Prep Course
Module 2
Lecture 3

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Ways to Multiply 2 Matrices

First way: dot product of row and column

Given matrices A, B, C with $AB = C$,

$$AB = \begin{bmatrix} \boldsymbol{a_1} \\ \vdots \\ \boldsymbol{a_n} \end{bmatrix} \begin{bmatrix} \boldsymbol{b_1} & \cdots & \boldsymbol{b_n} \end{bmatrix} = C$$

where $\boldsymbol{a_i}$ is the ith row of A and $\boldsymbol{b_i}$ is the ith column of B. Then each entry C_{ij} at row i and column j of C is

$$C_{ij} = \boldsymbol{a_i} \cdot \boldsymbol{b_j}$$

Second way: linear combinations of columns of A

Given matrices A, B, C with $AB = C$,

$$AB = \begin{bmatrix} \boldsymbol{a_1} & \cdots & \boldsymbol{a_n} \end{bmatrix} \begin{bmatrix} \boldsymbol{b_1} & \cdots & \boldsymbol{b_n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c_1} & \cdots & \boldsymbol{c_n} \end{bmatrix} = C$$

where $\boldsymbol{a_i}$ is the ith column of A and $\boldsymbol{b_i}$ is the ith column of B. Let $\boldsymbol{b_i} \coloneqq \left(b_{i,1}, b_{i,2}, \ldots, b_{i,n}\right)$. Then $\boldsymbol{c_i}$, the ith column of C, is a linear combination of all columns of A with respect to $\boldsymbol{b_i}$.

$$\boldsymbol{c_i} = b_{i,1}\boldsymbol{a_1} + b_{i,2}\boldsymbol{a_2} + \cdots + b_{i,n}\boldsymbol{a_n}$$

Third way: linear combinations of rows of B

Given matrices A, B, C with $AB = C$,

$$AB = \begin{bmatrix} \boldsymbol{a_1} \\ \vdots \\ \boldsymbol{a_n} \end{bmatrix} \begin{bmatrix} \boldsymbol{b_1} \\ \vdots \\ \boldsymbol{b_n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c_1} \\ \vdots \\ \boldsymbol{c_n} \end{bmatrix} = C$$

where $\boldsymbol{a_i}$ is the ith row of A and $\boldsymbol{b_i}$ is the ith row of B. Let $\boldsymbol{a_i} \coloneqq (a_{i,1}, a_{i,2}, \dots, a_{i,n})$. Then $\boldsymbol{c_i}$, the ith row of C, is a linear combination of all rows of B with respect to $\boldsymbol{a_i}$.

$$\boldsymbol{c_i} = a_{i,1}\boldsymbol{b_1} + a_{i,2}\boldsymbol{b_2} + \cdots + a_{i,n}\boldsymbol{b_n}$$

Important properties of matrix multiplication

Given matrices A, B, C with compatible dimensions:

- Associative Law is valid:

$$(AB)C = A(BC)$$

- Commutative Law is NOT true in general:

$$AB \neq BA$$

- Distributive Law is valid:

$$A(B + C) = AB + AC, \qquad (A + B)C = AC + BC$$

- Scalar multiplication: given scalar c

$$c(AB) = (cA)B = (Ac)B = A(cB) = A(Bc) = (AB)c$$

- Transpose:

$$(AB)^T = B^T A^T$$

- Inverse:

$$(AB)^{-1} = B^{-1} A^{-1}$$

Elimination with Matrices

Elimination: to produce an upper triangular matrix

Given the following system of linear equations

$$\begin{aligned} x + 2y + z &= 2 \\ 3x + 8y + z &= 12 \\ 4y + z &= 2 \end{aligned} \iff \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix} \iff A\boldsymbol{x} = \boldsymbol{b}$$

Elimination is done by a series of row operations

Augmented Matrix $[A|\mathrm{b}]$

Upper Triangular Matrix U

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow[\textit{entry}\ (2,1)]{\textit{eliminate}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow[\textit{entry}\ (3,2)]{\textit{eliminate}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

Coefficient Matrix A

All entries in green have to be 0.

If all diagonal entries are non-zero, the elimination is successful.

Backward substitution

Having done a series of row operations, we transform $A\boldsymbol{x} = \boldsymbol{b}$ into $U\boldsymbol{x} = \boldsymbol{c}$, where $\boldsymbol{c} = (2,6,-10)$. With the upper triangular U, we can easily solve the equations in reverse order, meaning, from bottom to top.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right] \Rightarrow z = -2, y = 1, x = 2$$

Pivots

A pivot is an entry in a matrix that satisfies the following:

- It's the first non-zero entry at its row;
- All entries below it, if any, are zero.

For example, the upper triangular U has three pivots on the diagonal:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

If we can successfully transform a matrix K to an upper triangular matrix where every entry on the diagonal is a pivot, then we say the upper triangular matrix has a full set of pivots and the original matrix K is invertible.

Elimination matrices

Previously in matrix A, we subtracted 3 times row 1 from row 2 to eliminate entry (2,1):

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[]{\substack{eliminate \\ entry\ (2,1)}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

The same operation can be done by multiplication by an elimination matrix $E_{2,1}$ as follows:

$$E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where the subscript 2,1 means we are eliminating entry (2,1)

$$E_{2,1}A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

Elimination by matrix multiplication

Now we want to express the following row operations using matrix multiplication:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[\text{entry } (2,1)]{\text{eliminate}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[\text{entry } (3,2)]{\text{eliminate}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

In addition to $E_{2,1}$, let's define $E_{3,2}$ as follows:

$$E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

where the subscript 2,1 means we are eliminating entry (2,1)

$$E_{3,2}E_{2,1}A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

Observations about $E_{3,2}E_{2,1}$

Basically the matrix $E_{3,2}E_{2,1}$ does all the row operations in one go.

$$E_{3,2}E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -2 & 1 \end{bmatrix}$$

Note that

- Entry (2,1) is -3
- Entry (3,2) is -2
- Entry (3,1) is 6
- the product of : $E_{3,2}$ and $E_{2,1}$ is still lower triangular
- Don’t be confused. An elimination matrix does not have to be lower triangular.

Properties of triangular matrices

- The sum of two lower triangular matrices is lower triangular.
- The product of two lower triangular matrices is lower triangular.
- The inverse of an invertible lower triangular matrix is lower triangular.
- The product of an lower triangular matrix by a scalar is an lower triangular matrix.

The above properties also hold for upper triangular matrices.

Inverse Matrices

Invertible matrices

A n by n matrix A is invertible if there exists a matrix A^{-1} such that

$$A^{-1}A = I \text{ or } AA^{-1} = I$$

- $A^{-1}A = I$ if and only $AA^{-1} = I$
- The inverse exists if and only if elimination produces n pivots.
- The inverse is unique.
- If A is invertible, then the only solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.
- If there exists a non-zero vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \mathbf{0}$, then A is not invertible, because no matrix A^{-1}can bring $\mathbf{0}$ back to $\boldsymbol{x}$.

Inverse of a diagonal matrix

A n by n diagonal matrix A is invertible if no diagonal entries are zero.

$$\text{If } A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}, \text{ then } A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}$$

Inverse of product AB

(Theorem) If A and B are invertible then so is AB. Moreover, the inverse of product AB is

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

(Corollary) If A, B, C are all invertible, then

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Inverse of elimination matrices

Basically the inverse of an elimination matrix is to recover what has been eliminated. Take $E_{2,1}$ and $E_{3,2}$ as example:

$$E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

And the inverses of them are as follows:

$$E_{2,1}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E_{3,2}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

However, let's compare $E_{3,2}E_{2,1}$ and its inverse $E_{2,1}^{-1}E_{3,2}^{-1}$

$$E_{3,2}E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -2 & 1 \end{bmatrix} \text{ and } E_{2,1}^{-1}E_{3,2}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Gauss-Jordan Elimination

Finding an inverse is like solving linear equations

In $\mathbb{R}^n$, let e_i denote a vector which has 1 at the ith component and 0 at the rest. For example, in $\mathbb{R}^3$, $\boldsymbol{e_1}$ = (1,0,0), $\boldsymbol{e_2}$ = (0,1,0), $\boldsymbol{e_3}$ = (0,0,1). Given a $n\ by\ n$ matrix A, finding its inverse is equivalent to solving the following equations:

$$AA^{-1} = A\begin{bmatrix}\boldsymbol{x_1} & \dots & \boldsymbol{x_n}\end{bmatrix} = \begin{bmatrix}\boldsymbol{e_1} & \dots & \boldsymbol{e_n}\end{bmatrix} = I$$

where $\boldsymbol{x_i}$'s represents the columns of A^{-1} and $\boldsymbol{e_i}$'s are the columns of identity matrix I. Previously we have only one system of equations $A\boldsymbol{x} = \boldsymbol{b}$. Now we have n systems of equations. Gauss-Jordan elimination helps us solve the n systems together.

Example of Gauss-Jordan elimination (1/3)

Given matrix A as follows:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Initially the augmented matrix $=[A|I]$.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{entry } (2,1)]{\text{eliminate}} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{entry } (3,2)]{\text{eliminate}} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

Example of Gauss-Jordan elimination (2/3)

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \xrightarrow{\substack{\text{eliminate} \\ \text{entry } (2,3)}} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\text{eliminate} \\ \text{entry } (1,2)}} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\text{divide row} \\ \text{by pivot}}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$$

Example of Gauss-Jordan elimination (3/3)

Here are the matrices corresponding the elimination steps

$$E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; E_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix} \; E_{2,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{1,2} = \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; D = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 3/4 \end{bmatrix}$$

Initially we have

$$AA^{-1} = I$$

Then we multiply both sides by the matrices above:

$$DE_{1,2}E_{2,3}E_{3,2}E_{2,1}AA^{-1} = DE_{1,2}E_{2,3}E_{3,2}E_{2,1}I$$

$$\Rightarrow A^{-1} = DE_{1,2}E_{2,3}E_{3,2}E_{2,1}$$