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Problem Set 4

Due Date: 9/24 at 11:35 AM

Throughout, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

(P.1) *Let $(A_n)_{n\geq 1}$ be a sequence of pairwise disjoint events in $\mathcal{A}$. Show that* $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$.

Proof. Consider a sequence $B_n \in \mathcal{A}$ such that $B_n = \bigcup_{m\geq n}^{\infty} A_m$ for $m \geq n$. Since the events A_n are pairwise disjoint, then by definition of countable additivity we have

$$\mathbb{P}(B_n) = \mathbb{P}(\bigcup_{m=n}^{\infty} A_m) = \sum_{m=n}^{\infty} \mathbb{P}(A_m). \tag{1}$$

Notice that

$$B_{n+1} = \bigcup_{m=n+1}^{\infty} A_m = (A_{n+1} \cup A_{n+2} \cup \dots) \subseteq B_n = \bigcup_{m=n}^{\infty} A_m = (A_n \cup A_{n+1} \cup A_{n+2} \cup \dots).$$

Since $B_{n+1} \subseteq B_n$ for all n, then B_n is a decreasing sequence and thus we define the limit of B_n to be

$$\lim_{n\to\infty} B_n = \bigcap_{n=1}^{\infty} B_n.$$

By definition of pairwise disjoint, we have $\bigcap_{n=1}^{\infty} B_n = \emptyset$. Thus

$$\lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) = \mathbb{P}(\emptyset) = 0. \tag{2}$$

Therefore we use (1) and (2) to show

$$\lim_{n\to\infty} \left(\sum_{m=n}^{\infty} \mathbb{P}(A_m)\right) = \lim_{n\to\infty} \mathbb{P}(B_n) = 0.$$

This implies $\sum_{m=n}^{\infty} \mathbb{P}(A_m) = \emptyset$ for all $m \geq n$. We can extend this to all $n \geq 1$ to show $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \emptyset$. Hence $\mathbb{P}(A_n) = \emptyset$ for all n. Therefore we show

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \emptyset = 0.$$

$\square$

(P.2) *Let $\mathcal{B}$ denote the Borel σ-algebra. Show that $\{x\} \in \mathcal{B}$ for all $x \in \mathbb{R}$, i.e., show that $\{x\}$ is a Borel set.*

Proof. By Theorem 2.1, we know that $\mathcal{B}$ is generated by open intervals of the form $(-\infty, a]$, where $a \in \mathbb{Q}$. Let $(x - \frac{1}{n}, x + \frac{1}{n}) \in \mathcal{B}$ be an open interval containing x. Then

$$\begin{aligned}\{x\} \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right) &= \cup_{n=N}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n} - \frac{1}{n}\right] \\ &= \cup_{n=N}^{\infty} \left(x - \frac{1}{n}, x\right]\end{aligned}$$

for some N large enough, so $\mathcal{B}$ contains all open intervals of x. Since the Borel σ-algebra is closed under countable open intervals, we can therefore express the singleton set $\{x\}$ as the intersection of decreasing open intervals containing x, where

$$\{x\} = \cap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \in \mathcal{B}.$$

Thus $\{x\}$ is a Borel set. $\square$

(P.3) *Consider a probability space* $(\mathbb{R}, \mathcal{B}, \mathbb{P})$ *and let* $F(x)$ *be the distribution function, i.e.,*

$$F(x) \quad := \quad \mathbb{P}((-\infty,\, x]), \qquad x \in \mathbb{R}.$$

Assume that F *is a continuous function and show the following:*

(a) $\mathbb{P}(\{x\}) = 0$ *for all* $x \in \mathbb{R}$*.*

Proof. Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$ such that $x_n \downarrow x$. Then

$$(-\infty, x) = \bigcup_{n=1}^{\infty} (-\infty, x_n].$$

Hence by Theorem 2.3,

$$\mathbb{P}((-\infty, x)) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} (-\infty, x_n]\right) = \lim_{z\to x^-} F(x) = F(x-). \tag{3}$$

Similarly, consider a sequence $(y_n)_{n\geq 1}$ in $\mathbb{R}$ such that $y_n \uparrow x$. Then

$$(-\infty, x] = \bigcap_{n=1}^{\infty} (-\infty, y_n].$$

Thus we also show that

$$\mathbb{P}((-\infty, x]) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} (-\infty, y_n]\right) = \lim_{z\to x^+} F(x) = F(x). \tag{4}$$

Using Corollary 7.1, we show

$$\begin{aligned} \mathbb{P}(\{x\}) &= F(x) - F(x-) \\ &= \mathbb{P}((-\infty, x]) - \mathbb{P}((-\infty, x)) \\ &= \mathbb{P}((x, x]) \\ &= \mathbb{P}(\emptyset) \\ &= 0. \end{aligned} \tag{5}$$

$\square$

(b) If $S \subset \mathbb{R}$ *is countable, then* S *is a null set.*

Proof. Let $x_1, x_2, ... \in S$. From (a) we know that $\mathbb{P}(\{x_i\}) = 0$ for each $x_i \in S$. We also know that $\{x_i\} \cap \{x_j\} = \emptyset$ for any two singleton sets where $i \neq j$. Hence each $\{x_i\} \subseteq S$ is pairwise disjoint. Therefore we can use Axiom (2) of Definition 2.3 to show

$$\mathbb{P}(S) = \mathbb{P}(\cup_{i=1}^{\infty}\{x_i\}) = \sum_{i=1}^{\infty} \mathbb{P}(\{x_i\}) = 0.$$

By Theorem 2.2, $\mathbb{P}(S) = 0$ implies $S = \emptyset$. Thus S is a null set. $\square$

(c) *Show that there exists a median of the distribution, i.e., a point* $m_0 \in \mathbb{R}$ *such that* $F(m_0) = .5$.

By definition of probability distribution, we know that F is non-decreasing and $\lim_{n\to-\infty} F(x) = 0$ and $\lim_{n\to\infty} F(x) = 1$. Then for any $\epsilon > 0$, $\epsilon \in \mathbb{R}$ there exists an $a \in \mathbb{R}$ such that

$$0 < F(a) < \epsilon.$$

We manipulate the above equation to show

$$1 > 1 - F(a) > 1 - \epsilon.$$

Then for any $\delta = 1 - \epsilon < 1$, there exists an $a \in \mathbb{R}$ such that

$$\delta < 1 - F(a) < 1.$$

Let $\epsilon = \frac{1}{3}$. Then

$$0 < F(a) < \frac{1}{3}$$

and

$$\frac{2}{3} < 1 - F(a) < 1.$$

Since F is continuous on $\mathbb{R}$, then there exists a point $c \in \mathbb{R}$ such that $\frac{1}{3} \leq c \leq \frac{2}{3}$. Consider $c = \frac{1}{2}$. Therefore we apply the Intermediate Value Theorem to conclude that there exists a point $c \in [\frac{1}{3}, \frac{2}{3}]$ such that $F(c) = \frac{1}{2}$.

(P.4) *Consider a probability space $(\mathbb{R}, \mathcal{B}, \mathbb{P})$ and let $F(x)$ be the distribution function. Show that if F has discontinuities then all are jump discontinuities and show that F can have at most a countably infinite number of jump discontinuities.*

Proof. First, let $x \in \mathbb{R}$ such that F has a discontinuity at x, and consider the following:

Case 1: Let x be a removable discontinuity. Then $\lim_{y \to x^+} F(y) \neq F(x)$. This leads to a contradiction, since F is right continuous and therefore

$$\lim_{y \to x^+} F(y) = F(x).$$

Case 2: Let x is an essential discontinuity. If $\lim_{y \to x^+} F(y)$ does not exist, then F is not right-continuous, which leads to a contradiction. Then we must have that $\lim_{y \to x^+} F(y)$ exists. Then if $\lim_{y \to x^-} F(y)$ does not exist, this also leads to a contradiction, since F is non-decreasing.

Case 3: Let x is a jump discontinuity. Since F is right continuous, then

$$\lim_{y \to x^+} F(y) = F(x)$$

and it must be true that

$$F(x^-) = \lim_{y \to x^-} F(y) \neq F(x).$$

Because F is non-decreasing, we can write

$$F(x^-) < F(x).$$

Applying this to Corollary 7.1, we show

$$\mathbb{P}(\{x\}) = F(x) - F(x^-) > 0. \tag{6}$$

Thus we have shown that any point of discontinuity x on F must be a jump discontinuity.

Next, let (a, b) be an open interval in $\mathbb{R}$. Then for any discontinuity $x \in (a, b)$, we use (6) to show

$$F(a) \leq F(x^-) < F(x) \leq F(b).$$

Since all the discontinuities of F are jump discontinuities, then we can define a set J such that

$$J := \{\frac{1}{n} : n \in \mathbb{N}, x \in (a, b), F(x) - F(x^-) = \frac{1}{n}\}.$$

Then each $j \in J$ is a rational number defined on the interval (a, b), corresponding to each jump discontinuity x. Since rational numbers are countable, then J has a countably infinite number of elements, which implies a countably infinite number of jump discontinuities in F.

$\square$

(P.5) *Consider a probability space $(\mathbb{R}, \mathcal{B}, \mathbb{P})$ and let $F(x)$ be the distribution function. Show that there are no atoms in $\mathcal{B}$ if the distribution function F is continuous.*

Proof. Suppose the distribution function F is continuous and that there exists an atom $A \in \mathcal{B}$. We will show this leads to a contradiction.

Case 1: Consider the event $A = \{x\}$, such that A is an atom with $P(A) = c > 0$.
Since F is continuous, then as proven in **(P.3)(a)** we have

$$\mathbb{P}(A) = \mathbb{P}(\{x\}) = 0$$

for all $x \in \mathbb{R}$. This contradicts the fact that an atom must have positive probability, i.e. $P(A) > 0$.

Case 2: Consider the sets $B_n := \{(-\infty, x_n] \cap A : x_n \in A\}$, where $B_n \subset A$. Since closure under unions is assumed for a Borel σ-algebra and $P(A) = c > 0$, then we define

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} ((-\infty, x_n] \cap A) = [0, c].$$

Thus $B_n \in \mathcal{B}$ and we can define a new continuous probability distribution $F_B(x) \in [0, c]$. By the Intermediate Value Theorem, we know that there exists a $b \in B_n$, such that $0 < \mathbb{P}(b) < c$. Then for any $B_n \subset A$, we have $0 < \mathbb{P}(b) < \mathbb{P}(A)$, which defines $\mathbb{P}$ to be a non-atomic measure. This leads to a contradiction, since we assumed there exists an atom in F when F is continuous. $\square$

(P.6) *Consider a probability space* $(\mathbb{R}, \mathcal{B}, \mathbb{P})$ *and let* $F(x)$ *be the distribution function. Show that there is at least one atom in* $\mathcal{B}$ *if the distribution function* F *has a jump discontinuity.*

Proof. Suppose F has a jump discontinuity at a point $x \in \mathbb{R}$. Then in **(P.4)** we defined

$$F(x) - F(x^-) > 0$$

and showed that

$$F(x) - F(x^-) = \mathbb{P}(\{x\}) > 0.$$

To show that the singleton set $\{x\}$ of the jump discontinuity is an atom, let $A = \{x\}$. Then for any subset $B \subset A$ that exists, it must be true that $B = \emptyset$. We know that $\mathbb{P}(B) = \mathbb{P}(\emptyset) = 0$. Thus we show that at any point x where a jump discontinuity occurs, we can define a set $A = \{x\}$, where $\mathbb{P}(\{x\}) > 0$ and for any $B \subset A$, it is true that $\mathbb{P}(B) = 0$.

□