Name: Jenny Petrova

Problem Set 9

Due Date: 11/7 at 11:35 AM

(These problems are harder, so each problem is worth 20 points)

(P.1) *An n-dimensional (closed) rectangle $\mathcal{R}$ in $\mathbb{R}^n$ is represented by*

$$\begin{aligned}\mathcal{R} &= [a_1,\, b_1] \times [a_2,\, b_2] \times \cdots \times [a_n,\, b_n] \\ &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_i \in [a_i, b_i],\, i \in \{1, \ldots, n\}\},\end{aligned}$$

for n pairs $(a_i, b_i) \in \mathbb{R}^2$ $(i \in \{1, \ldots, n\})$ satisfying $a_i \leq b_i$. The volume is computed by

$$\mathcal{V}(\mathcal{R}) = (b_1 - a_1)\,(b_2 - a_2) \cdots (b_n - a_n).$$

Let $\mathscr{R}_n \equiv \mathscr{R}(\mathbb{R}^n)$ denote the set of all n-dimensional rectangles $\mathcal{R}$ in $\mathbb{R}^n$ and define a set-valued map $m_n : \mathscr{R}_n \mapsto [0, \infty)$ by

$$m_n(\mathcal{R}) \coloneqq \mathcal{V}(\mathcal{R}) = (b_1 - a_1)\,(b_2 - a_2) \cdots (b_n - a_n), \qquad \mathcal{R} \in \mathscr{R}_n.$$

Show that the Borel σ-algebra on $\mathbb{R}^n$ denoted by $\mathcal{B}_n \equiv \mathcal{B}(\mathbb{R}^n)$ is generated by $\mathscr{R}_n$ and show that the n-dimensional Lebesgue measure m_n exists and is unique on $\mathscr{R}_n$ and extends to $\mathcal{B}_n$.

(Hint: Try to extend the exercises outlined in the proofs of Theorem 11.1 and Theorem 11.2 in the text of Jacod and Protter).

Proof. We know that the Borel σ-algebra $\mathcal{B}_n$ on $\mathbb{R}^n$ is generated by "quadrants" of the form

$$\prod_{i=1}^{n} (-\infty, x_i],\ x_i \in \mathbb{Q},$$

where $\mathcal{B}_n$ is also the smallest σ-algebra generated by the n-fold Cartesian product of the Borel sets on $\mathbb{R}$. The proof of this result will allow us to show $\mathcal{B}$ is generated by $\mathscr{R}_n$.

Let O denote all open intervals in $\mathbb{R}^n$. Since every set in $\mathbb{R}$ is the countable union of open intervals, then $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}_n = \sigma(O)$. Consider the open rectangle generated by the open sets such that

$$\mathcal{R}_O = (a_1 - \frac{1}{k}, b_1 + \frac{1}{k}) \times (a_2 - \frac{1}{k}, b_2 + \frac{1}{k}) \times \cdots \times (a_n - \frac{1}{k}, b_n + \frac{1}{k}).$$

For each quadrant, we have

$$\bigcap_{k=1}^{\infty} (a_i - \frac{1}{k}, b_i + \frac{1}{k}) = [a_i, b_i].$$

Hence we can write

$$\mathcal{R} = \prod_{i=1}^{n} [a_i, b_i] = \bigcap_{k=1}^{\infty} \prod_{i=1}^{n} (a_i - \frac{1}{k}, b_i + \frac{1}{k}) = \bigcap_{k=1}^{\infty} \mathcal{R}_O.$$

Therefore $\mathcal{R}$ can be expressed as a countable intersection of open rectangles. Since $\mathcal{R} \in \mathscr{R}_n$ by assumption and $\mathcal{R}_O \in \mathcal{B}_n$, then $\mathcal{B}_n \subseteq \sigma(\mathscr{R}_n)$. Additionally, we know $\mathscr{R}_n$ consists of closed rectangles in $\mathbb{R}^n$, which are Borel sets. Since $\sigma(\mathscr{R}_n)$ is the smallest $\sigma-$algebra containing $\mathscr{R}_n$, then $\sigma(\mathscr{R}_n) \subseteq \mathcal{B}_n$. Thus $\sigma(\mathscr{R}_n) \subseteq \mathcal{B}_n \subseteq \sigma(\mathscr{R}_n)$, and therefore $\sigma(\mathscr{R}_n) = \mathcal{B}_n$, i.e. $\mathcal{B}_n$ is generated by $\mathscr{R}_n$.

To show the Lebesgue measure m_n is unique, we fix $a_i < b_i$ in $\mathbb{R}$ for all $i \in \{1, \ldots, n\}$ and define

$$m_{a_i,b_i}(\mathcal{R}) = \frac{m_n(\mathcal{R} \cap \prod_{i=1}^n [a_i, b_i])}{\prod_{i=1}^n (b_i - a_i)},$$

for all $\mathcal{R} \in \mathcal{B}_n$. Then m_{a_i,b_i} is a probability measure on $(\mathbb{R}^n, \mathcal{B}_n)$, and we construct the corresponding distribution function F_n by

$$\begin{aligned}
F_{a_i,b_i}(x_1, x_2, \ldots, x_n) &= m_{a_i,b_i}(\prod_{i=1}^n (-\infty, x_i]) \\
&= \frac{m_n(\prod_{i=1}^n ((-\infty, x_i] \cap [a_i, b_i]))}{\prod_{i=1}^n (b_i - a_i)} \\
&= \begin{cases} 0 \text{ if } x_i < a_i \\ \prod_{i=1}^n \frac{(x_i - a_i)}{(b_i - a_i)} \text{ if } a_i \leq x_i < b_i \\ 1 \text{ if } b_i \leq x_i. \end{cases}
\end{aligned}$$

Therefore m_{a_i,b_i} is uniquely determined (since F_{a_i,b_i} has a given formula and is therefore unique). By Theorem 7.2, we know F_{a_i,b_i} exists since the function is non-decreasing, continuous, and 0 for x_i small enough ($x_i < a_i$) and 1 for x_i large enough $x_i \geq b_i$. Hence m_{a_i,b_i} exists. Therefore the Lebesgue measure m_n exists and is unique.

□

(P.2) *Derive the standard multivariate Gaussian probability measure space (i.e., zero mean vector, identity covariance matrix) by specifying a density $f : \mathbb{R} \mapsto (0, \infty)$ with respect to the n-dimensional Lebesgue measure.*

Proof. The density function f for the standard multivariate Gaussian random vector $X \sim \mathcal{N}(0, I)$ is defined by

$$\begin{aligned}
f_X(x) &= \frac{1}{(2\pi)^{\frac{n}{2}} |I|^{\frac{1}{2}}} e^{-\frac{1}{2} x^T I^{-1} x} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} x^T x} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} ||x||^2}
\end{aligned}$$

Using Definition 12.2, we derive the measure $\mathbb{P}$ on $(\mathbb{R}^n, \mathcal{B}_n)$ with density f, with respect to the n-dimensional Lebesgue measure, to be

$$\begin{aligned}
\mathbb{P}(A) &= \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) \mathbb{1}_A(x_1, \ldots, x_n) m_n(x) \\
&= \int_A \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} ||x||^2} m(x).
\end{aligned}$$

for all $A \in \mathcal{B}_n$. Consider $A = \mathbb{R}^n$. Since the f is nonnegative and continuous on $(0, \infty)$, we use Riemann integration to solve

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}||x||^2} dx_1 \dots dx_n &= \int\limits_{-\infty}^{\infty} \dots \int\limits_{\infty}^{\infty} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}||x||^2} dx_1 \dots dx_n \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{-\infty}^{\infty} \dots \int\limits_{\infty}^{\infty} \prod_{i=1}^{n} e^{-\frac{1}{2}x_i^2} dx_1 \dots dx_n \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \prod_{i=1}^{n} \int\limits_{-\infty}^{\infty} \dots \int\limits_{\infty}^{\infty} e^{-\frac{1}{2}x_i^2} dx_1 \dots dx_n \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} (\sqrt{2\pi})^n \\
&= 1,
\end{aligned}$$

where $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x_i^2} dx = \sqrt{2\pi}$ is the Gaussian integral. Hence $\mathbb{P}$ is a probability measure on $(\mathbb{R}^n, \mathcal{B}_n)$ and since $\int f(x)dx = 1$, then by Theorem 12.1 f is the density of a probability measure on $(\mathbb{R}^n, \mathcal{B}_n)$.

$\square$

(P.3) *Derive the characteristic function of a multivariate Gaussian random vector.*

Proof. We know that the characteristic function ρ_X for a univariate Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ is

$$\varphi_X(u) = e^{iu\mu - \frac{\sigma^2 u^2}{2}}.$$

Suppose we have $\mathbb{R}$-valued independent Gaussian random vectors $X_1, \dots, X_n$ with laws $\mathcal{N}(\mu_j, \sigma_j^2)$. By Corollary 14.1, we show

$$\begin{aligned}
\varphi_X(u_1, \dots, u_n) &= \prod_{j=1}^{n} \varphi_{X_j}(u_j) \\
&= \prod_{j=1}^{n} e^{iu_j\mu_j - \frac{\sigma_j^2 u_j^2}{2}} \\
&= \exp\left(i \sum_{j=1}^{n} u_j \mu_j - \frac{1}{2} \sum_{j=1}^{n} u_j^2 \sigma_j^2 \right) \\
&= \exp\left(i \sum_{j=1}^{n} u_j \mu_j - \frac{1}{2} \sum_{j=1}^{n} u_j \sigma_j^2 u_j \right) \\
&= e^{i\langle u, \mu \rangle - \frac{1}{2}\langle u, Qu \rangle},
\end{aligned}$$

where $\mu \in \mathbb{R}^n$ and Q is a diagonal matrix such that $Q_{ij} = \sigma_j^2$ for $i = j$ and $Q_{ij} = 0$ for $i \neq j$. Then we satisfy Theorem 16.1, and hence $X = X_1, \dots, X_n$ is a multivariate Gaussian random vector with characteristic function

$$\varphi_X(u) = \varphi_X(u_1, \dots, u_n) = e^{i\langle u, \mu \rangle - \frac{1}{2}\langle u, Qu \rangle}.$$

$\square$

(P.4) *An undirected simple graph is a double $(\mathscr{N}, \mathscr{E})$ where $\mathscr{N}$ is a non-empty set called the* **node set** *and $\mathscr{E}$ is a subset of unordered pairs of nodes called the* **edge set***, i.e., $\mathscr{E} \subseteq \{\{v, w\} : v \in \mathscr{N}, w \in \mathscr{N}\}$. An Erdős-Rényi random graph is an undirected simple random graph where the edge set $\mathscr{E}$ is random and where the cardinality $|\mathscr{E}|$ of $\mathscr{E}$ follows a Binomial$(\binom{N}{2}, p)$ distribution where $N = |\mathscr{N}|$ and $p \in (0, 1)$. Derive a probability space for the random graph.*

Proof. We define the set $\mathcal{D}(\mathscr{N})$ to be a set of all unordered pairs of nodes in the graph such that

$$\mathcal{D}(\mathscr{N}) = \{\{v, w\} : v \in \mathscr{N}, w \in \mathscr{N}\},$$

where $\mathscr{E} \subseteq \mathcal{D}(\mathscr{N})$ by assumption, since it is possible that an edge exists between any two nodes. Then the state space Ω will be the set of all possible edge sets for any pairs of $\binom{N}{2}$ nodes in the graph, meaning $\Omega = \mathcal{P}(\mathcal{D}(\mathscr{N}))$, where $|\Omega| = 2^{\binom{N}{2}}$, since each edge may or may not exist. We define the σ-algebra $\mathcal{A}$ to be the power set of Ω, hence $\mathcal{A} = \mathcal{P}(\Omega) = \mathcal{P}(\mathcal{P}(\mathcal{D}(\mathscr{N})))$. Since each edge exists independently, the number of edges $|\mathscr{E}|$ follows a Binomial $(\binom{N}{2}, p)$ distribution. Therefore the number of possible edge sets with exactly M edges is given by

$$|\{\omega \in \Omega : |\omega| = M\}| = \binom{\binom{N}{2}}{M}.$$

We define the probability mass function of having exactly M edges in an edge set as

$$f(M) = \binom{\binom{N}{2}}{M} p^M (1-p)^{\binom{N}{2} - M}.$$

Therefore the probability of generating any graph with N nodes and exactly M edges is

$$\mathbb{P}(\{\omega\}) = \frac{f(M)}{|\{\omega\}|} = p^M (1-p)^{\binom{N}{2} - M},$$

for any edge set $\omega \in \Omega$. Thus we have defined the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ for the random graph. □

(P.5) *Consider the previous problem. Define a random vector $\boldsymbol{X}$ and corresponding probability space $(\mathbb{R}^{\binom{N}{2}}, \mathcal{B}_{\binom{N}{2}}, \mathbb{P})$ to represent the preceding example of an Erdős-Rényi random graph. Define the probability measure $\mathbb{P}_{\boldsymbol{X}}$ through integration of a density and give the specific density. Show that the "edge variables" which encode whether an edge is present/absent in the edge set $\mathcal{E}$ are independent Bernoulli random variables.*

Proof. Consider any node set $\{v, w\} \in \mathcal{D}(\mathscr{N})$. We can define a random vector

$$X = (X_{\{1,2\}}, X_{\{1,3\}}, \ldots, X_{\{1,N\}}, X_{\{2,3\}}, X_{\{2,4\}}, \ldots, X_{\{N-1,N\}})$$

such that

$$X_{\{v,w\}} = \begin{cases} 1 & \text{if } \{v, w\} \in \mathcal{E}, \\ 0 & \text{if } \{v, w\} \notin \mathcal{E}. \end{cases},$$

where each $X_{\{v,w\}}$ is an indicator variable for the presence of an edge between nodes v and w. Thus, each $X_{\{v,w\}}$ is a Bernoulli random variable with parameter p and

$X \in \{0,1\}^{\binom{N}{2}}$, hence we have the state space $\mathbb{R}^{\binom{N}{2}}$. We denote $\mathcal{B}_{\binom{N}{2}}$ to be the Borel σ-algebra on the open sets of $\mathbb{R}^{\binom{N}{2}}$. Since

$$|\mathcal{E}| = \sum_{\{v,w\} \in \mathcal{D}(\mathscr{N})} X_{\{v,w\}} = ||X||_1,$$

then $||X||_1 \sim$ Binomial $(\binom{N}{2}, p)$. To specify $\mathbb{P}$, we first define the density (probability mass function) for the function $||X||_1$ to be

$$\begin{aligned} f(x) &= p^{||X||_1} (1-p)^{\binom{N}{2} - ||X||_1} \mathbb{1}_{\{0,1\}^{\binom{N}{2}}}(x) \\ &= \sum_{y \in \{0,1\}^{\binom{N}{2}}} p^{||y||_1} (1-p)^{\binom{N}{2} - ||y||_1} \mathbb{1}_y(x) \end{aligned}$$

for $x \in \mathbb{R}^{\binom{N}{2}}$. Therefore for any $A \in \mathbb{R}^{\binom{N}{2}}$ we can construct the probability measure

$$\begin{aligned} \mathbb{P}(A) &= \int_A f(x) \,\mathrm{d}\,\nu(x) \\ &= \int_A \sum_{y \in \{0,1\}^{\binom{N}{2}}} p^{||y||_1} (1-p)^{\binom{N}{2} - ||y||_1} \mathbb{1}_y(x) \,\mathrm{d}\,\nu(x) \\ &\vdots \\ &= \sum_{y \in \{0,1\}^{\binom{N}{2}}} p^{||y||_1} (1-p)^{\binom{N}{2} - ||y||_1} \nu(\{y\} \cap A) \\ &= \sum_{x \in A \cap \{0,1\}^{\binom{N}{2}}} p^{||y||_1} (1-p)^{\binom{N}{2} - ||y||_1}. \end{aligned}$$

$\square$

(P.6) *Derive the characteristic function of the Geometric(p) distribution (on support $\{0, 1, 2, \ldots\}$).*

Proof. If a random variable X is Geometric with parameter p, we define the characteristic function as

$$\begin{aligned}\phi_X(u) &= \mathbb{E}\{e^{iuX}\}\\ &= \sum_{k=0}^{\infty} e^{iuk}\mathbb{P}(X = k)\\ &= \sum_{k=0}^{\infty} e^{iuk}(1-p)^k p\\ &= p\sum_{k=0}^{\infty} e^{iuk}(1-p)^k\\ &= p\sum_{k=0}^{\infty}(e^{iu}(1-p))^k.\end{aligned}$$

Since $|e^{iu}| = 1$ and $0 < 1 - p < 1$, then $|e^{iu}(1-p)| < 1$ and thus the infinite series $\sum_{k=0}^{\infty}(e^{iu}(1-p))^k$ converges to the limit value $\frac{1}{1-e^{iu}(1-p)}$. Hence

$$\phi_X(u) = p\sum_{k=0}^{\infty}(e^{iu}(1-p))^k = \frac{p}{1-e^{iu}(1-p)}.$$

□

(P.7) *Derive the characteristic function of the Uniform(a, b) distribution.*

Proof. If a random variable X is Uniform on (a,b), we define the characteristic function as

$$\begin{aligned}\phi_X(u) &= \mathbb{E}\{e^{iuX}\}\\ &= \int_b^a e^{iux}\frac{1}{b-a}dx\\ &= \frac{1}{b-a}\int_b^a e^{iux}dx\\ &= \frac{1}{b-a}\left(\frac{e^{iub}-e^{iua}}{iu}\right)\\ &= \frac{e^{iub}-e^{iua}}{iu(b-a)}.\end{aligned}$$

□