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Problem Set 6
Due Date: 10/10 at 11:35 AM

Throughout, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{B}$ denote the Borel σ-algebra on $\mathbb{R}$.

(P.1) *Consider a $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable and differentiable function $f : \mathbb{R} \mapsto \mathbb{R}$. Show that the derivative f' of f is also $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable.*

Proof. Since f is differentiable on $\mathbb{R}$, then f' exists for every $x \in \mathbb{R}$, and differentiability implies f is continuous on $\mathbb{R}$. The derivative $f^{'}(x)$ at a point $x \in \mathbb{R}$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Consider an arbitrary sequence $\{h_n\}$ in $\mathbb{R}$ such that $h_n \to 0$ as $n \to \infty$. We rewrite the above equation to show

$$f_n'(x) = \lim_{n \to \infty} \frac{f(x+h_n) - f(x)}{h_n}$$

for $n \in \mathbb{N}$. Since $f(x)$ is continuous on $\mathbb{R}$ for all $x \in \mathbb{R}$, and $x + h_n$ is defined on $\mathbb{R}$, then we have $f(x+h_n)$ is continuous on $\mathbb{R}$. Because h_n is a scalar and continuity is preserved under linear transformations, then $\frac{f(x+h_n)-f(x)}{h_n}$ is continuous and therefore Borel measurable (by Theorem 8.3). Since f' exists and measurability is preserved under limits, then f' is $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable. $\square$

(P.2) *Consider a measurable function $f : \mathbb{R} \times \Omega \mapsto \mathbb{R}$. Assume that*

- *The function f is integrable with respect to $\mathbb{P}$ for all $t \in \mathbb{R}$, i.e.,*

$$\int_{\Omega} |f(t,\omega)| \, d\mathbb{P}(\omega) \quad < \quad \infty, \qquad t \in \mathbb{R}.$$

- *The partial derivatives $\frac{\partial f(t,\omega)}{\partial t}$ exist, are continuous in t, and are bounded (all statement $\mathbb{P}$-.a.s.).*

Prove that

$$\frac{\partial}{\partial t} \int_{\Omega} f(t,\omega) \, d\mathbb{P}(\omega) \quad = \quad \int_{\Omega} \frac{\partial f(t,\omega)}{\partial t} \, d\mathbb{P}(\omega),$$

i.e., you can interchange the order of differentiation and integration for all $t \in \mathbb{R}$.

Proof. Consider an arbitrary sequence $\{h_n\}$ such that $h_n \to 0$ as $n \to \infty$. Since the partial derivatives are continuous in t, we can use the Mean Value Theorem to express the partial derivatives as

$$\frac{\partial f(t,\omega)}{\partial t} = \lim_{n\to\infty} \frac{f(t+h_n,\omega) - f(t,\omega)}{h_n} = \lim_{n\to\infty} f_n(t,\omega).$$

Then for each $\omega \in \Omega$ we have defined a sequence $f_n(t,\omega)$ that converges to $\frac{\partial f(t,\omega)}{\partial t}$ as $n \to \infty$. Since the partial derivatives are integrable and bounded, then there exists an integrable function $g(\omega) \in \mathcal{L}^1$ such that

$$|f_n(t,\omega)| \leq g(\omega)$$

for all $n \in \mathbb{N}$, $t \in \mathbb{R}$, $\omega \in \Omega$. Therefore by the Dominated Convergence Theorem we show

$$\begin{aligned}
\frac{\partial}{\partial t} \int_{\Omega} f(t,\omega) \, \mathrm{d}\,\mathbb{P}(\omega) &= \lim_{n\to\infty} \int_{\Omega} f_n(t,\omega) d\mathbb{P}(\Omega) \\
&= \int_{\Omega} \lim_{n\to\infty} f_n(t,\omega) d\mathbb{P}(\Omega) \\
&= \int_{\Omega} \frac{\partial f(t,\omega)}{\partial t} d\mathbb{P}(\omega).
\end{aligned}$$

□

(P.3) *Let $X : \Omega \mapsto (0, \infty)$ be a random variable. Prove that*

$$\lim_{n\to\infty} \int_{\Omega} n \, \log\left(1 + \frac{X(\omega)}{n}\right) \, d\mathbb{P}(\omega) \quad = \quad \int_{\Omega} X(\omega) \, d\mathbb{P}(\omega).$$

Proof. We can define the sequence

$$X_n = n \log\left(1 + \frac{X(\omega)}{n}\right) = \log\left(1 + \frac{X(\omega)}{n}\right)^n$$

Taking the limit of the expression, we show

$$\lim_{n\to\infty} \left(1 + \frac{X(\omega)}{n}\right)^n = e^{X(\omega)}$$

Thus as $n \to \infty$, we have

$$\lim_{n\to\infty} X_n = \lim_{n\to\infty} \log\left(1 + \frac{X(\omega)}{n}\right)^n = \log e^{X(\omega)} = X(\omega).$$

Since log functions are non-negative, then $X_n \geq 0$ for each random variable X_n. We know $e^{X(\omega)}$ increases as each n increases, meaning $X_n \uparrow X$ as $n \to \infty$. Thus we satisfy the conditions for the Monotone Convergence Theorem, which we apply to show

$$\lim_{n\to\infty} \int_{\Omega} X_n(\omega) d\mathbb{P}(\omega) = \lim_{n\to\infty} \int_{\Omega} n \log\left(1 + \frac{X(\omega)}{n}\right) d\mathbb{P}(\omega) = \int_{\Omega} X(\omega) d\mathbb{P}(\omega).$$

$\square$

(P.4) *Let $X : \Omega \mapsto \mathbb{R}$ be a random variable such that $X \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{E}\,X = 1$. Define a set function $\mathbb{Q} : \mathcal{A} \mapsto \mathbb{R}$ by $\mathbb{Q}(A) := \mathbb{E}[X\,\mathbb{1}_A]$ for all $A \in \mathcal{A}$. Show that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{A})$.*

Proof. First, by assumption we show

$$\mathbb{Q}(\Omega) = \mathbb{E}\{X\mathbb{1}_\Omega\} = \mathbb{E}\{X\} = 1.$$

Next, we define a countable sequence $(A_n)_{n\geq 1}$ of pairwise disjoint events of $\mathcal{A}$ such that $A_n \cap A_m = \emptyset$ for $n \neq m$. Then

$$\begin{aligned}
\mathbb{Q}(\cup_{n=1}^{\infty} A_n) &= \mathbb{E}\{X\mathbb{1}_{\cup_{n=1}^{\infty} A_n}\} \\
&= \mathbb{E}\{X\mathbb{P}(\cup_{n=1}^{\infty} A_n)\} \text{ and since the } A_n \text{ are pairwise disjoint then} \\
&= \mathbb{E}\{X\sum_{n=1}^{\infty}\mathbb{P}(A_n)\} = \sum_{n=1}^{\infty}\mathbb{E}\{X\mathbb{P}(A_n)\} = \sum_{n=1}^{\infty}\mathbb{E}\{X\mathbb{1}_{A_n}\} = \sum_{n=1}^{\infty}\mathbb{Q}(A_n).
\end{aligned}$$

Thus the axiom of countable additivity is satisfied and therefore $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{A})$. □

(P.5) *Consider the previous problem. Show that if $\mathbb{P}(A) = 0$, then $\mathbb{Q}(A) = 0$ for all $A \in \mathcal{A}$. Give an example which shows that $\mathbb{Q}(A) = 0$ does not in general imply that $\mathbb{P}(A) = 0$.*

Proof. If $\mathbb{P}(A) = 0$, then

$$\mathbb{Q}(A) = \mathbb{E}\{X\mathbb{1}_A\} = \mathbb{E}\{X\mathbb{P}(A)\} = \mathbb{E}\{0\} = 0.$$

Now suppose $\Omega = [0, 1]$ such that $X : [0, 1] \mapsto \mathbb{R}$. Let $A = [\frac{1}{2}, 1]$, $A \in \mathcal{A}$ and define

$$X(\omega) = \begin{cases} 2 \text{ if } \omega \notin A \\ 0 \text{ if } \omega \in A \end{cases}.$$

Then $\mathbb{P}(A) = \frac{1}{2}$ and we can calculate the expectation of X to be

$$\mathbb{E}\{X\} = \int\limits_A X(\omega)d\mathbb{P}(\omega) + \int\limits_{\Omega\backslash A} X(\omega)d\mathbb{P}(\omega) = \int\limits_{1/2}^{1} 0d\mathbb{P}(\omega) + \int\limits_{0}^{1/2} 2d\mathbb{P}(\omega) = 0 + 2(\frac{1}{2} - 0) = 1.$$

Therefore the assumption $\mathbb{E}\{X\} = 1$ still holds. Note that $\mathbb{1}_A(\omega) = 1$ when $\omega \in A$. We can solve for $\mathbb{Q}(A)$ to show

$$\mathbb{Q}(A) = \mathbb{E}\{X\mathbb{1}_A\} = \int\limits_A X(\omega)\mathbb{1}_A d\mathbb{P}(\omega) = \int\limits_A (0)(1)d\mathbb{P}(\omega) = 0.$$

Thus $\mathbb{Q}(A) = 0$ when $\mathbb{P}(A) \neq 0$. Therefore we have shown that $\mathbb{Q}(A) = 0$ does not necessarily imply that $\mathbb{P}(A) = 0$.

□

(P.6) *Let $(A_n)_{n\in\mathbb{Z}^+}$ be a sequence of events in $\mathcal{A}$ with $\mathbb{P}(A_n) \to 0$ as $n \to \infty$. Let $X : \Omega \mapsto \mathbb{R}$ be an integrable function, i.e., $X \in \mathcal{L}_1$. Show that $\mathbb{E}[X\, \mathbb{1}_{A_n}] \to 0$ as $n \to \infty$.*

Proof. If $X \in \mathcal{L}_1$, we know

$$\mathbb{E}\{X^+\} < \infty \text{ and } \mathbb{E}\{X^-\} < \infty,$$

Since expectation is linear, we have

$$\mathbb{E}\{X\} = \mathbb{E}\{X^+\} - \mathbb{E}\{X^-\},$$

where $\mathbb{E}\{X\} \geq 0$, and X^+, X^- are positive and integrable. Thus we can write

$$\mathbb{E}\{X\mathbb{1}_{A_n}\} = \mathbb{E}\{X^+\mathbb{1}_{A_n}\} - \mathbb{E}\{X^-\mathbb{1}_{A_n}\}.$$

Since $0 \leq \mathbb{1}_{A_n} \leq 1$, then

$$0 \leq X^+\mathbb{1}_{A_n} \leq X^+ \text{ and } 0 \leq X^-\mathbb{1}_{A_n} \leq X^-.$$

By assumption, $\mathbb{P}(A_n) \to 0$ implies $\mathbb{1}_{A_n} \to 0$ as $n \to \infty$. Therefore by the Dominated Convergence Theorem we show

$$\mathbb{E}\{X^+\mathbb{1}_{A_n}\} \to \mathbb{E}\{0\} = 0 \text{ and } \mathbb{E}\{X^-\mathbb{1}_{A_n}\} \to \mathbb{E}\{0\} = 0$$

as $n \to \infty$. Hence $\mathbb{E}\{X\mathbb{1}_{A_n}\} \to \infty$ as $n \to \infty$.

□

(P.7) *Consider the previous problem. Is the proof easier if we know $X \in \mathcal{L}_2$, as opposed to just $X \in \mathcal{L}_1$?*

Proof. Yes. If $X \in \mathcal{L}_2$, then we can simplify the proof by using the Cauchy-Schwarz inequality to show

$$|\mathbb{E}\{X\mathbb{1}_{A_n}\}| \leq \sqrt{\mathbb{E}\{X^2\}\mathbb{E}\{(\mathbb{1}_{A_n})^2\}} = \sqrt{\mathbb{E}\{X^2\}\mathbb{E}\{\mathbb{1}_{A_n}\}} = \sqrt{\mathbb{E}\{X^2\}\mathbb{P}(A_n)}.$$

Since $|\mathbb{E}\{X\mathbb{1}_{A_n}\}|$ is bounded and $\mathbb{P}(A_n) \to 0$ as $n \to \infty$ by assumption, then

$$\mathbb{E}\{X\mathbb{1}_{A_n}\} \to 0.$$

□