Math 352 Final Project

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Jacobi and Gauss-Seidel Iteration

Introduction

Iteration is a useful technique finding roots of equations. Generalization of fixed-point iteration could be applied to solve systems of linear equations with the desired accuracy.

In this project, we examine the following two iterative methods: Gauss-Seidel iteration, attributed to Johann Gauss(1777-1855) and Philip Seidel (1821-1896), and the Jacobi iteration, attributed to Carl Jacobi (1804-1851).

Jacobi method is used to determining the solutions of a diagonally dominant system of linear equations. Every diagonal element is solved for and an approximate value would be plugged in. This process is repeated until it converges.

The Jacobi method solves equations in order and is derived by examining every of the n equations in the linear system of equations Ax = b. In the *i* th equation, $\sum_{j=1}^{n} a_{ij} = b_i$ we solve for *i* th unknown term, we obtain the equation of $x_i^{(k)} = [-\sum_{j=1}^{n} \left(\frac{a_{ij}}{a_{ii}}\right) x_j^{(k-1)} + \frac{b_i}{a_{ii}}]$ for 1⩽ i ⩽ n. We assume that all diagonal elements are non-zero. If this is not the case, rearrange the equations so that it is.

Similar to the Jacobi method, Gauss-Seidel iteration is applied to any matrix with non-zero elements on the diagonals. The convergence is guaranteed only if the matrix is either diagonally dominant or symmetric and positive definite.

$x_i^{(k)} = \left[-\sum_{j=1}^{n} \left(\frac{a_{ij}}{a_{ii}}\right) x_j^{(k)} - \sum_{j=1}^{n} \left(\frac{a_{ij}}{a_{ii}}\right) x_j^{(k-1)} + \frac{b_i}{a_{ii}}\right]$ The Gauss-Seidel solves linear system of equations Ax = b one at a time in sequence. The computations are serial. Each component of the new iterate depends upon all previously computed ones. Therefore, if the ordering is changed, the components of the new iterates would also change.

Let’s take a closer look on how we program the methods in Mathematica followed with some examples.

Diagonally dominant

Using Jacobi method, and Gauss-Seidel methd, iteration does not always converge. Both are the iterative algorithm for determining the solutions of a diagonally dominant system of linear equations. So before the algorithm to be computed, we need to check if the matrix is diagonally dominant.

Algorithm of Jacobi Method

Input:
-matrix *A0*, *B0*, (qualifying AX = B).
-*epsilon*:In order to have a tolerance in the algorithm allowing to break the loop when the convergence is reached for efficiency.
-*iter*: Max iterations to perform.
Output:
- returns the roots and prints the iteration roots that have been calculated by loop
or it returns null, and breaks the loop when matrix A0 is not diagonally dominant and prints that it cannot be printed.

FYI:

The initial value of the method is set to all 0.

```
Clear[Jacobi]
Clear[GaussSeidel]
Clear[A, B]
```

```
Jacobi[A0_, B0_, epsilon_, iter_] :=
  Module[{bool = 1, n = Length[A0], A = N[A0], B = N[B0], eps,
    i, j, k = 0, P = Table[0, {i, 0, Length[A] - 1}], len = Length[A0]},

   root = Table["x"_i, {i, 1, n}];
   Print["Solve using Jacobi, AX = B"];
   Print[MatrixForm[A0], MatrixForm[root], " = ", MatrixForm[B0]];
   For[i = 1, i ≤ Length[A0], i++,
    If[∑_{j=1}^{n} Abs[A[[i,j]]] > 2 Abs[A[[i,i]]], bool = 0;];];
   If[bool == 1,
    Print["Diagonally dominant. \n Starting Jacobi iteration..."];,
    Print["Failing in Jacobi iteration."];
    Return[Null];
    Break;];
   (*When A is not diagonally dominant, then print failure and Break*)
   Print["Initial values are defaulted ", root, " = ", P];
   X = P;
   eps = 1; (*Setting eps as 1*)
   While[And[eps > epsilon, iter > k], (*while loop*)
    X = P;
    For[i = 1, i ≤ len, i++,
     P[[i]] = 1/A[[i,i]] (B[[i]] + A[[i,i]] X[[i]] - ∑_{j=1}^{len} A[[i,j]] X[[j]])];
    eps = Sqrt[(P - X).(P - X)];
    Print["P"_{k+1} " = ", P];
    X = P;
    (*Print[X];*)
    k += 1];
   Print["Results: \n For ", k, " iterations, "];
   Print["A X = ", MatrixForm[A0], " ", MatrixForm[X],
    " = ", MatrixForm[A.X], " ≈ ", MatrixForm[B], " = B"];
   Return[
    P];];

(*Definition[Jacobi]*)
```

Algorithm of Gauss-Seidel Method

Input:

-matrix *A0*, *B0*, (qualifying AX = B).

-*epsilon*:In order to have a tolerance in the algorithm allowing to break the loop when the convergence is reached for efficiency.

-*iter*: Max iterations to perform.

Output:

- returns the roots and prints the iteration roots that have been calculated by loop

or it returns null, and breaks the loop when matrix A0 is not diagonally dominant and prints that it cannot be printed.

FYI:

The initial value of the method is set to all 0.

```
GaussSeidel[A0_, B0_, epsilon_, iter_] :=
  Module[{bool = 1, A = N[A0], B = N[B0], n = Length[A0], eps,
    i, j, k = 0, P = Table[0, {i, 0, Length[A] - 1}], len = Length[A0]},

   root = Table["x"_i, {i, 1, n}];
   Print["Solve using Gauss-Seidel Method, Ax = B."];
   Print[MatrixForm[A0], MatrixForm[root], " = ", MatrixForm[B0]];
   (*Checking for the diagonally dominant*)
   For[i = 1, i ≤ Length[A0], i++,
    If[∑_{j=1}^{n} Abs[A[[i,j]]] > 2 Abs[A[[i,i]]], bool = 0;];];
   If[bool == 1,
    Print["Diagonally dominant. \n Starting Gauss-Seidel..."];,
    Print["Failing in Gauss-Seidel."];
    Return[Null];
    Break;];
   Print["Initial values are defaulted ", root, " = ", P];
   eps = 1;
   While[And[eps > epsilon, iter > k],
    X = P;
    For[i = 1, i ≤ len, i++,
     P[[i]] = 1/A[[i,i]] (B[[i]] + A[[i,i]] P[[i]] - ∑_{j=1}^{len} A[[i,j]] P[[j]])];
    eps = Sqrt[(P - X).(P - X)];
    Print["P"_{k+1} " = ", P];
    k += 1];
   Print["Results: \n For ", k, " iterations, "];
   Print["A X = ", MatrixForm[A0], " ", MatrixForm[X],
    " = ", MatrixForm[A.X], " ≈ ", MatrixForm[B], " = B"];
   Return[
    P];];
```

Examples

1)

```
A = ( 7  -2  1  2
      2   8  3  1
     -1   0  5  2
      0   2 -1  4 );
B = {3, -2, 5, 4};

Jacobi[A, B, 0.000001, 30];
Print["\n\n"]
GaussSeidel[A, B, 0.000001, 20];
```

Solve using Jacobi, AX = B

$$\begin{pmatrix} 7 & -2 & 1 & 2 \\ 2 & 8 & 3 & 1 \\ -1 & 0 & 5 & 2 \\ 0 & 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \\ 4 \end{pmatrix}$$

Diagonally dominant.
Starting Jacobi iteration...

Initial values are defaulted $\{x_1, x_2, x_3, x_4\} = \{0, 0, 0, 0\}$

$= P_1\{0.428571, -0.25, 1., 1.\}$

$= P_2\{-0.0714286, -0.857143, 0.685714, 1.375\}$

$= P_3\{-0.307143, -0.661161, 0.435714, 1.6\}$

$= P_4\{-0.279719, -0.536607, 0.298571, 1.43951\}$

$= P_5\{-0.178686, -0.471973, 0.368253, 1.34295\}$

$= P_6\{-0.142585, -0.511291, 0.427084, 1.32805\}$

$= P_7\{-0.157967, -0.540517, 0.440263, 1.36242\}$

$= P_8\{-0.178019, -0.545909, 0.42344, 1.38032\}$

$= P_9\{-0.182272, -0.536826, 0.412267, 1.37881\}$

$= P_{10}\{-0.17765, -0.531384, 0.41202, 1.37148\}$

$= P_{11}\{-0.173964, -0.53153, 0.415878, 1.3687\}$

$= P_{12}\{-0.173762, -0.53355, 0.417729, 1.36973\}$

$= P_{13}\{-0.1749, -0.534425, 0.417354, 1.37121\}$

$= P_{14}\{-0.175517, -0.534184, 0.416537, 1.37155\}$

$= P_{15}\{-0.175429, -0.533766, 0.416276, 1.37123\}$

$= P_{16}\{-0.17518, -0.53365, 0.416424, 1.37095\}$

$= P_{17}\{-0.17509, -0.533733, 0.416583, 1.37093\}$

$= P_{18}\{-0.17513, -0.533813, 0.41661, 1.37101\}$

$= P_{19}\{-0.17518, -0.533823, 0.416569, 1.37106\}$

$= P_{20}\{-0.17519, -0.533801, 0.416541, 1.37105\}$

$= P_{21}\{-0.175178, -0.533787, 0.41654, 1.37104\}$

$= P_{22}\{-0.175169, -0.533788, 0.41655, 1.37103\}$

$= P_{23}\{-0.175169, -0.533793, 0.416555, 1.37103\}$

$= P_{24}\{-0.175172, -0.533795, 0.416554, 1.37103\}$

$= P_{25}\{-0.175173, -0.533794, 0.416552, 1.37104\}$

$= P_{26}\{-0.175173, -0.533793, 0.416551, 1.37103\}$

$= P_{27}\{-0.175172, -0.533793, 0.416551, 1.37103\}$

$= P_{28}\{-0.175172, -0.533793, 0.416552, 1.37103\}$

Results:

For 28 iterations,

$$A\ X = \begin{pmatrix} 7 & -2 & 1 & 2 \\ 2 & 8 & 3 & 1 \\ -1 & 0 & 5 & 2 \\ 0 & 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} -0.175172 \\ -0.533793 \\ 0.416552 \\ 1.37103 \end{pmatrix} = \begin{pmatrix} 3. \\ -2. \\ 5. \\ 4. \end{pmatrix} \approx \begin{pmatrix} 3. \\ -2. \\ 5. \\ 4. \end{pmatrix} = B$$

Solve using Gauss-Seidel Method, $Ax = B$.

$$\begin{pmatrix} 7 & -2 & 1 & 2 \\ 2 & 8 & 3 & 1 \\ -1 & 0 & 5 & 2 \\ 0 & 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \\ 4 \end{pmatrix}$$

Diagonally dominant.

Starting Gauss-Seidel...

Initial values are defaulted $\{x_1, x_2, x_3, x_4\} = \{0, 0, 0, 0\}$

$= P_{1}\{0.428571, -0.357143, 1.08571, 1.45\}$

$= P_{2}\{-0.242857, -0.777679, 0.371429, 1.4817\}$

$= P_{3}\{-0.270026, -0.506991, 0.353316, 1.34182\}$

$= P_{4}\{-0.150136, -0.512688, 0.433243, 1.36465\}$

$= P_{5}\{-0.169704, -0.540622, 0.420197, 1.37536\}$

$= P_{6}\{-0.17888, -0.534774, 0.41408, 1.37091\}$

$= P_{7}\{-0.175063, -0.532878, 0.416625, 1.37059\}$

$= P_{8}\{-0.174796, -0.53386, 0.416803, 1.37113\}$

$= P_{9}\{-0.175255, -0.533879, 0.416497, 1.37106\}$

$= P_{10}\{-0.175197, -0.53377, 0.416535, 1.37102\}$

$= P_{11}\{-0.175159, -0.533788, 0.416561, 1.37103\}$

$= P_{12}\{-0.175172, -0.533796, 0.416552, 1.37104\}$

$= P_{13}\{-0.175174, -0.533793, 0.416551, 1.37103\}$

$= P_{14}\{-0.175172, -0.533793, 0.416552, 1.37103\}$

$= P_{15}\{-0.175172, -0.533793, 0.416552, 1.37103\}$

Results:

For 15 iterations,

$$A\ X = \begin{pmatrix} 7 & -2 & 1 & 2 \\ 2 & 8 & 3 & 1 \\ -1 & 0 & 5 & 2 \\ 0 & 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} -0.175172 \\ -0.533793 \\ 0.416552 \\ 1.37103 \end{pmatrix} = \begin{pmatrix} 3. \\ -2. \\ 5. \\ 4. \end{pmatrix} \approx \begin{pmatrix} 3. \\ -2. \\ 5. \\ 4. \end{pmatrix} = B$$

2)

```
Clear[A, B];
```

$$A = \begin{pmatrix} 2 & 8 & 3 & 1 \\ 0 & 2 & -1 & 4 \\ 7 & -2 & 1 & 2 \\ -1 & 0 & 5 & 2 \end{pmatrix};$$

```
B = {-2, 4, 3, 5};
Jacobi[A, B, 0.000001, 30];
Print["\n \n"];
GaussSeidel[A, B, 0.000001, 30];
```

Solve using Jacobi, AX = B

$$\begin{pmatrix} 2 & 8 & 3 & 1 \\ 0 & 2 & -1 & 4 \\ 7 & -2 & 1 & 2 \\ -1 & 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 3 \\ 5 \end{pmatrix}$$

Failing in Jacobi iteration.

Solve using Gauss-Seidel Method, Ax = B.

$$\begin{pmatrix} 2 & 8 & 3 & 1 \\ 0 & 2 & -1 & 4 \\ 7 & -2 & 1 & 2 \\ -1 & 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 3 \\ 5 \end{pmatrix}$$

Failing in Gauss-Seidel.

3)

```
Clear[A, B];
```

$$A = \begin{pmatrix} 7 & -2 & 1 \\ 2 & 8 & 3 \\ -1 & 0 & 5 \end{pmatrix};$$

```
B = {3, -2, 5};
Jacobi[A, B, 0.000001, 20];
Print["\n \n \n"];
GaussSeidel[A, B, 0.000001, 20];
```

Solve using Jacobi

$$\begin{pmatrix} 7 & -2 & 1 \\ 2 & 8 & 3 \\ -1 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

Diagonally dominant.
Starting Jacobi iteration...

$= P_1\{0.428571, -0.25, 1.\}$

$= P_2\{0.214286, -0.732143, 1.08571\}$

$= P_3\{0.0642857, -0.710714, 1.04286\}$

$= P_4\{0.0765306, -0.657143, 1.01286\}$

$= P_5\{0.0961224, -0.648954, 1.01531\}$

$= P_6\{0.0981122, -0.65477, 1.01922\}$

$= P_7\{0.0958907, -0.656737, 1.01962\}$

$= P_8\{0.0952719, -0.656331, 1.01918\}$

$= P_9\{0.0954514, -0.65601, 1.01905\}$

$= P_{10}\{0.0955609, -0.656008, 1.01909\}$

$= P_{11}\{0.0955562, -0.656049, 1.01911\}$

$= P_{12}\{0.0955414, -0.656056, 1.01911\}$

$= P_{13}\{0.0955395, -0.656052, 1.01911\}$

$= P_{14}\{0.0955411, -0.65605, 1.01911\}$

$= P_{15}\{0.0955416, -0.656051, 1.01911\}$

Checking AX = b

$$A\ X = \begin{pmatrix} 7 & -2 & 1 \\ 2 & 8 & 3 \\ -1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 0.0955416 \\ -0.656051 \\ 1.01911 \end{pmatrix} = \begin{pmatrix} 3. \\ -2. \\ 5. \end{pmatrix} \approx \begin{pmatrix} 3. \\ -2. \\ 5. \end{pmatrix} = B$$

Solve using Gauss-Seidel Method

$$\begin{pmatrix} 7 & -2 & 1 \\ 2 & 8 & 3 \\ -1 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

Diagonally dominant.
Starting Gauss-Seidel...

Initial values are defaulted $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$= P_1\{0.428571, -0.357143, 1.08571\}$

$= P_2\{0.171429, -0.7, 1.03429\}$

$= P_3\{0.0808163, -0.658061, 1.01616\}$

$= P_4\{0.0953878, -0.654908, 1.01908\}$

$= P_5\{0.0958723, -0.656122, 1.01917\}$

$= P_6\{0.0955116, -0.656068, 1.0191\}$

$= P_7\{0.0955373, -0.656048, 1.01911\}$

$= P_8\{0.0955425, -0.656051, 1.01911\}$

$= P_9\{0.0955414, -0.656051, 1.01911\}$

$= P_{10}\{0.0955414, -0.656051, 1.01911\}$

Checking AX = b

$$A\ X = \begin{pmatrix} 7 & -2 & 1 \\ 2 & 8 & 3 \\ -1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 0.0955414 \\ -0.656051 \\ 1.01911 \end{pmatrix} = \begin{pmatrix} 3. \\ -2. \\ 5. \end{pmatrix} \approx \begin{pmatrix} 3. \\ -2. \\ 5. \end{pmatrix} = B$$

4)

$$\mathbf{A} = \begin{pmatrix} 8 & 1 & -2 & 1 & 0 & -1 \\ 2 & 9 & 2 & -1 & 2 & -2 \\ -1 & -2 & 12 & 3 & 2 & 0 \\ 0 & 2 & 3 & -34 & -4 & 2 \\ 1 & -2 & 1 & 0 & 4 & 0 \\ 0 & 1 & -5 & 3 & -3 & -17 \end{pmatrix};$$

```
B = {1, 0, 3, 6, 3, 9};
```

```
Jacobi[A, B, 0.000001, 30];
Print["\n \n \n"];
GaussSeidel[A, B, 0.000001, 30];
```

Solve using Jacobi, AX = B

$$\begin{pmatrix} 8 & 1 & -2 & 1 & 0 & -1 \\ 2 & 9 & 2 & -1 & 2 & -2 \\ -1 & -2 & 12 & 3 & 2 & 0 \\ 0 & 2 & 3 & -34 & -4 & 2 \\ 1 & -2 & 1 & 0 & 4 & 0 \\ 0 & 1 & -5 & 3 & -3 & -17 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 6 \\ 3 \\ 9 \end{pmatrix}$$

Diagonally dominant.
Starting Jacobi iteration...

Initial values are defaulted $\{x_1, x_2, x_3, x_4, x_5, x_6\} = \{0, 0, 0, 0, 0, 0\}$

$= P_1\{0.125, 0., 0.25, -0.176471, 0.75, -0.529412\}$

$= P_2\{0.143382, -0.387255, 0.179534, -0.273789, 0.65625, -0.766436\}$

$= P_3\{0.15671, -0.418333, 0.156478, -0.305699, 0.475643, -0.76912\}$

$= P_4\{0.158484, -0.380178, 0.190488, -0.288472, 0.462537, -0.737927\}$

$= P_5\{0.163962, -0.376371, 0.194872, -0.27985, 0.472668, -0.740332\}$

$= P_6\{0.163204, -0.380391, 0.192119, -0.280572, 0.472106, -0.741664\}$

$= P_7\{0.162942, -0.379862, 0.191661, -0.281064, 0.470974, -0.741119\}$

$= P_8\{0.162891, -0.379384, 0.192039, -0.280908, 0.471418, -0.74084\}$

$= P_9\{0.162941, -0.379476, 0.192001, -0.280882, 0.471576, -0.740974\}$

$= P_{10}\{0.162923, -0.379541, 0.191957, -0.280918, 0.471527, -0.740991\}$

$= P_{11}\{0.162923, -0.379524, 0.191962, -0.280921, 0.47151, -0.74098\}$

$= P_{12}\{0.162924, -0.379519, 0.191968, -0.280916, 0.471517, -0.740978\}$

= P_{13} {0.162924, -0.379521, 0.191967, -0.280916, 0.471518, -0.74098}

= P_{14} {0.162924, -0.379522, 0.191966, -0.280917, 0.471517, -0.74098}

= P_{15} {0.162924, -0.379521, 0.191966, -0.280917, 0.471517, -0.74098}

Results:

For 15 iterations,

$$A\ X = \begin{pmatrix} 8 & 1 & -2 & 1 & 0 & -1 \\ 2 & 9 & 2 & -1 & 2 & -2 \\ -1 & -2 & 12 & 3 & 2 & 0 \\ 0 & 2 & 3 & -34 & -4 & 2 \\ 1 & -2 & 1 & 0 & 4 & 0 \\ 0 & 1 & -5 & 3 & -3 & -17 \end{pmatrix} \begin{pmatrix} 0.162924 \\ -0.379521 \\ 0.191966 \\ -0.280917 \\ 0.471517 \\ -0.74098 \end{pmatrix} = \begin{pmatrix} 1. \\ -1.72519 \times 10^{-7} \\ 3. \\ 6. \\ 3. \\ 9. \end{pmatrix} \approx \begin{pmatrix} 1. \\ 0. \\ 3. \\ 6. \\ 3. \\ 9. \end{pmatrix} = B$$

Solve using Gauss-Seidel Method, Ax = B.

$$\begin{pmatrix} 8 & 1 & -2 & 1 & 0 & -1 \\ 2 & 9 & 2 & -1 & 2 & -2 \\ -1 & -2 & 12 & 3 & 2 & 0 \\ 0 & 2 & 3 & -34 & -4 & 2 \\ 1 & -2 & 1 & 0 & 4 & 0 \\ 0 & 1 & -5 & 3 & -3 & -17 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 6 \\ 3 \\ 9 \end{pmatrix}$$

Diagonally dominant.

Starting Gauss-Seidel...

Initial values are defaulted $\{x_1, x_2, x_3, x_4, x_5, x_6\} = \{0, 0, 0, 0, 0, 0\}$

= P_1 {0.125, -0.0277778, 0.255787, -0.155535, 0.640914, -0.746827}

= P_2 {0.118507, -0.408845, 0.123799, -0.30893, 0.485001, -0.729978}

= P_3 {0.154424, -0.366148, 0.198243, -0.280516, 0.478759, -0.743246}

= P_4 {0.162488, -0.382888, 0.190062, -0.282268, 0.470419, -0.740662}

= P_5 {0.163077, -0.378968, 0.192592, -0.280681, 0.471599, -0.741104}

= P_6 {0.162966, -0.379689, 0.191869, -0.280952, 0.471446, -0.740955}

= P_7 {0.162928, -0.379483, 0.191994, -0.280902, 0.471528, -0.740985}

= P_8 {0.162924, -0.379529, 0.19196, -0.280919, 0.471515, -0.740978}

= P_9 {0.162924, -0.379519, 0.191968, -0.280916, 0.471518, -0.74098}

= P_{10} {0.162924, -0.379522, 0.191966, -0.280917, 0.471517, -0.74098}

= P_{11} {0.162924, -0.379521, 0.191967, -0.280917, 0.471517, -0.74098}

Results:

For 11 iterations,

$$A\ X = \begin{pmatrix} 8 & 1 & -2 & 1 & 0 & -1 \\ 2 & 9 & 2 & -1 & 2 & -2 \\ -1 & -2 & 12 & 3 & 2 & 0 \\ 0 & 2 & 3 & -34 & -4 & 2 \\ 1 & -2 & 1 & 0 & 4 & 0 \\ 0 & 1 & -5 & 3 & -3 & -17 \end{pmatrix} \begin{pmatrix} 0.162924 \\ -0.379522 \\ 0.191966 \\ -0.280917 \\ 0.471517 \\ -0.74098 \end{pmatrix} = \begin{pmatrix} 1. \\ -5.18348 \times 10^{-6} \\ 3. \\ 6. \\ 3. \\ 9. \end{pmatrix} \approx \begin{pmatrix} 1. \\ 0. \\ 3. \\ 6. \\ 3. \\ 9. \end{pmatrix} = B$$

Conclusion

Since the idea of the Gauss-Seidel iteration is simply to accelerate the convergence by incorporating each vector as soon as it has been compute, the convergence of the Gauss-Seidel method is approximately twice as fast as that of the Jacobi method. We could see this from the three examples above where Gauss-Seidel iteration will take less steps than the Jacobi method.

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