Formulation

$$\frac{dx}{dt} = a\left(1-\frac{x}{K}\right)x - bxy \rightarrow \text{prey}$$

$$\frac{dy}{dt} = cxy - dy \quad \longrightarrow \text{predator}$$

① Find equilibrium condition. $\rightarrow \frac{dx}{dt}=0,\ \frac{dy}{dt}=0$

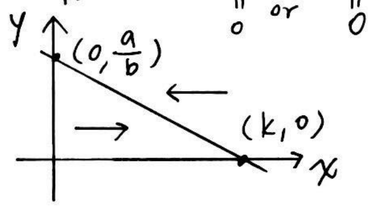
$$x_e\left(a - \frac{a x_e}{K} - b y_e\right) = 0$$

$$(c x_e - d) y_e = 0$$

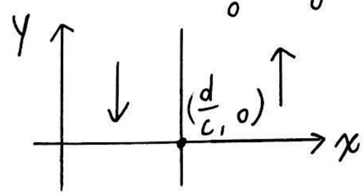
$$(x_e, y_e) = (0,0) \text{ or } (K,0) \text{ or } \left(\frac{d}{c}, \frac{a}{b} - \frac{ad}{bcK}\right)$$

② Plot isocline to determine the stability.

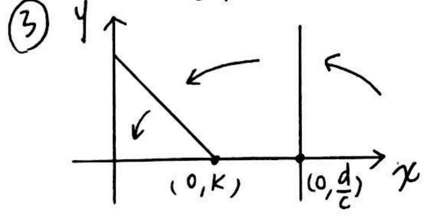
$$\frac{dx}{dt}=0 \Rightarrow \underset{0}{\underset{\shortparallel}{x}}\ \text{or}\ \underset{0}{\underset{\shortparallel}{\left(a-\frac{ax}{K}-by\right)}}=0 \Rightarrow \begin{cases} x=0 \\ \text{or} \\ a-\frac{ax}{K}-by=0 \Rightarrow \text{過 } \left(0,\frac{a}{b}\right),\ (k,0) \text{ 的直線} \end{cases}$$



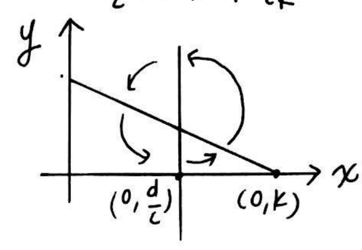
$$\frac{dy}{dt}=0 \Rightarrow \underset{0}{\underset{\shortparallel}{y}}\ \text{or}\ \underset{0}{\underset{\shortparallel}{(cx-d)}}=0 \Rightarrow \begin{cases} y=0 \\ \text{or} \\ x=\frac{d}{c} \end{cases}$$



③ $k < \frac{d}{c}$, 即 $\frac{d}{ck} > 1$ 時:



$\frac{d}{c} < k$, 即 $\frac{d}{ck} < 1$ 時:



④

$$\begin{bmatrix} \frac{d\Delta x}{dt} \\ \frac{d\Delta y}{\Delta t} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x}\Big|_{\substack{x_e \\ y_e}} & \frac{\partial f}{\partial y}\Big|_{\substack{x_e \\ y_e}} \\ \frac{\partial g}{\partial x}\Big|_{\substack{x_e \\ y_e}} & \frac{\partial g}{\partial g}\Big|_{\substack{x_e \\ y_e}} \end{bmatrix} = \lambda max \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$\lambda max = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a - \frac{2ax}{k} - by & -bx \\ cy & cx-d \end{bmatrix}$$

$\boxed{4-1}$ $(x_e, y_e) = \left(\frac{d}{c}, \frac{a}{b} - \frac{ad}{bck}\right)$

$$a - \frac{2ad}{kc} - a - \frac{ad}{ck} = \frac{ad-2ad}{ck}$$

$$\lambda max = \begin{bmatrix} a - \frac{2ax}{k} - by & -bx \\ cy & cx-d \end{bmatrix} \xrightarrow[y=y_e]{x=x_e \text{ 代入}} \begin{bmatrix} -\frac{ad}{ck} & -\frac{bd}{c} \\ \frac{a(c-\frac{d}{k})}{b} & 0 \end{bmatrix}$$

λmax is a maximum solution of:

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = 0$$

$$\rightarrow \lambda^2 - \left(-\frac{ad}{kc}\right)\lambda + 0 - \left(\frac{-bd}{c} \cdot \frac{a(c-\frac{d}{k})}{b}\right) = 0$$

$$\rightarrow \lambda^2 + \frac{ad}{kc}\lambda + 0 + \frac{ad(c-\frac{d}{k})}{c} = 0$$

$$\lambda = \lambda_1 = \frac{1}{2}\left(-\frac{ad}{kc} + \sqrt{\left(\frac{ad}{kc}\right)^2 - \frac{4ad(c-\frac{d}{k})}{c}}\right)$$

$$= \frac{1}{2}\left(-\frac{ad}{kc} + \sqrt{\frac{ad}{ck} \cdot \left(\frac{ad}{ck} - 4(ck-d)\right)}\right)$$

$$\lambda_2 = \frac{1}{2}\left(-\frac{ad}{kc} - \sqrt{\frac{ad}{ck} \cdot \left(\frac{ad}{ck} - 4(ck-d)\right)}\right)$$

當 $ck-d<0$, 即 $\frac{d}{ck}>1 \rightarrow \lambda_1>0, \lambda_2<0 \rightarrow$ 一正一負 unstable

當 $ck-d>0$, 即 $\frac{d}{ck}<1 \rightarrow \lambda_1<0, \lambda_2>0 \rightarrow$ 一正一負 unstable

$\boxed{4\text{-}2}$ $(X_e, Y_e) = (0, 0)$

$$\lambda_{max} = \begin{bmatrix} a - \frac{2ax}{K} - by & -bx \\ cy & cx-d \end{bmatrix} \xrightarrow[y=y_e]{x=x_e \text{代入}} \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$$

λ_{max} is a maximum solution of:

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = 0$$

$$\rightarrow \lambda - (a-d)\lambda + (-ad) - 0 = 0$$

$$\rightarrow \lambda = \begin{cases} \lambda_1 = \frac{1}{2}\left(a-d + \sqrt{(a-d)^2 + 4ad}\right) = \frac{1}{2}(a-d+a+d) = a \\ \lambda_2 = \frac{1}{2}\left(a-d - \sqrt{(a-d)^2 + 4ad}\right) = \frac{1}{2}(a-d-a-d) = -d \end{cases} \rightarrow \text{一正一負 unstable}$$

$\boxed{4\text{-}3}$ $(X_e, Y_e) = (K, 0)$

$$\lambda_{max} = \begin{bmatrix} a - \frac{2ax}{K} & -bx \\ cy & cx-d \end{bmatrix} \xrightarrow[y=y_e]{x=x_e \text{代入}} \begin{bmatrix} -a & -bK \\ 0 & cK-d \end{bmatrix}$$

λ_{max} is a maximum solution of:

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = 0$$

$$\rightarrow \lambda - (-a+cK-d)\lambda + (-a)(cK-d) - 0 = 0$$

$$\sqrt{(-a+cK-d)^2 - 4(-a)(cK-d)} = \sqrt{(-a-(cK-d))^2}$$

$$\rightarrow \lambda = \begin{cases} \lambda_1 = \frac{1}{2}\left(-a+cK-d + (-a-(cK-d))\right) = \frac{1}{2}(-2a) = -a \\ \lambda_2 = \frac{1}{2}\left(-a+cK-d - (-a-(cK-d))\right) = \frac{1}{2}(-2d) = cK-d \end{cases}$$

$cK-d < 0 \rightarrow \lambda_1 < 0, \lambda_2 < 0 \rightarrow$ stable

$cK-d > 0 \rightarrow \lambda_1 < 0, \lambda_2 > 0 \rightarrow$ unstable