IB120/201 - Lab 3

Ordinary Differential Equations & Systems of Equations

Due Date: February 14, 2020

University of California, Berkeley *GSI: Naveed Ziari*

In this lab, we will learn about how to implement and solve differential equations and systems of equations, which are applicable to many phenomena and modeling efforts in biology.

Background

Systems of Equations

Performing operations on large and multidimensional datasets involves solving a system of equations. Instead of a simple algebra problem where one must solve for *one* unknown (i.e. $2x + 1 = 7$), we must solve for multiple unknown variables, with more than one equation to guide us. Consider the following example:

$$5x_1 + 3.5x_2 = 4.38 \tag{1}$$

$$x_1 + x_2 = 1 \tag{2}$$

In order to solve this by hand, one can solve for x_1 in terms of x_2 and plug it into equation 1. This is feasible in the simple case of 2 equations and 2 unknowns, but what if we have more than 10 unknowns? Solving by hand all of a sudden seems like a daunting task. It is therefore necessary to employ linear algebra methods and represent the coefficients in matrix format. Now one can solve for $x = [x_1 x_2]$ with the following setup:

$$Ax = b$$

$$A = \begin{bmatrix} 5 & 3.5 \\ 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4.38 \\ 1 \end{bmatrix}$$

Differential Equations

Differential equations relate a function to its derivative. For instance, consider the following equation:

$$\frac{dx}{dt} = \alpha x \qquad x(0) = 1$$

This is a way of representing exponential growth - the slope of the function at any given time t is proportional to the value x takes. In order to solve this, we need to separate the variables and integrate:

$$\int \frac{dx}{x} = \int \alpha dt$$

Taking the integral yields the following solution:

$$\ln(x) + C = \alpha t$$

In order to resolve C, we must employ the initial condition $x(0) = 1$. Solving for C by plugging in 0 for t and 1 for x, we get $C = 0$, so the final solution is:

$$e^{ln(x)} = e^{\alpha t} \longrightarrow x = e^{\alpha t}$$

Logistic Growth

We can revisit the logistic growth model from last lab with a differential equation model, which still contains the parameters for reproductive rate and carrying capacity:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$$

Protocol & Questions

1. Come up with a system of equations with 3 unknowns and 3 equations (in contrast to the example with 2 unknowns and 2 equations) and solve it.

2. Solve the following differential equation exactly as was done in the example. You may do it by hand or with code:
$$\frac{dx}{dt} = \alpha x^2 \qquad x(0) = 1$$

3. Write code that models the predator-prey relationship (Lotka-Volterra model).
$$\frac{dV}{dt} = rV - \overset{k}{c}VP \qquad \text{c is actually k}$$
$$\frac{dP}{dt} = e\underset{k}{c}VP - dP$$

4. Solve it with the following parameters and plot the results: $r = 1.8, k = 0.4, e = 0.12, d = 1$

5. **BONUS:** The discrepancy between the timing of the oscillations in the predator-prey model is called the *phase*. Figure out how to solve for the phase and plot it.