# Mini Project Part 2

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## 1 Executive Summary

This project examines a sample portfolio, $P$, invested in the top $p = 400$ stocks (by market capitalization) selected from S&P500 5.1. The weekly closing prices for each stock are observed over a time period of $n = 26$ weeks, from 2024-08-02 to 2025-01-24. The portfolio's excess returns $r_P$ are determined with respect to a risk-free rate $r_F$, and the (weekly) sample covariance matrix $S \in \mathbb{R}^{400\times400}$ is computed. The leading eigenvalue $\lambda^2$ and corresponding eigenvector $\vec{v}$ of $S$ are used to construct a single factor model covariance matrix $\Sigma \in \mathbb{R}^{400\times400}$. The holdings vector $h_c$ is then calculated for the minimum risk, fully invested portfolio, $C$. The feasibility of this portfolio is assessed and remarks are given regarding the level of risk of these investments.

## 2 Description of the Mathematics

1. Let $P_t$ be the time $t$ value of portfolio $P$. Then the weekly return of each stock $i$ is given by

$$r_t^{(i)} = \frac{P_t^{(i)} - P_{t-1}^{(i)}}{P_{t-1}^{(i)}}, \quad 1 \le i \le 400, \quad 1 \le t \le 26 \tag{1}$$

The weekly excess return of each stock $i$ are then calculated by

$$r_P^{(i)}(t) = r_t^{(i)} - r_F, \tag{2}$$

where $r_F$ is the risk-free rate (we set $r_F$ to a constant rate for ease of computation). We construct a matrix $X \in \mathbb{R}^{p\times n}$ of weekly excess return such that $r_P^{(i)}(t) \in X$.

2. Let $f_P^{(i)} = \frac{1}{26}\sum_{t=1}^{26} r_P^{(i)}(t)$ be the expected (average) weekly excess return for each stock $i$. The de-meaned excess returns for each stock $i$ are found by subtracting $f_P^{(i)}$ from the stock's weekly excess return, such that

$$\tilde{r}_P^{(i)}(t) = r_P^{(i)}(t) - f_P^{(i)}, \quad 1 \le i \le 400, \quad 1 \le t \le 26 \tag{3}$$

for all $r_P^{(i)}(t) \in X$, where $\tilde{r}_P^{(i)}(t) \in Y$. The (weekly) sample covariance matrix $S$ is given by

$$S = \frac{Y \cdot Y^T}{n} \tag{4}$$

3. In this scenario, the matrix $S$ is not invertible. To proceed with our computations, we must first replace the sample covariance matrix $S$ with a single factor model covariance matrix $\Sigma$. Let $\lambda^2$ be the leading eigenvalue of $S$ and $\vec{v}$ be the corresponding unit eigenvector. Let $I$ denote the identity matrix, such that $I \in \mathbb{R}^{400\times400}$. Let $tr(S)$ denote the trace of $S$ (sum of the main diagonal entries of $S$). Compute the average of the non-zero eigenvalues less than $\lambda^2$ by

$$l^2 = \frac{tr(S) - \lambda^2}{n-1}. \tag{5}$$

Now we can construct

$$\Sigma = \left((\lambda^2 - l^2)\vec{v} \cdot \vec{v}^T\right) + \left(\frac{n}{p} l^2 \times I\right), \tag{6}$$

where $n = 26$ weeks and $p = 400$ stocks. Since the matrix $\Sigma$ is constructed using the leading eigenvalue of the matrix $S$ and the corresponding eigenvector, the $\Sigma$ now has the same leading eigenvector as $S$ and is invertible. We prove these results:

*Proof.* We want to show $\Sigma$ is invertible. Take $\vec{x}^T \cdot \Sigma \cdot \vec{x}$, where $\vec{x}$ is a nonzero real column vector. We show

$$\begin{aligned}
\vec{x}^T \, \Sigma \, \vec{x} &= \vec{x}^T \left[\left((\lambda^2 - l^2)\vec{v} \cdot \vec{v}^T\right) + \left(\frac{n}{p} l^2 \times I\right)\right] \vec{x} \\
&= (\lambda^2 - l^2)(\vec{x}^T \vec{v})(\vec{v}^T \vec{x}) + \frac{n}{p} l^2 (\vec{x}^T x) \\
&= (\lambda^2 - l^2)\langle x, v\rangle^2 + \frac{n}{p} l^2 ||\vec{x}||_2^2
\end{aligned}$$

The inner product of two real-valued vectors, $\langle x, v\rangle$ is a real-valued scalar. Therefore the squared value yields $\langle x, v\rangle^2 > 0$. Since all the eigenvalues of the sample covariance matrix $S$ are positive, then the average of the nonzero eigenvalues less than $\lambda^2$ will be positive ($l^2 > 0$) and smaller than $\lambda^2$, therefore $(\lambda^2 - l^2) > 0$. From this argument, it also follows that $\frac{n}{p} l^2 > 0$ for $n > 0$, $p > 0$, and the squared norm of the nonzero vector $\vec{v}$ will be positive such that $||\vec{x}||_2^2 > 0$. Since each value in the above equation is nonzero and positive, we then have

$$\vec{x}^T \, \Sigma \, \vec{x} > 0.$$

Therefore $\Sigma$ satisfies the conditions for a symmetric positive definite (SPD) matrix which is, by definition, invertible. For a SPD matrix, we also know that all the eigenvalues are real and strictly positive. This leads us into the second part of this prove, where we wish to show that $\Sigma$ will have the same leading eigenvector as $S$. Let $\lambda^2$ be the leading eigenvalue of the sample covariance matrix $S$, and let $\vec{v}$ be the corresponding unit eigenvector. Therefore, by definition of eigenvectors and eigenvalues, we know

$$S\vec{v} = \lambda^2\vec{v}.$$

We want to show $\Sigma\vec{v} = \lambda^2\vec{v}$. We solve to get

$$\begin{aligned}
\Sigma\vec{v} &= \left[\left((\lambda^2 - l^2)\vec{v}\cdot\vec{v}^T\right) + \left(\frac{n}{p}l^2 \times I\right)\right]\vec{v} \\
&= (\lambda^2 - l^2)\vec{v}(\vec{v}^T\vec{v}) + \frac{n}{p}l^2\vec{v} \\
&= (\lambda^2 - l^2)\vec{v} + \frac{n}{p}l^2\vec{v} \\
&= (\lambda^2 - l^2 + \frac{n}{p}l^2)\vec{v},
\end{aligned}$$

where the norm of the unit eigenvector is $\vec{v}^T\vec{v} = ||\vec{v}||_2^2 = 1$. Therefore $\Sigma\vec{v} = (\lambda^2 - l^2 + \frac{n}{p}l^2)\vec{v}$ satisfies that $\vec{v}$ is also an eigenvector of $\Sigma$, where $\lambda^2 - l^2 + \frac{n}{p}l^2$ is the correspondign eigenvalue. Now, let $\vec{y}$ denote any other eigenvector of $\Sigma$. Since $\Sigma$ is symmetric, then any two eigenvectors associated with different eigenvalues will be orthogonal, meaning $\vec{v}^T\vec{y} = 0$ for any $\vec{y}$. Then

$$\begin{aligned}
\Sigma\vec{y} &= \left[\left((\lambda^2 - l^2)\vec{v}\cdot\vec{v}^T\right) + \left(\frac{n}{p}l^2 \times I\right)\right]\vec{y} \\
&= (\lambda^2 - l^2)\vec{v}(\vec{v}^T\vec{y}) + \frac{n}{p}l^2\vec{y} \\
&= \frac{n}{p}l^2\vec{v}.
\end{aligned}$$

Therefore for any other eigenvector $\vec{y}$, the corresponding eigenvalue will be $\frac{n}{p}l^2$. We have shown $(\lambda^2 - l^2) > 0$, and thus is it true that

$$\lambda^2 - l^2 + \frac{n}{p}l^2 > \frac{n}{p}l^2,$$

meaning all the eigenvalues corresponding to eigenvectors $\vec{y} \neq \vec{v}$ will be smaller than the eigenvalue corresponding to $\vec{v}$. Therefore, the eigenvector $\vec{v}$ corresponds to the largest eigenvalue of $\Sigma$, meaning it is also the leading eigenvector of $\Sigma$. $\square$

Following from the above proof, we may now use $\Sigma$ as an estimator of the true covariance matrix in the equations that follow.

4. We use $\Sigma$ to compute the holdings vector for the minimum risk, fully invested portfolio $C$ by

$$\vec{h_C} = \frac{\Sigma^{-1} \cdot \vec{e}}{\vec{e}^T \cdot \Sigma^{-1} \cdot \vec{e}} \tag{7}$$

where $\vec{e}^T = [1\,1\,\ldots\,1]$. The value of the expected excess returns of the fully invested portfolio $C$ is

$$f_C = \vec{h_C^T} \cdot \vec{f} \tag{8}$$

where $\vec{f} = [f_P^{(1)}, f_P^{(2)}, \ldots, f_P^{(400)}]$ is the expected excess returns vector for all stocks in portfolio $P$. We compute the variance of portfolio $C$ by

$$\sigma_C^2 = \vec{h_C^T} \cdot \Sigma \cdot \vec{h_C}, \tag{9}$$

and the standard deviation (i.e. risk) is simply

$$\sigma_C = \sqrt{\vec{h_C^T} \cdot \Sigma \cdot \vec{h_C}}. \tag{10}$$

5. The variance of each asset $i$ is given by the diagonal entries of the covariance matrix $\Sigma$, where

$$[\sigma_1^2, \sigma_2^2, \ldots, \sigma_{400}^2] = diag(\Sigma). \tag{11}$$

Note that this portfolio is fully invested in risky assets (i.e. no cash assets). Any fully invested portfolio $P$ will have $\sigma_P = \sigma_C$, so in context of this portfolio we may use these variable notations interchangeably.

# 3 Numerical Results

All stock (closing) price data is accessed using the ’STOCKHISTORY’ command in Excel. The portfolio is constructed in Excel and exported as a .xlsx file into a Python program. All computations are performed using the NumPy and Pandas Python programming libraries on the PyCharm integrated development environment. In Excel, the weekly stock price data for each stock is obtained by:

=STOCKHISTORY(”AAPL”,”2024-7-26”, ”2025-2-1”,1,1,1)

where the data for each stock is retrieved by changing the stock’s corresponding ticker symbol. In the formula, first ”1” indicates ”weekly” data, the second ”1” indicates the inclusion of headers, and the third ”1” indicates retrieving the ”Close” prices for each stock. Therefore we obtain the weekly closing prices of stocks from week 2024-07-26 (time $t = 0$) to week 2025-01-24 (time $t = 26$).

From the close prices, we calculate the returns from week $t = 1$ (2024-08-02) to week $t = 26$ (2025-01-24) and subtract the risk-free rate $r_F$ from each cell to generate a $(400 \times 26)$ matrix $P$ of excess returns $r_P^{(i)}$ for each asset $i$ at each time $t$ (1). The data is then exported into Python and the following Python commands are executed:

```
prices = pd.read_excel("snp500.xlsx", index_col=0)

returns = prices.pct_change(fill_method=None).dropna()

excess_returns = returns - 0.00087
```

We set the risk-free rate to $r_F = 0.00087$. This value is determined by taking the average of daily 3-Month Treasury Bill (T-Bill) rates corresponding to our observed time period (2024-08-02 to 2025-01-24), obtained from public U.S. Department of Treasury records. These rates are often used by U.S.-based investors as a proxy for the true risk-free rate, since we expect there to be no risk of default on obligations made by the government. We found the average annualized 3-Month T-Bill rate to be $r_{F_{\text{annual}}} = .045$. The average weekly rate is found by taking $.045/52 \approx 0.00087$.

To compute the expected excess returns $f_P^{(i)}$ of each asset $i$ (2), we transpose the above table (i.e. our matrix $P$) and calculate the average (*expected* excess returns) for each row $i$ in $P$. Therefore we find the vector of expected excess returns to be

| Ticker | $f_P$ |
|---|---|
| AAPL | 0.000273 |
| NVDA | 0.010408 |
| MSFT | 0.001160 |
| AMZN | 0.009631 |
| META | 0.012552 |
| GOOGL | 0.006630 |
| TSLA | 0.027038 |
| AVGO | 0.021243 |
| GOOG | 0.006563 |
| BRK.B | 0.001576 |
| ⋮ | ⋮ |
| GNRC | 0.000108 |
| LKQ | -0.002310 |
| PNW | -0.000047 |
| GL | 0.010096 |
| LW | 0.003465 |
| HRL | -0.001919 |
| APA | -0.010625 |
| TFX | -0.007795 |
| MKTX | -0.000550 |
| MOS | -0.001591 |

The expected excess return for stock each $i$ is subtracted from the stock's excess return value at each time $t$ (3). This transforms matrix $P$ into the demeaned excess returns matrix $Y$. The following commands are executed:

```
excess_returns = excess_returns.T

expected_returns = excess_returns.mean(axis=1)
Y = excess_returns.sub(expected_returns, axis=0)
```

Then the weekly sample covariance matrix $S$ is calculated using matrix-matrix multiplication (4) such that the following code:

```
S = Y @ Y.T / 26
```

yields a $(400 \times 400)$ matrix. Then we find the trace of $S$, leading eigenvalue $\lambda^2$ and corresponding unit eigenvector $\vec{v}$ of $S$, and calculate $l^2$ by the following:

```
trace_S = np.trace(S)

eigvals, eigvecs = np.linalg.eigh(S)

lambdaS = eigvals[-1]
v = eigvecs[:, -1]

l = (trace_S - lambdaS) / (n-1)
```

The above Python command returns the eigenvalues of the matrix in ascending order, so we set the last eigenvalue (i.e. the largest eigenvalue) to $\lambda^2$, and we set the corresponding eigenvector to be our $\vec{v}$. Using these values, we compute the single factor model covariance matrix $\Sigma$ (6) by:

```
term1 = lambdaS - l

term2 = (n / p) * l

sigma = (term1 * np.outer(v, v)) + (term2 * np.eye(p))
```

We then create a $(400 \times 1)$ vector of 1's and find the holdings vector of excess returns $\vec{h_C}$ (7), using the following commands:

```
ones = np.ones(p)

sigma_inv = np.linalg.inv(sigma)

h_C = sigma_inv @ ones / (ones.T @ sigma_inv @ ones)
```

Our holdings vector yields 301 positive values (indicating long positions in 301 of stock) and 91 negative values (indicating short positions the remaining 91 stocks). The value of the expected excess returns $f_c$ (8) is computed by taking the inner product of the holdings vector $\vec{h_C}$ and the excess returns vector $\vec{f}$:

```
portfolio_expected_returns = h_C.T @ expected_returns
```

yielding

| | |
|---|---|
| $f_C$ | -0.0004455363547419587 |

Lastly, we are interested in the variance and standard deviation of the fully invested portfolio $C$ (9), as well as the variance and standard deviation of the excess returns for each individual stock $i$ in our original portfolio $P$, stored in the diagonal entries of the sample covariance matrix $\Sigma$ (11). We compute

```
portfolio_var = h_C.T @ V @ h_C

portfolio_std_dev = np.sqrt(portfolio_var)

stock_var = np.diag(V)
```

The resulting outputs are displayed in the following table:

| Variable | Variance ($\sigma^2$) | Standard Deviation ($\sigma$) |
|---|---|---|
| Portfolio $C$ | 0.00001058 | 0.00325319 |
| AAPL | 0.00129591 | 0.03599871 |
| NVDA | 0.00307025 | 0.05540986 |
| MSFT | 0.00151077 | 0.03886866 |
| AMZN | 0.00165787 | 0.04071693 |
| META | 0.00141264 | 0.03758515 |
| GOOGL | 0.00130778 | 0.03616328 |
| TSLA | 0.00259565 | 0.50947490 |
| AVGO | 0.00269895 | 0.05195140 |
| GOOG | 0.00131561 | 0.03627128 |
| BRK.B | 0.00144275 | 0.03798355 |
| ⋮ | ⋮ | ⋮ |
| GNRC | 0.00283635 | 0.05325736 |
| LKQ | 0.00159389 | 0.03992350 |
| PNW | 0.00143604 | 0.03789518 |
| GL | 0.00144415 | 0.03800200 |
| LW | 0.00181247 | 0.04257314 |
| HRL | 0.00127885 | 0.03576098 |
| APA | 0.00140975 | 0.03754662 |
| TFX | 0.00125925 | 0.03548590 |
| MKTX | 0.00126289 | 0.03553710 |
| MOS | 0.00202752 | 0.04502804 |

The above results for the portfolio return, variance, and standard deviation, along with the individual stock variances, are presented as *weekly* quantities. The annualized values are found by scaling the values by 52 weeks/year. Hence we compute

```
annualized_return = portfolio_expected_returns * 52
annualized_var = portfolio_var * 52
annualized_std_dev = portfolio_std_dev * np.sqrt(52)
annualized_stock_var = stock_var * 52
```

The resulting value of the annualized expected excess portfolio return is

| | |
|---|---|
| $f_{C_{annual}}$ | -0.02316789044658185 |

and the annualized variances and standard deviations are given by

| Variable | Annualized Variance ($\sigma^2$) | Annualized Standard Deviation ($\sigma$) |
|---|---|---|
| Portfolio $C$ | 0.00055033 | 0.02345911 |
| AAPL | 0.06738719 | 0.25959042 |
| NVDA | 0.15965314 | 0.39956619 |
| MSFT | 0.07856018 | 0.28028589 |
| AMZN | 0.08620918 | 0.29361399 |
| META | 0.07345746 | 0.27103036 |
| GOOGL | 0.06800471 | 0.26077712 |
| TSLA | 0.13497362 | 0.36738756 |
| AVGO | 0.06841150 | 0.37462690 |
| GOOG | 0.00131561 | 0.26155591 |
| BRK.B | 0.07502300 | 0.27390327 |
| ⋮ | ⋮ | ⋮ |
| GNRC | 0.14749002 | 0.38404430 |
| LKQ | 0.08288208 | 0.28789247 |
| PNW | 0.07467431 | 0.27326601 |
| GL | 0.07509589 | 0.27403630 |
| LW | 0.09424854 | 0.30699925 |
| HRL | 0.06650009 | 0.25787611 |
| APA | 0.07330693 | 0.27075252 |
| TFX | 0.06548094 | 0.25589244 |
| MKTX | 0.06567004 | 0.25626165 |
| MOS | 0.10543129 | 0.32470185 |

We use the above results to construct plots for further analysis. We first analyze the relationship between the expected excess returns for each stock $i$ ($f_P^{(i)}$) in the portfolio to the level of risk (the standard deviation $\sigma_P^{(i)}$). Examine the following:

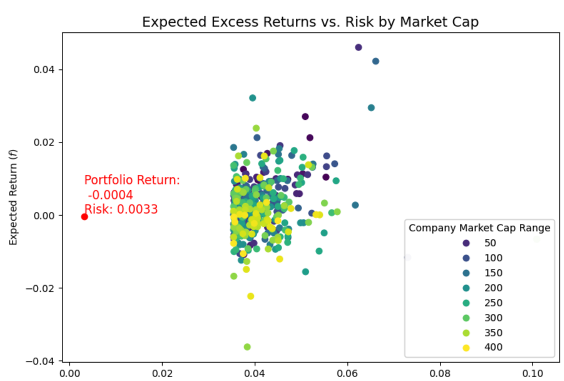


Figure 1: Comparing Expected Excess Returns to Risk

Since we selected the top 400 stocks ordered by market cap, we added a color gradient to provide an additional layer of comparison. Smaller market caps are represented by lighter colors (yellow to green), while larger market caps are represented by darker colors (purple to blue). We observe that the companies with the largest market caps exhibit more volatility, which correlates with higher expected excess returns. These companies, such as Apple, Nvidia, Microsoft, Amazon, Meta, Google, and Tesla, may experience higher risk due to their reliance on innovation, market trends, and competition. However, despite this higher volatility, most of these assets show relatively low expected excess returns, with both positive and negative expected returns centered around 0. Since these companies make up the S&P500 index, their movements reflect the stability of the broader market. The risk (volatility) associated with these stocks is generally low, making them stable long-term investments. The red node on the plot represents the portfolio's total expected excess returns. As expected from the clustering of the assets, the portfolio's expected excess returns are nearly 0. Despite this, the fully invested portfolio has a significantly lower level of risk compared to the individual assets. This can be attributed to the large diversification of the portfolio in stable assets, which results in a more stable investment overall.

We also compare the relationship between the portfolio holdings ($h_C$) for the minimum-risk fully invested portfolio $C$ and the level of risk associated with these assets. A particularly interesting phenomenon is observed:

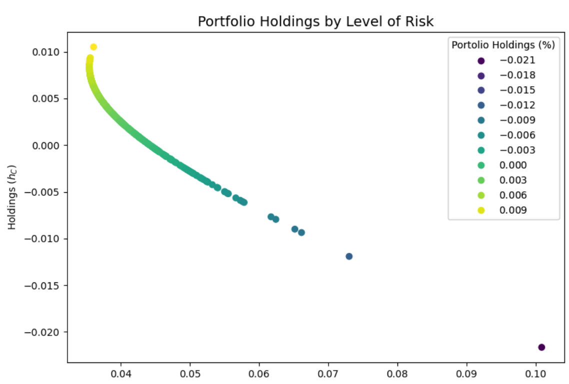


Figure 2: Comparing Portfolio Holdings to Risk

The color gradient is an added visual component to help differentiate between the negative (purple) holdings and the positive (yellow) holdings. Negative holdings indicate short positions in the assets, whereas positive indicate long positions in the assets. In general, we observe an inverse relationship between risk and portfolio holdings: as risk increases, the portfolio holdings decrease.

Therefore, for lower risk investments, we hold more assets, and the portfolio is more diversified. For higher risk investments, we bet against a handful of stocks' performance relative to the market. This plot represents the portfolio optimization process, where our holdings are adjusted based on our desired risk-level (or the desired risk-level of the investor).

# 4 Analysis of Results and Conclusion

The S&P500 is a stock index that tracks the performance of 500 of the largest publicly traded companies in the U.S. and is often regarded as the best overall measure of the U.S. stock market's performance. For our constructed portfolio, which is comprised of risky assets, we found the annualized portfolio variance and standard deviation (a measure of risk) to be approximately 0.00055 and 0.02346, respectively. These relatively low values suggest that the portfolio's returns are unlikely to deviate significantly from their average, indicating a relatively low risk investment. However, the annualized expected excess return is approximately -0.02317, which is negative. This implies that, relative to risk-free assets, we expect the portfolio to underperform, which signals a potential downturn in the market. Given this result, investing in this portfolio may not be advisable, as it is expected to provide negative returns. A better investment option, based on these findings, would be in riskless assets, which offer a nearly guaranteed return (though, often smaller return) without the uncertainty associated with a risky portfolio.

# 5 Appendix

## 5.1 Complete Stock List

| # | Ticker | Company |
|---|---|---|
| 1 | AAPL | Apple Inc. |
| 2 | NVDA | Nvidia Corp |
| 3 | MSFT | Microsoft Corp |
| 4 | AMZN | Amazon.com Inc |
| 5 | META | Meta Platforms, Inc. Class A |
| 6 | GOOGL | Alphabet Inc. Class A |
| 7 | TSLA | Tesla, Inc. |
| 8 | AVGO | Broadcom Inc. |
| 9 | GOOG | Alphabet Inc. Class C |
| 10 | BRK.B | Berkshire Hathaway Class B |
| 11 | JPM | Jpmorgan Chase & Co. |
| 12 | LLY | Eli Lilly & Co. |
| 13 | V | Visa Inc. |
| 14 | XOM | Exxon Mobil Corporation |
| 15 | UNH | Unitedhealth Group Incorporated |
| 16 | COST | Costco Wholesale Corp |
| 17 | WMT | Walmart Inc. |
| 18 | HD | Home Depot, Inc. |
| 19 | PG | Procter & Gamble Company |
| 20 | NFLX | Netflix Inc |
| 21 | ABBV | Abbvie Inc. |
| 22 | CRM | Salesforce, Inc. |
| 23 | BAC | Bank of America Corporation |
| 24 | ORCL | Oracle Corp |
| 25 | MRK | Merck & Co., Inc. |
| 26 | CVX | Chevron Corporation |
| 27 | WFC | Wells Fargo & Co. |
| 28 | KO | Coca-Cola Company |
| 29 | ACN | Accenture Plc |
| 30 | NOW | Servicenow, Inc. |
| 31 | TMO | Thermo Fisher Scientific, Inc. |
| 32 | MCD | Mcdonald's Corporation |
| 33 | IBM | International Business Machines Corporation |
| 34 | DIS | The Walt Disney Company |
| 35 | PEP | Pepsico, Inc. |
| 36 | LIN | Linde Plc |
| 37 | ISRG | Intuitive Surgical Inc. |
| 38 | PM | Philip Morris International Inc. |
| 39 | GE | Ge Aerospace |

| | | |
|---|---|---|
| 40 | ADBE | Adobe Inc. |
| 41 | TXN | Texas Instruments Incorporated |
| 42 | CAT | Caterpillar Inc. |
| 43 | INTU | Intuit Inc |
| 44 | AXP | American Express Company |
| 45 | VZ | Verizon Communications |
| 46 | T | At&t Inc. |
| 47 | MS | Morgan Stanley |
| 48 | SPGI | S&p Global Inc. |
| 49 | PFE | Pfizer Inc. |
| 50 | DHR | Danaher Corporation |
| 51 | RTX | Rtx Corporation |
| 52 | NEE | Nextra Energy, Inc. |
| 53 | PLTR | Palantir Technologies Inc. Class A |
| 54 | HON | Honeywell International, Inc. |
| 55 | AMGN | Amgen Inc |
| 56 | CMCSA | Comcast Corp |
| 57 | LOW | Lowe's Companies Inc. |
| 58 | UNP | Union Pacific Corp. |
| 59 | ETN | Eaton Corporation, Plcs |
| 60 | BSX | Boston Scientific Corp. |
| 61 | TJX | Tjx Companies, Inc. |
| 62 | COP | Conocophillips |
| 63 | BA | Boeing Company |
| 64 | BX | Blackstone Inc. |
| 65 | SYK | Stryker Corporation |
| 66 | ANET | Arista Networks |
| 67 | ADP | Automatic Data Processing |
| 68 | FI | Fiserv, Inc. |
| 69 | BMY | Bristol-Myers Squibb Co. |
| 70 | PANW | Palo Alto Networks, Inc. |
| 71 | GILD | Gilead Sciences Inc |
| 72 | SCHW | The Charles Schwab Corporation |
| 73 | ADI | Analog Devices, Inc. |
| 74 | SBUX | Starbucks Corp |
| 75 | VRTX | Vertex Pharmaceuticals Inc |
| 76 | TMUS | T-Mobile Us, Inc. |
| 77 | MMC | Marsh & Mclennan Companies, Inc. |
| 78 | MDT | Medtronic Plc |
| 79 | KKR | Kkr & Co. Inc. |
| 80 | PLD | Prologis, Inc. |
| 81 | LMT | Lockheed Martin Corp. |
| 82 | UPS | United Parcel Service, Inc. Class B |
| 83 | EQIX | Equinix, Inc. Reit |
| 84 | SO | The Southern Company |
| 85 | ELV | Elevance Health, Inc. |

| | | |
|---|---|---|
| 86 | PYPL | Paypal Holdings, Inc. |
| 87 | MO | Altria Group, Inc. |
| 88 | INTC | Intel Corp |
| 89 | NKE | Nike, Inc. |
| 90 | TT | Trane Technologies Plc |
| 91 | ICE | Intercontinental Exchange Inc. |
| 92 | CRWD | Crowdstrike Holdings, Inc. Class A |
| 93 | CDNS | Cadence Design Systems |
| 94 | DUK | Duke Energy Corporation |
| 95 | PH | Parker-Hannifin Corporation |
| 96 | CME | Cme Group Inc. |
| 97 | CI | The Cigna Group |
| 98 | MDLZ | Mondelez International, Inc. Class A |
| 99 | SHW | The Sherwin-Williams Company |
| 100 | PNC | Pnc Financial Services Group |
| 101 | USB | U.S. Bancorp |
| 102 | AON | Aon Plc Class A |
| 103 | SNPS | Synopsys Inc |
| 104 | MCK | Mckesson Corporation |
| 105 | WM | Waste Management, Inc. |
| 106 | ZTS | Zoetis Inc. |
| 107 | APO | Apollo Global Management, Inc. |
| 108 | MMM | 3m Company |
| 109 | EOG | Eog Resources, Inc. |
| 110 | CL | Colgate-Palmolive Company |
| 111 | AJG | Arthur J. Gallagher & Co. |
| 112 | WMB | Williams Companies Inc. |
| 113 | BDX | Becton, Dickinson and Co. |
| 114 | GD | General Dynamics Corporation |
| 115 | CTAS | Cintas Corp |
| 116 | TGT | Target Corporation |
| 117 | ADSK | Autodesk Inc |
| 118 | MAR | Marriot International Class A |
| 119 | NOC | Northrop Grumman Corp. |
| 120 | FDX | Fedex Corporation |
| 121 | FTNT | Fortinet, Inc. |
| 122 | OKE | Oneok, Inc. |
| 123 | ECL | Ecolab, Inc. |
| 124 | CVS | Cvs Health Corporation |
| 125 | PCAR | Paccar Inc |
| 126 | GM | General Motors Company |
| 127 | CARR | Carrier Global Corporation |
| 128 | BK | Bank of New York Mellon Corporation |
| 129 | DLR | Digital Realty Trust, Inc. |
| 130 | RCL | Royal Caribbean Group |

| | | |
|---|---|---|
| 131 | FCX | Freeport-Mcmoran Inc. |
| 132 | VST | Vistra Corp. |
| 133 | TRV | The Travelers Companies, Inc. |
| 134 | ROP | Roper Technologies, Inc. |
| 135 | NXPI | Nxp Semiconductors N.v. |
| 136 | KMI | Kinder Morgan, Inc. |
| 137 | SPG | Simon Property Group, Inc. |
| 138 | SRE | Sempra |
| 139 | WDAY | Workday, Inc. Class A |
| 140 | AMP | Ameriprise Financial, Inc. |
| 141 | ROST | Ross Stores Inc |
| 142 | CMI | Cummins Inc. |
| 143 | CPRT | Copart Inc |
| 144 | ALL | The Allstate Corporation |
| 145 | MET | Metlife, Inc. |
| 146 | PWR | Quanta Services, Inc. |
| 147 | GWW | W.W. Grainger, Inc. |
| 148 | PSA | Public Storage |
| 149 | MSCI | Msci, Inc. |
| 150 | O | Realty Income Corporation |
| 151 | MPC | Marathon Petroleum Corporation |
| 152 | D | Dominion Energy, Inc |
| 153 | HWM | Howmet Aerospace Inc. |
| 154 | AIG | American International Group, Inc. |
| 155 | URI | United Rentals, Inc. |
| 156 | PAYX | Paychex Inc |
| 157 | DFS | Discover Financial Services |
| 158 | LULU | Lululemon Athletica Inc. |
| 159 | PCG | Pg&e Corporation |
| 160 | NEM | Newmont Corporation |
| 161 | EW | Edwards Lifesciences Corp |
| 162 | BKR | Baker Hughes Company |
| 163 | FIS | Fidelity National Information Services, Inc. |
| 164 | PEG | Public Service Enterprise Group Incorporated |
| 165 | KMB | Kimberly-Clark Corp. |
| 166 | PRU | Prudential Financial, Inc. |
| 167 | RSG | Republic Services Inc. |
| 168 | AME | Ametek, Inc. |
| 169 | COR | Cencora, Inc. |
| 170 | TRGP | Targa Resources Corp. |
| 171 | AXON | Axon Enterprise, Inc. |
| 172 | KVUE | Kenvue Inc. |
| 173 | DHI | D.R. Horton Inc. |
| 174 | A | Agilent Technologies Inc. |
| 175 | KR | The Kroger Co. |

| | | |
|---|---|---|
| 176 | VLO | Valero Energy Corporation |
| 177 | DAL | Delta Air Lines, Inc. |
| 178 | CTVA | Corteva, Inc. |
| 179 | CBRE | Cbre Group, Inc. |
| 180 | VRSK | Verisk Analytics, Inc. |
| 181 | LHX | L3harris Technologies, Inc. |
| 182 | HES | Hess Corporation |
| 183 | F | Ford Motor Company |
| 184 | GEHC | Ge Healthcare Technologies Inc. |
| 185 | CCI | Crown Castle Inc. |
| 186 | IT | Gartner, Inc. |
| 187 | XEL | Xcel Energy, Inc. |
| 188 | EXC | Exelon Corporation |
| 189 | CTSH | Cognizant Technology Solutions |
| 190 | GLW | Corning Incorporated |
| 191 | SYY | Sysco Corporation |
| 192 | MNST | Monster Beverage Corporation |
| 193 | IR | Ingersoll Rand Inc. |
| 194 | IDXX | Idexx Laboratories Inc |
| 195 | IQV | Iqvia Holdings Inc. |
| 196 | OXY | Occidental Petroleum Corporation |
| 197 | RMD | Resmed Inc. |
| 198 | ACGL | Arch Capital Group Ltd |
| 199 | KDP | Keurig Dr Pepper Inc. |
| 200 | EA | Electronic Arts Inc |
| 201 | STZ | Constellation Brands, Inc. |
| 202 | ODFL | Old Dominion Freight Line |
| 203 | GIS | General Mills, Inc. |
| 204 | CHTR | Charter Comm Inc Del Cl a |
| 205 | UAL | United Airlines Holdings, Inc. |
| 206 | VMC | Vulcan Materials Company |
| 207 | ETR | Entergy Corporation |
| 208 | WAB | Wabtec Inc. |
| 209 | HPQ | Hp Inc. |
| 210 | FANG | Diamondback Energy, Inc. |
| 211 | GRMN | Garmin Ltd |
| 212 | MTB | M&t Bank Corp. |
| 213 | LEN | Lennar Corporation Class A |
| 214 | ROK | Rockwell Automation, Inc. |
| 215 | CNC | Centene Corporation |
| 216 | NDAQ | Nasdaq, Inc. |
| 217 | DD | Dupont De Nemours, Inc. |
| 218 | DECK | Deckers Outdoor Corp |
| 219 | WTW | Willis Towers Watson Public Limited Companys |
| 220 | MLM | Martin Marietta Materials |

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|---|---|---|
| 221 | DXCM | Dexcom, Inc. |
| 222 | MCHP | Microchip Technology Inc |
| 223 | VICI | Vici Properties Inc. |
| 224 | ED | Consolidated Edison, Inc. |
| 225 | MPWR | Monolithic Power Systems, Inc. |
| 226 | EFX | Equifax, Incorporated |
| 227 | EBAY | Ebay Inc |
| 228 | AVB | Avalonbay Communities, Inc. |
| 229 | TTWO | Take-Two Interactive Software Inc |
| 230 | IRM | Iron Mountain Inc. |
| 231 | WEC | Wec Energy Group, Inc. |
| 232 | CAH | Cardinal Health, Inc. |
| 233 | ANSS | Ansys Inc |
| 234 | RJF | Raymond James Financial, Inc. |
| 235 | HPE | Hewlett Packard Enterprise Company |
| 236 | CSGP | Costar Group Inc |
| 237 | KEYS | Keysight Technologies, Inc. |
| 238 | EQT | Eqt Corp |
| 239 | STT | State Street Corporation |
| 240 | GPN | Global Payments, Inc. |
| 241 | NUE | Nucor Corporation |
| 242 | XYL | Xylem Inc |
| 243 | DOW | Dow Inc. |
| 244 | GDDY | Godaddy Inc |
| 245 | PPG | Ppg Industries, Inc. |
| 246 | ON | On Semiconductor Corp |
| 247 | FTV | Fortive Corporation |
| 248 | BR | Broadridge Financial Solutions Inc |
| 249 | SYF | Synchrony Financial |
| 250 | DOV | Dover Corporation |
| 251 | SW | Smurfit Westrock Plc |
| 252 | CHD | Church & Dwight Co., Inc. |
| 253 | VLTO | Veralto Corporation |
| 254 | TROW | T Rowe Price Group Inc |
| 255 | CCL | Carnival Corporation |
| 256 | HSY | The Hershey Company |
| 257 | HAL | Halliburton Company |
| 258 | CPAY | Corpay, Inc. |
| 259 | CDW | Cdw Corporation |
| 260 | TYL | Tyler Technologies, Inc. |
| 261 | WST | West Pharmaceutical Services, Inc. |
| 262 | HBAN | Huntington Bancshares Inc |
| 263 | BRO | Brown & Brown, Inc. |
| 264 | VTR | Ventas, Inc. |
| 265 | AWK | American Water Works Company, Inc |

| | | |
|---|---|---|
| 266 | ADM | Archer Daniels Midland Company |
| 267 | PPL | Ppl Corporation |
| 268 | AEE | Ameren Corporation |
| 269 | WBD | Warner Bros. Discovery, Inc. Series a |
| 270 | HUBB | Hubbell Incorporated |
| 271 | EQR | Equity Residential |
| 272 | DVN | Devon Energy Corporation |
| 273 | TER | Teradyne, Inc. |
| 274 | WDC | Western Digital Corp. |
| 275 | EXPE | Expedia Group, Inc. |
| 276 | WAT | Waters Corp |
| 277 | CINF | Cincinnati Financial Corp |
| 278 | PHM | Pultegroup, Inc. |
| 279 | PTC | Ptc, Inc |
| 280 | K | Kellanova |
| 281 | RF | Regions Financial Corp. |
| 282 | DRI | Darden Restaurants, Inc. |
| 283 | TDY | Teledyne Technologies Incorporated |
| 284 | IFF | International Flavors & Fragrances Inc. |
| 285 | SBAC | Sba Communications Corp |
| 286 | ES | Eversource Energy |
| 287 | ZBH | Zimmer Biomet Holdings, Inc. |
| 288 | ZBRA | Zebra Technologies Corporation |
| 289 | STE | Steris Plc |
| 290 | NTRS | Northern Trust Corp |
| 291 | WY | Weyerhaeuser Company |
| 292 | LYV | Live Nation Entertainment Inc. |
| 293 | FE | Firstenergy Corp. |
| 294 | ULTA | Ulta Beauty, Inc. |
| 295 | PKG | Packaging Corp of America |
| 296 | CNP | Centerpoint Energy, Inc. |
| 297 | CBOE | Cboe Global Markets, Inc. |
| 298 | CLX | Clorox Company |
| 299 | LUV | Southwest Airlines Co. |
| 300 | CFG | Citizens Financial Group, Inc. |
| 301 | LH | Labcorp Holdings Inc. |
| 302 | LDOS | Leidos Holdings, Inc. |
| 303 | CMS | Cms Energy Corporation |
| 304 | LII | Lennox International Inc. |
| 305 | CTRA | Coterra Energy Inc. |
| 306 | LYB | Lyondellbasell Industries N.v. Class A |
| 307 | STX | Seagate Technology Holdings Plcs |
| 308 | MKC | Mccormick & Company, Incorporated Non-Vtg Cs |
| 309 | PODD | Insulet Corporation |
| 310 | IP | International Paper Co. |

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|---|---|---|
| 311 | COO | The Cooper Companies, Inc. |
| 312 | INVH | Invitation Homes Inc. |
| 313 | TRMB | Trimble Inc. |
| 314 | SNA | Snap-on Incorporated |
| 315 | ESS | Essex Property Trust, Inc |
| 316 | MAA | Mid-America Apartment Communities, Inc. |
| 317 | WRB | W.R. Berkley Corporation |
| 318 | EL | The Estee Lauder Companies Inc. Class A |
| 319 | SMCI | Super Micro Computer, Inc. |
| 320 | VRSN | Verisign Inc |
| 321 | JBL | Jabil Inc. |
| 322 | LVS | Las Vegas Sands Corp. |
| 323 | KEY | Keycorp |
| 324 | MOH | Molina Healthcare, Inc. |
| 325 | STLD | Steel Dynamics Inc |
| 326 | HOLX | Hologic Inc |
| 327 | NI | Nisource Inc. |
| 328 | OMC | Omnicom Group Inc. |
| 329 | BLDR | Builders Firstsource, Inc. |
| 330 | PFG | Principal Financial Group, Inc. |
| 331 | J | Jacobs Solutions Inc. |
| 332 | DG | Dollar General Corp. |
| 333 | GPC | Genuine Parts Company |
| 334 | MRNA | Moderna, Inc. |
| 335 | BALL | Ball Corporation |
| 336 | EG | Everest Group, Ltd. |
| 337 | TSN | Tyson Foods, Inc. |
| 338 | IEX | Idex Corporation |
| 339 | TPR | Tapestry, Inc. |
| 340 | EXPD | Expeditors International of Washington, Inc. |
| 341 | MAS | Masco Corporation |
| 342 | CF | Cf Industries Holding, Inc. |
| 343 | ALGN | Align Technology Inc |
| 344 | DLTR | Dollar Tree Inc. |
| 345 | ARE | Alexandria Real Estate Equities, Inc. |
| 346 | BAX | Baxter International Inc. |
| 347 | AVY | Avery Dennison Corp. |
| 348 | L | Loews Corporation |
| 349 | FFIV | F5, Inc. |
| 350 | SWKS | Skyworks Solutions Inc |
| 351 | LNT | Alliant Energy Corporation |
| 352 | GEN | Gen Digital Inc. |
| 353 | APTV | Aptiv Plc |
| 354 | VTRS | Viatris Inc. |
| 355 | RVTY | Revvity, Inc. |

| | | |
|---|---|---|
| 356 | DPZ | Domino's Pizza Inc. |
| 357 | JBHT | Jb Hunt Transport Services Inc |
| 358 | TXT | Textron, Inc. |
| 359 | EVRG | Evergy, Inc. |
| 360 | DOC | Healthpeak Properties, Inc. |
| 361 | AKAM | Akamai Technologies Inc |
| 362 | EPAM | Epam Systems, Inc. |
| 363 | ROL | Rollins, Inc. |
| 364 | CAG | Conagra Brands, Inc. |
| 365 | JNPR | Juniper Networks Inc |
| 366 | SWK | Stanley Black & Decker, Inc. |
| 367 | POOL | Pool Corporation |
| 368 | UDR | Udr, Inc. |
| 369 | KMX | Carmax Inc. |
| 370 | CHRW | C.H. Robinson Worldwide, Inc. |
| 371 | HST | Host Hotels & Resorts, Inc. |
| 372 | CPT | Camden Property Trust |
| 373 | INCY | Incyte Genomics Inc |
| 374 | SJM | The J.M. Smucker Company |
| 375 | NCLH | Norwegian Cruise Line Holdings Ltd.s |
| 376 | NDSN | Nordson Corp |
| 377 | ALLE | Allegion Public Limited Company |
| 378 | UHS | Universal Health Services, Inc. Class B |
| 379 | FOXA | Fox Corporation Class A |
| 380 | BG | Bunge Global Sa |
| 381 | ALB | Albemarle Corporation |
| 382 | NWSA | News Corporation Class A |
| 383 | IPG | The Interpublic Group of Companies, Inc. |
| 384 | EMN | Eastman Chemical Company |
| 385 | BXP | Bxp, Inc. |
| 386 | ENPH | Enphase Energy, Inc. |
| 387 | PAYC | Paycom Software, Inc. |
| 388 | RL | Ralph Lauren Corporation |
| 389 | CRL | Charles River Laboratories International, Inc. |
| 390 | TAP | Molson Coors Beverage Company Class B |
| 391 | GNRC | Generac Holdings Inc |
| 392 | LKQ | Lkq Corporation |
| 393 | PNW | Pinnacle West Capital Corporation |
| 394 | GL | Globe Life Inc. |
| 395 | LW | Lamb Weston Holdings, Inc. |
| 396 | HRL | Hormel Foods Corporation |
| 397 | APA | Apa Corporation |
| 398 | TFX | Teleflex Incorporated |
| 399 | MKTX | Marketaxess Holdings Inc. |
| 400 | MOS | The Mosaic Company |

### 5.2 Complete Code

```
import numpy as np
import pandas as pd

p = 400
n = 26

# Get weekly closing price data
prices = pd.read_excel("snp500.xlsx", index_col=0)
print("\nPortfolio Closing Prices:\n",prices)

# Compute weekly returns
# Drop first row (t = 0)
returns = prices.pct_change(fill_method=None).dropna()
print("\nPortfolio Weekly Returns:\n",returns)

# Compute excess returns
excess_returns = returns - 0.00087
print("\nPortfolio Excess Returns\n",excess_returns)

# Transpose excess returns matrix
excess_returns = excess_returns.T # (p x n) matrix of p=400 stocks and n=26 weeks

# Compute de-meaned excess returns matrix Y
expected_returns = excess_returns.mean(axis=1)
print("\nExpected Returns:\n",expected_returns)
Y = excess_returns.sub(expected_returns, axis=0)
print("\nY:\n", Y)

# Compute sample covariance matrix S
S = Y @ Y.T / n
print("\nS:\n", S)

# Find trace, leading eigenvalue, and corresponding eigenvector of S
trace_S = np.trace(S)
print("\ntr(S):\n", trace_S)
eigvals, eigvecs = np.linalg.eigh(S)
lambdaS = eigvals[-1]
print("\nleading eigenvalue:\n", lambdaS)
v = eigvecs[:, -1]
print("\ncorresponding eigenvector:\n", v)
l = (trace_S - lambdaS) / (n-1)
print("\nl:\n", l)
```

```python
# Compute single factor model covariance matrix sigma
term1 = lambdaS - 1
term2 = (n / p) * 1
sigma = (term1 * np.outer(v, v)) + (term2 * np.eye(p))
print("\nSigma:\n", sigma)
print(sigma.shape)

# Verify leading eigenvector is the same as S (note:
    floating point error is possible)
eigvals1, eigvecs1 = np.linalg.eigh(sigma)
lambda1 = eigvals1[-1]
print("\nlambda1:\n", lambda1)

# Compute holdings vector h_C for minimum variance,
    fully invested portfolio C
ones = np.ones(p)
sigma_inv = np.linalg.inv(sigma)
h_C = sigma_inv @ ones / (ones.T @ sigma_inv @ ones)
print("\nh_C:\n", h_C)

# Compute portfolio expected excess return, variance,
    and standard deviation
portfolio_expected_returns = h_C.T @ expected_returns
print("\nPortfolio Expected Returns\n",
    portfolio_expected_returns)
portfolio_var = h_C.T @ sigma @ h_C
print("\nPortfolio Variance:\n", portfolio_var)
portfolio_std_dev = np.sqrt(portfolio_var)
print("\nPortfolio Standard Deviation:\n",
    portfolio_std_dev)
stock_var = np.diag(sigma)
print("\nStock Variance:\n", stock_var)

# Scale by 52 for annualized results
annualized_return = portfolio_expected_returns * 52
print("\nAnnualized Return:\n", annualized_return)
annualized_var = portfolio_var * 52
print("\nAnnualized Variance:\n", annualized_var)
annualized_std_dev = portfolio_std_dev * np.sqrt(52)
print("\nAnnualized Standard Deviation:\n",
    annualized_std_dev)
annualized_stock_var = stock_var * 52
print("\nAnnualized Stock Variance:\n",
    annualized_stock_var)


```

```
# PLOT 1
row = prices.iloc[0]
new_df = pd.DataFrame([row])
print(new_df)
for i in range(p):
    new_df.iloc[0, i] = i+1
print(new_df)

x = stock_std_dev
y = expected_returns
z = new_df.iloc[0, :]
fig, ax = plt.subplots(figsize=(9, 6))
scatter = ax.scatter(x, y, c=z, cmap='viridis')
point = ax.plot(portfolio_std_dev, portfolio_expected_returns, 'ro')
ax.text(portfolio_std_dev, portfolio_expected_returns,
        f'Portfolio Return:\n {portfolio_expected_returns:.4f}\nRisk: {portfolio_std_dev:.4f}',
        fontsize=12, color='red', ha='left', va='bottom')
legend = ax.legend(*scatter.legend_elements(), loc="lower right", title="Company Market Cap Range")
ax.add_artist(legend)
ax.set_xlabel('Risk ($\sigma$)')
ax.set_ylabel('Expected Return ($f$)')
ax.set_title('Expected Excess Returns vs. Risk by Market Cap', size=14)
plt.show()

# PLOT 2
x = stock_std_dev
y = h_C
fig, ax = plt.subplots(figsize=(9, 6))
scatter = ax.scatter(x, y, c=y, cmap='viridis')
ax.text(portfolio_std_dev, portfolio_expected_returns,
        f'Portfolio Return:\n {portfolio_expected_returns:.4f}\nRisk: {portfolio_std_dev:.4f}',
        fontsize=14, color='red', ha='left', va='bottom')
legend = ax.legend(*scatter.legend_elements(), loc="upper right", title="Portolio Holdings (%)")
ax.add_artist(legend)
ax.set_ylabel('Holdings ($h_C$)')
ax.set_xlabel('Risk ($\sigma$)')
```

```
ax.set_title('Portfolio Holdings by Level of Risk',
    size=14)
plt.show()
```