MARIE CURIE ACTIONS

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# A filtering framework for Finite Volume / Element schemes

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# Post-processing data with SIAC filters

❶ **Superconvergence:** for a DG/FEM solution of degree $p$

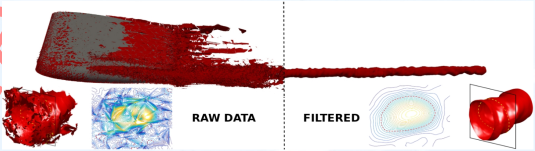
- Proven $2p+1$ order ($L^2$ and $L_\infty$ norms) for <u>linear</u> PDEs
- Observed $2p+1$ order for <u>non-linear</u> PDEs

❷ **Smoothness:** the filtered data is a (local) $2p+1$ polynomial

- Removes oscillations in the error
- Recovers continuity levels across element interfaces

**Applications**

Flow visualization
Shock detection
Multiresolution analysis
Cut cells




*Jallepalli, Docampo, Ryan, Haimes, Kirby* (TVCG 2017)

# Post-processing data with SIAC filters

**Goal:**
- Establish a filtering framework for general purpose
- Create a standalone tool for general applications

- **Overview of the SIAC kernel**

  Basis functions, moment preservation and superconvergence

- **Filtering challenges**
- **The software package**

# Overview of the Smoothness-Increasing Accuracy-Conserving (SIAC) filter

**We post-process our data via convolution:**

$$\text{data}^{\star}(x) = \int_{\mathbb{R}} \boldsymbol{K}(y-x) \cdot \text{data}(y)\,dy$$

$$\textbf{SIAC kernel:} \quad \boldsymbol{K}^{(r+1,n)}(\cdot) = \sum_{\gamma=1}^{r+1} \boldsymbol{c}_{\gamma} \cdot \boldsymbol{B}_{\mathsf{T}_{\gamma},n}(\cdot)$$

- $\boldsymbol{c}$: kernel **weights** chosen to maintain $r$ moments
- $\boldsymbol{B}_{\mathsf{T},n}$: $n^{th}$-order central **B-spline** with knot sequence $\mathbf{T}$

# SIAC kernel weights: $K^{(r+1,n)} = \sum_{\gamma=1}^{r+1} c_\gamma \cdot B_{\tau_\gamma,n}$

**Filtering principle:** preserve the accuracy in the data

Choose the $c_\gamma$'s to satisfy

$$\int_{\mathbb{R}} K(x)dx = 1, \qquad \int_{\mathbb{R}} K(x)x^j dx = 0, \quad j = 1, 2, \ldots, r$$

**Consistency** + **Moment Conditions**

This is equivalent to impose **polynomial reproduction**

$$\int_{\mathbb{R}} K(x-y) \cdot y^j dx = x^j, \quad j = 0, 1, \ldots, r$$

**To extract $2p+1$ order, the kernel must satisfy $2p$ moments**

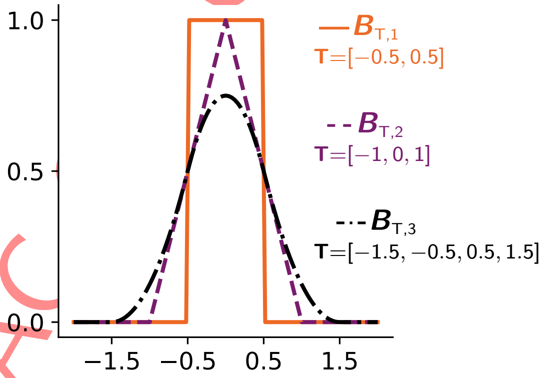
# SIAC kernel basis functions: $K^{(r+1,n)} = \sum_{\gamma=1}^{r+1} c_\gamma \cdot B_{T_\gamma,n}$

## B-splines

$B_{T,1} = \chi_{[-\frac{1}{2}, \frac{1}{2})}$

$B_{T,n} = B_{T,n\text{-}1} \star B_{T,1}$

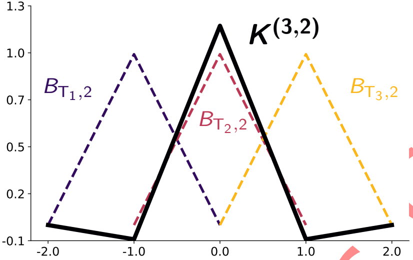
$\mathbf{T}$ (uniform) knot-sequence




**Properties:**

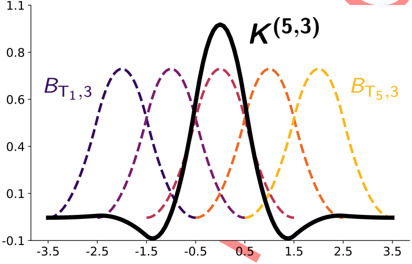
- Compact support (smaller integral region)
- Smoothness of $n-2$ (remove oscillations)
- Derivatives as divided differences (superconvergence theory)

# SIAC Kernel: $K^{(r+1,n)}(\cdot) = \sum_{\gamma=1}^{r+1} c_\gamma B_{T_\gamma,n}(\cdot)$




**$r = 2,\ n = 2$**

→ Preserves 2 moments
→ Continuity: $\mathcal{C}^0$
→ Optimal order: 3




**$r = 4,\ n = 3$**

→ Preserves 4 moments
→ Continuity: $\mathcal{C}^1$
→ Optimal order: 5

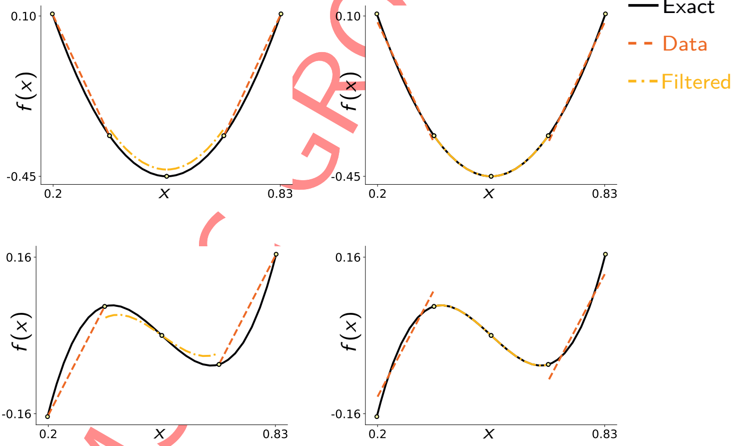
# Polynomial functions: filtering linear data ($p = 1$)

*Docampo, Jacobs, Li, Ryan* (CAF 2020)

**Target** | **Interpolant** **(No Galerkin orthogonality)** | **$L^2$-projection**

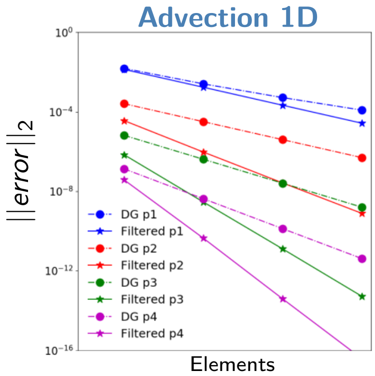
**Quadratic Function**

**Cubic Function**




**The SIAC kernel recovers the exact function for the $L^2$-projected data!**

# SIAC filtering: superconvergence for DG solutions



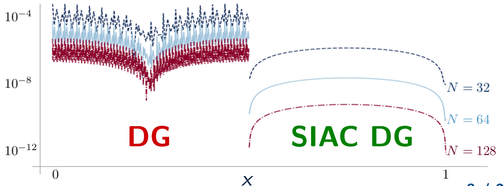

→ DG approximation order: $p+1$

→ **SIAC DG order: $2p+1$**

✔ Reduces oscillations

✔ General error reduction

**Point-wise errors (log) for $p = 2$**



# Post-processing data with SIAC filters

✔ **Overview of the SIAC kernel**

Moment preservation + B-splines + $L^2$-initialization
$\Rightarrow$ **provable** $2p+1$ **superconvergence**

➢ **Filtering challenges**

Multidimensional data, domain boundaries and kernel scaling

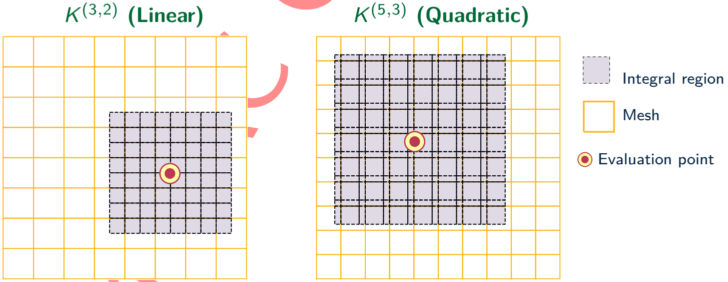
➢ **The software package**

# Traditional SIAC filter: tensor product structure

**2D data: $K = k_x \otimes k_y$**

$$data^{\star}(\overline{x,y}) = \frac{1}{h_x \cdot h_y} \int_{\mathbb{R}} \int_{\mathbb{R}} k_x(\overline{x} - x) \cdot k_y(\overline{y} - y) data(x,y) dydx$$

**Computation**: split integral based on elements and spline breaks




| | $K^{(3,2)}$ (Linear) | $K^{(5,3)}$ (Quadratic) |
|---|---|---|
| **Filter Footprint** | | |
| **Total number of Integrals** | **64** | **196** |

# LSIAC filter: a computationally efficient kernel

*Docampo, Mirzargar, Ryan, Kirby (SISC 2017)*

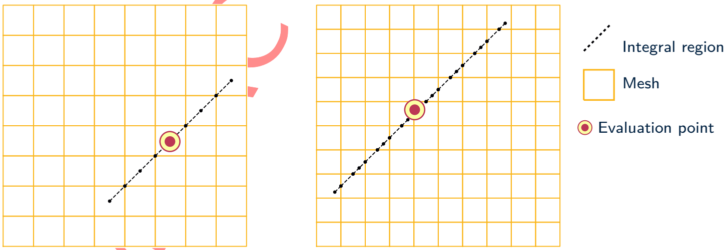
**2D case: $K = k_\Gamma$,**


$$data^\star(\overline{x,y}) = \frac{1}{h}\int_{\mathbb{R}} k_\Gamma(t)\,data(\Gamma(t))dt, \quad \Gamma = \overline{(x,y)} + h(\cos\theta, \sin\theta)$$

**Computation**: split integral based on elements and spline breaks

$K^{(3,2)}$ **(Linear)** $K^{(5,3)}$ **(Quadratic)**

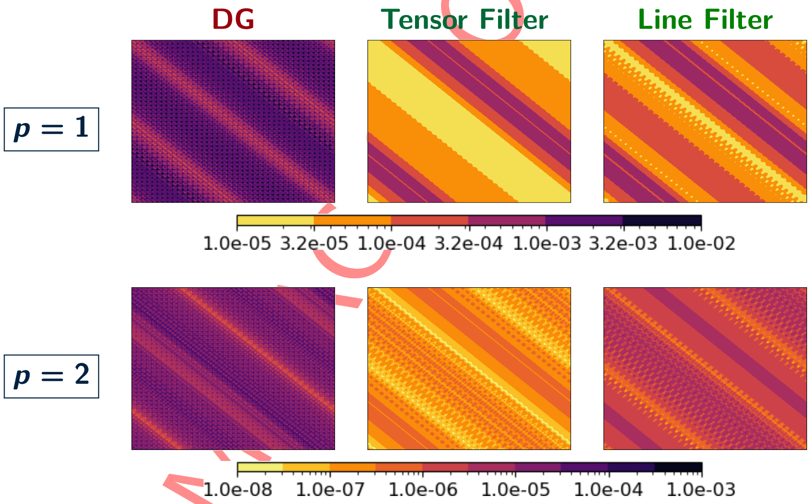
**Filter Footprint**



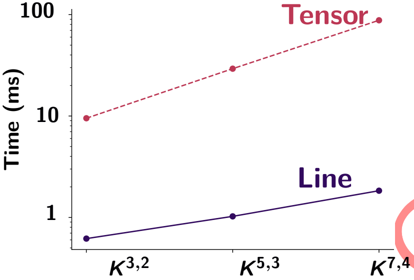

| | | |
|---|---|---|
| **Total number of Integrals** | **8** (tensor: 64) | **21** (tensor: 196) |

# Filters performance: errors for a non-linear equation

**Error contours** for inviscid Burgers equation with exact solution $u(x, y, t = 1.0) = \sin(x + y - t)$ using $40^2$ elements:



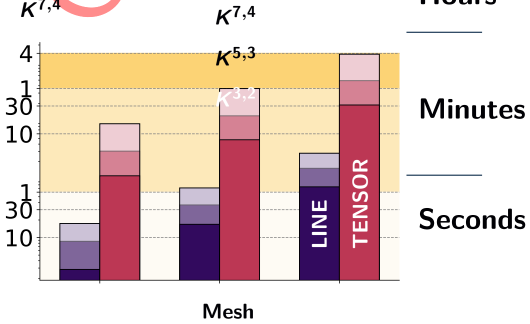
# Filter performance: CPU times




**Single point**

**Entire field**

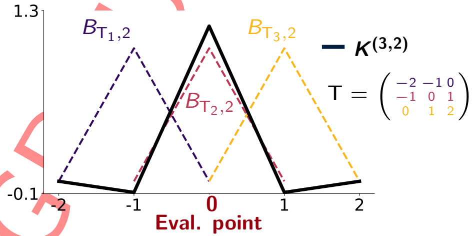
25 points per element



# The non-symmetric RKLV (SIAC) kernel

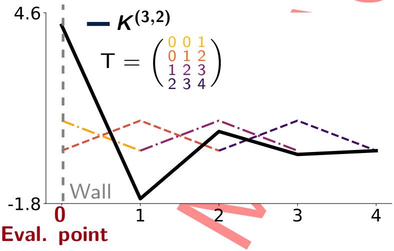
*Ryan, Li, Kirby, Vuik* (SISC 2014)

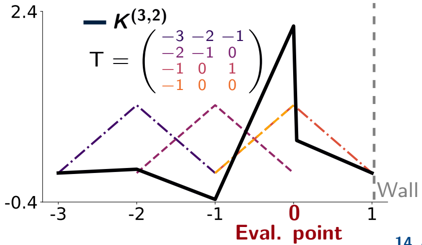
Symmetric filter: equal amount of information from both sides.




**What if the filter doesn't fit?**

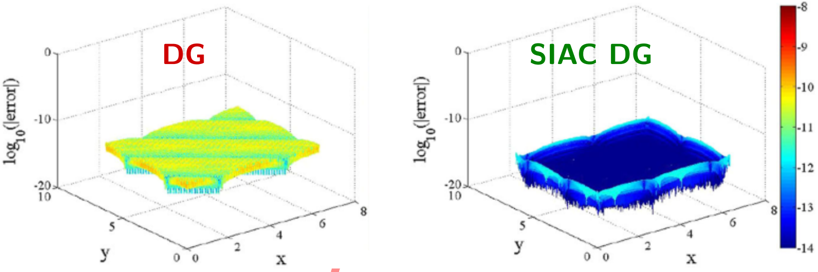
**One-sided filters: shift the knot matrix**



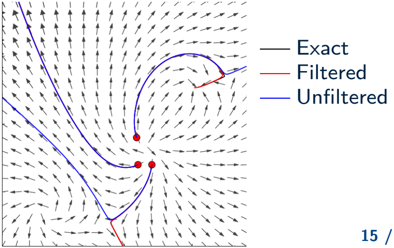
# RKLV kernel: filtering near boundaries

Less effective than the symmetric kernel. Yet, **increased order and error reduction**.



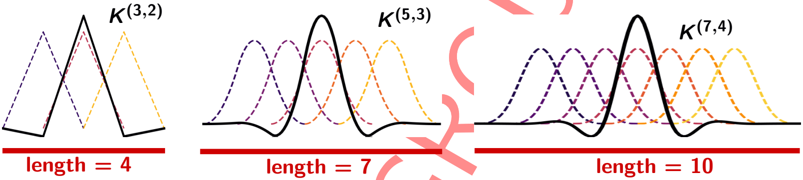

2D advection for $p = 4$ and $80 \times 80$ elements, *Ryan, Li, Kirby, Vuik, (SISC 2014).*

**Flow visualization:**
filtering (backwards) along streamlines
*Docampo (PhD thesis 2017)*

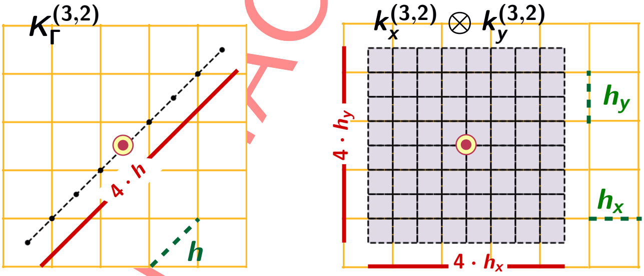


# Footprint: kernel structure and scaling

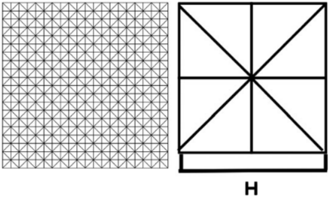
**Kernel size:** determined by the number & order of the B-splines




**Theoretical scaling ($h$) for uniform meshes: element size**

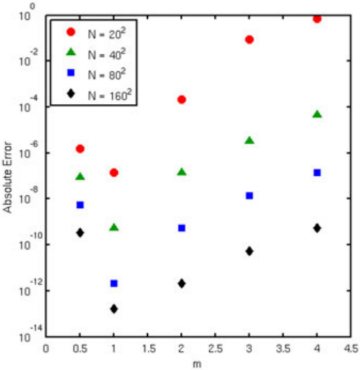


# Filter performance linked to the underlying geometry




Scaling: $h = m{\cdot}\mathrm{H}$

King, Mirzaee, Ryan, Kirby (SISC 2012)




**For non-uniform meshes remains an open problem:**

- Optimal scaling only found numerically
- In practise: use maximum / local element size

# Post-processing data with SIAC filters

✔ **Overview of the SIAC kernel**

Moment preservation + B-splines + $L^2$-initialization
$\Rightarrow$ **provable** $2p + 1$ **superconvergence**

✔ **Filtering challenges**

Higher dimension: tensor (accuracy) vs. line (CPU efficiency)
Domain boundaries: shifted kernels (accuracy loss)
Filter scaling: non-trivial for non-uniform meshes

➢ **The software package**

Code structure, geometry and tool capabilities

# A standalone tool written in julia

Mesh file + Data file + Filter options = Filtered file

```
function filter_data (mesh, data, parameters)
    modes  = l2_projection(data)
    kernel = set_kernel(parameters)
    for point in data
        map = find_kernel_breaks(mesh, point, kernel)   # Footprint
        point* = sum( gauss(map, kernel, modes))        # Convolution
    end
end
```

➢ Need data file sampled at quadrature points to recover the modal form ($L^2$-projection)

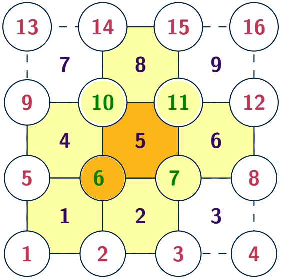
➢ **Most CPU time spent finding the filter footprint**

# Handling the geometry: the mesh data structure

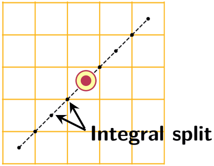
```
struct element
       nodes :: List of ordered vertex indexes forming element
       neigh :: List element neighbors
       type  :: 4 (quad) 3 (triangle)
end

struct node
       xyz  :: vertex coordinates
       ele  :: List of elements surrounding vertex
       type :: 1 (interior) 0 (boundary edge) -1 (boundary corner)
end
```

```
--------------------------------
elmt[5].nodes = [6,7,11,10]
elmt[5].neigh = [2,6,8,4]
elmt[5].type  = 4
--------------------------------
node[6].xyz  = [0.2,0.2,0.0]
node[6].ele  = [1,2,5,4]
node[6].type = 1
--------------------------------
```



# Task: collect all spline breaks and element interfaces

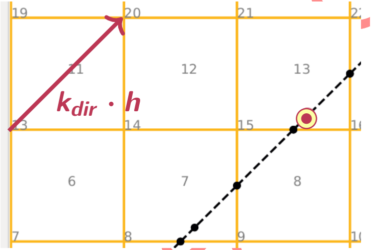



**LSIAC example**

$K^{(3,2)}$

**Sorted knot matrix**

$$\mathrm{T} = \begin{pmatrix} -2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \mathrm{T}_- = (-2,-1) \quad \mathrm{T}_+ = (0,1,2)$$



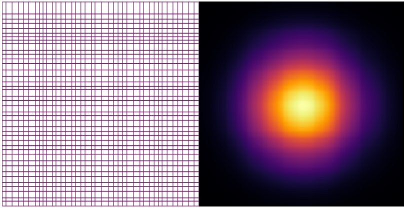

**for** $t \in (\mathrm{T}_-, \mathrm{T}_+)$ **do**
    $x_p$ = eval point, $e = 13$
    **for** $i \in t$ **do**
        $x_n = k_{dir} \cdot h \cdot i$
        $S_{pn} = segment(x_p, x_n)$
        **if** $x_n \in elmt[e]$ **then**
            $\boldsymbol{store(x_n, elmt[e])}$
            $x_p = x_n$
        **else**
            $(j, x_p) = intersect(S_{pn}, elmt[13])$
            $\boldsymbol{store(x_p, e, elmt[e].neigh[j])}$
            $e = elmt[e].neigh[j]$ #8
        **end if**
    **end for**
**end for**

**Avoids querying point location (computationally expensive)**

# Examples on non-uniform meshes: quads

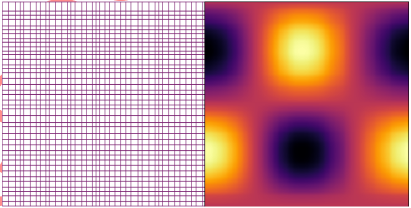
$\boxed{p = 1}$

**Mesh:** $40 \times 40$ $\quad u = e^{-x^2-y^2}$

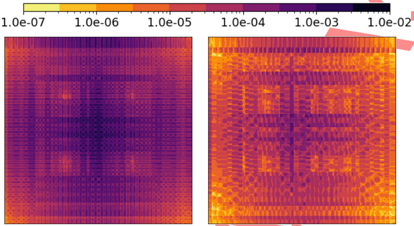


$\boxed{p = 2}$

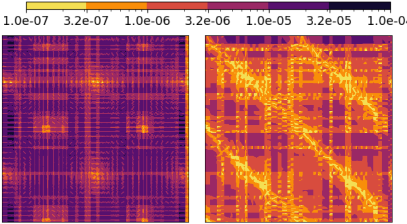
**Mesh:** $40 \times 40$ $\quad u = \sin(x)\cos(y)$



**Error contours** (log)




$L^2$**-projection** **Tensor Filter**




$L^2$**-projection** **Line Filter**

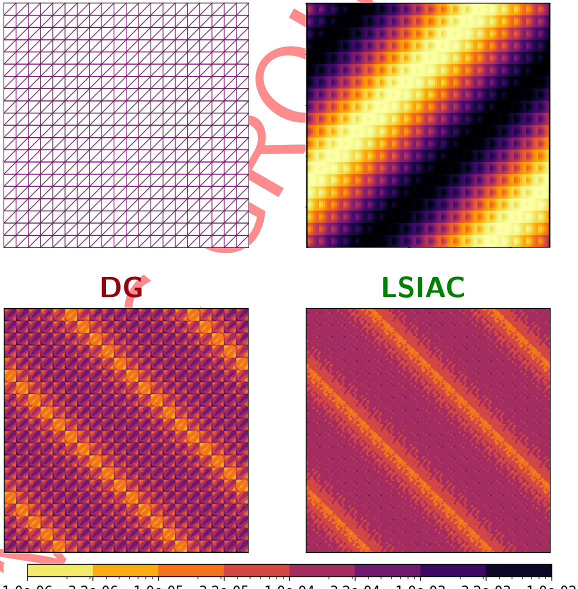
# LSIAC filtering: triangular meshes (Matt Picklo)

**Mesh:** $40 \times 20$ | $u = \sin(x + y)$

**Solution**

**DG** | **LSIAC**

**Error Contours**



# Wrapping up: hopefully you will remember...

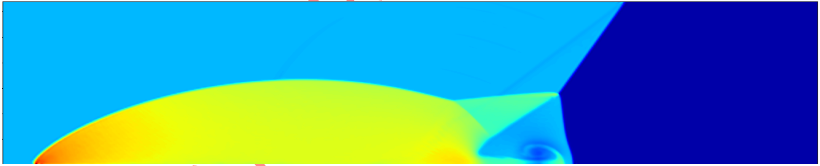
**SIAC filtering: increased smoothness and reduced errors**

- ➢ Optimal superconvergent rate: $2p + 1$
- ➢ Theory: equations (linearity), mesh type & domain boundaries

**Challenging implementation. Our tool currently supports:**

| | | | Mesh | | | |
|---|---|---|---|---|---|---|
| | Kernel | | Uniform | | Non-uniform | |
| **Filter** | Symmetric | Shifted | Quads | Triangles | Quads | Triangles |
| **Tensor** | | | | | | |
| **Line** | | | | | | |

Theory & implemented    Implemented but no theory    Testing



Post-processing the DG solution ($p = 2$) to the double Mach reflection problem. Raw data: **Théa Vuik** and **Soraya Terrab**.

# Grazas !!

## Acknowledgements

**Contributors to this talk:** **Matthew Picklo** , Jennifer K. Ryan, Soraya Terrab and Abel Gargallo.

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