MARIE CURIE ACTIONS

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A filtering framework for Finite Volume / Element schemes

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Post-processing data with SIAC filters

Bramble, Schatz (Math Comp 1977), *Cockburn et al.* (Math Comp 2002)

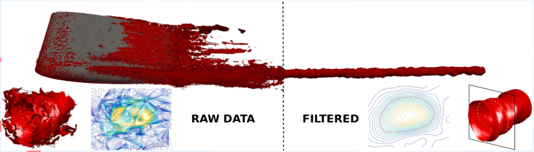
❶ **Superconvergence:** for a DG/FEM solution of degree p

- Proven $2p+1$ order (L^2 and L^∞ norms) for linear PDEs
- Observed $2p+1$ order for non-linear PDEs

❷ **Smoothness:** the filtered data is a (local) $2p+1$ polynomial

- Removes oscillations in the error
- Recovers continuity levels across element interfaces

Applications
Flow visualization
Shock detection
Multiresolution analysis
Cut cells



Jallepalli, Docampo, Ryan, Haimes, Kirby (TVCG 2017)

Post-processing data with SIAC filters

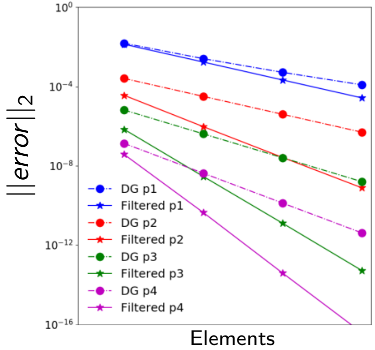
Goal:

- Establish a filtering framework for general purpose
- Create a standalone tool for general applications

- **SIAC filters overview**
  - → Superconvergence, kernel structure and filter properties
  - → Applications to multidimensional data
- **The software package**

SIAC filtering: superconvergence for DG solutions

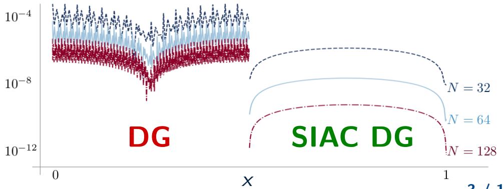
Advection 1D



→ DG approximation order: $p + 1$

→ **SIAC DG order: $2p + 1$**

Point-wise errors (log) for $p = 2$



✔ Reduces oscillations
✔ General error reduction

Smoothness-Increasing Accuracy-Conserving filters

We post-process our data via convolution:

$$\text{data}^{\star}(x) = \int_{\mathbb{R}} \boldsymbol{K}(y-x) \cdot \text{data}(y)\,dy$$

SIAC kernel: weighted sum of B-spline functions.

$$\boldsymbol{K}^{(r+1,n)}(\cdot) = \sum_{\gamma=1}^{r+1} \boldsymbol{c}_{\gamma} \cdot B_{\mathsf{T}_{\gamma},n}(\cdot)$$

Choose $\boldsymbol{c}_{\gamma}'\boldsymbol{s}$ to satisfy **consistency + r-moments:**

$$\int_{\mathbb{R}} K(x)dx = 1, \qquad \int_{\mathbb{R}} K(x)x^j dx = 0, \quad j = 1, 2, \ldots, r.$$

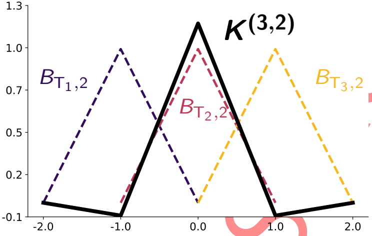
- ➢ **This ensures data accuracy is preserved**
- ➢ **For $2p+1$ superconvergence, we need $2p$ moments**

SIAC kernel: $K^{(r+1,n)} = \sum_{\gamma=1}^{r+1} c_\gamma \cdot B_{T_\gamma,n}$

(Some) B-splines properties:

- Compact support (smaller integral region)
- Smoothness of $n-2$ (remove oscillations)
- Derivatives as divided differences (superconvergence theory)

Linear kernel

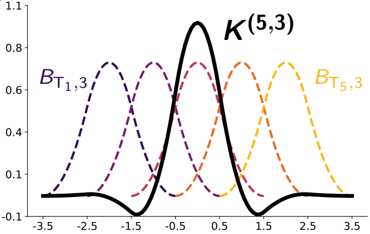


→ Moments: $r = 2$

→ Continuity: $\mathcal{C}^0$ ($n = 2$)

→ Optimal order: 3

Quadratic kernel



→ Moments: $r = 4$

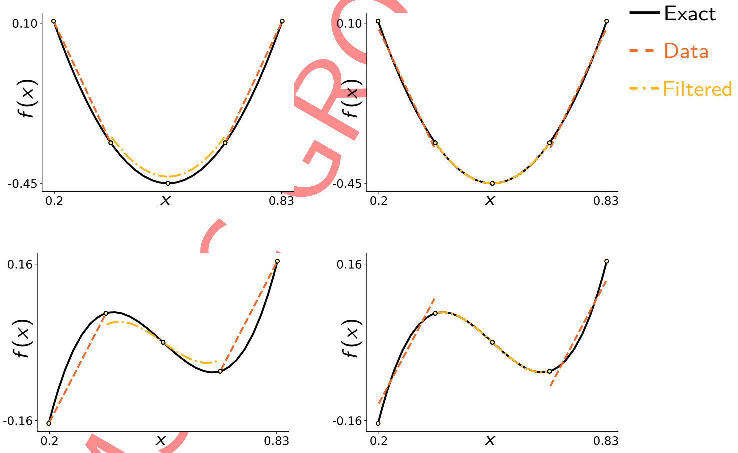
→ Continuity: $\mathcal{C}^1$ ($n = 3$)

→ Optimal order: 5

Polynomial functions: filtering linear data ($p = 1$)

Docampo, Jacobs, Li, Ryan (CAF 2020)

	Interpolant (No Galerkin orthogonality)	**L^2-projection**
Target		
Quadratic Function		
Cubic Function		



We can recover the exact function for the L^2-projected data!

Filtering in multidimensions

➢ **Tensor filter:** natural extension to higher dimension

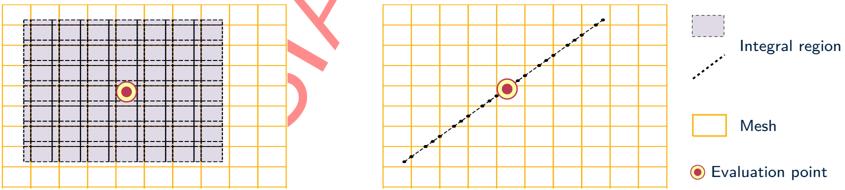
$$K = k_x \otimes k_y \Rightarrow u^\star(\overline{x}, \overline{y}) = \int_{\infty}^{\infty}\int_{\infty}^{\infty} k_x\left(\frac{\overline{x} - x}{h_x}\right) \cdot k_y\left(\frac{\overline{y} - y}{h_y}\right) u(x, y)dxdy$$

➢ **Line filter:** choose angle of rotation and perform a line integral

Docampo, Mirzargar, Ryan, and Kirby (SISC 2017)

$$K = k_\Gamma \Rightarrow u^\star(\overline{x}, \overline{y}) = \int_\Gamma k_\Gamma\left(\frac{\Gamma(0) - \Gamma(t)}{h_t}\right) u(\Gamma(t))dt$$

Footprints for a uniform mesh and quadratic kernels $K^{(5,3)}$



Tensor: 196 $2d$**-integrals** **Line: 21** $1d$**-integrals**

2D filters: superconvergence & error reduction

- **Provable** $2p+1$ order for linear problems + uniform meshes
- **Observed** $2p+1$ order also for non-linear equations

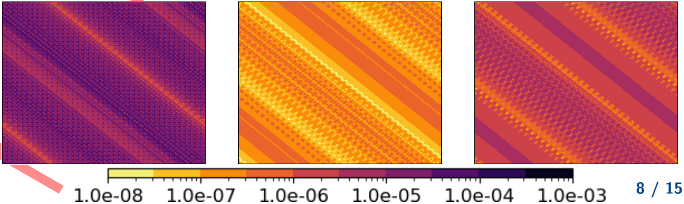
Example: Burgers eq. with source term and uniform quad mesh

		DG		LSIAC		TSIAC	
	N	$\|\|e\|\|_2$	Rate	$\|\|e\|\|_2$	Rate	$\|\|e\|\|_2$	Rate
$p=1$	40^2	3.02e-03	2.01	3.34e-04	3.32	1.37e-04	3.05
	80^2	7.52e-04	2.00	4.74e-05	2.82	1.63e-05	3.07
$p=2$	40^2	4.01e-05	3.00	9.37e-06	6.04	4.66e-07	4.65
	80^2	4.93e-06	3.02	2.95e-07	4.99	1.52e-08	4.93

Error contours

$p=2,\ N=40^2$

DG — Tensor Filter — Line Filter



Post-processing data with SIAC filters

✔ **SIAC filters overview**

Moment preservation + B-splines + L^2-initialization
$\Rightarrow$ **provable** $2p+1$ **superconvergence**

Multidimensional data

Line filter: computationally efficient

Tensor filter: theory extends naturally from 1d

➢ **The software package**

- Structure and compatibility
- Examples of applications

A standalone tool written in julia

Mesh file + **Data file** + **Filter options** = **Filtered file**

- **Mesh:** element indices, connectivity and (boundary map)
- **Data:** quadrature points or a modal file
- **Pipe** with .vtu files using VTKDataIO (needs Python)
- **Parallel** implementation with *Julia multi-threading*

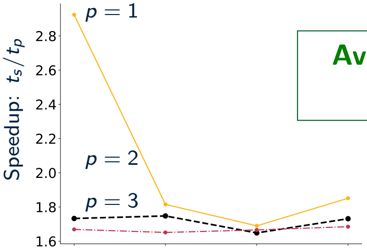
Currently supporting 1D & 2D data:

			Mesh			
	Kernel		Uniform		Non-uniform	
Filter	Symmetric	Shifted	Quads	Triangles	Quads	Triangles
Tensor						
Line						

Theory & implemented   Implemented but no theory

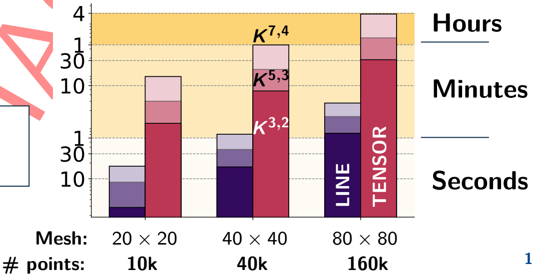
Filters performance: CPU times

Parallelization using a dell XPS + Intel i7 with 4 cores:



Average speedup
1.8

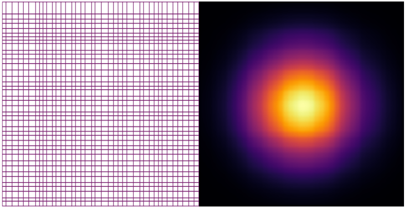
CPU times (parallel):
25 points per element



Examples on non-uniform meshes: quads

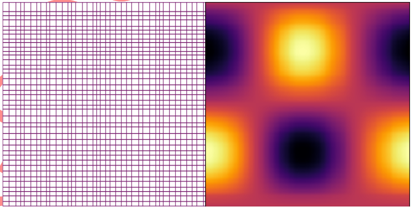
$\boxed{p = 1}$

Mesh: 40×40 $\quad u = e^{-x^2-y^2}$

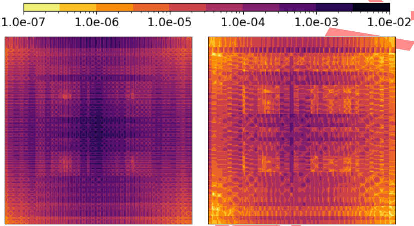


$\boxed{p = 2}$

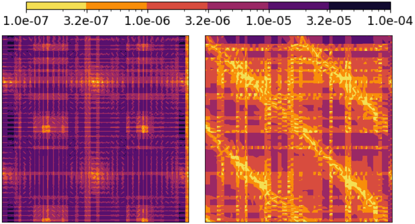
Mesh: 40×40 $\quad u = \sin(x)\cos(y)$



Error contours (log)



L^2-**projection** **Tensor Filter**



L^2-**projection** **Line Filter**

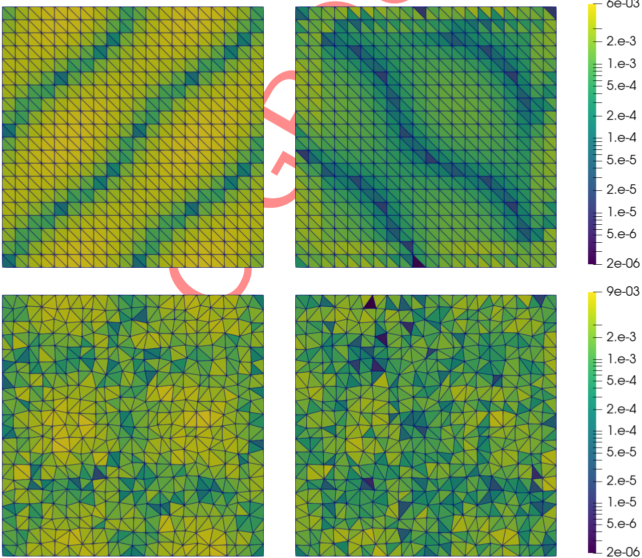
LSIAC filtering: triangular meshes and Paraview

Error contours ($p = 1, n = 20^2$) for a sine wave (DG at $t = 0$)

L^2 **projection** | **LSIAC filter**

Structured

Unstructured



Wrapping up: hopefully you will remember...

SIAC filters: increase smoothness and reduce errors

- → **$2p+1$ superconvergence**

  moment conditions + Bsplines + L^2-initialization

- → **Theoretical estimates**

  equations (linearity), mesh type & domain boundaries

- → **Multidimensional data**

  tensor filter (accuracy) vs. line filter (CPU efficiency)

The Julia Package: currently supporting 1D & 2D data

			Mesh			
	Kernel		Uniform		Non-uniform	
Filter	Symmetric	Shifted	Quads	Triangles	Quads	Triangles
Tensor						
Line						

Theory & implemented   Implemented but no theory

Grazas !!

The MSIAC project is a joint initiative with Prof. Jennifer K. Ryan.

Visit us !

https://siac_magic.gitlab.io/siac-magic-tools

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