \mid_I$. If a language $\mathcal{L}$ has an order on it, we call $\mathcal{L}$ an **ordered language**.

Remark. Let $d = 9$ and $n = 1$. To see an example of how the subsequence order works, consider words $s = 1304, t = 5163074, t' = 1034$. Then $s \leq t$ since 1304 can be embedded into t, but $s \not\leq t$ since there is not subset of 1034 for which 1304 can be embedded into.

$\mathcal{L}$ may also be equipped with the **Lexicographic order**, using the standard order on Σ. For two words $w, u \in \mathcal{L}$, we say that $w \leq u$ if w is a prefix of u, or if there exist (possibly empty) words $a, b, c \in \mathcal{L}$ and elements x, y of Σ such that $x \leq y$ and

$$w = axb,$$
$$u = ayc.$$

To avoid confusion, we will always specify whether we are using the subsequence order or lexicographic order on $\mathcal{L}$.

Now, we can see that $\mathcal{L}$, with the subsequence order, is a Noetherian poset. *Higman's Lemma* states:

$$\text{If } \Sigma \text{ is a Noetherian poset, so is } \Sigma^* \tag{6.1}$$

(see [2] Theorem 2.5 for a proof). Thus so is any subset $\mathcal{L}$ of Σ^*. With this result in hand, we can now prove the following crucial lemma.

Lemma 6.2.1. The category $\mathbf{OI}_d$ is Gröbner.

Proof. Recall from definition 5.3.2 that a category $\mathcal{C} = \mathbf{OI}_d$ is a Gröbner category if (i) $\mathcal{C}$ is a directed category, (ii) S_x is orderable, and (iii) $|S_x|$ is Noetherian (note that the order referred to in (ii) is not equal to the order in (iii); in part (iii), we order $|S_x|$ with the order defined in §5.1). The proof of (i) is obvious, given that the only injective, order preserving endomorphism is the identity. (iii) can be proven by constructing an isomorphism between the posets $|S_x|$ and $\mathcal{L}$, where we use the subword order on $\mathcal{L}$: Let $x = [n]$. For an object of S_x, $(f, g) \in \mathrm{Hom}_{\mathcal{C}}([n], [m])$, define a map $h : [m] \to \Sigma$ such that

$$h(t) = \begin{cases} 0, & t \in f([n]); \\ g(t), & t \in [m] \setminus f([n]). \end{cases} \tag{6.2}$$

It is not difficult to show that the pair (f, g) can be recovered from the function h since f is injective and order preserving, and h is a word in $\mathcal{L}$ since it has precisely n zeros. Thus, this construction gives rise to the isomorphism $i : |S_x| \to \mathcal{L}$ where $i((f_i, g_j)) = h_{i,j}$, proving the claim of (iii). For (ii), the lexicographic order on $\mathcal{L}$ restricts to an admissible order on $|S_x|$, as this order is preserved under morphisms of S_x. Thus S_x is orderable, completing our proof. $\square$

Theorem 6.2.2. The forgetful functor $\phi : \mathbf{OI}_d \to \mathbf{FI}_d$ is an $\mathcal{F}$-functor. In particular, $\mathbf{FI}_d$ is quasi-Gröbner.

Proof. ϕ sends a totally ordered set to its underlying set, forgetting the order on it. Thus, for an object $(y, \preceq)$ of $\mathbf{OI}_d$, we have

$$\phi((y, \preceq)) = y.$$

Let $x = [n]$ be an object of $\mathbf{FI}_d$ and let $f : x \to y$ be a morphism. To demonstrate that ϕ is an $\mathcal{F}$-functor, we must show that we can factor f through finitely many morphisms $f_i : x \to y_i$. Consider the permutation $\sigma_i \in S_n$, the symmetric group on n elements. Then $\sigma_i : x \to x$ acts as an injective function, permuting the elements of x. There are $n!$ such permutations in S_n, so by letting $y_1, ..., y_{n!}$ all be equal to x, we see that f factors through σ_i:

$$\begin{tikzcd} x \arrow[r, "\sigma_i"] \arrow[dr, "f"] & x = y_i \arrow[d, "f'"] \\ & y \end{tikzcd}$$

where f' is an order preserving injection. Thus, $f = \phi(f') \circ \sigma_i$, showing that ϕ is an $\mathcal{F}$-functor. Finally, by definition 5.3.2 and the previous lemma, $\mathbf{FI}_d$ is quasi-Gröbner. [2, Proposition 7.1.4] $\square$

We may now state the final result of the paper, which follows directly from Theorem 6.2.2 and Corollary 4.4.1.

Corollary 6.2.1. If $\mathbf{k}$ is left-Noetherian, then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_d)$ is Noetherian.

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