NCERT Solutions for Class 11

Physics

Chapter 6 - Systems of Particles and Rotational Motion

1. Give the location of the centre of mass of a

(a) sphere,

(b) cylinder,

(c) ring, and

(d) cube, each of uniform mass density.

Does the centre of mass of a body necessarily lie inside the body?

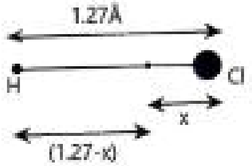
Ans: The centre of mass (C.M.) can be defined as a point where the mass of a body is supposed to be concentrated.

For the above listed geometric shapes having a uniform mass density, the centre of mass lies at their respective geometric centres.

The centre of mass of a specific body need not necessarily lie inside of the body. For example, the centre of mass of bodies such as a ring, a hollow sphere, etc., lie outside the respective body.

2. In the HCl molecule, the separation between the nuclei of the two atoms is about $1.27Å$ $20kg$. Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

Ans: The provided situation can be expressed as:



Distance between H and Cl atoms $= 1.27\text{Å}$

Mass of H atom $= m$

Mass of Cl atom $= 35.5m$

Let the centre of mass of the given system be at a distance x from the Cl atom.

Distance between the centre of mass and the H atom $= (1.27 - x)$

Let us suppose that the centre of mass of the given molecule lies at the origin. Therefore, it can be written as:

$$\frac{m(1.27-x)+35.5mx}{m+35.5m}=0$$

$$\Rightarrow m(1.27-x)+35.5mx=0$$

$$\Rightarrow 1.27-x=-35.5x$$

$$\Rightarrow x=\frac{-1.27}{(35.5-1)}=-0.037\text{Å}$$

Here, the negative sign gives an indication that the centre of mass lies at the left side of the molecule.

Therefore, the centre of mass of the HCl molecule lies 0.037Å from the Cl atom.

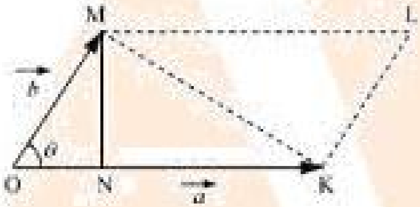
3. A child sits stationary at one end of a long trolley moving uniformly with a speed v on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system?

Ans: There will not be any change in the speed of the centre of mass of the given system.

The child is running arbitrarily on a trolley that is moving forward with velocity v. However, the running of the child will have no effect on the velocity of the centre of mass of the trolley. This happens because of the force due to the motion of child is purely internal. Internal forces in a body produce no effect on the motion of the bodies on which they are acting. Because there is no external force involved in the (child + trolley) system, the child's motion will not produce any change in the speed of the centre of mass of the trolley.

4. Show that the area of the triangle contained between the vectors a and b is one half of the magnitude of $a \times b$.

Ans: Let us consider two vectors $OK = |a|$ and $\overline{OM} = |\overline{b}|$, which are inclined at an angle θ, as shown in the following figure.



In ΔOMN, we can express the relation:

$$\sin\theta = \frac{MN}{OM} = \frac{MN}{|\overline{b}|}$$

$$\Rightarrow MN = |b|\sin\theta$$

Now,

$$|a \times \overline{b}| = |a||\overline{b}|\sin\theta$$

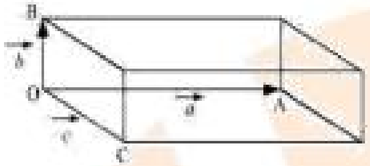
$$\Rightarrow |\overline{a} \times b| = OK \cdot MN \times \frac{2}{2}$$

$$\Rightarrow |\overline{a} \times b| = 2 \times \text{Area of } \Delta OMK$$

$$\Rightarrow \text{Area of } \Delta OMK = \frac{1}{2}|a \times b|$$

5. Show that $a \cdot (b \times c)$ is equal in magnitude to the volume of the parallelepiped formed on the three vectors, a, b and c.

Ans: A parallelepiped with origin O and sides a, b, and c is depicted in the following figure.



Volume of the given parallelepiped $= abc$

And

$\dot{OA} = \bar{a}$

$\dot{OB} = \dot{b}$

$\overline{OC} = \dot{c}$

Let us suppose that $\hat{n}$ be a unit vector perpendicular to both $\bar{b}$ and c. Therefore, $\hat{n}$ and $\bar{a}$ have the same direction.

$\dot{b} \times c = bc\sin\theta\hat{n}$

$\Rightarrow \bar{b} \times \dot{c} = bc\sin 90^{\circ}\,\hat{n}$

$\Rightarrow \dot{b} \times \bar{c} = bc\hat{n}$

Now,

$\acute{a}\left(\dot{b} \times \dot{c}\right) = a \cdot (bc\hat{n})$

$\Rightarrow \acute{a}\left(\bar{b} \times \dot{c}\right) = abc\cos\theta\hat{n}$

$\Rightarrow \bar{a}\left(\bar{b} \times \bar{c}\right) = abc\cos 0\ \hat{n}$

$\Rightarrow \bar{a}\left(\dot{b} \times \bar{c}\right) = abc\cos 0^{\circ}$

$$\Rightarrow \bar{a}\left(\dot{b}\times\bar{c}\right) = abc$$

$$\Rightarrow \dot{a}\left(\dot{b}\times\dot{c}\right) = abc = \text{Volume of the parallelepiped}$$

6. Find the components along the x, y, z axes of the angular momentum l of a particle, whose position vector is r with components x, y, z and momentum is p with components p_x, p_y and p_z. Show that if the particle moves only in the x-y plane the angular momentum has only a z-component.

Ans: Linear momentum of the particle, $\dot{p} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$

Position vector of the particle, $\dot{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Angular momentum,

$$l = \bar{r}\times\bar{p}$$

$$\Rightarrow \dot{l} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\times\left(p_x\hat{i} + p_y\hat{j} + p_z\hat{k}\right)$$

$$\Rightarrow \dot{l} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{pmatrix}$$

Now,

$$l_x\hat{i} + l_y\hat{j} + l_z\hat{k} = \hat{i}\left(yp_z - zp_y\right) - \hat{j}(xp_z - zp_x) + \hat{k}\left(xp_y - zp_x\right)$$

On comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we can write:

$$l_x = yp_z - zp_y,$$

$$l_y = zp_x - xp_z,$$

$$l_z = xp_y - yp_x \quad \ldots\ldots (1)$$

The particle is moving in the x-y plane. Therefore, the z-component of the position vector and linear momentum vector is becoming zero, i.e., $z = p_z = 0$

Thus, equation (1) reduces to:

$l_x = 0$

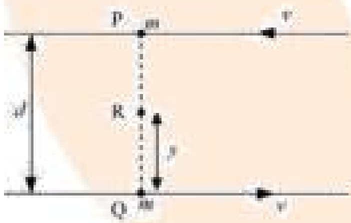
$l_y = 0$

$l_z = xp_y - yp_x$

Hence, when the particle is subject to move in the x-y plane, the direction of angular momentum will be along the z-direction.

7. Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of the two-particle system is the same whatever be the point about which the angular momentum is taken.

Ans: Let us suppose that at a certain instant two particles be at points P and Q, as shown in the given figure.



Angular momentum of the system about point P can be given as:

$L_P^{'} = mv \times 0 + mv \times d$

$\Rightarrow L_P^{'} = mvd$ ...... (1)

Angular momentum of the system about point Q can be given as:

$L_Q^{'} = mv \times d + mv \times 0$

$\Rightarrow \overline{L_Q} = mvd$ ...... (2)

Let us consider a point R, which is at a distance y from point Q, i.e.,

$$QR = y$$
$$\Rightarrow PR = d - y$$

Angular momentum of the system about point R can be given as:

$$\overline{L_R} = mv \times (d - y) + mv \times y$$

$$\Rightarrow L_R = mvd - mvy + mvy$$

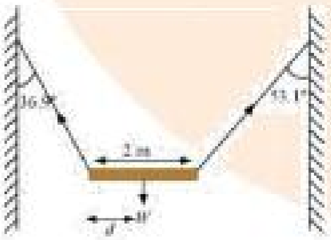
$$\Rightarrow \overline{L_R} = mvd \ \ldots\ldots (3)$$

On comparing equations (1), (2), and (3), we get:

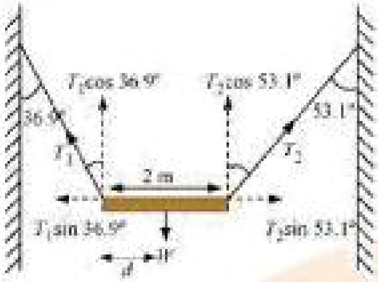
$$\overline{L_P} = \overline{L_Q} = \overline{L_R} \ \ldots\ldots (4)$$

We can hence infer from equation (4) that the angular momentum of a system is independent of the point about which it is taken.

8. A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in Fig.7.39. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is $2\,m$ long. Calculate the distance d of the centre of gravity of the bar from its left end.



Ans: The free body diagram of the bar can be drawn as shown in the given figure.



Length of the bar is given, $l = 2m$

T_1 and T_2 are the tensions generated in the left and right strings respectively.

At translational equilibrium, we can express:

$$T_1 \sin 36.9° = T_2 \sin 53.1°$$

$$\Rightarrow \frac{T_1}{T_2} = \frac{\sin 53.1°}{\sin 36.9°}$$

$$\Rightarrow \frac{T_1}{T_2} = \frac{0.800}{0.600} = \frac{4}{3}$$

$$\Rightarrow T_1 = \frac{4}{3} T_2$$

On taking the torque about the centre of gravity, for rotational equilibrium, we can write:

$$T_1 \cos 36.9° \times d = T_2 \cos 53.1° (2 - d)$$

$$\Rightarrow T_1 \times 0.800 \times d = T_2 0.600 (2 - d)$$

$$\Rightarrow \frac{4}{3} \times T_2 \times 0.800 \times d = T_2 (0.600 \times 2 - 0.600d)$$

$$\Rightarrow 1.067d + 0.6d = 1.2$$

$$\Rightarrow d = \frac{1.2}{1.67}$$

$$\Rightarrow d = 0.72m$$

Therefore, the centre of gravity of the given bar lies 0.72m from the left end of the bar.

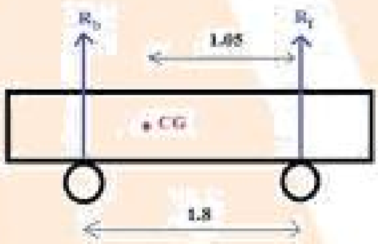
9. A car weighs 1800kg. The distance between its front and back axles is 1.8m. Its centre of gravity is 1.05m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

Ans: Mass of the car is given as, $m = 1800\text{kg}$

Distance between the front and back axles, $d = 1.8\text{m}$

Distance between the centre of gravity and the back axle $= 1.05\text{m}$

The different forces acting on the car are shown in the given figure:



The forces in the figure, R_f and R_b are the forces exerted by the level ground on the front wheels and back wheels respectively.

At translational equilibrium we can write:

$R_f + R_b = mg$

$\Rightarrow R_f + R_b = 1800 \times 9.8$

$\Rightarrow R_f + R_b = 17640\text{N}$ ...... (1)

For rotational equilibrium, on taking the torque about the centre of gravity, we can write:

$R_f(1.05) = R_b(1.8 - 1.05)$

$\Rightarrow R_f \times (1.05) = R_b \times (0.75)$

$$\Rightarrow \frac{R_f}{R_b} = \frac{0.75}{1.05} = \frac{5}{7}$$

$$\Rightarrow \frac{R_b}{R_f} = \frac{7}{5}$$

$$\Rightarrow R_b = 1.4R_f \quad \ldots\ldots (2)$$

Solving equations (1) and (2), we obtain:

$$1.4R_f + R_f = 17640\text{N}$$

$$\Rightarrow R_f = \frac{17640}{2.4}\text{N} = 7350\text{N}$$

$$\Rightarrow R_b = (17640 - 7350)\text{N} = 10290\text{N}$$

Therefore, the force exerted on each front wheel can be given as

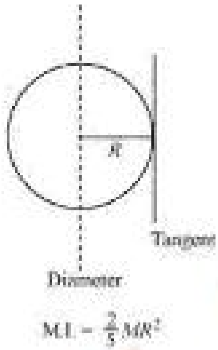
$\frac{7350}{2}\text{N} = 3675\text{N}$ and

the force exerted on each back wheel can be given as $\frac{10290}{2}\text{N} = 5145\text{N}$

10.

(a) Find the moment of inertia of a sphere about a tangent to the sphere, given the moment of inertia of the sphere about any of its diameters to be $\frac{2MR^2}{5}$ where M is the mass of the sphere and R is the radius of the sphere.

Ans: The moment of inertia (M.I.) of a sphere about its diameter can be given as: $\frac{2MR^2}{5}$



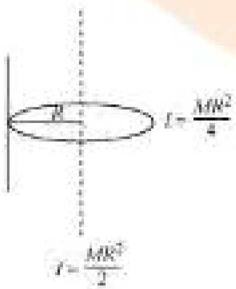
According to the theorem of parallel axes, the moment of inertia of a body about any axis is same as the sum of the moment of inertia of a certain body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.

The moment of inertia about a tangent of the sphere can be expressed as:

$$\frac{2MR^2}{5}+MR^2=\frac{7}{5}MR^2$$

(b) Given the moment of inertia of a disc of mass M and radius R about any of its diameters to be $\frac{1}{4}MR^2$, find its moment of inertia about an axis normal to the disc and passing through a point on its edge.

Ans: The moment of inertia of a disc about its diameter can be given as: $\frac{1}{4}MR^2$.



We can infer that, according to the theorem of perpendicular axis, the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is same as the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

The moment of inertia of the disc about its centre is

$$\frac{1}{4}MR^2 + \frac{1}{4}MR^2 = \frac{1}{2}MR^2$$

The position of the perpendicular axis is shown in the following figure.

On application of the theorem of parallel axes:

The moment of inertia about an axis normal to the disc and passing through a point on its edge is $\frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$.

11. Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time?

Ans: Let us assume that m and r be the respective mass and radius of the hollow cylinder and the solid sphere. The moment of inertia of the hollow cylinder about its standard axis can be given as,

$$I_1 = mr^2$$

The moment of inertia of the solid sphere about an axis that passes through its centre can be given as,

$$I_2 = \frac{2}{5}mr^2$$

The formula for torque in terms of angular acceleration and moment of inertia can be expressed as:

$$\tau = I\alpha$$

Where,

$\tau =$ Torque

$\alpha =$ Angular acceleration

$I =$ Moment of inertia

For the hollow cylinder the expression can be given as,

$\tau_1 = I_1\alpha_1$

For the solid sphere the expression can be given as,

$\tau_2 = I_2\alpha_2$

As an equal amount of torque is applied to both the bodies it can be stated as,

$$\frac{\alpha_2}{\alpha_1} = \frac{I_1}{I_2} = \frac{mr^2}{\frac{2}{5}mr^2} = \frac{2}{5}$$

$\alpha_2 > \alpha_1$ ...... (1)

Using the relation $\omega = \omega_0 + \alpha t$

Where,

$\alpha =$ Angular acceleration

$t =$ Time of rotation

$\omega_0 =$ Initial angular velocity

$\omega =$ Final angular velocity

For equal ω_0 and t, we have:

$\omega = \alpha$ ...... (2)

From equations (1) and (2), we can conclude:

$\omega_2 > \omega_1$

Therefore, the angular velocity (ω) of the solid sphere will be greater than that of the hollow cylinder.

12. A solid cylinder of mass 20kg rotates about its axis with angular speed 100rads^{-1}. The radius of the cylinder is 0.25m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

Ans: Mass of the cylinder is given, $m = 20\text{kg}$

Angular speed of the cylinder, $\omega = 100\text{rad s}^{-1}$

Radius of the solid cylinder, $r = 0.25\text{m}$

The moment of inertia of the solid cylinder can be expressed as:

$$I = \frac{1}{2}mr^2$$

$$\Rightarrow I = \frac{1}{2} \times 20\text{kg} \times (0.25)^2$$

$$\Rightarrow I = 6.25\text{kgm}^2$$

Kinetic energy of the cylinder $= \frac{1}{2}I\omega^2$

$$\Rightarrow \text{K.E.} = \frac{1}{2} \times 6.25 \times (100)^2$$

Angular Momentum of the cylinder,

$$L = I\omega$$

$$\Rightarrow L = 6.25 \times 100$$

$$\Rightarrow L = 62.5\text{Js}$$

13.

(a) A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of $40\text{rev}/\text{min}$. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to $\frac{2}{5}$ times the initial value? Assume that the turntable rotates without friction.

Ans: Given that,

Initial angular velocity of turntable, $\omega_1 = 40 \text{rev / min}$

Final angular velocity of the given turntable is ω_2

The moment of inertia of the child with stretched hands can be given as I_1

The moment of inertia of the child with folded hands can be given as I_2

The two moments of inertia are related to each other as follows:

$$I_2 = \frac{2}{5} I_1$$

Since no external force acts on the child, the angular momentum L is not varying.

Therefore, for the two circumstances, we can write:

$$I_1 \omega_1 = I_2 \omega_2$$

$$\Rightarrow \omega_2 = \frac{I_1}{I_2} \omega_1$$

$$\Rightarrow \omega_2 = \frac{I_1}{\frac{2}{5} I_1} \times 40 = \frac{5}{2} \times 40$$

$$\Rightarrow \omega_2 = 100 \text{rev / min}$$

(b) **Show that the child's new kinetic energy of rotation is more than the** initial kinetic energy of rotation. How do you account for this increase in kinetic energy?

Ans: Given that,

Initial Kinetic energy of rotation of the turntable, $E_i = \frac{1}{2} I_1 \omega^2_1$

Final kinetic energy of rotation of the turntable, $E_f = \frac{1}{2} I_2 \omega^2_2$

$$\frac{E_f}{E_i} = \frac{\frac{1}{2} I_2 \omega^2_2}{\frac{1}{2} I_1 \omega^2_1}$$

$$\Rightarrow \frac{E_f}{E_i} = \frac{\frac{1}{2} \times \frac{2}{5} \times I_1 \omega^2_2}{\frac{1}{2} I_1 \omega^2_1}$$

$$\Rightarrow \frac{E_f}{E_i} = \frac{2}{5} \times \frac{\omega^2_2}{\omega^2_1}$$

$$\Rightarrow \frac{E_f}{E_i} = \frac{2}{5} \times \frac{(100)^2}{(40)^2}$$

$$\Rightarrow \frac{E_f}{E_i} = 2.5$$

$$\Rightarrow E_f = 2.5 \times E_i$$

The increase in the rotational kinetic energy is related to the internal energy of the boy on the turntable.

14. A rope of negligible mass is wound round a hollow cylinder of mass 3kg and radius 40cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30N? What is the linear acceleration of the rope? Assume that there is no slipping.

Ans: Given that,

Mass of the hollow cylinder is given as, $m = 3\text{kg}$

Radius of the hollow cylinder is given as, $r = 40cm = 0.4m$

Applied force on the given rope is given as, $F = 30N$

The moment of inertia of the hollow cylinder about its geometric axis can be given as:

$$I = mr^2$$
$$\Rightarrow I = 3 \times (0.4)^2$$
$$\Rightarrow I = 0.48kgm^2$$

Torque acting on the rope,

$$\tau = F \times r$$
$$\Rightarrow \tau = 30 \times 0.4$$
$$\Rightarrow \tau = 12Nm$$

For angular acceleration α , torque can also be given by the expression:

$$\tau = I\alpha$$
$$\Rightarrow \alpha = \frac{\tau}{I} = \frac{12}{0.48}$$
$$\Rightarrow \alpha = 25rad\,s^{-2}$$

Linear acceleration of the rope can be stated as $= ra = 0.4 \times 25 = 10ms^{-2}$.

15. To maintain a rotor at a uniform angular speed of $200rad\,s^{-1}$ an engine needs to transmit a torque of $180Nm$. What is the power required by the engine? (Note: uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque). Assume that the engine is 100% efficient.

Ans: Given that,

Angular speed of the rotor is given as, $200rad / s$

Torque required by the rotor of the engine is given as $180Nm$.

The power of the rotor (P) can be expressed in the relation of torque and angular speed by the formula $P = \tau\omega$

$\Rightarrow P = 180 \times 200 = 30 \times 10^3$

$\Rightarrow P = 36kW$

Therefore, the power required by the engine is $36kW$.

16. From a uniform disk of radius R, a circular hole of radius $\frac{R}{2}$ is cut out. The centre of the hole is at $\frac{R}{2}$ from the centre of the original disc. Locate the centre of gravity of the resulting flat body.

Ans: Given that,

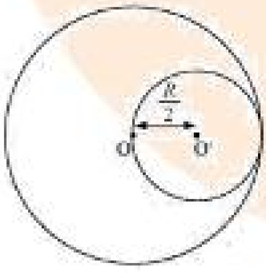
Mass per unit area of the original disc can be given as σ.

Radius of the original disc $= R$

Mass of the original disc,

$M = \pi R^2 \sigma$

The disc with the cut portion is shown in the given figure:



Radius of the smaller disc is given $= \frac{R}{2}$

Mass of the smaller disc is given as $M' = \pi\left(\frac{R}{2}\right)^2 \sigma = \frac{1}{4}\pi R^2 \sigma = \frac{M}{4}$

Let us suppose that O and O' be the respective centres of the original disc and the disc cut off from the original. As per definition of the centre of mass, the centre of mass of the original disc is assumed to be concentrated at O, while that of the smaller disc is assumed to be concentrated at O'.

It is provided that:

$$OO' = \frac{R}{2}$$

After the smaller disc has been cut from the original disc, the remaining portion left over after cutting is considered to be a system of two masses. The two masses can be expressed as:

$$M(\text{concentrated at } O) - M' = \left(\frac{M}{4}\right) \text{concentrated at } O'$$

(The negative sign in the above statement indicates that this portion has been removed from the original disc.)

Let us suppose that x be the distance through which the centre of mass of the remaining portion shifts from point O.

The relation between the centres of masses of two masses is given as:

$$x = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

For the given system, it can be written as:

$$\Rightarrow x = \frac{M \times 0 - M' \times \left(\frac{R}{2}\right)}{M + (-M')}$$

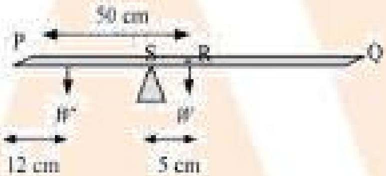
$$\Rightarrow x = \frac{\frac{-M}{4} \times \frac{R}{2}}{M - \frac{M}{4}} = \frac{-MR}{8} \times \frac{4}{3M} = \frac{-R}{6}$$

(The negative sign in the above statement indicates that the centre of mass gets shifted toward the left of point O.)

The centre of gravity of the resulting flat body can be located from the original centre of the body and opposite to the centre of the cut portion.

17. A meter stick is balanced on a knife edge at its centre. When two coins, each of mass $5g$ are put one on top of the other at the $12cm$ mark, the stick is found to be balanced at $45cm$. What is the mass of the meter stick?

Ans: Let us assume that W and W' be the respective weights of the meter stick and the coin.



The mass of the meter stick is supposed to be concentrated at its mid-point, i.e., at the $50cm$ mark.

Mass of the meter stick is m'

Mass of each coin is $m = 5g$

When the coins are placed $12cm$ away from the end P, the centre of mass gets shifted by $5cm$ from point R toward the end P. The centre of mass is located at a distance of $45cm$ from point P.

The net torque will be thus, conserved for rotational equilibrium about point R.

This can be expressed by the equation,

$$10 \times g(45 - 12) - m'g(50 - 45) = 0$$

$$\Rightarrow m' = \frac{10 \times 33}{5} = 66g$$

Therefore, the mass of the meter stick is $66g$.

18. A solid sphere rolls down two different inclined planes of the same heights but different angles of inclination.

(a) Will it reach the bottom with the same speed in each case?

Ans: Mass of the sphere $= m$

Height of the plane $= h$

Velocity of the sphere at the bottom of the plane is given as $= v$

At the top of the plane, the total energy of the sphere i.e., Potential energy $(P.E.) = mgh$

At the bottom of the plane, the sphere has both translational and rotational kinetic energies which can be expressed as,

Therefore, total energy $T.E. = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

Using the law of conservation of energy, we can state that:

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh \quad \ldots\ldots (1)$$

For a solid sphere, the moment of inertia about its centre can be given as,

$$I = \frac{2}{5}mr^2$$

Therefore, equation (1) becomes:

$$\frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\omega^2 = mgh$$

$$\Rightarrow \frac{1}{2}v^2 + \left(\frac{1}{5}r^2\right)\omega^2 = gh$$

But we have the formula,

$$v = r\omega$$

$$\frac{1}{2}v^2 + \frac{1}{5}v^2 = gh$$

$$\Rightarrow v^2\left(\frac{7}{10}\right)=gh$$

$$\Rightarrow v=\sqrt{\frac{10}{7}gh}$$

Therefore, the velocity of the sphere at the bottom depends only on height (h) and acceleration due to gravity (g). Both values are constants and do not change. Therefore, the velocity at the bottom remains the same from whichever inclined plane the sphere is rolled.

(b) Will it take longer to roll down one plane than the other?

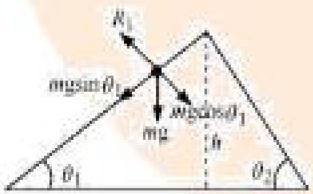
Ans: Let us consider two inclined planes with inclinations θ_1 and θ_2 respectively related as:

$\theta_1 < \theta_2$

The acceleration generated in the sphere when it rolls down the plane inclined at θ_1 is:

$g\sin\theta_1$

The different forces acting on the sphere are shown in the given figure.

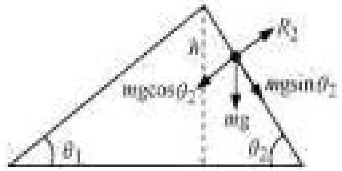


R_1 is the normal reaction to the sphere as shown in the above figure.

Similarly, the acceleration generated in the sphere when it rolls down the plane inclined at θ_2 is:

$g\sin\theta_2$

The different forces that act on the sphere are shown in the given figure.



R_2 is the normal reaction to the sphere as given in the figure.

$\theta_1 < \theta_2$, $\sin\theta_2 > \sin\theta_1$ …… (1)

$a_2 > a_1$ …… (2)

Initial velocity of sphere, $u = 0$

Final velocity of sphere, $v =$ constant

Now, by using the first equation of motion, we can obtain the time of roll as:

$v = u + at$

$$t \propto \frac{1}{a}$$

For inclination of angle θ_1:

$$t_1 \propto \frac{1}{a_1}$$

For inclination of angle θ_2:

$$t_2 \propto \frac{1}{a_2} \text{ …… (3)}$$

(c) If so, which one and why?

Ans: From equations (2) and (3), we obtain:

$t_2 < t_1$

Therefore, conclude that the sphere will take a longer time to reach the bottom of the inclined plane having the smaller inclination.

19. A hoop of radius 2m weighs 100kg. It rolls along a horizontal floor so that its centre of mass has a speed of $20\text{cm}/\text{s}$. How much work has to be done to stop it?

Ans: Radius of the hoop is given as, $r = 2\text{m}$

Mass of the hoop is, $m = 100\text{kg}$

Velocity of the hoop is,

$v = 20\text{cm}/\text{s} = 0.2\text{m}/\text{s}$

Total energy of the hoop $=$ Translational KE $+$ Rotational KE

$$E_r = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Moment of inertia of the hoop about its centre can be given as $I = mr^2$

$$E_r = \frac{1}{2}mv^2 + \frac{1}{2}\left(mr^2\right)\omega^2$$

But we have the formula, $v = r\omega$

$$\Rightarrow E_1 = \frac{1}{2}mv^2 + \frac{1}{2}mr^2\omega^2$$

$$\Rightarrow E_1 = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$$

The work needed to be done for halting the hoop is the same as the total energy of the hoop.

Hence, required work to be done can be given as,

$$W = mv^2 = 100 \times (0.2)^2 = 4\text{J}$$

20. The oxygen molecule has a mass of $5.30 \times 10^{26}\text{kg}$ and a moment of inertia of $1.94 \times 10^{-46}\text{kgm}^2$ about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is $500\text{m}/\text{s}$ and that its kinetic energy

of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.

Ans: Given that,

Mass of an oxygen molecule is given as, $m = 5.30 \times 10^{26}\,kg$

Moment of inertia of oxygen molecule is given as, $I = 1.94 \times 10^{-46}\,kgm^2$

Velocity of the oxygen molecule is given as, $v = 500m/s$

Let the separation between the two atoms of the oxygen molecule be $2r$

Mass of each oxygen atom in the oxygen molecule $= \frac{m}{2}$

Therefore, moment of inertia I, can be calculated as:

$$\left(\frac{m}{2}\right)r^2 + \left(\frac{m}{2}\right)r^2 = mr^2$$

$$r = \sqrt{\frac{I}{m}}$$

$$\Rightarrow \sqrt{\frac{1.94 \times 10^{-46}}{5.36 \times 10^{26}}} = 0.60 \times 10^{-10}\,m$$

It is provided that:

$$KE_{rot} = \frac{2}{3} KE_{trans}$$

$$\Rightarrow \frac{1}{2} I\omega^2 = \frac{2}{3} \times \frac{1}{2} mv^2$$

$$\Rightarrow mr^2\omega^2 = \frac{2}{3} mv^2$$

$$\Rightarrow \omega = \sqrt{\frac{2}{3}} \times \frac{v}{r}$$

$$\Rightarrow \omega = \sqrt{\frac{2}{3}} \times \frac{500}{0.6 \times 10^{-10}}$$

$\Rightarrow \omega = 6.80 \times 10^{12}\ \text{rad/s}$, which is the required average angular velocity.

21. A solid cylinder rolls up an inclined plane of angle of inclination 30°. At the bottom of the inclined plane the centre of mass of the cylinder has a speed of 5m/s. How far will the cylinder go up the plane? How long will it take to return to the bottom?

Ans: A solid cylinder rolling up an inclination is pictured in the following figure.

Initial velocity of the solid cylinder on the inclined plane, $v = 5\text{m/s}$

Angle of inclination is 30°

Height reached by the cylinder on the inclined plane $= h$

Energy of the cylinder on the inclined plane at point A:

$$KE_{rot} = KE_{trans}$$

$$\Rightarrow \frac{1}{2} I\omega^2 = \frac{1}{2} mv^2$$

Energy of the cylinder at point $B = mgh$

Let us use the law of conservation of energy, we can express:

$$\frac{1}{2} I\omega^2 = \frac{1}{2} mv^2 = mgh$$

Moment of inertia of the solid cylinder is $I = \frac{1}{2}mr^2$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2 + \frac{1}{2}mv^2 = mgh$$

$$\Rightarrow \frac{1}{4}mr^2\omega^2 + \frac{1}{2}mv^2 = mgh$$

$$\Rightarrow \frac{1}{4}r^2\omega^2 + \frac{1}{2}v^2 = gh$$

But we have the expression, $v = r\omega$

$$\Rightarrow \frac{1}{4}v^2 + \frac{1}{2}v^2 = gh$$

$$\Rightarrow \frac{3}{4}v^2 = gh$$

$$\Rightarrow h = \frac{3}{4} \times \frac{v^2}{g}$$

$$\Rightarrow h = \frac{3}{4} \times \frac{5 \times 5}{9.8} = 1.91\text{m}$$

In ΔABC,

$$\sin\theta = \frac{BC}{AB}$$

$$\Rightarrow \sin 30^\circ = \frac{h}{AB}$$

$$AB = \frac{1.91}{0.5} = 3.82\text{m}$$

Therefore, the cylinder will move 3.82m up the inclined plane.

For radius of gyration K, the velocity of the cylinder at the instance when it rolls back to the bottom is given by the formula:

$$v=\left(\frac{2gh}{1+\frac{K^2}{R^2}}\right)^{\frac{1}{2}}$$

$$\Rightarrow v=\left(\frac{2gAB\sin\theta}{1+\frac{K^2}{R^2}}\right)^{\frac{1}{2}}$$

For the solid cylinder we can write $K^2=\frac{R^2}{2}$

$$\Rightarrow v=\left(\frac{2gAB\sin\theta}{1+\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$\Rightarrow v=\left(\frac{4}{3}gAB\sin\theta\right)^{\frac{1}{2}}$$

The time taken to return to the bottom can be given as:

$$t=\frac{AB}{v}$$

$$\Rightarrow t=\frac{AB}{\left(\frac{4}{3}gAB\sin\theta\right)^{\frac{1}{2}}}$$

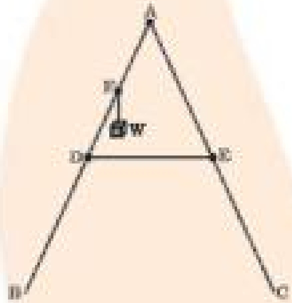
$$\Rightarrow t=\left(\frac{3AB}{4g\sin\theta}\right)^{\frac{1}{2}}$$

$$\Rightarrow t=\left(\frac{11.46}{19.6}\right)^{\frac{1}{2}}=0.764s$$

Therefore, the total time taken by the cylinder to return to the bottom is $(2 \times 0.764) = 1.53\text{s}$

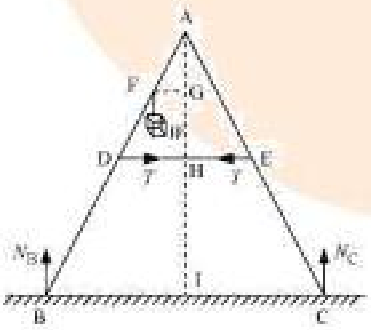
Additional Exercise

22. As shown in figure, the two sides of a step ladder BA and CA are 1.6m long and hinged at A. A rope DE, 0.5m is tied halfway up. A weight 40kg is suspended from a point F, 1.2m from B along the ladder BA. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (Take $g = 9.8\text{m}/\text{s}^2$)



(Hint: Consider the equilibrium of each side of the ladder separately.)

Ans: The given situation can be depicted as follows:



N_B = Force exerted on the ladder by the floor point B

N_C = Force exerted on the ladder by the floor point C

T = Tension in the given rope

$BA = CA = 1.6m$

$DE = 0.5m$

$BF = 1.2m$

Mass of the given weight, $m = 40kg$

Draw a perpendicular line from A on the floor BC. This line intersects DE at mid-point H.

In ΔABI and ΔAIC are similar

$BI = IC$

Therefore, I is the mid-point of BC.

$DE \quad BC$

$BC = 2 \times DE = 1m$

$AF = BA - BF = 0.4m \; \ldots\ldots (1)$

It can be said that D is the mid-point of AB.

Therefore, we can express:

$$AD = \frac{1}{2} \times BA = 0.8m \; \ldots\ldots. (2)$$

Using equations (1) and (2), we get:

$FE = 0.4m$

Therefore, F is the mid-point of AD.

$FG \,|\, DH$ and F is the mid-point of AD. Therefore, G will also be the mid-point of AH.

ΔAFG and ΔADH are similar triangles.

$$\frac{FG}{DH} = \frac{AF}{AD}$$

$$\Rightarrow \frac{FG}{DH}=\frac{0.4}{0.8}=\frac{1}{2}$$

$$\Rightarrow FG=\frac{1}{2}DH$$

$$\Rightarrow FG=\frac{1}{2}\times 0.25=0.125m$$

In ΔADH we can state,

$$AH=\sqrt{AD^2-DH^2}$$

$$\Rightarrow AH=\sqrt{0.8^2-0.25^2}=0.75m$$

For translational equilibrium of the ladder, the upward force should be the same as the downward force.

$$N_C+N_B=mg=392 \quad ...... (3)$$

For rotational equilibrium of the ladder, the net moment about A can be given as:

$$\Rightarrow -N_B\times 0.5+40\times 9.8\times 0.125+N_C\times 0.5=0$$

$$\Rightarrow (N_C-N_B)\times 0.5=49$$

$$\Rightarrow N_C-N_B=98 \quad ...... (4)$$

Solving equations (3) and (4), we can write:

$$N_C=245N$$

$$N_B=147N$$

For rotational equilibrium of the side AB, let us consider the moment about A.

$$-N_B\times BI+mg\times FG+T\times AG=0$$

$$\Rightarrow -245\times 0.5+40+9.8\times 0.125+T\times 0.76=0$$

$$\Rightarrow 0.76T=122.5-49$$

$\Rightarrow T=96.7N$, which is the required tension.

23. A man stands on a rotating platform, with his arms stretched horizontally holding a $5kg$ weight in each hand. The angular speed of the platform is 30 revolutions per minute. The man then brings his arms close to his body with the distance of each weight from the axis changing from $90cm$ to $20cm$. The moment of inertia of the man together with the platform may be taken to be constant and equal to $7.6kgm^2$.

(a) What is his new angular speed? (Neglect friction.)

Ans: Moment of inertia of the man-platform system is given as,

$7.6kgm^2$

Moment of inertia when the man stretches his hands to a distance of 90 cm:

$MI_{HandsStreched} = 2 \times mr^2$

$\Rightarrow MI_{HandsStreched} = 2 \times 5 \times (0.9)^2$

$\Rightarrow MI_{HandsStreched} = 8.1kgm^2$

Initial moment of inertia of the system can be given as,

$I_i = 7.6 + 8.1 = 15.7kgm^2$

Angular speed can be expressed as,

$\omega_1 = 300rev / min$

Angular momentum can be given as,

$L_i = I_i\omega_i = 15.7 \times 30 \; \ldots\ldots (1)$

Moment of inertia when the man folds his hands to a distance of 20 cm becomes:

$MI_{Hands at 20cm} = 2mr^2$

$\Rightarrow MI_{Hands at 20cm} = 2 \times 5(0.2)^2 = 0.4kgm^2$

Final moment of inertia is given as,

$I_f = 7.6 + 0.4 = 8kgm^2$

Final angular speed can be given as, ω_f

Final angular momentum can be expressed as,

$L_f = I_f\omega_f = 0.79\omega$ …… (2)

From the conservation of angular momentum, we can write:

$I_i\omega_i = I_f\omega_f$

$\omega_f = \dfrac{15.7 \times 30}{8} = 58.88 rev / min$, which is the new angular speed.

(b) Is kinetic energy conserved in the process? If not, from where does the change come about?

Ans: Kinetic energy is not conserved in the mentioned process. With the decrease in the moment of inertia, there is an increase in kinetic energy. The additional kinetic energy is generated from the work done by the man to fold his hands toward himself.

24. A bullet of mass $10g$ and speed $500m/s$ is fired into a door and gets embedded exactly at the centre of the door. The door is $1.0m$ wide and weighs $12kg$. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it. (Hint: The moment of inertia of the door about the vertical axis at one end is $\dfrac{ML^2}{3}$.)

Ans: Given that,

Mass of the bullet is given as, $m = 10g = 10 \times 10^{-3} kg$

Velocity of the bullet is given as, $v = 500m/s$

Width of the door, $L = 1.0m$

Radius of the door, $r = \frac{1}{2}m$

Mass of the door is given, $M = 12kg$

Angular momentum transmitted by the bullet on the door:

$\alpha = mvr$

$\Rightarrow \alpha = (100 \times 10^{-3}) \times (500) \times \frac{1}{2} = 2.5kgm^2s^{-1}$

Moment of inertia of the door can be given as:

$I = \frac{ML^2}{3}$

$\Rightarrow I = \frac{1}{3} \times 12 \times (1)^2 = 4kgm^2$

But we have the relation,

$\alpha = I\omega$

$\Rightarrow \omega = \frac{\alpha}{I} = \frac{2.5}{4} = 0.625rads^{-1}$, which is the required angular speed.

25. Two discs of moments of inertia I_1 and I_2 about their respective axes (normal to the disc and passing through the centre), and rotating with angular speeds ω_1 and ω_2 are brought into contact face to face with their axes of rotation coincident.

(a) What is the angular speed of the two-disc system?

Ans: Given that,

Moment of inertia of disc 1 is I_1

Angular speed of disc 1 is ω_1

Moment of inertia of disc 2 is I_2

Angular speed of disc 2 is ω_2

Angular momentum of disc 1 is $L_1 = I_1\omega_1$

Angular momentum of disc 2 is $L_2 = I_2\omega_2$

Total initial angular momentum is $L_i = I_1\omega_1 + I_2\omega_2$

When the two discs are joined together, their moments of inertia get summed up.

Moment of inertia of the system of two discs can be given as,

$I = I_1 + I_2$

Let ω be the angular speed of the system.

Total final angular momentum is given as, $L_f = (I_1 + I_2)\omega$

Let us use the law of conservation of angular momentum,

$L_i = L_f$

$\Rightarrow I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$

$\Rightarrow \omega = \dfrac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$, which is the required angular speed.

(b) Show that the kinetic energy of the combined system is less than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy? Take $\omega_1 \neq \omega_2$.

Ans: Kinetic energy of disc 1 is given as, $E_1 = \frac{1}{2} I_1 \omega^2{}_1$

Kinetic energy of disc 2, $E_2 = \frac{1}{2} I_2 \omega^2{}_2$

Total initial kinetic energy can be given as, $E_i = \frac{1}{2}\left(I_1\omega^2{}_1 + I_2\omega^2{}_2\right)$

When the discs are joined, their moments of inertia get summed up.

Moment of inertia of the system can be given as, $I = I_1 + I_2$

Angular speed of the system can be given as: ω

Final kinetic energy $E_f = \frac{1}{2}(I_1 + I_2)\omega^2$

$$\Rightarrow E_f = \frac{1}{2}(I_1 + I_2)\left(\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}\right)^2 = \frac{1}{2} \times \frac{(I_1\omega_1 + I_2\omega_2)^2}{I_1 + I_2}$$

And $E_i = E_f$

$$\Rightarrow E_i = \frac{1}{2}\left(I_1\omega^2{}_1 + I_2\omega^2{}_2\right) - \frac{(I_1\omega_1 + I_2\omega_2)^2}{2(I_1 + I_2)}$$

$$\Rightarrow E_i = \frac{1}{2}I_1\omega^2{}_1 + \frac{1}{2}I_2\omega^2{}_2 - \frac{1}{2}\frac{I^2{}_1\omega^2{}_1}{(I_1 + I_2)} - \frac{1}{2}\frac{I^2{}_2\omega^2{}_2}{(I_1 + I_2)} - \frac{1}{2}\frac{2I_1I_2\omega_1\omega_2}{(I_1 + I_2)}$$

$$\Rightarrow E_i = \frac{1}{(I_1 + I_2)}\left[\frac{1}{2}I_2\omega^2{}_1 + \frac{1}{2}I_1I_2\omega^2{}_1 + \frac{1}{2}I_1I_2\omega^2{}_2 + \frac{1}{2}I_2\omega^2{}_2 - \frac{1}{2}I_1\omega^2{}_1 - \frac{1}{2}I_2\omega^2{}_2 - I_1I_2\omega_1\omega_1\right]$$

$$\Rightarrow E_i = \frac{I_1I_2}{2(I_1 + I_2)}\left(\omega_1{}^2 + \omega^2{}_2 - 2\omega_1\omega_2\right)$$

$$\Rightarrow E_i = \frac{I_1I_2(\omega_1 - \omega_2)^2}{2(I_1 + I_2)}$$

All the quantities on right hand side are positive

$E_i - E_f > 0$

$E_i > E_f$

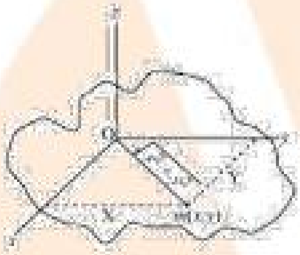
The loss of K.E. can be attributed to the frictional force that comes into play when the two discs come in contact with each other.

26.

(a) Prove the theorem of perpendicular axes. (Hint: Square of the distance of a point (x,y) in the x–y plane from an axis through the origin perpendicular to the plane is x^2+y^2)

Ans: It is stated by the theorem of perpendicular axes states that the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

A physical body with centre O and a point mass m, in the x–y plane at (x, y) is shown in the following figure.



Moment of inertia about x-axis can be given as, $I_x = mx^2$

Moment of inertia about y-axis can be given as, $I_y = my^2$

Moment of inertia about z-axis can be given as, $I_z = m\left(\sqrt{x^2+y^2}\right)^2$

$I_x + I_y = mx^2 + my^2 = m\left(x^2+y^2\right)$

$\Rightarrow I_x + I_y = m\left(\sqrt{x^2+y^2}\right)^2$

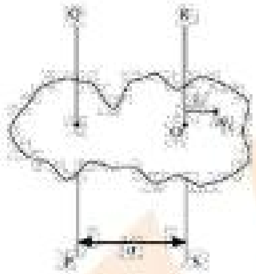
$\Rightarrow I_x + I_y = I_z$

Therefore, the theorem is proved.

(b) Prove the theorem of parallel axes.

(Hint: If the centre of mass is chosen to be the origin $\sum m_i r_i = 0$)

Ans: The theorem of parallel axes states that the moment of inertia of a body about any axis is same as the sum of the moment of inertia of the body about a parallel axis that passes through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.



Let us suppose that a rigid body is made up of n particles, having masses at $m_1, m_2, m_3, ..., m_n$ perpendicular distances $r_1, r_2, r_3, ..., r_n$ respectively from the centre of mass O of the rigid body.

The moment of inertia about axis RS passing through the point O can be given as:

$$I_{RS} = \sum_{i=1}^{n} m_i r_i^2$$

The perpendicular distance of mass m_i, from the axis

$$QP = a + r_i$$

Therefore, the moment of inertia about axis QP can be given as:

$$I_{QP} = \sum_{i=1}^{n} m_i (a + r_i)^2$$

$$\Rightarrow I_{QP} = \sum_{i=1}^{n} m_i \left(a^2 + r^2{}_i + 2ar_i\right)$$

$$\Rightarrow I_{QP} = \sum_{i=1}^{n} m_i \left(a^2\right) + \sum_{i=1}^{n} m_i r^2{}_i + \sum_{i=1}^{n} m_i 2ar_i$$

$$\Rightarrow I_{QP} = I_{RS} + \sum_{i=1}^{n} m_i \left(a^2\right) + 2\sum_{i=1}^{n} m_i ar_i$$

Now, at the centre of mass, the moment of inertia of all the particles about the axis that passes through the centre of mass is zero, that is,

$$2\sum_{i=1}^{n} m_i a r_i = 0$$

Where, $a \neq 0$

$$\sum m_i r_i = 0$$

$$\sum_{i=1}^{n} m_i = M$$

$M =$ Total mass of the rigid body

$$I_{QP} = I_{RS} + Ma^2$$

Therefore, the theorem is proved.

27. Prove the result that the velocity v of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height h is given by $v^2 = \dfrac{2gh}{\left(1+\dfrac{k^2}{R^2}\right)}$ using dynamical consideration (i.e. by consideration of forces and torques). Note k is the radius of gyration of the body about its symmetry axis, and R is the radius of the body. The body starts from rest at the top of the plane.

Ans: A body rolling on an inclined plane of height h, is depicted in the given figure:

$m =$ Mass of the body

$R =$ Radius of the body

K = Radius of gyration of the body

v = Translational velocity of the body

h = Height of the inclined plane

g = Acceleration due to gravity

Total energy at the top of the plane is given as,

$E_1 = mgh$

Total energy at the bottom of the plane can be given as,

$E_b = KE_{rot} + KE_{trans}$

$$\Rightarrow E_b = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

But $I = mk^2$ and $\omega = \frac{v}{R}$

$$\Rightarrow E_b = \frac{1}{2}mk^2\left(\frac{v^2}{R^2}\right) + \frac{1}{2}mv^2$$

$$\Rightarrow E_b = \frac{1}{2}mv^2\frac{k^2}{R^2} + \frac{1}{2}mv^2$$

$$\Rightarrow E_b = \frac{1}{2}mv^2\left(1 + \frac{k^2}{R^2}\right)$$

From the law of conservation of energy, we can write:

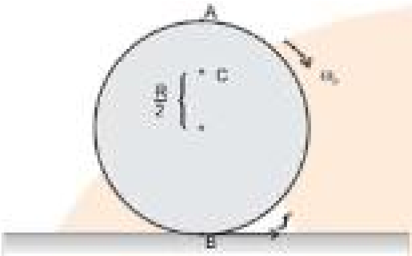
$E_T = E_B$

$$\Rightarrow mgh = \frac{1}{2}mv^2\left(1 + \frac{k^2}{R^2}\right)$$

$$\Rightarrow v = \frac{2gh}{\left(1 + \frac{k^2}{R^2}\right)}$$

Therefore, the given result is proved.

28. A disc rotating about its axis with angular speed ω_0 is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the disc shown in figure? Will the disc roll in the direction indicated?



Ans: From the question we can infer that:

$v_A = R\omega_0$,

$v_B = R\omega_0$,

$v_C = \left(\frac{R}{2}\right)\omega_0$

The rolling of the disc will not take place.

Angular speed of the disc is given $= \omega_0$

Radius of the disc is given $= R$

Let us use the relation for linear velocity, $v = \omega_0 R$

For point A we can write:

$v_A = R\omega_0$ in the direction tangential to the right

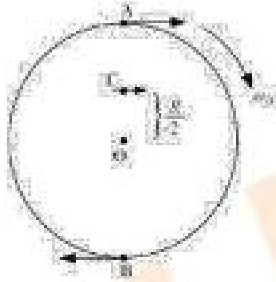
For point B we can write:

$v_B = R\omega_0$ in the direction tangential to the left

For point C it can be written as:

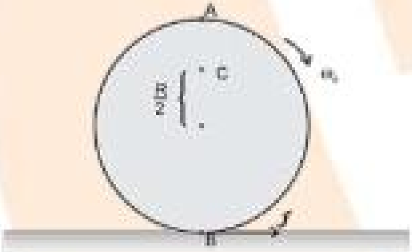
$v_C = \left(\frac{R}{2}\right)\omega_0$ in the direction same as that of v_A.

The directions of motion of points A, B, and C on the disc are depicted in the following figure



Because the disc is placed on a frictionless table, the disc will not roll. This is due to the presence of friction is essential for the rolling of a body.

29. Explain why friction is necessary to make the disc in figure given roll in the direction indicated.



(a) Give the direction of frictional force at B, and the sense of frictional torque, before perfect rolling begins.

Ans: To roll the given disc, some torque is necessary. As per the definition of torque, the rotating force must be tangential to the disc. Since the frictional force at point B is along the tangential force at point A, a frictional force is necessary for making the disc roll.

Force of friction will act in the opposite direction to the direction of velocity at point B. The direction of linear velocity at point B can be pointed tangentially leftward. Therefore, frictional force will act tangentially rightward. The frictional torque before the start of perfect rolling is perpendicular to the plane of the disc in the outward direction.

(b) What is the force of friction after perfect rolling begins?

Ans: Since frictional force will act opposite to the direction of velocity at point B, perfect rolling will start when the velocity at that point becomes equal to zero. This will make the frictional force that acts on the disc as zero.

30. A solid disc and a ring, both of radius 10cm are placed on a horizontal table simultaneously, with initial angular speed equal to $10\pi \text{ rad s}^{-1}$. Which of the two will start to roll earlier? The co-efficient of kinetic friction is $\mu_k = 0.2$.

Ans: Given that,

Radii of the ring and the disc are given as, r = 10 cm = 0.1 m

Initial angular speed is given, $u = 0$

Coefficient of kinetic friction is, $\mu_k = 0.2$

Initial velocity of both the objects, $u = 0$

Motion of the two objects is a result of frictional force. As per Newton's second law of motion,

we have frictional force, $f = ma$

$\mu_k mg = ma$

Where,

$a =$ Acceleration produced in the objects

$m =$ Mass

$a = \mu_k g \;\ldots\ldots (1)$

From the first equation of motion, the final velocity of the objects can be obtained as:

$v = u + at$

$\Rightarrow v = 0 + \mu_k gt$

$v = \mu_k gt$ ...... (2)

The torque applied by the frictional force will act in perpendicularly outward direction and cause reduction in the initial angular speed.

Torque, $\tau = -I\alpha$

$\alpha =$ Angular acceleration

$\mu_k mgr = -I\alpha$

$$\Rightarrow \alpha = \frac{-\mu_k mgr}{I} \text{ ...... (3)}$$

Let us use the first equation of rotational motion to obtain the final angular speed:

$\omega = \omega_0 + \alpha t$

$$\Rightarrow \omega = \omega_0 + \frac{-\mu_k mgr}{I} t \text{ ...... (4)}$$

Rolling starts when linear velocity, $v = r\omega$

$$v = r\left(\omega_0 - \frac{\mu_k mgrt}{I}\right) \text{ ...... (5)}$$

Equating equations (2) and (5), we can write:

$$\mu_k gt = r\left(\omega_0 - \frac{\mu_k mgrt}{I}\right)$$

$$\Rightarrow \mu_k gt = r\left(\omega_0 - \frac{\mu_k mgrt}{I}\right) \text{ ...... (6)}$$

For the ring:

$I = mr^2$

$$\mu_k gt = r\left(\omega_0 - \frac{\mu_k mgrt}{mr^2}\right)$$

$\mu_k gt = r\omega_0 - \mu_k gt$

$\Rightarrow 2\mu_k gt = r\omega_0$

$\Rightarrow t_r = \dfrac{r\omega_0}{2\mu_k g}$

$\Rightarrow t_r = \dfrac{0.1\times10\times3.14}{2\times0.2\times9.8} = 0.80s$ ...... (7)

For the disc:

$I = \dfrac{1}{2}mr^2$

$\mu_k gt_d = r\omega_0 - \dfrac{\mu_k gmr^2 t_d}{\frac{1}{2}mr^2}$

$\Rightarrow \mu_k gt_d = r\omega_0 - \dfrac{\mu_k gt_d}{\frac{1}{2}}$

$\Rightarrow \mu_k gt_d = r\omega_0 - 2\mu_k gt_d$

$\Rightarrow 3\mu_k gt_d = r\omega_0$

$t_d = \dfrac{r\omega_0}{3\mu_k g}$

$\Rightarrow t_d = \dfrac{0.1\times10\times3.14}{3\times0.2\times9.8} = 0.53s$ ...... (8)

Since, $t_d > t_r$ the disc will start rolling before the ring.

31. A cylinder of mass $10kg$ and radius $15cm$ is rolling perfectly on a plane of inclination $30°$. The coefficient of static friction $\mu_k = 0.25$

(a) How much is the force of friction acting on the cylinder?

Ans: Given that,

Mass of the cylinder is given as, $m = 10kg$

Radius of the cylinder is given as, $r = 15\text{cm} = 0.15\text{m}$

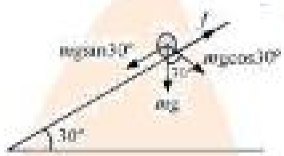
Co-efficient of kinetic friction $\mu_k = 0.25$

Angle of inclination is given as, $\theta = 30^\circ$

Moment of inertia of a solid cylinder about its geometric axis is,

$$I = \frac{1}{2}mr^2$$

The various forces acting on the cylinder are depicted in the given figure:



The acceleration of the cylinder is given as:

$$a = \frac{mg\sin\theta}{m + \frac{I}{r^2}}$$

$$\Rightarrow a = \frac{mg\sin\theta}{m + \frac{mr^2}{2r^2}} = \frac{2}{3}g\sin 30^\circ$$

$$\Rightarrow a = \frac{2}{3} \times 9.8 \times 0.5 = 3.27\text{m}/\text{s}^2$$

Let us use Newton's second law of motion, we can express net force as:

$$f_{net} = ma$$

$$mg\sin 30^\circ - f = ma$$

$$\Rightarrow f = mg\sin 30^\circ - ma$$

$$\Rightarrow f = 10 \times 9.8 \times 0.5 - 10 \times 3.27$$

$\Rightarrow f = 49 - 32.7 = 16.3\text{N}$, which is the frictional force.

(b) What is the work done against friction during rolling?

Ans: During rolling, the instantaneous point of contact with the plane will come to rest. Therefore, the work done against frictional force will be zero.

(c) If the inclination of the plane is increased, at what value of angle does the cylinder begin to skid, and not roll perfectly?

Ans: For rolling without skidding, we have the formula:

$$\mu = \frac{1}{3}\tan\theta$$

$$\Rightarrow \tan\theta = 3\mu = 3 \times 0.25 = 0.75$$

$\Rightarrow \tan^{-1} 0.75 = 36.87^\circ$, which is the required value of angle.

32. Read each statement below carefully, and state, with reasons, if it is true or false;

(a) During rolling, the force of friction acts in the same direction as the direction of motion of the CM of the body.

Ans: False.

Frictional force acts in the opposite direction of motion of the centre of mass of a body. In the case of rolling, the direct point of motion of the centre of mass is in backward direction. Therefore, frictional force acts in the forward direction.

(b) The instantaneous speed of the point of contact during rolling is zero.

Ans: True.

Rolling can be considered as the rotation of a body about an axis that passes through the point of contact of the body with the ground. Therefore, its instantaneous speed is zero.

(c) The instantaneous acceleration of the point of contact during rolling is zero.

Ans: False.

When a body is rolling, its instantaneous acceleration is not equal to zero. It has some value.

(d) For perfect rolling motion, work done against friction is zero.

Ans: True.

When perfect rolling begins, the frictional force that acts at the lowermost point becomes zero. Therefore, the work done against friction is also zero.

(e) A wheel moving down a perfectly frictionless inclined plane will undergo slipping (not rolling) motion.

Ans: True.

The rolling of a body occurs when a frictional force will act between the body and the surface. This frictional force will give the torque necessary for rolling. When the frictional force is not present, the body slips from the inclined plane under the effect of its own weight.

33. Separation of Motion of a system of particles into motion of the centre of mass and motion about the centre of mass:

(a) Show $p_i = p_i' + m_i V$ Where p_i is the momentum of the i^{th} particle (of mass m_i) and $p_i' = m_i v_i'$. Note v_i' is the velocity of the i^{th} particle relative to the centre of mass. Also, prove using the definition of the centre of mass $\sum_i p_i' = 0$.

Ans: Let us take a system of i moving particles.

Mass of the i^{th} particle $= m_i$

Velocity of the i^{th} particle $= v_i$

Therefore, momentum of the i^{th} particle, $p_i = m_i v_i$

Velocity of the centre of mass is V

The velocity of the i^{th} particle with respect to the centre of mass of the system is given as: $v_i' = v_i - V$ …… (1)

Multiplying m_i throughout equation (1),

we can write:

$$m_i v_i' = m_i v_i - m_i V$$

$$p_i' = p_i - m_i V$$

Where,

$p_i' = m_i v_i'$ is the Momentum of the i^{th} particle with respect to the centre of mass of the system.

Hence, $p_i = p_i' + m_i V$

We have the formula: $p_i' = m_i v_i'$

Taking the summation of momentum of all the particles with respect to the centre of mass of the system, we can write:

$$\sum_i p_i' = \sum_i m_i v_i' = \sum_i m_i \frac{dr_i'}{dt}$$

Where

r_i' is the position vector of i^{th} particle with respect to the centre of mass

$$v_i' = \frac{dr_i'}{dt}$$

As per the definition of the centre of mass, we have:

$$\sum_i m_i v_i' = 0$$

$$\Rightarrow \sum_i m_i \frac{dr_i'}{dt} = 0$$

$$\sum_i p_i' = 0$$

Hence proved.

(b) Show $K = K' + \frac{1}{2}MV^2$ Where K is the total kinetic energy of the system of particles, K' is the total kinetic energy of the system when the particle velocities are taken with respect to the centre of mass and $\frac{1}{2}MV^2$ is the kinetic energy of the translation of the system as a whole (i.e. of the centre of mass motion of the system). The result has been used in Sec. 7.14.

Ans: We have the formula for velocity of the i^{th} particle as:

$$v_i = v_i' + V$$

$$\sum m_i v_i = \sum m_i v_i' + \sum m_i V$$

Taking the dot product of equation (2) with itself, we can write:

$$\sum_i m_i v_i \sum_i m_i v_i' + \sum_i m_i \left(v_i' + V\right) \sum_i m_i \left(v_i' + v\right)$$

$$M^2 \sum_i v_i^2 = M^2 \sum_i v_i^2 + M^2 \sum_i v_i v_i' + M^2 \sum_i v_i' v_i + M^2 V^2$$

Now, for the centre of mass of the system of particles, $\sum_i v_i' v_i = -\sum_i v_i' v_i$,

$$M^2 \sum_i v_i^2 = M^2 \sum_i v_i^2 + M^2 V^2$$

$$\frac{1}{2} M \sum_i v_i^2 = \frac{1}{2} M \sum_i v_i'^2 + \frac{1}{2} MV^2$$

$K = K' + \frac{1}{2}MV^2$

Where,

$K = \frac{1}{2}M\sum_i v_1^2$ is the total kinetic energy of the system of particles

$K' = \frac{1}{2}M\sum_i v_1'^2$ is the total kinetic energy of the system of particles with respect to the centre of mass

$\frac{1}{2}MV^2$ is the kinetic energy of the translation of the system as a whole.

(c) Show $L = L' + R \times MV$, Where $L' = \sum r_i' \times p_i'$ is the angular momentum of the system about the centre of mass with velocities taken relative to the centre of mass. Remember $r_i' = r_i - R$ rest of the notation is the standard notation used in the chapter. Note L' and $R \times MV$ can be said to be angular momenta, respectively, about and of the centre of mass of the system of particles.

Ans: Position vector of the i^{th} particle with respect to origin can be given $= r_i$

Position vector of the i^{th} particle with respect to the centre of mass is given $= r_i'$

With respect to the origin the position vector of the centre of mass $= R$

It is provided that:

$r_i' = r_i - R$

$r_i = r_i' + R$

We have the following relation from part (a),

$p_i = p_i' + m_i V$

Taking the cross product of this relation by r_i we can write:

$$\sum_i r_i \times p_i = \sum_i r_i \times p_i' = \sum_i r_i \times m_i V$$

$$L = \sum_i (r_i \times R) \times p_i' + \sum_i (r_i' \times R) \times m_i V$$

$$\Rightarrow L = \sum_i r_i' \times p_i' + \sum_i R \times p_i' + \sum_i r_i' \times m_i V + \sum_i R \times m_i V$$

$$\Rightarrow L = L' + \sum_i R_i \times p_i' + \sum_i r_i' \times m_i V + \sum_i R \times m_i V$$

where,

$$R \times \sum_i p_i' = 0 \text{ and } \left(\sum_i r_i'\right) \times MV = 0$$

Hence, $\sum_i m_i = M$

$$L = L' + R \times MV$$

Hence proved.

(d) Show $\frac{dL'}{dt} = \sum_i r_i' \times \frac{d}{dt}\left(P_i'\right)$. Further, show that $\frac{dL'}{dt} = \tau'_{ext}$ where τ'_{ext} is the sum of all external torques acting on the system about the centre of mass. **(Hint: Use the definition of centre of mass and Newton's Third** Law. Assume the internal forces between any two particles act along the line joining the particles.)

Ans: We have the following relation:

$$L' = \sum_i r_i' \times p_i'$$

$$\frac{dL'}{dt} = \frac{d}{dt}\left(\sum_i r_i' \times p_i'\right)$$

$$\Rightarrow \frac{dL'}{dt} = \frac{d}{dt}\left(\sum_i r_i'\right) \times p_i' + \sum_i r_i' \times \frac{d}{dt}\left(p_i'\right)$$

$$\Rightarrow \frac{dL'}{dt} = \frac{d}{dt}\left(\sum_i m_i r_i'\right) \times v_i' + \sum_i r_i' \times \frac{d}{dt}\left(p_i'\right)$$

Where,

r_i' is the position vector with respect to the centre of mass of the system of particles.

$$\sum_i m_i r_i' = 0$$

$$\frac{dL'}{dt} = \sum_i r_i' \times \frac{d}{dt}\left(p_i'\right)$$

We have the following relation:

$$\Rightarrow \frac{dL'}{dt} = \sum_i r_i' \times m_i \frac{d}{dt}\left(v_i'\right)$$

Where,

$\frac{d}{dt}\left(v_i'\right)$ is the rate of change of velocity of the i^{th} particle with respect to the centre of mass of the system.

Therefore, according to Newton's third law of motion, we can express:

$m_i \frac{d}{dt}\left(v_i'\right)$ is the external force acting on the i^{th} particle.

$\sum_i \left(\tau_i'\right)_{ext}$ i.e., $\sum_i r_i' \times m_i \frac{d}{dt}\left(v_i'\right) = \tau'_{ext}$ is the external torque acting on the system as a whole.

Hence proved that $\frac{dL'}{dt} = \tau'_{ext}$.