# Subband Adaptive Filters

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***Abstract*—A transform-domain adaptive filter allows the error signal being projected on an orthogonal basis, and thus separates the convergence into decorrelated components which would give an improvement of convergence rate. A transform being fixed, in other words, a predefined basis set, yields a filter bank, so we derive the equivalence between the transform-domain and subband methods using diagonalization of the correlation matrix. By exploiting the properties of both input signal and the target system, the convergence behavior of the adaptive filter can be highly enhanced. Our simulation results speak for themselves.**

## I. Introduction

The motivation of using subband adaptive filters (SAFs) based on least mean squares (LMS) algorithm is to allow better convergence rate [1]. Intuitively, the better convergence rate can be achieved by matching the energy of the subband signal to the adaptation step size in each subband. As we increase the number of subbands, the convergence rate increases and approaches the rate that can be obtained with a flat input spectrum. Details of SAFs are presented in the following sections. Section II designs popular filter banks called cosine modulated filter banks which will be used in SAFs. Next, in section III, we start from some well-known LMS algorithms and explain the intuitions behind them in order to bring up the subband method. We will see that subband method is an extension or a generalization to color input signal. Finally, section IV shows the simulation results, the conclusion is made in section V and the discussion is given by section VI.

## II. Maximally Decimated Cosine Modulated Filter Banks

A filter bank block diagram is given by Fig. 1. In general, we only apply the signal processing techniques or the algorithms to modify the transform coefficients in the processing part. As we know, when there is no modification being made on the transform coefficients, the system remains perfect reconstruction. However, the system cannot guarantee perfect reconstruction if there are some algorithms which modify or change the transform coefficients. Fig. 2 shows an M-channel maximally decimated cosine modulated filter bank (CMFB). There are two types of CMFB, pseudo quadrature-mirror filter (QMF) banks and perfect reconstruction systems, respectively. For the pseudo QMF, we can only design a prototype filter; however, perfection reconstruction systems need to satisfy a set of quadratic constraints.

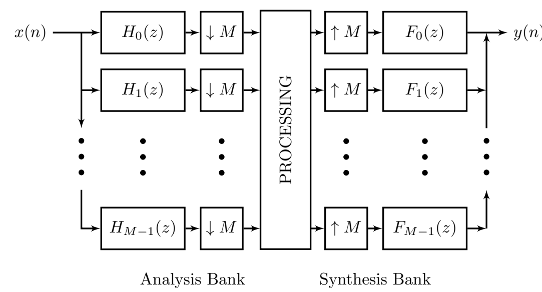


Fig. 1. A M-channel maximally decimated filter bank.

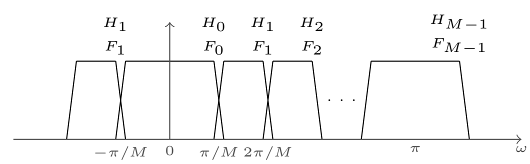


Fig. 2. A M-channel maximally decimated CMFB.

### A. Cosine Modulated Pseudo QMF Banks

Given a linear-phase, low-pass prototype filter $P_0(z) = \sum_{n=0}^{N-1} p_0(n)z^{-n}$, the analysis filters $H_k(z)$ and the synthesis filters $F_k(z)$ are obtained by

$$h_k(n) = 2p_0(n)\cos\left((2k+1)\frac{\pi}{2M}(n-\frac{N-1}{2})+\theta_k\right), \tag{1}$$

$$f_k(n) = h_k(N-n), \tag{2}$$

for $0 \le n \le N-1$ and $0 \le k \le M-1$ where $h_k(n)$ and $f_k(n)$ are the impulse responses of $H_k(z)$ and $F_k(z)$, respectively. We choose the $\theta_k$ as $\theta_k = (-1)^k \frac{\pi}{4}$.

### B. Cosine Modulated Perfect Reconstruction Systems

The following is a theorem for forcing losslessness in causal FIR systems [2].

*Theorem 1:* Let the prototype filter $P_0(z)$ be a linear phase filter with length $N+1 = 2mM$ for some integer $m$. Let $G_k(z), 0 \leq k \leq 2M-1$ be the $2M$ polyphase components of $P_0(z)$. Then the $M \times M$ polyphase component matrix $\mathbf{E}(z)$ is paraunitary if and only if $G_k(z)$ satisfy the pairwise power complementary conditions

$$\tilde{G}_k(z)G_k(z) + \tilde{G}_{M+k}(z)G_{M+k}(z) = \alpha \qquad (3)$$

for $0 \leq k \leq M-1$ and some $\alpha > 0$.
A very simple prototype filter [3] within feasible region is given by

$$P_0(n) = \begin{cases} \frac{1}{\sqrt{4M}}, & (mM - M \leq k \leq (mM + M - 1) \\ 0, & \text{otherwise} \end{cases}. \qquad (4)$$

However, the quality of the prototype filter in the above example is not desired in terms of the stopband attenuation and the sharpness of the transition band. Therefore, we resort to some optimization techniques to find a good quality prototype filter within the feasible region. Note that the feasible region is formed by all prototype filters which satisfies the perfect reconstruction conditions stated by Theorem 1. One very popular objective function is the stopband energy where the constraints are quadratic because of the pairwise power complementary conditions. To sum up, we can design a perfect reconstruction cosine modulated filter bank by using quadratic constraints least-squares (QCLS) [4]. The optimization problem is given by

$$\text{minimize } \Phi(\mathbf{p}) \text{ subject to } \mathbf{p}^T\mathbf{Q}_{l,n}\mathbf{p} = c_n \qquad (5)$$

where $0 \leq n \leq 2m-2$, $0 \leq l \leq \frac{M}{2} - 1$, and

$$\Phi(\mathbf{p}) = \int_{\omega_s}^{\pi} |P(e^{j\omega})|^2 d\omega. \qquad (6)$$

To derive the matrices $\mathbf{Q}_{l,n}$, firstly note that the polyphase components $G_l(z)$ can be written as

$$G_l(z) = \mathbf{p}^T\mathbf{V}_l\mathbf{e}, \qquad (7)$$

where $\mathbf{p} = \begin{bmatrix} p(0) & p(1) & \cdots & p(2mM-1) \end{bmatrix}^T$, $\mathbf{e} = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(m-1)} \end{bmatrix}^T$ and

$$[\mathbf{V}_l]_{i,j} = \begin{cases} 1, & i = l + 2jM \\ 0, & \text{otherwise} \end{cases}, \mathbf{V}_l \in \mathbb{R}^{2mM \times m}. \qquad (8)$$

Then, Eq. 3 can be expressed as

$$\mathbf{p}^T \left[ \mathbf{V}_l\mathbf{e}\mathbf{e}^T\mathbf{V}_l^T + \mathbf{V}_{M+l}\mathbf{e}\mathbf{e}^T\mathbf{V}_{M+l}^T \right] \mathbf{p} = \alpha, \qquad (9)$$

where we let $\mathbf{Q}_{l,n} = \mathbf{V}_l\mathbf{e}\mathbf{e}^T\mathbf{V}_l^T + \mathbf{V}_{M+l}\mathbf{e}\mathbf{e}^T\mathbf{V}_{M+l}^T$.

## III. Subband Adaptive Filters

### A. Fullband LMS Algorithms

*1) Least Mean Square:* For a system identification problem in Fig. 3, the adaptation of the filter is based on the error signal $e(n)$. The algorithm used for adaptation is generally a gradient type. The least mean square (LMS) algorithm in the following [5] has been used widely.

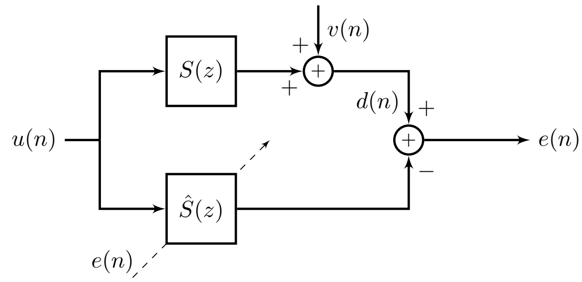


Fig. 3. A system identification problem.

$$\boxed{\hat{\mathbf{s}}(n+1) = \hat{\mathbf{s}}(n) + \mu\mathbf{u}(n)e(n)}. \qquad (10)$$

By exploiting the properties of the input signal $u(n)$ and the target system $S(z)$, the convergence behavior can be largely improved.

*2) Proportionate LMS:* Proportionate adaptation [6] for LMS is given by

$$\boxed{\hat{\mathbf{s}}(n+1) = \hat{\mathbf{s}}(n) + \mu\mathbf{G}(n)\mathbf{u}(n)e(n)}, \qquad (11)$$

where

$$\mathbf{G}(n) = \text{diag}\left(g_0(n), g_1(n), \ldots, g_{L-1}(n)\right) \qquad (12)$$

is called the "step-size control matrix" and $g_i(n)$ is a function of the current $\hat{\mathbf{s}}_i(n)$. Note that different algorithms have been proposed to update $\mathbf{G}(n)$ which utilizes the properties of the target system $S(z)$. It does not exploit the properties of the input signal $\mathbf{u}(n)$.

*3) Modified LMS:* Modified LMS [7] exploits the power of the input signal $u(n)$, so the convergence behavior can be slightly enhanced. An explanation of the term $\hat{\sigma}^2(n)$ in modified LMS can be seen by the power level estimate in Fig. 4 where the assumption here is that the power spectrum of the input signal is flat.

$$\boxed{\hat{\mathbf{s}}(n+1) = \hat{\mathbf{s}}(n) + \frac{\mu}{\hat{\sigma}^2(n) + \delta}\mathbf{u}(n)e(n)}. \qquad (13)$$

*4) Transform-Domain LMS:* The basic idea behind transform-domain LMS [8] is to modify the input signal $\mathbf{u}(n)$ to be applied to the adaptive filter by

$$\mathbf{u}'(n) = \mathbf{A}\mathbf{u}(n) \text{ where } \mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A} = \mathbf{I}, \qquad (14)$$

such that the conditioning number $\kappa(\mathbf{\Phi}_{\mathbf{u}'})$ of the corresponding correlation matrix $\mathbf{\Phi}_{\mathbf{u}'}$ is smaller than $\kappa(\mathbf{\Phi}_{\mathbf{u}})$. Consequently, the best scenario can be immediately found by the elements of $\mathbf{u}'(n)$ are uncorrelated, namely, the matrix $\mathbf{\Phi}_{\mathbf{u}'}$ is diagonal. Hence, the question will be how to find a transformation $\mathbf{A}$ which diagonalizes the correlation matrix $\mathbf{\Phi}_{\mathbf{u}}$ of the input signal $\mathbf{u}(n)$. Since the

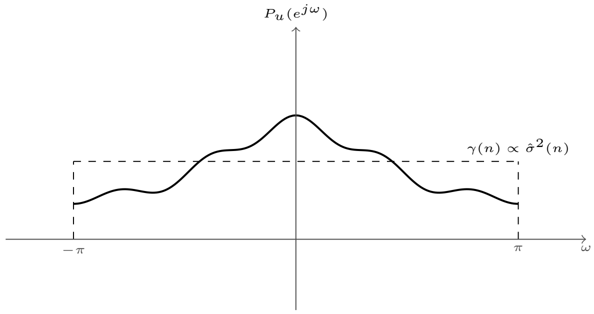


Fig. 4. The input spectrum is estimated by a scalar.

analysis filter banks can be viewed as transformations, the transform-domain LMS is actually deeply connected to the subband adaptive filters.

### B. Subband LMS

Firstly, note that the adaptive filter $\hat{S}(z)$ can be represented by its polyphase components as

$$\hat{S}(z) = \sum_{k=0}^{M-1} z^{-k} \hat{S}_k(z^M), \tag{15}$$

where we denote the z-transform of the impulse response $\hat{\mathbf{s}}(n)$ as $\hat{S}(z)$. In fullband LMS, the filter is adapted by $e(n)\mathbf{u}(n)$. However, subband LMS [9] adapts each polyphase component $\hat{S}_k(z^M)$ according to a linear combination of $e_l(n)\mathbf{x}_{lk}(n)$ where $e_l(n)$ is the subband error signal and $\mathbf{x}_{lk}(n)$ is the transform coefficients.

Then, by inserting a M-band filter bank before the adder which produces the error signal $e(n)$ and applying the noble identities to the polyphase components $\hat{S}_k(z^M)$ of the adaptive filter $\hat{S}(z)$ for the system identification problem in Fig. 3, we get a structure of the subband adaptive filter illustrated by Fig. 5. Hence, the

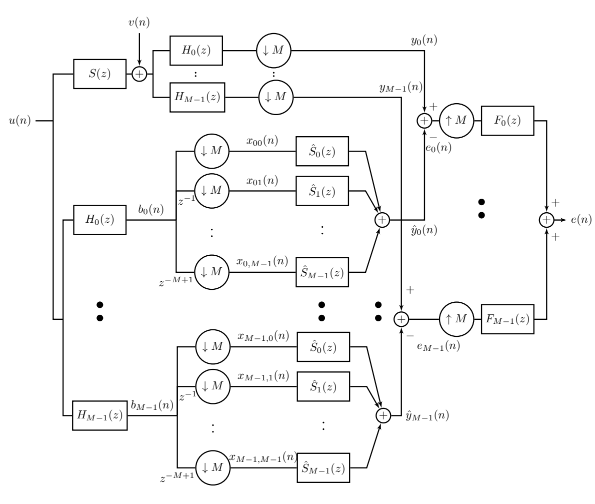


Fig. 5. Structure of the subband adaptive filter for the M-band case.

cost function $J(n)$ can be defined as

$$J(n) = E\left[\sum_{l=0}^{M-1} \alpha_l(n) e_l^2(n)\right], \tag{16}$$

where $e_l(n)$ is the error signal in $l^{th}$ subband and $\boldsymbol{\alpha}(n) = \begin{bmatrix}\alpha_0(n) & \alpha_1(n) & \cdots & \alpha_{M-1}(n)\end{bmatrix}^T \in \mathbb{R}^M, \alpha_l(n) \geq 0, \forall k$ is the error weighting coefficients. By choosing the $\boldsymbol{\alpha}(n)$ in a proper way, we are able to decorrelate the input signal $u(n)$; thus, yield a better convergence behavior. Then, using the same techniques in deriving LMS algorithm, the subband LMS algorithm is given by

$$\boxed{\hat{\mathbf{s}}_k(n+1) = \hat{\mathbf{s}}_k(n) + \mu \sum_{l=0}^{M-1} \alpha_l(n) e_l(n) \mathbf{x}_{lk}(n)}, \tag{17}$$

where $\hat{\mathbf{s}}_k(n)$ is the impulse response of the $k^{th}$ polyphase component of $\hat{\mathbf{s}}(n)$ and $\mathbf{x}_{lk}(n)$ is the output of a M-fold decimator whose input is a subband signal $\mathbf{b}_l(n)$ being delayed by $k$ samples at $l^{th}$ subband, namely, the transform coefficients.

### C. The Improvement of Convergence Behavior

Assuming that the spectrum $P_u(e^{j\omega})$ of the input signal $u(n)$ is piecewise flat, that is, the dotted line in Fig. 6. Then, the spectrum can be written as

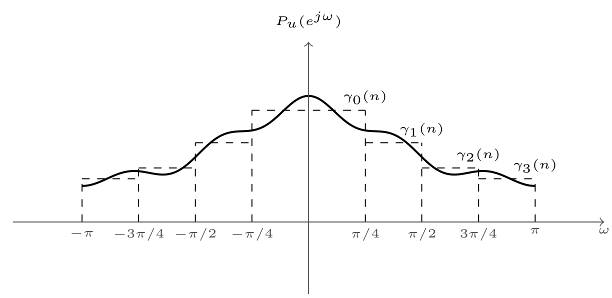


Fig. 6. Using 4-level piecewise flat spectrum to approximate the spectrum $P_u(e^{j\omega})$ of the input signal $u(n)$.

$$P_u(e^{j\omega}) = \sum_{k=0}^{M-1} \gamma_k(n) \left[P_0(e^{j(\omega-\omega_k)}) + P_0(e^{j(\omega+\omega_k)})\right] \tag{18}$$

where $P_0(e^{j\omega})$ is the spectrum of the prototype filter for the cosine modulated filter bank where we assume that the prototype filter is an ideal low-pass filter. Therefore, the spectrum of the subband signal $b_k(n)$ is given by

$$\begin{aligned} P_{b_k}(e^{j\omega}) &= P_u(e^{j\omega}) |H_k(e^{j\omega})|^2 \\ &= \gamma_k(n) \left[P_0(e^{j(\omega-\omega_k)}) + P_0(e^{j(\omega+\omega_k)})\right], \end{aligned} \tag{19}$$

where the autocorrelation function $r_k(m) = 2\gamma_k(n) p_0(m) \cos(\omega_k m), m \in \mathbb{Z}$ can be easily found by inverse Fourier transform. However, since $p_0(m)$ is a sinc function which is evaluated on integers, we have the correlation function $r_k(m) = 0, \forall m \neq 0$; thus the correlation matrix $\mathbf{R}_{b_k}(n) = 2\gamma_k(n)\mathbf{I}$. Finally, the correlation matrix $\mathbf{R}_u(n)$ of the input signal $u(n)$ is given by $\mathbf{R}_u(n) = \text{diag}(\alpha_0(n)\mathbf{R}_{b_0}(n), \cdots, \alpha_{M-1}(n)\mathbf{R}_{b_{M-1}}(n))$ where it is obvious that we can choose $\alpha_k(n) = 1/\gamma_k(n)$ to diagonalize the correlation matrix.

## IV. Simulation Results

Table I elaborates the experimental setup for the subband adaptive filters using LMS ans sparsity-promoting LMS [10] adaptations. Fig. 7 shows the impulse response of the target system.

In experiment I, we aim to show that the subband method gives no benefit since the input signal is white (the spectrum is flat). The property we can exploit in this case is the sparsity of the target system, so the SLMS should outperform the LMS. On the other hand, in experiment II, when the input signal is correlated (color), the subband method should improve the convergence behavior according to the theoretical analysis. Notice that the input spectrum can be approximated by a piecewise flat spectrum or a combination of ideal band-pass filters much more accurate when we increase $M$, namely, the number of subbands. The results are given by Fig. 8 and Fig. 9 for experiment I and II, respectively.

TABLE I
Experimental setup for the SAF using fullband, $M = 2$, $M = 4$ and $M = 8$.

| Experiment | I | II |
|---|---|---|
| Target system $S(z)$ | Quasi-Sparse, $L = 256$ (length) | |
| Input signal $y(n)$ | White | Color (first-order AR process) |
| System noise $N(n)$ | White Gaussian noise (WGN) | |
| System noise level | $-30$ dB | |
| Adaptation | LMS and SLMS ($p = 1.5$) [10] | |
| Filter Bank | Pseudo QMF Bank | |
| Monte Carlo Runs | 2000 | |

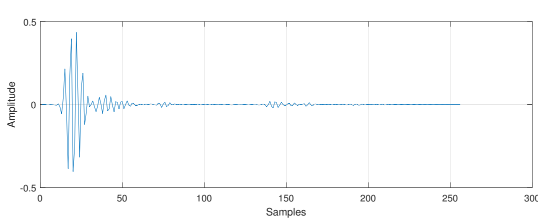


Fig. 7. The target system $\mathbf{s}$.

## V. Conclusion

### A. By Kuan-Lin Chen

Increasing the number of subbands in subband adaptive filters decorrelates the input signal and yields a better convergence behavior when the input spectrum is color

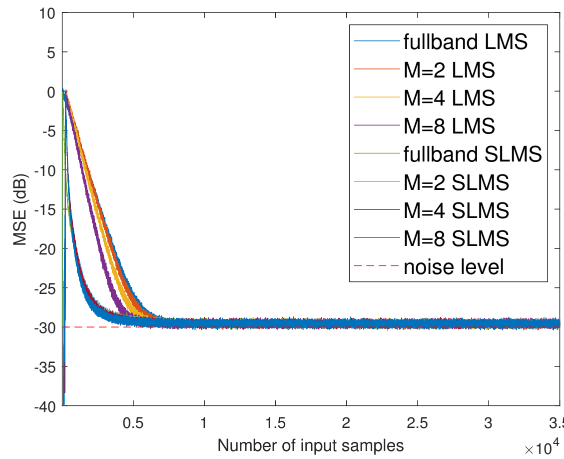


Fig. 8. Convergence behavior for experiment I. The behavior of the fullband, $M = 2$, $M = 4$ and $M = 8$ LMS are slightly different (We may choose wrong step sizes for the comparison). The behavior of the fullband, $M = 2$, $M = 4$ and $M = 8$ SLMS are exactly the same.

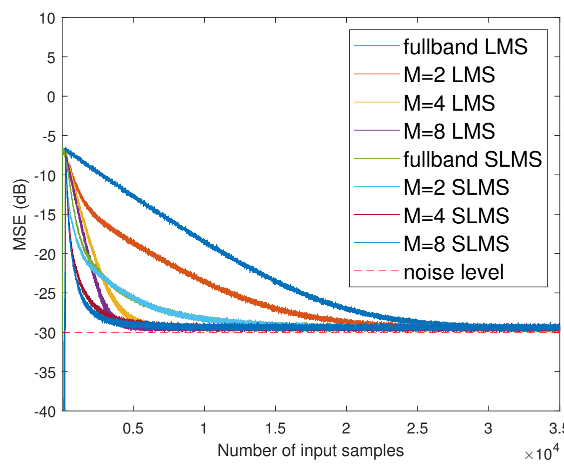


Fig. 9. Convergence behavior for experiment II. Notice that "M=8 SLMS" outperforms every other one.

and can be approximated by a piecewise flat spectrum. Mathematically, subband method can be viewed by a transform-domain LMS algorithm where the analysis filters diagonalize the correlation matrix of the input signal.

Subband adaptive filters can be incorporated with sparsity-promoting LMS algorithm even though the length of the filters are shorter than the fullband case. According to the simulation results, it seems to be no conflict between sparsity promoting and subband decomposition.

### B. By Hsiao-Chen Huang

As we can see in Fig 9, the case of "M=8 LMS" converges faster than the case of "fullband SLMS". This may be resulted from the high correlation of the input signal since the all-pole model we used is given by $\frac{1}{1-0.8z^{-1}}$. On the other hand, the target system is not extremely sparse so the SLMS algorithm cannot fully exploit its objective.

The same observation can be made by seeing from "M=4 LMS" outperforms "M=2 SLMS". Even though the subband SLMS with $M = 2$ is used, the desired approximation of the input spectrum is not satisfied.

## VI. Discussion

### A. By Kuan-Lin Chen

The filter bank is usually predefined which yields a fixed basis set. If the eigenvectors of the input correlation matrix has changed, then the filter bank (transform) needs to be updated for better convergence.

We also apply the SAFs for the adaptive feedback cancellation (AFC) problem [11] but the results are somehow disappointing that the speech quality is degraded even for low gain. This may be caused from using the maximally decimated filter banks. A non-maximally decimated filter banks may resolve this problem.

### B. By Hsiao-Chen Huang

In order to further enhance the decorrelation of the input color signal, the future work will be designing an adaptive filter bank or a time-varying orthogonal transformation to track the eigenvectors of the corrrelation matrix. For instance, the Gram-Schmidt procedure may be used to find a set of orthonormal basis where the diagonalization of the correlation matrix can be realized in real-time.

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