**Name:** Jenny Petrova

**Problem Set 7**

**Due Date: 10/15 at 11:35 AM**

Throughout, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{B}$ denote the Borel $\sigma$-algebra on $\mathbb{R}$.

**(P.1)** *Given $A \in \mathcal{A}$, we define the integral of a measurable function $f : \Omega \mapsto \mathbb{R}$ over $A$ to be*

$$\int\limits_A f(\omega)\ d\mathbb{P}(\omega) \ \ := \ \ \int\limits_\Omega f(\omega)\, \mathbb{1}_A(\omega)\ d\mathbb{P}(\omega).$$

*Show that if $f$ is a simple function of the form*

$$f(\omega) \ \ = \ \ \sum_{k=1}^{m} \alpha_k\, \mathbb{1}_{A_k}(\omega),$$

*with $A_k \in \mathcal{A}$ ($k \in \{1, \ldots, m\}$), then*

$$\int\limits_A f(\omega)\ d\mathbb{P}(\omega) \ \ = \ \ \sum_{k=1}^{m} \alpha_k\, \mathbb{P}(A_k \cap A), \qquad A \in \mathcal{A}.$$

*Prove that if $f$ is a non-negative function (not necessarily simple), then there exists a sequence of non-negative simple functions $(f_n)_{n\in\mathbb{Z}^+}$ such that $f_n \uparrow f$ and*

$$\int\limits_A f_n(\omega)\ d\mathbb{P}(\omega) \ \ \uparrow \ \ \int\limits_A f(\omega)\ d\mathbb{P}(\omega), \qquad A \in \mathcal{A}.$$

*Discuss the extension to general functions $f$ which are not necessarily non-negative.*

*Proof.* We substitute the expression for a simple function $f$ into the integral of $f$:

$$\begin{aligned}\int\limits_\Omega f(\omega)\mathbb{1}_A(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) &= \int\limits_\Omega (\sum_{k=1}^{m} \alpha_k \mathbb{1}_{A_k}(\omega))\mathbb{1}_A(\omega)\,\mathrm{d}\,\mathbb{P}(\omega)\\ &= \sum_{k=1}^{m} \alpha_k \int\limits_\Omega \mathbb{1}_{A_k}(\omega)\mathbb{1}_A(\omega)\,\mathrm{d}\,\mathbb{P}(\omega)\end{aligned}$$

Note $\mathbb{1}_{A_k}\mathbb{1}_A = 1$ if and only if $\omega \in (A_k \cap A)$. So $\mathbb{1}_{A_k}\mathbb{1}_A = \mathbb{1}_{A_k \cap A}$ and we can write

$$\sum_{k=1}^{m} \alpha_k \int\limits_\Omega \mathbb{1}_{A_k \cap A}\,\mathrm{d}\,\mathbb{P}(\omega) = \sum_{k=1}^{m} \alpha_k \mathbb{P}(A_k \cap A).$$

Since $f$ is a non-negative (not necessarily simple) function, then by Theorem 9.1 we know that for every non-negative measurable function we can construct a sequence of non-negative simple functions $(f_n)_{n\in\mathbb{Z}^+}$ such that $f_n \uparrow f$ as $n \to \infty$. Thus by the Monotone Convergence Theorem we directly satisfy

$$\int\limits_A f_n(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) \uparrow \int\limits_A f(\omega)\,\mathrm{d}\,\mathbb{P}(\omega).$$

Let $f = f^+ - f^-$ for a general function $f$ (that is not necessarily non-negative). Then we can construct

$$\int\limits_A f(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) = \int\limits_A f^+(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) - \int\limits_A f^-(\omega)\,\mathrm{d}\,\mathbb{P}(\omega)$$

where $f^+$, $f^-$ are non-negative functions. We can extend the result above to define sequences of non-negative simple functions $(f_n^+)_{n\in\mathbb{Z}^+}$, $(f_n^-)_{n\in\mathbb{Z}^+}$ such that $f_n^+ \uparrow f^+$ and $f_n^- \uparrow f^-$ as $n \to \infty$. We apply the Monotone Convergence Theorem again to show

$$\left(\int\limits_A f_n^+(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) - \int\limits_A f_n^-(\omega)\,\mathrm{d}\,\mathbb{P}(\omega)\right) \uparrow \left(\int\limits_A f^+(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) - \int\limits_A f^-(\omega)\,\mathrm{d}\,\mathbb{P}(\omega)\right),$$

hence

$$\int\limits_A f_n(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) \uparrow \int\limits_A f(\omega)\,\mathrm{d}\,\mathbb{P}(\omega)$$

when $f$ is not necessarily non-negative. □

**(P.2)** *Let $f : \Omega \mapsto \mathbb{R}$ be an integrable function and $N \in \mathcal{A}$ be a null set of $\mathbb{P}$. Show that*

$$\mathbb{E}\,f \;=\; \int\limits_{\Omega\setminus N} f(\omega)\;d\mathbb{P}(\omega).$$

*Note: You must show this for a general $f$.*

*Proof.* Since $N \in \mathcal{A}$ is a null set of $\mathbb{P}$, then $\mathbb{P}(N) = 0$. Hence

$$\int\limits_N f(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) = \int\limits_\Omega f(\omega)\mathbb{1}_N\,\mathrm{d}\,\mathbb{P}(\omega) = 0.$$

Since $\Omega = (\Omega \setminus N) \cup N$, we use the definition of the expectation of $f$ to show

$$\mathbb{E}f = \int\limits_\Omega f(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) = \int\limits_{\Omega\setminus N} f(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) + \int\limits_N f(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) = \int\limits_{\Omega\setminus N} f(\omega)\,\mathrm{d}\,\mathbb{P}(\omega).$$

□

**(P.3)** *Let $(X_n)_{n\in\mathbb{Z}^+}$ be a sequence of random variables in $\mathcal{L}_1$ which converges ($\mathbb{P}$-a.s.) to a random variable $X \in \mathcal{L}_1$. Show, for all $A \in \mathcal{A}$, that*

$$\int\limits_A \left(\lim_{n\to\infty} X_n(\omega)\right) \, d\mathbb{P}(\omega) \quad = \quad \int\limits_A X(\omega) \, d\mathbb{P}(\omega).$$

*Note: You cannot assume that $X_n(\omega) \to X(\omega)$ for all $\omega \in A$.*

*Proof.* For any fixed set $A$, we can define the integral of $X_n$ over $A$ as

$$\int\limits_A X_n(\omega) \, \mathrm{d}\,\mathbb{P}(\omega) = \int\limits_\Omega X_n(\omega) \mathbb{1}_A \, \mathrm{d}\,\mathbb{P}(\omega) = \int\limits_\Omega X_n^A(\omega) \, \mathrm{d}\,\mathbb{P}(\omega)$$

and similarly for $X$,

$$\int\limits_A X(\omega) \, \mathrm{d}\,\mathbb{P}(\omega) = \int\limits_\Omega X(\omega) \mathbb{1}_A \, \mathrm{d}\,\mathbb{P}(\omega) = \int\limits_\Omega X^A(\omega) \, \mathrm{d}\,\mathbb{P}(\omega).$$

Since $X_n \to X$ in $\mathcal{L}_1$ ($\mathbb{P}-$a.s.), then $X_n^A \to X^A$ ($\mathbb{P}$ a.s.) for all $A \in \mathcal{A}$. Since $X_n, X \in \mathcal{L}_1$, then the expectations are finite, i.e. $\mathbb{E}|X| < \infty$ and $\mathbb{E}|X_n| < \infty$ for each $n$. Since $X_n$ converges to $X$ in $\mathcal{L}_1$, then by definition of convergence in $\mathcal{L}_1$

$$\lim_{n\to\infty} \mathbb{E}|X_n - X| = 0.$$

By the triangle inequality,

$$\mathbb{E}|X_n| - \mathbb{E}|X| \leq \mathbb{E}|X_n - X|$$

Since $\mathbb{E}|X_n - X| \to 0$, then $\mathbb{E}|X_n| \to \mathbb{E}|X|$. Additionally, by rearrangement we show

$$\mathbb{E}|X_n| \leq \mathbb{E}|X_n - X| + \mathbb{E}|X|.$$

This implies $\mathbb{E}|X_n|$ is uniformly bounded, since $\mathbb{E}|X_n - X| \to 0$ and $\mathbb{E}|X| < \infty$. Since $\mathbb{E}|X_n|$ is uniformly bounded, there exists an integrable function $g(\omega) \in \mathcal{L}_1$ such that $|X_n(\omega)| \leq g(\omega)$ for all $n$ and almost every $\omega$. Hence $|X_n^A(\omega)| \leq g(\omega)$ and by the Dominated Convergence Theorem:

$$\lim_{n\to\infty} \int\limits_\Omega X_n^A(\omega) \, \mathrm{d}\,\mathbb{P}(\omega) = \int\limits_\Omega X^A(\omega) \, \mathrm{d}\,\mathbb{P}(\omega)$$

for all $A$. The above expression is equivalent to

$$\lim_{n\to\infty} \int\limits_A X_n(\omega) \, \mathrm{d}\,\mathbb{P}(\omega) = \int\limits_A \left(\lim_{n\to\infty} X_n(\omega)\right) \mathrm{d}\,\mathbb{P}(\omega) = \int\limits_A X(\omega) \, \mathrm{d}\,\mathbb{P}(\omega).$$

$\square$

**(P.4)** *Let $X$ be an integrable random variable and assume that $X \geq 0$ ($\mathbb{P}$-a.s.). Show that if $\mathbb{E}\,X = 0$, then $X = 0$ ($\mathbb{P}$-a.s.).*

*Proof.* Since $\mathbb{E}X = 0$ and $X$ is an integrable random variable, then by definition of the expectation of $X$:

$$\mathbb{E}X = \int\limits_{\Omega} X(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) = 0.$$

We will do a proof by contradiction. Suppose $\mathbb{E}X$ and $X(\omega) > 0$. Consider the case where $\mathbb{P}(A) > 0$. We define $A \subseteq \Omega$ as the set of all $X(\omega) > 0$:

$$A := \{\omega \in \Omega : X(\omega) > 0\}.$$

If $\mathbb{P}(A) > 0$, then

$$\mathbb{E}X = \int\limits_{A} X(\omega)\,\mathrm{d}\,\mathbb{P}(\omega) = \int\limits_{\Omega} X(\omega)\mathbb{1}_A\,\mathrm{d}\,\mathbb{P}(\omega) > 0.$$

This contradicts the assumption that $\mathbb{E}X = 0$. Then we must have $\mathbb{P}(A) = 0$ for all $X(\omega) \in A$. Hence the set of all $X > 0$ has measure zero. Therefore $X = 0$ $\mathbb{P}-$a.s. when $X \geq 0$ and $\mathbb{E}X = 0$.

$\square$

**(P.5)** *Let $X \in \mathcal{L}_1$ and $Y \in \mathcal{L}_1$ be random variables. Prove that*

$$\int_\Omega |X(\omega) - Y(\omega)| \; d\mathbb{P}(\omega) \quad = \quad 0$$

*if and only if $X = Y$ ($\mathbb{P}$-a.s.).*

*Proof.* For the forward direction we assume $\int_\Omega |X(\omega) - Y(\omega)| \,\mathrm{d}\,\mathbb{P}(\omega) = 0$. We will show $X = Y$. By definition of expectation,

$$\int_\Omega |X(\omega) - Y(\omega)| \,\mathrm{d}\,\mathbb{P}(\omega) = \mathbb{E}|X(\omega) - Y(\omega)| = 0.$$

By the proof in **(P.4)**, it follows that $|X(\omega) - Y(\omega)| = 0$ ($\mathbb{P}-$a.s.). Hence $X = Y$ ($\mathbb{P}-$a.s.). Next, for the backward direction we assume $X = Y$ ($\mathbb{P}-$a.s.). We will show $\int_\Omega |X(\omega) - Y(\omega)| \,\mathrm{d}\,\mathbb{P}(\omega) = 0$. If $X = Y$ ($\mathbb{P}-$a.s.), then

$$\mathbb{P}(X = Y) = \mathbb{P}(\{\omega : X(\omega) = Y(\omega)\}) = \mathbb{P}(\{\omega : X(\omega) - Y(\omega) = 0\}) = 1.$$

We can define $N := \{\omega : X(\omega) = Y(\omega)\}^c = \{\omega : X(\omega) \neq Y(\omega)\}$ to be the null set of $\mathbb{P}$. Then we use the proof in **(P.2)** to show

$$\begin{aligned}
\int_\Omega |X(\omega) - Y(\omega)| \,\mathrm{d}\,\mathbb{P}(\omega) &= \int_{\Omega \setminus N} |X(\omega) - Y(\omega)| \,\mathrm{d}\,\mathbb{P}(\omega) + \int_N |X(\omega) - Y(\omega)| \,\mathrm{d}\,\mathbb{P}(\omega) \\
&= \int_{\Omega \setminus N} |X(\omega) - Y(\omega)| \,\mathrm{d}\,\mathbb{P}(\omega) + 0, \quad \text{and since } |X(\omega) - Y(\omega)| = 0 \\
&= 0.
\end{aligned}$$

Then $\int_\Omega |X(\omega) - Y(\omega)| \,\mathrm{d}\,\mathbb{P}(\omega) = 0$ when $X = Y$ ($\mathbb{P}-$a.s.).

$\square$

**(P.6)** *Let $f = (f_1, f_2) : \Omega \mapsto E \times F$. Show that $f : (\Omega, \mathcal{A}) \mapsto (E \times F, \mathcal{E} \otimes \mathcal{F})$ is measurable if and only if $f_1$ is $\langle \mathcal{A}, \mathcal{E} \rangle$-measurable and $f_2$ is $\langle \mathcal{A}, \mathcal{F} \rangle$-measurable.*

*Proof.* First, for the forward direction we assume $f : (\Omega, \mathcal{A}) \mapsto (E \times F, \mathcal{E} \otimes \mathcal{F})$ is measurable. We will show $f_1$ is $\langle \mathcal{A}, \mathcal{E} \rangle$-measurable and $f_2$ is $\langle \mathcal{A}, \mathcal{F} \rangle$-measurable. Since $f$ is measurable, then for any $A_1 \in \mathcal{E}$ we know $A_1 \times F \in \mathcal{E} \otimes \mathcal{F}$ and by definition of measurability we have

$$f^{-1}(A_1 \times F) \in \mathcal{A}.$$

Then by definition of Cartesian Product, we know

$$f^{-1}(A_1 \times F) = f_1^{-1}(A_1).$$

Hence $f_1^{-1}(A_1) \in \mathcal{A}$ for all $A_1 \in \mathcal{E}$ and we satisfy the definition of $\langle \mathcal{A}, \mathcal{E} \rangle$-measurability for $f_1$. Similarly, for any $A_2 \in \mathcal{F}$ we know $E \times A_2 \in \mathcal{E} \otimes \mathcal{F}$ and by definition of measurability we have

$$f^{-1}(E \times A_2) \in \mathcal{A}.$$

Then by definition of Cartesian Product, we know

$$f^{-1}(E \times A_2) = f_2^{-1}(A_2).$$

Hence $f_2^{-1}(A_2) \in \mathcal{A}$ for all $A_2 \in \mathcal{F}$ and we satisfy the definition of $f_2$ is $\langle \mathcal{A}, \mathcal{F} \rangle$-measurability for $f_2$.
For the backward direction we assume $f_1$ is $\langle \mathcal{A}, \mathcal{E} \rangle$-measurable and $f_2$ is $\langle \mathcal{A}, \mathcal{F} \rangle$-measurable. We will show $f : (\Omega, \mathcal{A}) \mapsto (E \times F, \mathcal{E} \otimes \mathcal{F})$ is measurable. Let $B_1 \times B_2$ be any product set such that $B_1 \in \mathcal{E}$, $B_2 \in \mathcal{F}$. Then $B_1 \times B_2 \in \mathcal{E} \otimes \mathcal{F}$. By definition of Cartesian product, we define the pre-image of $f$ as

$$f^{-1}(B_1 \times B_2) = f_1^{-1}(B_1) \cap f_2^{-1}(B_2).$$

Since $f_1$ is measurable, then $f_1^{-1}(B_1) \in \mathcal{A}$ for all $B_1 \in \mathcal{E}$. Similarly, since $f_2$ is measurable, then $f_2^{-1}(B_2) \in \mathcal{A}$ for all $B_2 \in \mathcal{F}$. Therefore

$$f_1^{-1}(B_1) \cap f_2^{-1}(B_2) \in \mathcal{A}.$$

by definition of $\sigma$-algebra. Since $f^{-1}(B_1 \times B_2) \in \mathcal{A}$ for all $B_1 \times B_2 \in \mathcal{E} \otimes \mathcal{F}$, then we satisfy the definition of measurability. Therefore $f : (\Omega, \mathcal{A}) \mapsto (E \times F, \mathcal{E} \otimes \mathcal{F})$ is measurable. $\square$

**(P.7)** *Consider a sequence* $(X_n)_{n\in\mathbb{Z}^+}$ *of random variables. Prove that* $X_n \to X$ *(*$\mathbb{P}$*-a.s.), i.e.,*

$$\mathbb{P}\left(\left\{\omega \in \Omega\ :\ \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) \;=\; 1,$$

*if there exists a sequence of positive numbers* $(\epsilon_n)_{n\in\mathbb{Z}^+}$ *with* $\epsilon_n \downarrow 0$ *such that*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\{|X_n - X| \,>\, \epsilon_n\}\right) \;<\; \infty.$$

*Proof.* Since $\epsilon_n \downarrow 0$ as $n \to \infty$, then $|X_{n+1} - X| \subseteq |X_n - X|$ for all $n$. Define the set of decreasing sequences to be $A_n := \{|X_n - X| > \epsilon_n\}$. Since $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ by assumption, then by the Borel-Cantelli Theorem for measurable spaces we have that $\mathbb{P}(\lim_{n\to\infty} \sup A_n) = 0$. We use the definition of limit superior to show

$$\mathbb{P}(\lim_{n\to\infty} \sup A_n) = \lim_{n\to\infty}\lim_{k\to\infty} \mathbb{P}(\cup_{m=n}^{k} A_m) = \lim_{n\to\infty}\lim_{k\to\infty}(1 - \mathbb{P}(\cap_{m=n}^{k} A_m^c)) = 0.$$

From the equation above, it follows that have $\lim_{n\to\infty}\lim_{k\to\infty} \mathbb{P}(\cap_{m=n}^{k} A_m^c) = 1$, where $A^c := \{0 \leq |X_n - X| \leq \epsilon_n\}$. Since $|X_n - X| \downarrow \epsilon_n$ as $n \to \infty$, then for any $k \geq n$ we have $|X_n - X| \to 0$. Therefore $X_n \to X$ ($\mathbb{P}-$a.s.), and we have

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

$\square$