Finding optimal goals in card games with uncertainty

1 Introduction

Our central question is the following: given the history of the game, what is the optimal goal to pursue?

There are 52 possible goals (i.e. straight flushes of 6 cards). In our goal-focused model, we will give them each different weights in the weight vector $\vec{\mathbf{w}} \in \mathbb{R}^{52}$. At the beginning before any cards are dealt, every goal is equally (un)optimal, and $\vec{\mathbf{w}} := \vec{\mathbf{0}}$.

2 The Metrics

To begin with a simple model, I propose to evaluate goals with three metrics: overlap, distance, and likelihood.

2.1 Overlap

Overlap measures how many cards in common a certain goal G has with a set of cards C, typically the ones visible (in hands and on the table). Formally, Overlap is defined as the following:

$$\textsc{Overlap}(C, G) = |C \cap G| \tag{1}$$

Overlap takes on values in the interval $[0, 6]$. It is independent of game history.

2.2 Distance

Distance measures how close a certain goal G is from a set of cards C. It is independent of game history.

2.3 Likelihood

Likelihood measures how likely a certain goal G can be obtained given all previous history H. Likelihood is defined as the following:

$$\mathcal{L}(G, H) = \sum_{g \in G} (1 - P(g \text{ has been discarded} | H)) \tag{2}$$

Given a card g and history H, we can find $P(g \text{ has been discarded}|H)$ in the following way. First, we assume that the history H contains information about r_i, the number of cards reshuffled at round i for all i, as well as which cards we have seen and not seen. Let s be the index of the round that g was last seen, let n be the number of rounds that have occurred between s and now, and let D_i be the size of the deck at round i. Then we have:

$$P(g \text{ has been discarded}|H) = \begin{cases} 0 & g \in \text{ hands or table} \\ \frac{4-r_s}{4-r_s+r_s\prod_{j=1}^{n}(1-4/D_{s+j})} & \text{o.w.} \end{cases} \tag{3}$$

Proof. It is easy to see that a card g cannot have been discarded if it is currently in the hands or on the table. Now, consider the case where g has not been seen for n rounds (since round s). Let F be the event that g was discarded at around s, and let U be the event that we haven't seen g for n rounds. By Bayes' Rule, we have:

$$\begin{aligned} P(F|U) &= \frac{P(U|F)P(F)}{P(U)} && (4) \\ &= \frac{(1)\left(\frac{4-r_s}{4}\right)}{P(U|F)P(F)+P(U|F^c)P(F^c)} && (5) \\ &= \frac{(4-r_s)/4}{(4-r_s)/4+(r_s/4)P(U|F^c)} && (6) \end{aligned}$$

Let's look at $P(U|F^c)$. Suppose g is known to be reshuffled in the previous round, and the size of the deck was updated to D. The probability of not drawing g in this round is

$$\frac{\binom{D-1}{4}}{\binom{D}{4}} = \frac{D-4}{D}, \tag{7}$$

and this pattern continues for every subsequent round. This gives us

$$P(U|F^c) = \prod_{j=1}^{n} \frac{D_{s+j}-4}{D_{s+j}}. \tag{8}$$

Using this result and simplifying, we finally obtain

$$P(F|U) = \frac{4-r_s}{4-r_s+r_s\prod_{j=1}^{n}(1-\frac{4}{D_{s+j}})}. \tag{9}$$

$\square$

Note that we can also write the size of the deck at round k as

$$D_k = 46 + \sum_{i=1}^{k-1} r_i - 4k. \tag{10}$$

Proof. In the base case, $k = 1$ yields $46 - 4(1) = 42$, which is correct since we deal 10 cards from the original 52 in the first round. Suppose now that (10) is true for $k = n$. Then we have:

$$\begin{aligned} D_{n+1} &= D_n + r_n - 4 && (11) \\ &= (46 + \sum_{i=1}^{n-1} r_i - 4n) + r_n - 4 && (12) \\ &= 46 + \sum_{i=1}^{n} r_i - 4(n+1) && (13) \end{aligned}$$

By induction, (10) holds for all values of k. □