Data Science 07-09-15: Game Theory

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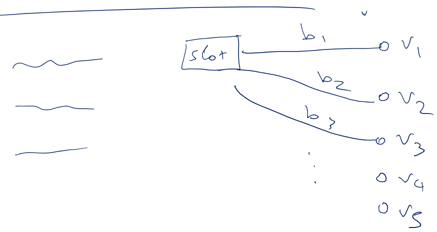
Prisoner's Dilemma

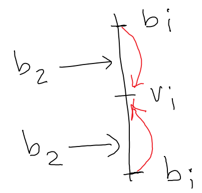
	Confess	Silent
Confess	4, 4 ~~1~~	1, 5
Silent	5	2, 2

Dominant Strategy:
- Strategy s.t no matter what the other player does you are better off playing it.

Second Price Auction



$$u_i = v_i x_i - p_i$$



Tragedy of the Commons

- n players
- $x_i \in [0,1]$
- $u_i = x_i \left(1 - \sum_j x_j\right)$

$$t = \sum_{j \neq 1} x_j$$

$$u_i = x_i (1 - t - x_i)$$

$$u_i'(x_i) = 1 - t - 2x_i = 0$$

$$x_i = \frac{1 - \sum_{j \neq 1} x_j}{2}$$

$$\Downarrow$$

$$x_i = \frac{1}{n+1}$$

Sanity Check

$$x_i = \frac{1 - \frac{n-1}{n+1}}{2} = \frac{\frac{n+1-(n-1)}{n+1}}{2}$$

$$= \frac{2}{2(n+1)} = \frac{1}{n+1}$$

Nash Equilibrium

A profile of strategies $s_1, s_2, \ldots, s_n$

$$u_i(s_1, \ldots, s_n) \geq u_i(\underbrace{s_i', s_{-i}}_{(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)})$$

Social Inefficiency

$$u_i = \frac{1}{n+1}\left(1 - \frac{n}{n+1}\right) = \frac{1}{(n+1)^2}$$

$$\sum_i u_i = \frac{n}{(n+1)^2} \approx \frac{1}{n}$$

But: if $x_i' = \frac{1}{2n} \Rightarrow$

$$SW(x') = n \frac{1}{2n}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

Selfish Outcome very inefficient $\frac{1}{n}$ vs $\frac{1}{4}$

Battle of the Sexes

M \ W	Shop	Foot
Shop	3, 2	1, 1
Foot	0, 0	2, 3

Two Nash Equilibria
Non-unique prediction

Does Nash Equilibrium always exist?

Matching Pennies

Match \ Mismatch	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

No equilibrium in pure strategies

What if players can randomize?
If both play H or T w.p. ½ then noone wants to deviate

Mixed Nash Equilibrium

A profile of randomized strategies $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ s.t.

$$\mathbb{E}_{\vec{s} \sim \vec{x}}\left[u_i(\vec{s})\right] \geq \mathbb{E}_{\vec{s}_{-i} \sim \vec{x}_{-i}}\left[u_i(s_i', s_{-i})\right]$$

Thm A mixed Nash Equilibrium always exists in any finite strategy game.

Zero-Sum Games

- Two players: Row Player (R)
  Column Player (C)
- Game matrices

$$A = \begin{array}{c c} & \begin{array}{ccccc} 1 & \cdots & j & \cdots & m \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ i \\ \vdots \\ n \end{array} & \begin{bmatrix} & & & & \\ & & u_{ij} & & \\ & & & & \end{bmatrix} \end{array}$$

utility of row player if
R plays i
C plays j

$$B = -A = \begin{bmatrix} & & \\ & -u_{ij} & \\ & & \end{bmatrix}$$

utility of column player

- Mixed strategies: $x = (x_1, \ldots, x_n)$
  $y = (y_1, \ldots, y_n)$

x_i : probability that R plays i

y_j : — // — — // — C — // — j

- Expected utilities

expected utility of R : $u_R = x^T A y$

— // — C : $u_C = -x^T A y$

- Explanation

$$\vec{u}_R = Ay = \begin{bmatrix} u_{11} & \cdots & u_{1m} \\ \vdots & \ddots & \\ \vdots & & \\ u_{n1} & \cdots & u_{nm} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_m \end{bmatrix}$$

$$\begin{bmatrix} u_{11} \cdot y_1 + \ldots + u_{1m} y_m \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} \cdot y_1 + \dots + u_{1m} y_m \\ \vdots \\ u_{n1} \cdot y_1 + \dots + u_{nm} \cdot y_{nm} \end{bmatrix}$$

$$u_R^i = (Ay)_i = u_{i1} y_1 + \dots + u_{im} \cdot y_m$$

$$= \left\{ \begin{array}{l} \text{expected utility} \\ \text{of } R \text{ if he plays } i \end{array} \right\}$$

- Overall expected utility is

$$x_1 u_R^1 + x_2 u_R^2 + \dots + x_n u_R^n = \sum_i x_i (Ay)_i$$

$$= x^T A y$$

- Nash Equilibrium

A pair $(\tilde{x}, \tilde{y})$ s.t.

$$\tilde{x}^T A \tilde{y} \geq (A \tilde{y})_i \quad \forall i$$

$$\tilde{x}^T A \tilde{y} \leq (\tilde{x}^T A)_j \quad \forall j$$

- Such pair always exists. Why?

$$\left. \begin{array}{l} \tilde{x} = \arg\max_x \min_y x^T A y \\ \tilde{y} = \arg\min_y \max_x x^T A y \end{array} \right\} \begin{array}{l} (\tilde{x}, \tilde{y}) \text{ defined} \\ \text{by this equations} \\ \text{is a Nash eq.} \end{array}$$

- Intuition: by minimax principle

$$\max_x \min_y x^T A y = \min_y \max_x x^T A y$$

Computing a Nash Equilibrium

Fictitious Play

At every iteration t each player forms a belief about mixed strategy of opponent, based on history of play.

$$\hat{x}^i_t = \frac{1}{t-1} \sum_{\tau=1}^{t-1} \mathbb{1}\{i_\tau = i\}$$: Belief of column player about row player's strategy

$$\hat{y}^j_t = \frac{1}{t-1} \sum_{\tau=1}^{t-1} \mathbb{1}\{j_\tau = j\}$$: Belief of row player about column player's strategy.

At each iteration each player best-responds to his belief:

$$i_t = \arg\max_i \left(A \hat{y}_t\right)_i$$

$$j_t = \arg\min_j \left(\hat{x}_t^T A\right)_j$$

Theorem $(\hat{y}_t, \hat{x}_t)$ converge to $(\tilde{x}, \tilde{y})$ s.t. $(\tilde{x}, \tilde{y})$ is a Nash Equilibrium of the zero sum game.

No-Regret Learning

$$R: \quad \frac{1}{T} \sum_{t=1}^{T} x_t^T A y_t \geq \frac{1}{T} \sum_{t=1}^{T} \left(A \hat{y}_t\right)_i$$

C: ____ " -

Small alteration of fictitious play leads to no-regret:

$$x_{t+1}^{i} = \underset{x}{\operatorname{argmax}} \quad x^T A \hat{y}_t - \frac{1}{\eta_{t+1}} R(x)$$

$$\underbrace{\sum_i x_i \log(x_i)}_{\text{entropy.}}$$

$$x_{t+1} = \frac{e^{\eta_t (A\hat{y}_t)_i}}{\sum_{i'} e^{\eta_t (A\hat{y}_t)_{i'}}}$$

Multiplicative weight updates. Algorithm.

$$x_{t+1} = \operatorname{argmax} \quad \frac{1}{t} \sum_{\tau=1}^{t} x^T A y_\tau - \frac{1}{\eta_t} R(x)$$

$$= \operatorname{argmax} \quad \sum_{\tau=1}^{t} x^T A y_\tau - \frac{t}{\eta_t} R(x)$$

Theorem:

If we run algorithm for T iterations and we set $\frac{t}{\eta_t} = \sqrt{T} \Rightarrow \eta_t = \frac{t}{\sqrt{T}}$

then

$$\forall i: \text{Regret}_R^i = \frac{1}{T} \sum_{t=1}^{T} \left[(Ay_t)_i - x_t^T A y_t \right] \leq \frac{1}{\sqrt{T}} \xrightarrow{T\to\infty} 0$$

$$\forall j: \text{Regret}_R^j = \frac{1}{T} \sum_{t=1}^{T} \left[x_t^T A y_t - (x_t^T A)_j \right] \leq \frac{1}{\sqrt{T}} \to 0$$