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Problem Set 0

(P.1) *Prove Chebyshev's inequality: Consider a random variable X with finite expectation $\mathbb{E}X$ and variance $\mathbb{V}X$. Then*

$$\mathbb{P}(|X - \mathbb{E}X| \geq t) \quad \leq \quad \frac{\mathbb{V}X}{t^2},$$

for all $t > 0$.

First, recall Markov's inequality which provides, for a nonnegative random variable Y and positive real number $t > 0$,

$$\mathbb{P}(Y \geq t) \quad \leq \quad \frac{\mathbb{E}Y}{t}.$$

Since $|X - \mathbb{E}X| \geq 0$ and $t > 0$, the transformation $h(z) = z^2$ on $z \in [0, \infty)$ is a monotone transformation, implying

$$\mathbb{P}(|X - \mathbb{E}X| \geq t) \quad = \quad \mathbb{P}((X - \mathbb{E}X)^2 \geq t^2). \tag{1}$$

Applying Markov's inequality to the right-hand side of (1), we obtain

$$\mathbb{P}((X - \mathbb{E}X)^2 \geq t^2) \quad \leq \quad \frac{\mathbb{E}(X - \mathbb{E}X)^2}{t^2} \quad = \quad \frac{\mathbb{V}X}{t^2},$$

noting the definition of variance $\mathbb{V}X := \mathbb{E}(X - \mathbb{E}X)^2$, establishing the inequality

$$\mathbb{P}(|X - \mathbb{E}X| \geq t) \quad \leq \quad \frac{\mathbb{V}X}{t^2},$$

for all $t > 0$.

(P.2) *Prove the expected value of X when $X \sim$ Poisson(λ) for $\lambda \in (0, \infty)$.*

By definition,

$$\mathbb{E}X \quad = \quad \sum_{x=0}^{\infty} x \, \mathbb{P}(X = x) \quad = \quad \sum_{x=0}^{\infty} x \, \frac{\lambda^x \exp(-\lambda)}{x!}, \tag{2}$$

substituting in the form of the probability mass function for a Poisson random variable with rate parameter $\lambda \in (0, \infty)$. Next, we manipulate and re-arrange terms in (2):

$$\sum_{x=0}^{\infty} x \, \frac{\lambda^x \exp(-\lambda)}{x!} \quad = \quad \sum_{x=1}^{\infty} x \, \frac{\lambda^x \exp(-\lambda)}{x!} \quad = \quad \sum_{x=1}^{\infty} \frac{\lambda^x \exp(-\lambda)}{(x-1)!} \quad = \quad \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} \exp(-\lambda)}{(x-1)!}.$$

A change of index allows us to write

$$\sum_{x=1}^{\infty} \frac{\lambda^{x-1} \exp(-\lambda)}{(x-1)!} \quad = \quad \sum_{x=0}^{\infty} \frac{\lambda^{x} \exp(-\lambda)}{x!} \quad = \quad 1, \tag{3}$$

recognizing the right-hand side of (3) to be $\mathbb{P}(X \in \{0, 1, 2, \ldots\})$, which has probability 1 by the axioms of probability because the support of a Poisson random variable is the set $\{0, 1, 2, \ldots\}$. Altogether, we have shown

$$\mathbb{E}X \quad = \quad \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} \exp(-\lambda)}{(x-1)!} \quad = \quad \lambda,$$

establishing λ as the expected value of X.

(P.3) *Let $\{X_n\}_{n\geq 1}$ be a sequence of Bernoulli random variables such that $\mathbb{P}(X_n = 1) = n^{-1}$. Prove that $X_n \xrightarrow{P} 0$, i.e., prove that X_n converges in probability to 0.*

The sequence $\{X_n\}_{n\geq 1}$ converges in probability to a random variable $X^\star$ if

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X^\star| \geq \epsilon) \quad = \quad 0, \qquad \text{for all } \epsilon > 0. \tag{4}$$

At least this is one definition. Let $\epsilon > 0$ and $n \in \{1, 2, \ldots\}$ be arbitrary and consider the probability

$$\mathbb{P}(|X_n - 0| \geq \epsilon) \quad = \quad \mathbb{P}(|X_n| \geq \epsilon) \quad = \quad \mathbb{P}(X_n \geq \epsilon) \quad = \quad \mathbb{P}(X_n = 1),$$

which follows because $X_n \in \{0, 1\}$ so that $|X_n| = X_n$ and

$$\{x \in \{0,1\} : x \geq \epsilon\} \quad = \quad \begin{cases} \{1\} & \epsilon \in (0, 1] \\ \{\} & \epsilon > 1 \end{cases}.$$

Thus, for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - 0| \geq \epsilon) \quad = \quad \lim_{n\to\infty} \mathbb{P}(X_n = 1) \quad = \quad \lim_{n\to\infty} \frac{1}{n} \quad = \quad 0,$$

showing by (4) that $X_n \xrightarrow{P} 0$ as $n \to \infty$.