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Extra Credit

Due Date: 10/15 at 11:35 AM

(P.1) *Consider a probability space* $([0,\infty), \mathcal{B}([0,\infty]), \mathbb{P})$ *and a function* $f : [0,\infty) \to [0,\infty)$ *defined by*

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and assume that $\mathbb{P}$ *is the probability measure corresponding to the Exp(1) probability distribution, i.e., the distribution function* $F : \mathbb{R} \to [0,1]$ *is given by* $F(t) = 1 - e^{-t}$ *for all* $t \in \mathbb{R}$*, i.e., we are letting* X *be an Exp(1) random variable.*

Accomplish the following:

(a) Compute the theoretical expected value $\mathbb{E}f$ using the standard calculus methods and show your derivation step-by-step.

(b) Write a programming script that approximates the non-negative function $f(x) = \sqrt{x}$ from below using sequences of increasing simple functions $(f_n)_{n\geq 1}$ for $n \in \{1, 2, \ldots, 15\}$ and then compute the approximate expected values by $\mathbb{E}f_n$. Proposition 9.1 and the proof gives an exact method for doing this.

(c) Visualize the approximation by simple functions for $n \in \{1, 2, \ldots, 10\}$ and plot the sequence of expected values $\mathbb{E}f_n$ as a function of n and plot a horizontal line showing the expected value $\mathbb{E}f$. The latter is justified by Proposition 9.2.

Solutions

(a) We have $X \sim \text{Exp}(1)$ with distribution function $F(t) = 1 - e^{-t}$ for all $t \in \mathbb{R}$. The density function of X is given by

$$f(t) = \frac{d}{dt}F(t) = \frac{d}{dt}(1 - e^{-t}) = e^{-t}. \tag{1}$$

The function $f(x) = \sqrt{x}$ is positive, so we use the Expectation Rule (Corollary 9.1) to show

$$\mathbb{E}f(X) = \mathbb{E}\sqrt{X} = \int_0^\infty \sqrt{x}e^{-x}dx = \int_0^\infty x^{\frac{1}{2}}e^{-x}dx. \tag{2}$$

Note that this integral is in the form of the Gamma function, which is defined to be $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, for $\alpha > 0$. Note that $\Gamma(\alpha) = (\alpha - 1)!$ for $\alpha \in \mathbb{N}$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Then from equation (2) we have $x^{\alpha-1} = x^{\frac{1}{2}}$, hence $\alpha = \frac{3}{2}$ and we solve

$$\int_0^\infty x^{\frac{1}{2}}e^{-x}dx = \Gamma\left(\frac{3}{2}\right) = \left(\frac{1}{2}\right)! = \left(\frac{1}{2}\right) \times \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \times \sqrt{\pi} = \frac{\sqrt{\pi}}{2}. \tag{3}$$

Therefore $\mathbb{E}f(X) = \frac{\sqrt{\pi}}{2}$.

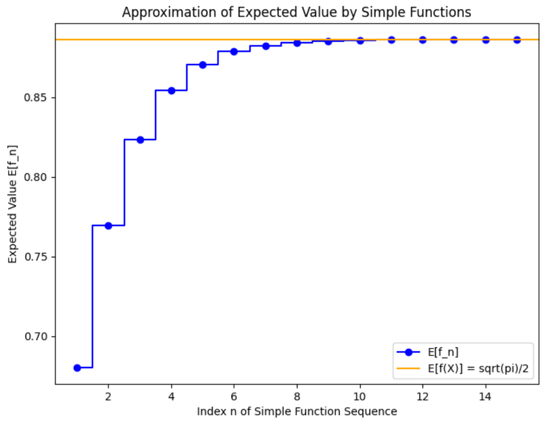


Figure 1: $\mathbb{E}f_n$ as $n \to \infty$

(b) By Proposition 9.1 we can define a sequence of positive increasing simple functions $(f_n)_{n\geq 1}$. Fix $n \in \mathbb{Z}^+$ and define the level sets

$$A_k^n = \begin{cases} \{k2^{-n} \leq x < (k+1)2^{-n}\} & , \ 0 \leq k \leq n2^n - 1 \\ \{x \geq n\} & , \ k = n2^n. \end{cases}$$

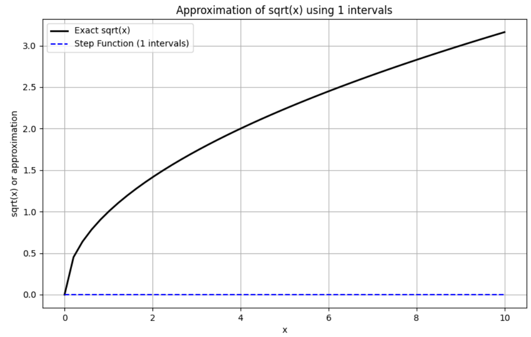
Thus we can define

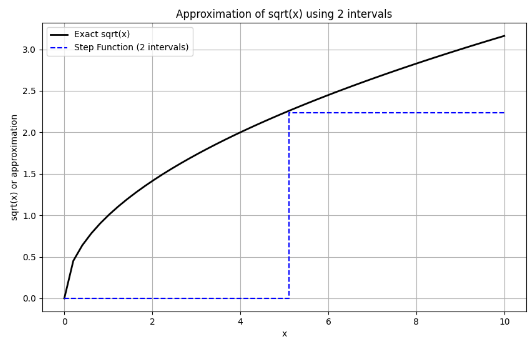
$$f_n(x) := \sum_{k=0}^{n2^n} \sqrt{k2^{-n}} \mathbb{1}_{A_k^n}(x).$$

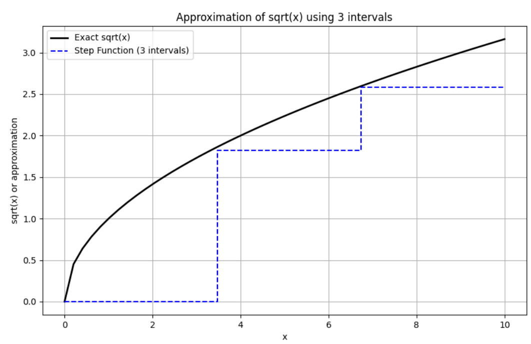
We will approximate $\mathbb{E}f_n$ by

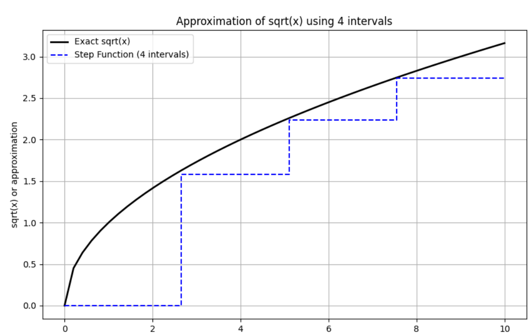
$$\mathbb{E}f_n(x) := \sum_{k=0}^{n2^n} \sqrt{k2^{-n}} \mathbb{P}(A_k^n).$$

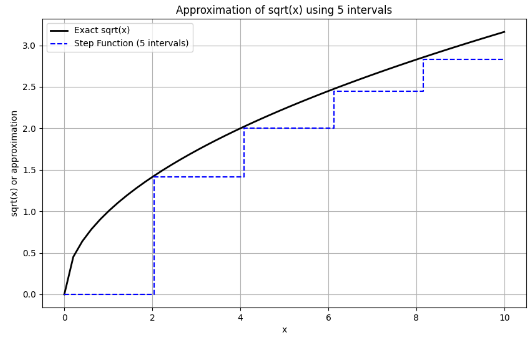
Figure 1 shows the expected values $\mathbb{E}f_n(x)$ for 100000 randomly generated samples of Exp(1) random variables. We see that $\mathbb{E}f_n(X) \to \mathbb{E}f(X) = \frac{\sqrt{\pi}}{2}$ as $n \to \infty$.
(c) Figures shown below.

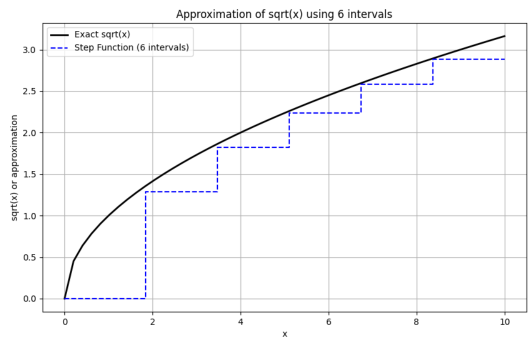

Approximation of sqrt(x) using 1 intervals
Exact sqrt(x)
Step Function (1 intervals)
sqrt(x) or approximation
x

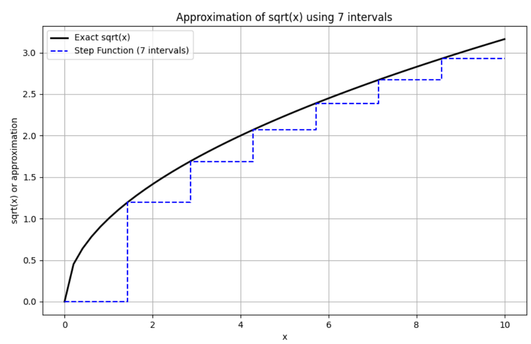

Approximation of sqrt(x) using 2 intervals
Exact sqrt(x)
Step Function (2 intervals)
sqrt(x) or approximation
x

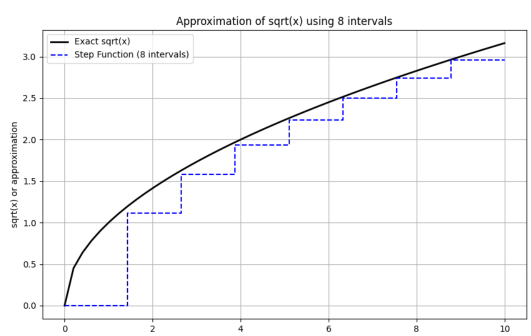

Approximation of sqrt(x) using 3 intervals
Exact sqrt(x)
Step Function (3 intervals)
sqrt(x) or approximation
x

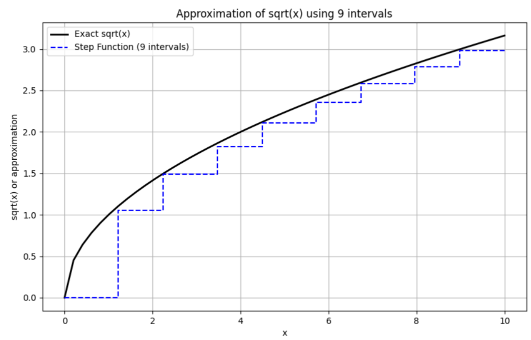

Approximation of sqrt(x) using 4 intervals
Exact sqrt(x)
Step Function (4 intervals)
sqrt(x) or approximation
x

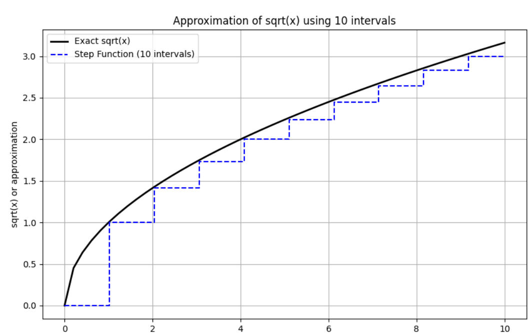

Approximation of sqrt(x) using 5 intervals
Exact sqrt(x)
Step Function (5 intervals)
sqrt(x) or approximation
x


Approximation of sqrt(x) using 6 intervals
Exact sqrt(x)
Step Function (6 intervals)
sqrt(x) or approximation
x


Approximation of sqrt(x) using 7 intervals
Exact sqrt(x)
Step Function (7 intervals)
sqrt(x) or approximation
x


Approximation of sqrt(x) using 8 intervals
Exact sqrt(x)
Step Function (8 intervals)
sqrt(x) or approximation
x


Approximation of sqrt(x) using 9 intervals
Exact sqrt(x)
Step Function (9 intervals)
sqrt(x) or approximation
x


Approximation of sqrt(x) using 10 intervals
Exact sqrt(x)
Step Function (10 intervals)
sqrt(x) or approximation
x