Name: Jenny Petrova
Problem Set 5
Due Date: 10/1 at 11:35 AM

Throughout, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{B}$ denote the Borel σ-algebra on $\mathbb{R}$.

(P.1) *Let $X : \Omega \mapsto \mathbb{R}$ be a random variable, i.e., a $\langle \mathcal{A}, \mathcal{B} \rangle$-measurable function.* Let

$$\mathcal{F} = \left\{A \,:\, A = X^{-1}(B),\ B \in \mathcal{B}\right\}.$$

Show that X is a $\langle \mathcal{F}, \mathcal{B} \rangle$-measurable function.

Proof. By definition of measurable function, we know that if X is $\langle \mathcal{A}, \mathcal{B} \rangle$-measurable then $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. By the given assumption, $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}$, and it follows from $\mathcal{A}$ that $\mathcal{F}$ is also closed under countable intersections, countable unions, and complements. Hence $\sigma(\mathcal{F}) \subseteq \mathcal{A}$ and $\mathcal{F} \subseteq \sigma(\mathcal{F})$. Therefore $\mathcal{F} = \mathcal{A}$ and $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}$, thus X is a $\langle \mathcal{F}, \mathcal{B} \rangle$-measurable function.

□

(P.2) *Let $X : \Omega \mapsto \mathbb{R}$ be a random variable and assume that $X(\Omega) = \{0, 1\}$. Define*

$$\sigma(X) \ := \ \{A \,:\, A = X^{-1}(B),\, B \in \mathcal{B}\}.$$

Show that $\sigma(X)$ is a σ-algebra and that it is a Bernoulli σ-algebra.

Proof. First, we know that $X(\Omega) = \{0, 1\}$. Hence $\{0, 1\} \in \mathcal{B}$ such that $X^{-1}(\{0, 1\}) = \Omega$. Thus $\Omega \in \sigma(X)$. Next, let $A_1, A_2, \cdots \in \sigma(X)$. Then there exists a $B_n \in \mathcal{B}$ such that $A_n = X^{-1}(B_n)$. Since $\mathcal{B}$ is a σ-algebra, we know $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$ and $\bigcap_{n=1}^{\infty} B_n \in \mathcal{B}$, by closure under countable unions and intersections. Thus we show

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} X^{-1}(B_n) = X^{-1}(\bigcup_{n=1}^{\infty} B_n) \in \sigma(X).$$

and

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} X^{-1}(B_n) = X^{-1}(\bigcap_{n=1}^{\infty} B_n) \in \sigma(X).$$

Thus $\sigma(X)$ is closed under countable unions and intersections. Lastly, since $\mathcal{B}$ is a σ-algebra we also know that $B^c \in \mathcal{B}$ by closure under complements. Therefore

$$A^c = (X^{-1}(B))^c = X^{-1}(B^c) \in \sigma(X).$$

Hence $\sigma(X)$ is closed under complements and it follows that $\sigma(X)$ is a σ-algebra.
If X is a Bernoulli random variable, then there exist singletons $\{0\}, \{1\} \in \mathcal{B}$ such that $A_0 = X^{-1}(\{0\})$ and $A_1 = X^{-1}(\{1\})$. Since $\sigma(X)$ is closed under unions and intersections, we show that $A_0 \cup A_1 = \Omega$ and $A_0 \cap A_1 = \emptyset$. Thus $\sigma(X)$ is the smallest σ-algebra that contains A_0 and A_1, and therefore we can define

$$\sigma(X) = \{\emptyset, A_0, A_1, \Omega\},$$

where $\sigma(X)$ is a Bernoulli σ-algebra generated by the Bernoulli random variable X. □

(P.3) *Let $f : \mathbb{R} \mapsto \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}$ be $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable functions, and $D \in \mathcal{B}$. Show that*

$$h(x) \quad = \quad \begin{cases} f(x) & \text{if } x \in D \\ g(x) & \text{if } x \in \mathbb{R} \setminus D \end{cases}$$

is likewise a $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable function.

Proof. For a $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable function, we know that $f^{-1}(B), g^{-1}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$. We can define the pre-image of B by h as

$$h^{-1}(B) = (f^{-1}(B) \cap D) \cup (g^{-1}(B) \cap \mathbb{R} \setminus D).$$

Since $D \in \mathcal{B}$, and $\mathcal{B}$ is closed under countable intersections, then

$$f^{-1}(B) \cap D \in \mathcal{B}$$

and

$$g^{-1}(B) \cap \mathbb{R} \setminus D \in \mathcal{B}.$$

Since $\mathcal{B}$ is closed under countable unions, then $h^{-1}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$. Therefore h is a $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable function.

$\square$

(P.4) *Let $(A_n)_{n \in \mathbb{Z}^+}$ be a countable partition of $\mathbb{R}$ such that $A_n \in \mathcal{B}$ for each $n \in \mathbb{Z}^+$, and let $f_n : A_n \mapsto \mathbb{R}$ $(n \in \mathbb{Z}^+)$ be $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable functions. Show that*

$$h(x) \quad = \quad \begin{cases} f_1(x) & \text{if } x \in A_1 \\ f_2(x) & \text{if } x \in A_2 \\ \vdots & \vdots \end{cases}$$

is a $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable function.

Proof. Since f_n is a $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable function, then $f_n^{-1}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$. Thus for all $A_n \in \mathcal{B}$, we know that $f_n^{-1}(B) \cap A_n \in \mathcal{B}$ since $\mathcal{B}$ is closed under countable intersections. We can define the pre-image of B by h as

$$h^{-1}(B) = \bigcup_{n \in \mathbb{Z}^+} (f_n^{-1}(B) \cap A_n).$$

Since $\mathcal{B}$ is closed under countable unions, then we have $h^{-1}(B) \in \mathcal{B}$ for any $B \in \mathcal{B}$. Therefore h is a $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable function.

$\square$

(P.5) *Consider the following functions $f : \mathbb{F} \mapsto \mathbb{R}$. For each, prove or disprove whether the function is $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable:*

(a) $f(x) = \sqrt{x}$ *where* $\mathbb{F} = (0, \infty)$.

(b) $f(x) = x^2$ *where* $\mathbb{F} = \mathbb{R}$.

(c) $f(x) = \log(1 + x)$ *where* $\mathbb{F} = [0, \infty)$.

(d) $f(x) = \exp(-x^2)$ *where* $\mathbb{F} = \mathbb{R}$.

Proof. We know $(\mathbb{F}, \mathcal{B})$ and $(\mathbb{R}, \mathcal{B})$ are topological spaces with with Borel σ-algebra $\mathcal{B}$.

(a) Since $f(x) = \sqrt{x}$ is continuous on domain $\mathbb{F} = (0, \infty)$ then by Theorem 8.3 it follows that $f(x)$ is $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable.

(b) Since $f(x) = x^2$ is continuous on domain $\mathbb{F} = \mathbb{R}$, then by Theorem 8.3 we have that $f(x)$ is $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable.

(c) Let $h(x) = \log(x)$ and let $g(x) = 1 + x$. We know that $h(x)$ is continuous on domain $\mathbb{F} = (0, \infty)$ and $h(x)$ is continuous on domain $F = \mathbb{R}$. Then by Theorem 8.3 we know that $h(x)$ and $g(x)$ are measurable functions. Since the composition of continuous functions is continuous, then $h(g(x)) = f(x) = \log(1 + x)$ is continuous on domain $\mathbb{F} = [0, \infty)$ and from Theorem 8.2 it follows that the composition of measurable functions is measurable. Hence $f(x)$ is $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable.

(d) Let $h(x) = e^x$ and let $g(x) = -x^2$. We know that $h(x)$ is continuous on domain $\mathbb{F} = \mathbb{R}$ and $h(x)$ is continuous on domain $F = \mathbb{R}$. Then by Theorem 8.3 we know that $h(x)$ and $g(x)$ are measurable functions. Since the composition of continuous functions is continuous, then $h(g(x)) = f(x) = e^{(-x^2)}$ is continuous on domain $\mathbb{F} = \mathbb{R}$ and from Theorem 8.2 it follows that the composition of measurable functions is measurable. Hence $f(x)$ is $\langle \mathcal{B}, \mathcal{B} \rangle$-measurable.

$\square$