**Name:** Jenny Petrova

**Problem Set 8**

**Due Date: 10/31 at 11:35 AM**

**(P.1)** *Let $p \in (0,1)$ and consider a simple function $f : (\mathbb{R}, \mathcal{B}) \mapsto (\mathbb{R}, \mathcal{B})$ defined by*

$$f(x) \quad = \quad \sum_{k=0}^{\infty} p\,(1-p)^k\, \mathbb{1}_{\{k\}}(x), \qquad x \in \mathbb{R}.$$

*Define a measure $\mu : \mathcal{B} \mapsto [0, \infty]$ by*

$$\mu(A) \quad := \quad \int_A f(x)\ d\nu(x), \qquad A \in \mathcal{B},$$

*where $\nu$ is the counting measure. Show that $(\mathbb{R}, \mathcal{B}, \mu)$ is a probability space.*

*Proof.* We use the defined measure $\mu$ and simple function $f$ and apply Lebesgue integration to show

$$\begin{aligned}
\mu(A) &= \int_A f(x)\,\mathrm{d}\,\nu(x) \\
&= \int_A \left( \sum_{k=0}^{\infty} p(1-p)^k \mathbb{1}_{\{k\}}(x) \right) \mathrm{d}\,\nu(x) \\
&= \sum_{k=0}^{\infty} \int_A p(1-p)^k \mathbb{1}_{\{k\}}(x)\,\mathrm{d}\,\nu(x) \\
&= \sum_{k=0}^{\infty} \int_{\mathbb{R}} p(1-p)^k \mathbb{1}_{A\cap\{k\}}(x)\,\mathrm{d}\,\nu(x) = \sum_{k=0}^{\infty} p(1-p)^k \int_{\mathbb{R}} \mathbb{1}_{A\cap\{k\}}(x)\,\mathrm{d}\,\nu(x) \\
&= \sum_{k=0}^{\infty} p(1-p)^k\ \nu(A \cap \{k\}) \\
&= \sum_{k=0}^{\infty} p(1-p)^k\ |A \cap \{k\}| \\
&= \sum_{k=0}^{\infty} p(1-p)^k \mathbb{1}_{\{A\}}(k) \\
&= \sum_{k\in A} p(1-p)^k,
\end{aligned}$$

where $v(x)$ is the counting measure, since $f$ is a discrete random variable. Thus by the definition of the sum of an infinite geometric series,

$$\mu(\mathbb{R}) = \sum_{k=0}^{\infty} p(1-p)^k \mathbb{1}_{\{\mathbb{R}\}}(k) = \sum_{k=0}^{\infty} p(1-p)^k = \frac{p}{1-(1-p)} = \frac{p}{p} = 1.$$

Hence by Definition 11.1 the measure $\mu$ is a probability measure defined on the space $(\mathbb{R}, \mathcal{B})$. Therefore $(\mathbb{R}, \mathcal{B}, \mu)$ is a probability space. $\square$

**(P.2)** *Let $\lambda > 0$ and consider the non-negative function $f : (\mathbb{R}, \mathcal{B}) \mapsto (\mathbb{R}, \mathcal{B})$ defined by*

$$f(x) \quad = \quad \lambda \exp(-\lambda\, x)\, \mathbb{1}_{[0,\infty)}(x), \qquad x \in \mathbb{R}.$$

*Define a measure $\mu : \mathcal{B} \mapsto [0, \infty]$ by*

$$\mu(A) \quad \coloneqq \quad \int\limits_A f(x)\ dm(x), \qquad A \in \mathcal{B},$$

*where $m$ is the Lebesgue measure. Show that $(\mathbb{R}, \mathcal{B}, \mu)$ is a probability space.*

*Proof.* Similar to the proof above, we use Lebesgue integration to solve

$$\begin{aligned}
\mu(A) &= \int\limits_A f(x)\,\mathrm{d}\,m(x) \\
&= \int\limits_A \lambda e^{-\lambda x} \mathbb{1}_{[0,\infty)}(x)\,\mathrm{d}\,m(x) \\
&= \int\limits_{\mathbb{R}} \lambda e^{-\lambda x} \mathbb{1}_{A\cap[0,\infty)}(x)\,\mathrm{d}\,m(x) \\
&= \int\limits_{A\cap[0,\infty)} \lambda e^{-\lambda x}\,\mathrm{d}\,m(x),
\end{aligned}$$

where $m(x)$ is the Lebesgue measure, since $f$ is a continuous random variable. Then

$$\mu(\mathbb{R}) = \int\limits_{[0,\infty)} \lambda e^{-\lambda x}\,\mathrm{d}\,m(x)$$

and since $\lambda e^{-\lambda x}$ is positive and continuous on $[0, \infty)$, by Theorem 12.2 we can evaluate using Riemann integration to show

$$\begin{aligned}
\int\limits_{[0,\infty)} \lambda e^{-\lambda x}\,\mathrm{d}\,m(x) &= \int\limits_0^\infty \lambda e^{-\lambda x}\,\mathrm{d}\,x \\
&= \lim_{x\to\infty} (-e^{-\lambda x}) - (-e^0) \\
&= 0 + 1 \\
&= 1.
\end{aligned}$$

Hence by Definition 11.1 the measure $\mu$ is a probability measure defined on the measurable space $(\mathbb{R}, \mathcal{B})$. Therefore $(\mathbb{R}, \mathcal{B}, \mu)$ is a probability space. $\square$

**(P.3)** *Construct the probability space for the Binomial distribution by defining the probability measure through Lebesgue integration and demonstrate how to calculate the probabilities of events from the Lebesgue integral. Explain each step.*

*Proof.* Consider the Binomial distribution $B(n,p)$ for a function $f:(\Omega,\mathcal{A})\to(\mathbb{R},\mathcal{B})$ defined by

$$f(x)=\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\mathbb{1}_{\{k\}}(x),$$

where $n=\{1,2,\dots\}$ is the number of trials, $k\in\{0,1,\dots,n\}$ is the number of successes, $p\in[0,1]$ is the success probability for each trial, and the indicator function is defined by

$$\mathbb{1}_{\{k\}}=\begin{cases}1 \text{ if } x=k\\ 0 \text{ otherwise}\end{cases}.$$

We define the state space to be the number of possible successes in $n$ trials, given by $\Omega=\{0,1,\dots,n\}$. The state space has a finite number of outcomes, since $|\Omega|=n+1$. We define the $\sigma-$algebra $\mathcal{A}$ to be the set of all subsets of $\Omega$, i.e. $\mathcal{A}=\mathcal{P}(\Omega)$. For any $A\in\mathcal{A}$, we define the probability measure $\mu:\mathcal{B}\mapsto[0,\infty]$ using Lebesgue integration:

$$\begin{aligned}
\mu(A)&=\int_A f(x)\,\mathrm{d}\,\nu(x)\\
&=\int_A\left(\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\mathbb{1}_{\{k\}}(x)\right)\mathrm{d}\,\nu(x)\\
&=\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\int_A \mathbb{1}_{\{k\}}(x)\,\mathrm{d}\,\nu(x)\\
&=\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\int_{\mathbb{R}} \mathbb{1}_{A\cap\{k\}}(x)\,\mathrm{d}\,\nu(x)\\
&=\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\ \nu(A\cap\{k\})\\
&=\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\ |A\cap\{k\}|\\
&=\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\mathbb{1}_{\{A\}}(k)\\
&=\sum_{k\in A}\binom{n}{k}p^k(1-p)^{n-k}.
\end{aligned}$$

where $\nu$ is the counting measure. Consider a sequence of countable pairwise disjoint sequence of events $(A_i)_{i\in I}$ in $\Omega$. We show

$$\mu(\cup_{i=1}^{\infty}A_i)=\sum_{k\in\cup_{i=1}^{\infty}A_i}\binom{n}{k}p^k(1-p)^{n-k}=\sum_{i=1}^{\infty}\sum_{k\in A_i}\binom{n}{k}p^k(1-p)^{n-k}=\sum_{i=1}^{\infty}\mu(A_i).$$

Therefore the measure $\mu$ satisfies countable additivity. When $A = \Omega$, we show

$$\mu(\Omega) = \sum_{k\in\Omega} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1.$$

Hence $\mu(\Omega) = 1$ and therefore $\mu$ is a probability measure on $(\mathbb{R}, \mathcal{B})$. Now, suppose $A = \{k\}$. We calculate the probability of event $A$ by

$$\mu(A) = \sum_{\{k\}} \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k},$$

which is defined to be the probability mass function of the Binomial distribution. If we suppose $A = \{k_1, k_2, \ldots, k_m\}$ has multiple outcomes, then the probability of event $A$ is

$$\mu(A) = \int\limits_{\{k_1,k_2,\ldots,k_n\}} f(x)\,\mathrm{d}\,\nu(X) = \sum_{i=1}^{m} \binom{n}{k_i} p^{k_i} (1-p)^{n-k_i},$$

the sum of probabilities of the individual outcomes in $A$. $\square$

**(P.4)** *Construct the probability space for the Laplace distribution by defining the probability measure through Lebesgue integration and demonstrate how to calculate the probabilities of events from the Lebesgue integral. Explain each step.*

*Proof.* Consider a Laplace$(a, b)$ distribution for a function $f : (\mathbb{R}, \mathcal{B}) \mapsto (\mathbb{R}, \mathcal{B})$ defined by

$$f(x) = \frac{1}{2b} e^{-\frac{|x-a|}{b}},$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}$, $b > 0$. For any $A \in \mathcal{A}$, we define the probability measure $\mu : \mathcal{B} \mapsto [0, \infty]$ of the Laplace distribution using Lebesgue integration:

$$\begin{aligned}\mu(A) &= \int_A f(x) \, \mathrm{d}\, m(x) \\ &= \int_A \frac{1}{2b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, m(x),\end{aligned}$$

where $m(x)$ is the Lebesgue measure. Consider a sequence of countable pairwise disjoint sequence of events $(A_i)_{i \in I}$ in $\mathbb{R}$. Then

$$\mu(\cup_{i=1}^{\infty} A_i) = \int_{\cup_{i=1}^{\infty} A_i} \frac{1}{2b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, m(x) = \sum_{i=1}^{\infty} \int_A \frac{1}{2b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, m(x) = \sum_{i=1}^{\infty} \mu(A_i).$$

Therefore the measure $\mu$ satisfies countable additivity. When $A = \mathbb{R}$, we show

$$\mu(\mathbb{R}) = \int_{\mathbb{R}} \frac{1}{2b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, m(x) = \int_{-\infty}^{\infty} \frac{1}{2b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, m(x).$$

Since $f(x)$ is a positive and continuous function on $[0, \infty)$, then $f(x)$ is Riemann integrable. We can split the integral to handle the cases where $x \geq a$ and $x < a$ and solve directly. When $x \geq a$:

$$\lim_{x \to \infty} \int_a^x \frac{1}{2b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, x = \frac{1}{2} \lim_{x \to \infty} \int_a^x \frac{1}{b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, x = \frac{1}{2} \left( \lim_{x \to \infty} (-e^{-\frac{x-a}{b}}) + 1 \right) = \frac{1}{2} (0 + 1) = \frac{1}{2}.$$

We also solve when $x < a$:

$$\lim_{x \to -\infty} \int_x^a \frac{1}{2b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, x = \frac{1}{2} \lim_{x \to -\infty} \int_x^a \frac{1}{b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, x = \frac{1}{2} \left( 1 - \lim_{x \to -\infty} (e^{-\frac{a-x}{b}}) \right) = \frac{1}{2} (1 - 0) = \frac{1}{2}.$$

Therefore

$$\mu(\mathbb{R}) = \int_{-\infty}^{\infty} \frac{1}{2b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, m(x) = \int_{-\infty}^{\infty} \frac{1}{2b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, x = \int_{-\infty}^{a} \frac{1}{2b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, x + \int_a^{\infty} \frac{1}{2b} e^{-\frac{|x-a|}{b}} \, \mathrm{d}\, x = \frac{1}{2} + \frac{1}{2} = 1.$$

Hence $\mu(\mathbb{R}) = 1$ and therefore $\mu$ is a probability measure on $(\mathbb{R}, \mathcal{B})$. Now, suppose $A = [c, d]$, where $c \leq d$ for $A \in \mathcal{B}$. If $a \leq c \leq d$, we calculate the probability of event $A$

by

$$\mu([c,d]) = \int\limits_c^d \frac{1}{2b} e^{-\frac{x-a}{b}} \,\mathrm{d}(x) = \left[-\frac{1}{2} e^{-\frac{x-a}{b}}\right]_c^d = -\frac{1}{2}\left(e^{-\frac{d-a}{b}} - e^{-\frac{c-a}{b}}\right).$$

If $c < a \leq d$, we calculate the probability of event $A$ by

$$\mu([c,d]) = \left[\frac{1}{2} e^{\frac{x-a}{b}}\right]_c^a + \left[-\frac{1}{2} e^{-\frac{x-a}{b}}\right]_a^d = 1 - \frac{1}{2}\left(e^{-\frac{c-a}{b}} + e^{-\frac{d-a}{b}}\right)$$

If $c \leq d \leq a$, we calculate the probability of event $A$ by

$$\mu([c,d]) = \left[\frac{1}{2} e^{\frac{x-a}{b}}\right]_c^d = \frac{1}{2}(e^{-\frac{d-a}{b}} - e^{-\frac{c-a}{b}}).$$

$\square$

**(P.5)** *Consider a measurable space* $(\mathbb{R}, \mathcal{B})$ *and let* $\mu : \mathcal{B} \mapsto [0, \infty]$ *and* $\nu : \mathcal{B} \mapsto [0, \infty]$ *be two measures defined on the space. Show that the set function* $\gamma : \mathcal{B} \mapsto [0, \infty]$ *defined by* $\gamma(A) := \mu(A) + \nu(A)$ *is a measure on* $(\mathbb{R}, \mathcal{B})$.

*Proof.* We need to prove that the set function $\gamma$ is a measure on $(\mathbb{R}, \mathcal{B})$.

First, we need to show $\gamma$ satisfies non-negativity. Since $\mu$ is a measure, then $\mu(A) \geq 0$ for all $A \in \mathcal{B}$. Likewise, since $\nu$ is a measure, then $\nu(A) \geq 0$ for all $A \in \mathcal{B}$. Therefore $\gamma(A) = \mu(A) + \nu(A) \geq 0$ for $A \in \mathcal{B}$.

Next, since $\mu$ is a measure, then $\mu(\emptyset) = 0$, and since $\nu$ is a measure then $\nu(\emptyset) = 0$. Hence $\gamma(\emptyset) = \mu(\emptyset) + \nu(\emptyset) = 0$.

Lastly, suppose we have a sequence of pairwise disjoint events $(A_n)_{n\geq 1}$ in $\mathcal{B}$. Since $\mu$ and $\nu$ are measures that satisfy countable additivity, then $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ and $\nu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n)$. Then

$$\begin{aligned}
\gamma(\cup_{n=1}^{\infty} A_n) &= \mu(\cup_{n=1}^{\infty} A_n) + \nu(\cup_{n=1}^{\infty} A_n) \\
&= \sum_{n=1}^{\infty} \mu(A_n) + \sum_{n=1}^{\infty} \nu(A_n) \text{ and by linearity of summation} \\
&= \sum_{n=1}^{\infty} (\mu(A_n) + \nu(A_n)) \\
&= \sum_{n=1}^{\infty} \gamma(A_n).
\end{aligned}$$

Therefore $\gamma$ is a measure on the space $(\mathbb{R}, \mathcal{B})$. $\square$

**(P.6)** *Consider modeling the amount of time spent per day calling your grandmother. Most people do not talk to their grandmother every day. As such, we need a probability model which places probability mass at 0 and also has a distribution function which is continuous on $(0, \infty)$ to model the amount of time spent on the phone. For concreteness, assume the amount of time you spend talking to your grandmother on any given call follows the exponential distribution with rate parameter .1 and that the probability you talk to your grandmother on a given day is .05.*

*To accomplish this, you will construct a probability measure by specifying a function $f : \mathbb{R} \mapsto [0, \infty)$ and then defining the desired probability measure through the Lebesgue integration of the function $f$ with respect to the proper choice of reference measure.*

*Proof.* The time you spend talking on the phone with your grandmother (per day) follows the exponential distribution. So we consider the function $f : \mathbb{R} \mapsto [0, \infty)$ defined by

$$f(x) = 0.1e^{-0.1x}\mathbb{1}_{(0,\infty)}.$$

The probability you talk to your grandmother on a given day is .05. Therefore, the probability you do *not* talk to your grandmother on a given day is .95. Hence we must place a probability mass at 0 (i.e. 0 time spent talking on the phone). Consider the measure $\gamma : \mathcal{B} \mapsto [0, \infty)$ defined by $\gamma(A) := .95\,\mathbb{1}_{\{0\}}(A) + .05\,\mu(A)$. We use the Lebesgue integration of the function $f$ to define

$$\begin{aligned}
\gamma(A) &= .95\,\mathbb{1}_{\{0\}}(A) + .05 \int_A 0.1e^{-0.1x}\mathbb{1}_{(0,\infty)}\,\mathrm{d}\,m(x) \\
&= .95\,\mathbb{1}_{\{0\}}(A) + .05 \int_{A\cap(0,\infty)} 0.1e^{-0.1x}\,\mathrm{d}\,m(x),
\end{aligned}$$

where $m(x)$ is the Lebesgue measure. Since $f$ is positive and continuous on $[0, \infty)$, we can solve using Riemann integration. Thus when $A = \mathbb{R}$, we show

$$\begin{aligned}
\gamma(\mathbb{R}) &= .95\,\mathbb{1}_{\{0\}}(\mathbb{R}) + .05 \int_{\mathbb{R}\cap(0,\infty)} 0.1e^{-0.1x}\,\mathrm{d}\,m(x) \\
&= .95\,(1) + .05 \int_0^\infty 0.1e^{-0.1x}\,\mathrm{d}\,x \\
&= .95 + .05\left(\lim_{x\to\infty}(-e^{-0.1x}) + 1\right) \\
&= .95 + .05(0 + 1) \\
&= 1.
\end{aligned}$$

Hence $\gamma(\mathbb{R}) = 1$ and therefore $\gamma$ is a probability measure on $(\mathbb{R}, \mathcal{B})$. $\square$