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Problem Set 11

Due Date: 11/19 at 11:35 AM

(P.1) *Show that if $X_n \xrightarrow{\mathcal{L}_p} X$ $(p \geq 1)$ then $X_n \xrightarrow{D} X$.*

Proof. If $X_n \xrightarrow{\mathcal{L}_p} X$, then by Definition 17.2 we know

$$\lim_{n\to\infty} \mathbb{E}|X_n - X|^p = 0. \tag{1}$$

Let $\epsilon > 0$. Then we can apply Markov's Inequality to the random variable $|X_n - X|^p$ to show

$$\frac{\mathbb{E}|X_n - X|^p}{\epsilon} \geq \mathbb{P}(|X_n - X| > \epsilon).$$

Then from equation (1) we show

$$\frac{1}{\epsilon} \lim_{n\to\infty} \mathbb{E}|X_n - X|^p \geq \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0. \tag{2}$$

Thus by Definition 17.3, equation (2) implies $X_n \xrightarrow{P} X$. To prove convergence in distribution, we must show $F_n(x)$ converges to $F(x)$ at every point x where $F(x)$ is continuous. For any $\varepsilon > 0$, we have

$$\begin{aligned} F_n(x) &= \mathbb{P}(X_n \leq x) \\ &= \mathbb{P}(X_n \leq x, X_n \leq x + \epsilon) + \mathbb{P}(X_n \leq x, X \geq x + \epsilon) \\ &\leq \mathbb{P}(X_n \leq x + \epsilon) + \mathbb{P}(|X_n - X| \geq \epsilon) \\ &= F(x + \epsilon) + \mathbb{P}(|X_n - X| \geq \epsilon) \end{aligned}$$

and

$$\begin{aligned} F(x - \epsilon) &= \mathbb{P}(X_n \leq x - \epsilon) \\ &= \mathbb{P}(X_n \leq x - \epsilon, X_n \leq x) + \mathbb{P}(X_n \leq x - \epsilon, X_n \geq x) \\ &\leq F_n(x) + \mathbb{P}(|X_n - X| \geq \epsilon) \end{aligned}$$

Combining these inequalities, we get

$$F(x + \epsilon) - \mathbb{P}(|X_n - X| \geq \epsilon) \leq F_n(x) \leq F(x + \epsilon) + \mathbb{P}(|X_n - X| \geq \epsilon).$$

Since $X_n \xrightarrow{P} X$ we know $\mathbb{P}(|X_n - X| \geq \epsilon) \to 0$ as $n \to \infty$. Therefore by taking the limit we get

$$F(x + \epsilon) \leq F_n(x) \leq F(x + \epsilon).$$

Since ϵ is arbitrary and $F(x)$ is continuous at x by assumption, it follows that $F_n(x) \to F(x)$ as $n \to \infty$. Therefore by Theorem 18.4 we can conclude $X_n \xrightarrow{D} X$. $\square$

(P.2) *Consider a probability space* $(\mathbb{R}, \mathcal{B}, \mathbb{P}_\theta)$ *indexed by a parameter* $\theta \in \Theta$ *and assume that* $\mathbb{P}_\theta$ *admits densities (with respect to the either the counting measure or the Lebesgue measure) in a parametric family* $\{f_\theta : \theta \in \Theta\}$*. Let* $(X_n)_{n\in\mathbb{Z}^+}$ *be a sequence of random variables whose induced probability distributions are given by* $\mathbb{P}_{\theta_n}$ *for a sequence* $(\theta_n)_{n\in\mathbb{Z}^+}$ *of points in* Θ*. If* $\theta_n \to \theta$ *as* $n \to \infty$*, what can be said about the sequence* $(X_n)_{n\in\mathbb{Z}^+}$*?*

Proof. Let us define each density in the parametric family as $f_\theta(x) = f_{X|\Theta}(x|\theta)$, the conditional probability density function of X given parameter $\theta \in \Theta$. If $f_\theta(x)$ is continuous in θ, and $\theta_n \to \theta$ as $n \to \infty$, then for every x we have

$$\lim_{n\to\infty} f_{\theta_n}(x) = \lim_{n\to\infty} f_{X|\Theta}(x|\theta_n) = f_{X|\Theta}(x|\theta) = f_\theta,$$

which implies pointwise convergence of the density functions. Then the sequence of probability distributions induced by density functions also converge, i.e.

$$\lim_{n\to\infty} \mathbb{P}_{\theta_n}(A) = \lim_{n\to\infty} \int\limits_A f_{X|\Theta}(x|\theta_n)\, dx = \int\limits_A f_{X|\Theta}(x|\theta)\, dx = \mathbb{P}_\theta(A)$$

for any measurable set A. By the Portmanteau Theorem, the above equation is equivalent to $X_n \xrightarrow{D} X$. Therefore it can be said that the sequence of random variables $(X_n)_{n\in\mathbb{Z}^+}$ converges in distribution to the random variable X. □

(P.3) *Let $(U_n)_{n\in\mathbb{Z}^+}$ be a sequence of independent and identically distributed Uniform$(0,1)$ random variables, i.e.,*

$$\mathbb{P}(U_i \le x) \;=\; x, \qquad x \in [0,1].$$

Show that $\prod_{i=1}^n U_i^{1/n}$ converges almost surely and give the precise limit. Next, center and scale $(\prod_{i=1}^n U_i^{1/n})_{n\in\mathbb{Z}^+}$ and show the resulting sequence converges in distribution to a non-degenerate limit, giving the precise distribution. Hint: You want to refresh your memory on the delta method.

Proof. First, note that the density function for $U_i \sim \text{Uniform}(0,1)$ taking values in the range $[0,1]$ is

$$f_{U_i}(x) = \begin{cases} 1 \text{ if } 0 \le x \le 1 \\ 0 \text{ otherwise.} \end{cases}$$

For simplicity, let us set $Z_n = \prod_{i=1}^n U_i^{1/n}$. We can take the logarithm of Z_n and simplify

$$\ln(Z_n) = \ln\left(\prod_{i=1}^n U_i^{1/n}\right) = \frac{1}{n}\sum_{i=1}^n \ln(U_i).$$

By the Strong Law of Large Numbers know

$$\frac{1}{n}\sum_{i=1}^n \ln(U_i) \overset{a.s.}{\to} \mathbb{E}[\ln(U_i)]$$

as $n \to \infty$. Applying the density function, we solve

$$\mathbb{E}[\ln(U_i)] = \int_0^1 \ln(x)(1)\,dx = [x\ln(x) - x]_0^1 = (0-1) - (\lim_{x\to 0^+} x\ln(x) - 0) = -1.$$

Therefore

$$\ln(Z_n) \overset{a.s.}{\to} -1$$

as $n \to \infty$, which implies

$$Z_n \overset{a.s.}{\to} e^{-1}.$$

Next, since $\ln(U_i)$ are i.i.d., then by the Central Limit Theorem we have

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \ln(U_i) - \mathbb{E}[\ln(U_i)]\right) \overset{D}{\to} \mathcal{N}(0,\sigma^2),$$

where $\sigma^2 := Var(\ln(U_i))$. We simply for the variance to get

$$\begin{aligned} Var(\ln(U_i)) &= \mathbb{E}[\ln(U_i)^2] - (\mathbb{E}[\ln(U_i)])^2 \\ &= \int_0^1 (\ln(x))^2 dx - (-1)^2 \\ &= 2 - 1 = 1 \end{aligned}$$

Then

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\ln(U_i)+1\right)=\sqrt{n}\,(\ln(Z_n)+1)\xrightarrow{D}\mathcal{N}(0,1),$$

meaning the sequence converges to a Normal distribution. This satisfies the assumption for the Delta-Method. Let $x=\ln(Z_n)$. Then $e^x=Z_n$ and we use the first order approximation of a Taylor series to show

$$e^x\approx e^{-1}+e^{-1}(x+1).$$

We substitute $x=\ln(Z_n)$ to get

$$Z_n\approx e^{-1}+e^{-1}(\ln(Z_n)+1).$$

Now, we can use the fact that $Z_n\xrightarrow{a.s.}e^{-1}$ and the above approximation to center and scale Z_n:

$$\sqrt{n}(Z_n-e^{-1})=\sqrt{n}(e^{-1}+e^{-1}(\ln(Z_n)+1)-e^{-1})=e^{-1}\sqrt{n}(\ln(Z_n)+1).$$

It follows from the Delta Method that

$$e^{-1}\sqrt{n}(\ln(Z_n)+1)\xrightarrow{D}\mathcal{N}(0,(e^{-1})^2).$$

Therefore

$$\sqrt{n}\,(\ln(Z_n)+1)\xrightarrow{D}\mathcal{N}(0,1)\text{ implies }\sqrt{n}(Z_n-e^{-1})\xrightarrow{D}\mathcal{N}(0,e^{-2}),$$

hence the sequence $Z_n=\prod_{i=1}^{n}U_i^{1/n}$ converges to a normal distribution with mean 0 and variance e^{-2}. $\square$

(P.4) *Consider sequences of independent Bernoulli random variables* $(X_{n,m})_{m\in\{1,\dots,n\}}$ *(*$n \in \mathbb{Z}^+$*) such that each* $X_{n,m} \sim$ *Bernoulli*(n^{-1})*. Show that the sum* $S_n = \sum_{m=1}^n X_{n,m}$ *converges in distribution to a random variable* S_∞ *as* $n \to \infty$ *and give the limiting distribution of* S_∞*. In other words,* S_n *is a sum of* n *independent and identically distributed Bernoulli*(n^{-1}) *random variables.*

Proof. If $S_n = \sum_{m=1}^n X_{n,m}$ is the sum of n i.i.d. Bernoulli random variables, then this defines a Binomial distribution. Hence $S_n \sim \text{Binomial}(n, n^{-1})$ such that

$$\mathbb{P}(S_n = k) = \binom{n}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{n-k}$$

for $k = 0, 1, \dots, n$. We expand the above equation and simplify:

$$\begin{aligned}\mathbb{P}(S_n = k) &= \frac{n!}{k!(n-k)!}\cdot\frac{1}{n^k}\cdot\left(1-\frac{1}{n}\right)^{n-k}\\ &= \frac{n(n-1)\cdots(n-k-1)}{k!}\cdot\frac{1}{n^k}\cdot\left(1-\frac{1}{n}\right)^{n-k}\\ &= \frac{1}{k!}\left(1-\frac{1}{n}\right)^n\frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{\left(1-\frac{1}{n}\right)^k}\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty}\mathbb{P}(S_n = k) = \frac{1}{k!}e^{-1}.$$

Notice that the above solution is simply the probability mass function of the Poisson distribution with parameter $\lambda = 1$. Then

$$\lim_{n\to\infty}\mathbb{P}(S_n = k) = \mathbb{P}(S_\infty = k),$$

where $S_\infty \sim \text{Poisson}(1)$. By Theorem 18.9, it immediately follows from the above equation that $S_n \xrightarrow{D} S_\infty$. And we have shown that the limiting distribution of S_∞ is a Poisson(1) distribution. $\square$

(P.5) ***True or False:*** *Consider a sequence of random variables $(X_n)_{n\in\mathbb{Z}^+}$ and X defined on a common measurable space $(\Omega, \mathcal{A})$. Assume that the induced probability distributions for the sequence $(X_n)_{n\in\mathbb{Z}^+}$ admit density functions $(f_n)_{n\in\mathbb{Z}^+}$ with respect to a given and fixed reference measure and denote the density function of X by f. Evaluate the statement "If $X_n \xrightarrow{D} X$, then $f_n \to f$" where for full credit, you must rigorously defend your answer. If the statement is true prove it; if the statement is false, demonstrate that it is false and outline conditions under which it would be true.*

Proof. The statement is false. Consider the following counterexample: Suppose the sequence $(X_n)_{n\in\mathbb{Z}^+}$ of random variables admit density functions

$$f_n(x) = (1 + \cos(2\pi nx))\mathbb{1}_{(0,1)}$$

defined on $[0, 1]$. Then the corresponding distribution functions are

$$F_n(x) = \int_0^x f_n(x)\,dx = \int_0^x (1 + \cos(2\pi nx))\mathbb{1}_{[0,1]} = x + \frac{\sin(2\pi nx)}{2\pi nx}$$

for each $n \in \mathbb{N}$. If we take the limit as $n \to \infty$, then the term $\frac{\sin(2\pi nx)}{2\pi nx} \to 0$, hence

$$F_{X_n}(x) \to F(x) = x$$

for all $x \in [0, 1]$. Thus by Theorem 18.4 we have that $X_n \xrightarrow{D} X$. Note that the distribution converges to a Uniform$(0, 1)$ distribution, where

$$F(x) = \begin{cases} 0 \text{ if } x < 0 \\ x \text{ if } 0 \le x \le 1 \\ 1 \text{ if } x > 1 \end{cases}.$$

However, since the density function $f_n(x)$ is an oscillating function, then it does not converge. Therefore $X_n \xrightarrow{D} X$ does not imply $f_n \to f$. For the statement to be true, we would need $f_n(x)$ to be continuous, bounded and converge pointwise to a function f on $[0, 1]$. Then by Dominated Convergence Theorem, we would have that $\mathbb{E}f_n \to \mathbb{E}f$. Thus Corollary 18.1 would allow us to directly show $X_n \xrightarrow{D} X$ implies $\mathbb{E}f_n \to \mathbb{E}f$ implies $f_n \to f$. $\square$