Finding optimal goals in card games with uncertainty

1   Introduction

Our central question is the following: given the history of the game, what is the optimal goal to pursue?

There are 52 possible goals (i.e. straight flushes of 6 cards). We give them each different weights in the weight vector $\vec{\mathbf{w}} \in \mathbb{R}^{52}$, which is updated every time we learn new information about the game. At the beginning before any cards are dealt, every goal is equally (un)optimal, and $\vec{\mathbf{w}} := \vec{\mathbf{0}}$.

2   The Metrics

I propose to evaluate goals with three metrics: OVERLAP, DISTANCE, and LIKELIHOOD.

2.1   Overlap

OVERLAP measures how many cards in common a certain goal G has with a set of cards C, typically the ones visible (in hands and on the table). Formally, OVERLAP is defined as the following:

$$\text{OVERLAP}(C, G) = |C \cap G| \tag{1}$$

OVERLAP takes on values in the interval $[0, 6]$. It is independent of game history.

2.2   Distance

DISTANCE measures how close a certain goal G is from a set of cards C. It is independent of game history.

IS DISTANCE JUST 6 - OVERLAP?!?!?!?!

2.3   Likelihood

LIKELIHOOD measures how likely a certain goal G can be obtained given all previous history H. LIKELIHOOD is defined as the following:

$$\mathcal{L}(G, H) = \sum_{g \in G} (1 - P(g \text{ has been discarded}|H)) \tag{2}$$

Given a card g and history H, we can find $P(g$ has been discarded$|H)$ in the following way. First, we assume that the history H contains information about r_i, the number of cards reshuffled at round i for all i, as well as which cards we have seen and not seen. Let s be the index of the round that g was last seen, let n be the number of rounds that have occurred between s and now, and let D_i be the size of the deck at round i. Then we have:

$$P(g \text{ has been discarded}|H) = \begin{cases} 0 & g \in \text{ hands or table, or unseen} \\ \frac{4-r_s}{4-r_s+r_s \prod_{j=1}^{n}(1-4/D_{s+j})} & \text{o.w.} \end{cases} \tag{3}$$

Proof. It is easy to see that a card g cannot have been discarded if it is currently in the hands, on the table, or unseen (in the deck). Now, consider the case where g has not been seen for n rounds (since round s). Let F be the event that g was discarded at around s, and let U be the event that we haven't seen g for n rounds. By Bayes' Rule, we have:

$$P(F|U) = \frac{P(U|F)P(F)}{P(U)} \tag{4}$$

$$= \frac{(1)\left(\frac{4-r_s}{4}\right)}{P(U|F)P(F) + P(U|F^c)P(F^c)} \tag{5}$$

$$= \frac{(4-r_s)/4}{(4-r_s)/4 + (r_s/4)P(U|F^c)} \tag{6}$$

Let's look at $P(U|F^c)$. Since we are conditioning on F^c, we know that g was reshuffled in the previous round, and the size of the deck has been updated to D. The probability of not drawing g in this round is

$$\frac{\binom{D-1}{4}}{\binom{D}{4}} = \frac{D-4}{D}, \tag{7}$$

and this pattern continues for every subsequent round. This gives us

$$P(U|F^c) = \prod_{j=1}^{n} \frac{D_{s+j}-4}{D_{s+j}}. \tag{8}$$

Using this result and simplifying, we finally obtain

$$P(F|U) = \frac{4-r_s}{4-r_s+r_s \prod_{j=1}^{n}(1-\frac{4}{D_{s+j}})}. \tag{9}$$

$\square$

Note that we can also write the size of the deck at round k as

$$D_k = 46 + \sum_{i=1}^{k-1} r_i - 4k. \tag{10}$$

Proof. In the base case, $k = 1$ yields $46 - 4(1) = 42$, which is correct since we deal 10 cards from the original 52 in the first round. Suppose now that (10) is true for $k = n$. Then we have:

$$D_{n+1} = D_n + r_n - 4 \tag{11}$$

$$= (46 + \sum_{i=1}^{n-1} r_i - 4n) + r_n - 4 \tag{12}$$

$$= 46 + \sum_{i=1}^{n} r_i - 4(n+1) \tag{13}$$

By induction, (10) holds for all values of k. $\qquad\square$

3   Evaluating Goals

Every time the history H is updated and we observe a new set of cards C, the weights w_i for each goal G_i are updated with a linear combination of the three metrics:

$$w_i := \alpha_0 \cdot \text{OVERLAP}(G_i, C) + \alpha_1 \cdot \text{DISTANCE}(G_i, C) + \alpha_2 \cdot \text{LIKELIHOOD}(G_i, H) \tag{14}$$

The values of $\{\alpha\}$ depend on how we want to weight the different metrics. A less rational player might weight OVERLAP more than LIKELIHOOD, for example.