Prep Course
Module 2
Lecture 2

Chao-Jen Chen
ccj@uchicago.edu

Vectors and Linear Combinations

Vectors

- A n-dimensional (column) vector is a tuple as follows:

$$\boldsymbol{v} \coloneqq \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}, where\ n \in \mathbb{N}\ and\ v_1, v_2, v_3, \dots, v_n \in \mathbb{R}$$

- When we mention a vector without specifying whether it's a column or row vector, we presume it's a column vector.
- For convenience, we may write a column vector as follows:

$$(v_1, v_2, v_3, \dots, v_n) \coloneqq \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$$

Vector addition and scalar multiplication

- Vector addition:

$$\boldsymbol{v} := \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \text{ and } \boldsymbol{w} := \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} \text{ add to } \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

- Scalar multiplication:

$$\text{Given } c \in \mathbb{R}, c\boldsymbol{v} := \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \\ \vdots \\ cv_n \end{bmatrix}$$

We use vectors to represent things

- In one-period 3-state model, there are two time points: t_0 (now) and t_1 (future). At t_1, there are 3 states: $\omega_1, \omega_2, \omega_3$. We could have the t_1-price vectors of bank account B and stock S as follows:

$$B_{t_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \; and \; S_{t_1} = \begin{bmatrix} 120 \\ 90 \\ 70 \end{bmatrix}$$

- If we have two assets, B and S, in our portfolio P, we could use a vector to represent the portfolio. Say, we have 10 units of B and 20 shares of S. Then our portfolio vector is (10,20) and all the possible t_1-values of P are

$$P_{t_1} = \begin{bmatrix} 1 & 120 \\ 1 & 90 \\ 1 & 70 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 2410 \\ 1810 \\ 1410 \end{bmatrix}$$

Linear combination

- Given vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}$, vector $\boldsymbol{x}$ defined as below is called a linear combination of $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}$

$$\boldsymbol{x} \coloneqq c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n} \text{ where } c_1, c_2, \ldots, c_n \in \mathbb{R}$$

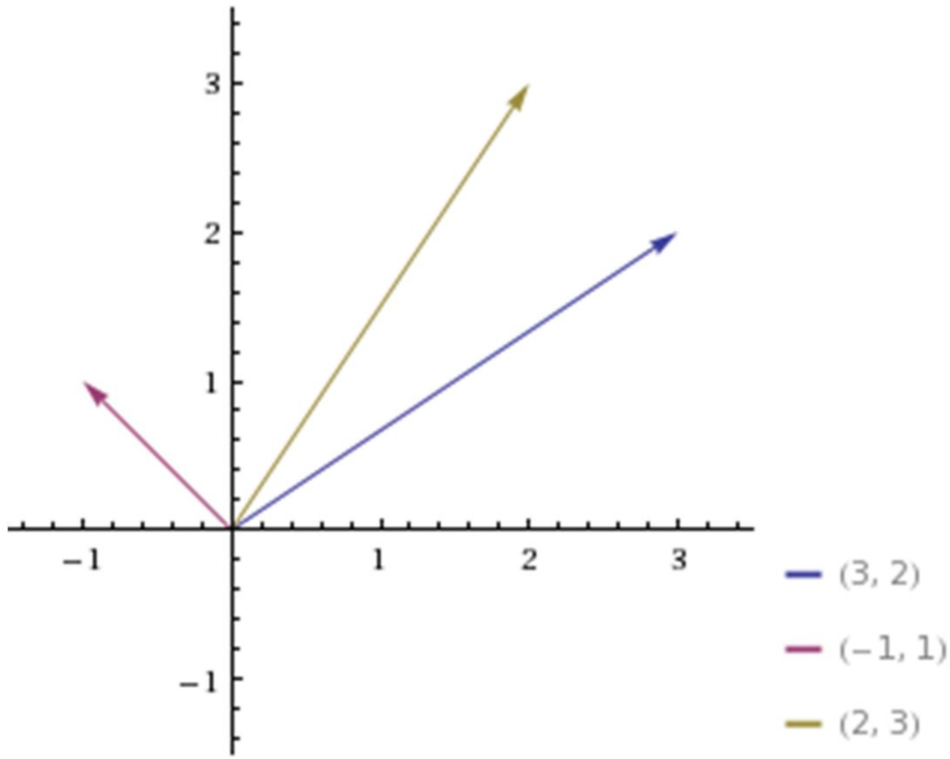
- In some sense, linear combination = vector addition + scalar multiplication. Both vector addition and scalar multiplication can be viewed as special cases of linear combination.
- For example, P_{t_1} can be viewed as a linear combination of B_{t_1} and S_{t_1}:

$$P_{t_1} = 10\begin{bmatrix}1\\1\\1\end{bmatrix} + 20\begin{bmatrix}120\\90\\70\end{bmatrix} = \begin{bmatrix}2410\\1810\\1410\end{bmatrix}$$

Visualize linear combinations in 2-dimensional space

In 2-dimensional space,

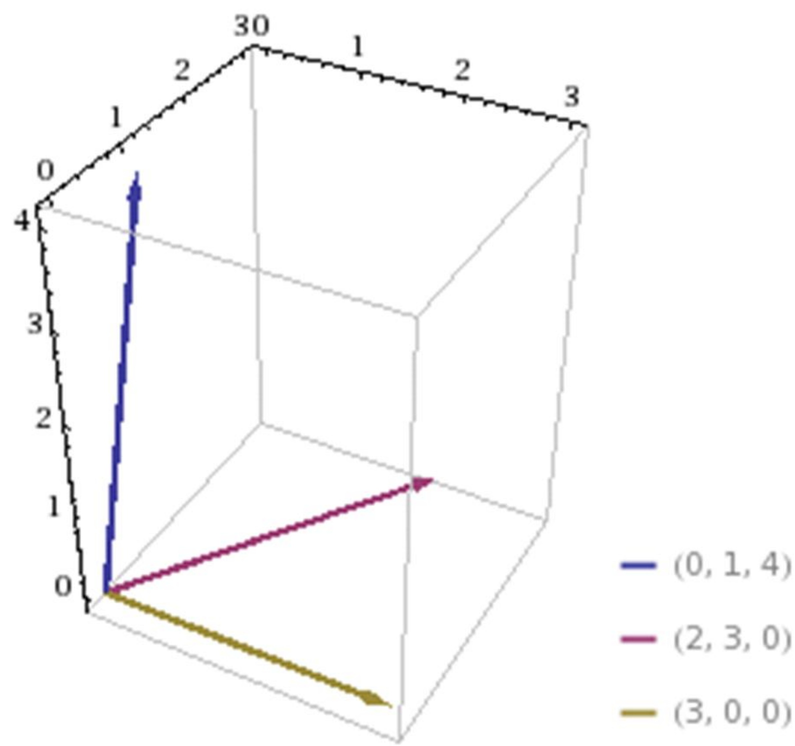
- Given vector $\boldsymbol{v}$, all linear combinations of $\boldsymbol{v}$ fill a line
- Given linearly independent vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, all linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill a plane, i.e., the 2-D space itself.



Visualize linear combinations in 3-dimensional space

In 3-dimensional space, given linearly independent vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$

- all linear combinations of $\boldsymbol{v}$ fill a line
- all linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill a plane
- all linear combinations of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ fill the 3-D space



Geometric examples of linear combinations

For $\boldsymbol{v}$ = (1,0) and $\boldsymbol{w}$ = (0,1)

- describe all points $c\boldsymbol{v}$ with integer c:
  - The vectors $c\boldsymbol{v} = (c, 0)$ are equally spaced points along the x axis. They include $(-2, 0)$, $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(2, 0)$.
- describe all points $c\boldsymbol{v}$ with $c \geq 0$:
  - The vectors $c\boldsymbol{v} = (c, 0)$ fill a half-line. It's the non-negative part of the x axis. It includes $(\pi, 0)$ but not $(-\pi, 0)$.
- describe all points $c\boldsymbol{v} + d\boldsymbol{w}$ with any integer c and real d:
  - Adding all vectors $d\boldsymbol{w}$ puts a vertical line through each of those points $c\boldsymbol{v}$. We have infinitely many parallel lines from $(integer\ c, any\ d)$.
- describe all points $c\boldsymbol{v} + d\boldsymbol{w}$ with $c \geq 0$ and real d:
  - Adding all vectors $d\boldsymbol{w}$ puts a vertical line through each of those points $c\boldsymbol{v}$ on the half-line. We have a half-plane, i.e., the right half of the xy plane $(x \geq 0, any\ y)$.

Lengths and Inner Products

Inner product (dot product)

The inner product or dot product of n-dimensional vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is the number $\boldsymbol{v} \cdot \boldsymbol{w}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} := \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{i=1}^{n} v_i w_i$$

Python command:

Code snippet M2L02-01

Length and Unit Vector

The length of a vector $\boldsymbol{v}$ is written as $||\boldsymbol{v}||$ or norm$(\boldsymbol{v})$, and defined as the square root of the dot product $\boldsymbol{v} \cdot \boldsymbol{v}$:

$$||\boldsymbol{v}|| := \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

A unit vector $\boldsymbol{u}$ is a vector whose length equals one. Then $\boldsymbol{u} \cdot \boldsymbol{u} = 1$.

Construct a unit vector

If we define a vector $\boldsymbol{u}$ as follows

$$\boldsymbol{u} := \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \left(\frac{v_1}{\|\boldsymbol{v}\|}, \frac{v_2}{\|\boldsymbol{v}\|}, \dots, \frac{v_n}{\|\boldsymbol{v}\|}\right)$$

then $\boldsymbol{u}$ is a unit vector in the same direction as $\boldsymbol{v}$.

For example, since (2,2,1) has length 3, the vector $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ has length 1. Check that $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) \cdot \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$.

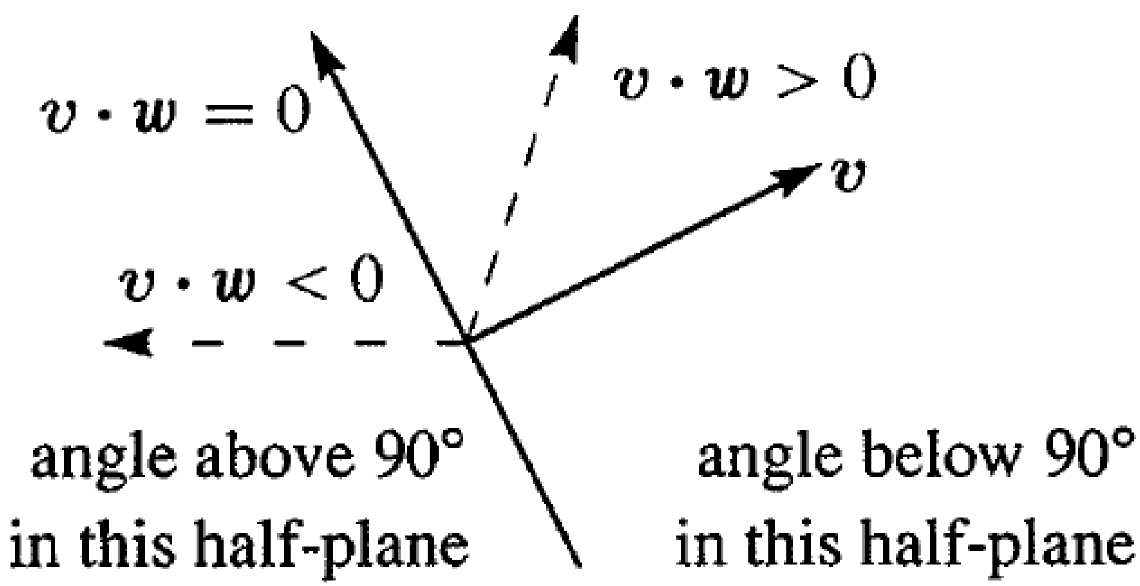
Python example:

Code snippet M2L02-02

θ: the angle between two vectors

Given that both $\boldsymbol{v}$ and $\boldsymbol{w}$ are non-zero vectors, we have the following formula:

$$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\cdot\|\boldsymbol{w}\|} \quad \Leftrightarrow \boldsymbol{v}\cdot\boldsymbol{w} = \|\boldsymbol{v}\|\cdot\|\boldsymbol{w}\|\cdot\cos\theta$$



Properties of inner product

- Positive Definiteness:

$$\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2 \geq 0$$

- Length of zero vector is zero:

$$\boldsymbol{v} \text{ is a zero vecctor } \mathbf{0} \coloneqq (0, 0, \dots, 0) \quad \Leftrightarrow \quad \|\boldsymbol{v}\| = 0$$

- Symmetry:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$$

- Linearity ($c \in \mathbb{R}$):

$$(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$$
$$(c\boldsymbol{v}) \cdot \boldsymbol{w} = c(\boldsymbol{v} \cdot \boldsymbol{w})$$

- Cauchy-Schwarz Inequality:

$$|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\|$$

- Triangle Inequality:

$$\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$$

- No cancellation law

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{w} \quad \text{does not guarantee} \quad \boldsymbol{v} = \boldsymbol{w}$$

Matrices

An example chosen deliberately

Given the following 3 vectors:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Their linear combinations in 3-D space are $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}$:

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

If we let the 3 vectors go into the columns of a matrix A and let vector $\boldsymbol{x} \coloneqq (x_1 \; x_2 \; x_3)$, we can then rewrite the same equation in matrix form:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

Matrix times vector: linear combination of columns

Matrix times vector produces a linear combination of the columns:

$$A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}$$

Let vector $\boldsymbol{b} \coloneqq (b_1\ b_2\ b_3)$ denote the result of $A\boldsymbol{x}$.

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \boldsymbol{b}$$

If we view matrix A as a function, the input is vector $\boldsymbol{x}$ and the output is vector $\boldsymbol{b}$.

Difference Matrix

Our matrix example A is called a difference matrix because the output vector $\boldsymbol{b}$ contains differences of the components of the input vector $\boldsymbol{x}$. Note that the top difference is $x_1 - 0 = x_1$.

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \boldsymbol{b}$$

Here is an example to show differences of numbers (squares in $\boldsymbol{x}$, odd numbers in $\boldsymbol{b}$.):

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \text{squares}, \qquad A\boldsymbol{x} = \begin{bmatrix} 1-0 \\ 4-1 \\ 9-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \boldsymbol{b}$$

Matrix times vector: dot products with rows

You may already have learned about multiplying $A\boldsymbol{x}$, by using the rows instead of the columns, i.e., taking the dot product of each row with $\boldsymbol{x}$:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0) \cdot (x_1, x_2, x_3) \\ (-1,1,0) \cdot (x_1, x_2, x_3) \\ (0,-1,1) \cdot (x_1, x_2, x_3) \end{bmatrix}$$

If we view matrix A as a function, the input is vector $\boldsymbol{x}$ and the output is vector $\boldsymbol{b}$.

Python example:

Code snippet M2L02-03

Linear equations

Previously, we were given $\boldsymbol{x}$ and want to find out $\boldsymbol{b} = A\boldsymbol{x}$. Now we want to deal with the inverse problem: given a particular $\boldsymbol{b}$, we want to look for $\boldsymbol{x}$:

$$Which\ combination\ of\ \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\ produces\ the\ vector\ \boldsymbol{b}?$$

$$\textbf{Solve}\ \begin{aligned} x_1 \qquad\qquad &= b_1 \\ -x_1 + x_2 \qquad &= b_2 \\ - x_2 + x_3 &= b_3 \end{aligned} \qquad \textbf{Solution}\ \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \end{aligned}$$

Most linear systems are not so easy to solve. In this example, the first equation decided $x_1 = b_1$. Then the second equation produced $x_2 = b_1 + b_2$. The equations could be solved in order (top to bottom) because the matrix A was selected to be low triangular.

Examples of Linear equations

Let's look at two specific choices (0,0,0) and (1,3,5) of $\boldsymbol{b}$:

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ gives } \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \text{ gives } \boldsymbol{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

The first solution (all zeros) is more important than it looks. In words, if the output is $\boldsymbol{b} = \mathbf{0}$, then the input must be $\boldsymbol{x} = \mathbf{0}$. That statement is true for this matrix A, but it is not true for all matrices. Actually, the fact that $\boldsymbol{x}$ has to be $\mathbf{0}$ in order for A to produce a $\mathbf{0}$ vector implies that

- This matrix A is invertible.
- From any given $\boldsymbol{b}$ we can recover $\boldsymbol{x}$.

The Inverse Matrix

The inverse matrix of A

Further to our difference matrix A,

$$A\boldsymbol{x} = \boldsymbol{b} \text{ is solved by } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If the differences of the x's are the b's, the sum of the b's are the x's. Let S denote the "sum" matrix we multiply by $\boldsymbol{b}$. Two observations here:

- For any given $\boldsymbol{b}$ there is one unique solution to $A\boldsymbol{x} = \boldsymbol{b}$.
- In this case, there exists another matrix S that produces $S\boldsymbol{b} = \boldsymbol{x}$. The sum matrix S is the inverse of the difference matrix A. Usually we denote the inverse as A^{-1}, so $A^{-1} = S$ in this case.

Example for matrices A and S

For example: let $\boldsymbol{x} \coloneqq (1,2,3)$ and $\boldsymbol{b} \coloneqq (1,1,1)$.

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } S\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Python example:

Code snippet M2L02-04

Cyclic Differences

Another example chosen deliberately

Given the following 3 vectors:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \qquad \boldsymbol{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now the linear combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ lead to a cyclic difference matrix C:

$$C\boldsymbol{x} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}$$

In this case, for any given $\boldsymbol{b}$, it is impossible to find one unique solution to $C\boldsymbol{x} = \boldsymbol{b}$, because the three equations either have infinitely many solutions or else no solution.

Infinitely many solutions and no solution at all

In this case, if $\boldsymbol{b} = (0,0,0)$, there are infinitely many solutions.

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ is solved by all vectors } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}, c \in \mathbb{R}$$

If $\boldsymbol{b} = (1,3,5)$, there are no solution at all. The following equation has no solution. One quick check is that the sum of left hand side does not equal to the sum of right hand sides

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

What kind of $\boldsymbol{b}$ can give us infinitely many solutions?

Any point $\boldsymbol{b} = (b_1, b_2, b_3)$ in the plane $b_1 + b_2 + b_3 = 0$ can equate the RHS with the LHS:

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

That's because the 3 columns of C can fill only the plane $b_1 + b_2 + b_3 = 0$ rather than the entire 3-D space. Any point which is not in the plane cannot be synthesized by the 3 columns.

Python example:

Code snippet M2L02-05