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Problem Set 2

Due Date: 9/12 at 11:35 AM

Throughout, assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space.

(P.1) *Show that if $A \cap B = \emptyset$, events A and B cannot be independent unless we have either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.*

Proof. If $A \cap B = \emptyset$, then $\mathbb{P}(A \cap B) = \mathbb{P}(\emptyset) = 0$. Suppose A and B are independent events. Then by Theorem 3.2 it must be true that $\mathbb{P}(A|B) = \mathbb{P}(A)$.
Suppose $\mathbb{P}(B) > 0$. By definition of conditional probability, we show

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0}{\mathbb{P}(B)} = 0 = \mathbb{P}(A).$$

Similarly, if A and B are independent it must also be true that $\mathbb{P}(B|A) = \mathbb{P}(B)$. By definition of conditional probability, we also show

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{0}{\mathbb{P}(A)} = 0 = \mathbb{P}(B).$$

Thus if $A \cap B = \emptyset$, we must have either $\mathbb{P}(A) = 0$, $\mathbb{P}(B) > 0$ or $\mathbb{P}(B) = 0$, $\mathbb{P}(A) > 0$ for A and B to be independent.

$\square$

(P.2) *Donated blood is screened for a disease. Suppose the test has 99% accuracy, and that one in ten thousand people in your age group are infected with the disease. The test has a 5% false positive rating, as well. Suppose the test screens you as positive. What is the probability you have the disease? Is it 99%?*

Proof. Suppose A is the event of a positive test. Let E_1 be the event that you have the disease and E_2 be the event that you do not have the disease. We are given

$$\begin{aligned}\mathbb{P}(A|E_1) &= 99\% = .99\\ \mathbb{P}(A|E_2) &= 5\% = .05\\ \mathbb{P}(E_1) &= \frac{1}{10000} = .0001.\end{aligned}$$

Then

$$\mathbb{P}(E_2) = 1 - \mathbb{P}(E_1) = 1 - .0001 = .9999.$$

Therefore by Bayes Theorem we calculate the probability that you have the disease given that you test positive to be

$$\begin{aligned}\mathbb{P}(E_1|A) &= \frac{\mathbb{P}(A|E_1)\mathbb{P}(E_1)}{\mathbb{P}(A|E_1)\mathbb{P}(E_1) + \mathbb{P}(A|E_2)\mathbb{P}(E_2)}\\ &= \frac{(.99)(.0001)}{(.99)(.0001) + (.05)(.9999)}\\ &= .001976\\ &\approx 0.20\%.\end{aligned}$$

□

(P.3) *Let $A, B \in \mathcal{A}$ with $\mathbb{P}(A) > 0$. Show that $\mathbb{P}(A \cap B \mid A \cup B) \leq \mathbb{P}(A \cap B \mid A)$.*

Proof. By definition of conditional probability, we have

$$\begin{aligned}
\mathbb{P}(A \cap B | A \cup B) &= \frac{\mathbb{P}((A \cap B) \cap (A \cup B))}{\mathbb{P}(A \cup B)} \\
&= \frac{\mathbb{P}((A \cap B \cap A) \cup (A \cap B \cap B))}{\mathbb{P}(A \cup B)} \\
&= \frac{\mathbb{P}((A \cap B) \cup (A \cap B))}{\mathbb{P}(A \cup B)} \\
&= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cup B)}.
\end{aligned}$$

We also have

$$\begin{aligned}
\mathbb{P}(A \cap B | A) &= \frac{\mathbb{P}((A \cap B) \cap A)}{\mathbb{P}(A)} \\
&= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.
\end{aligned}$$

We need to show $\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cup B)} \leq \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$.
Case 1: Let $B = \emptyset$. Then $\mathbb{P}(B) = 0$. Since $\mathbb{P}(A) > 0$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A)$.
Therefore

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cup B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

Case 2: Suppose $B \neq \emptyset$. Then $\mathbb{P}(A \cup B) > \mathbb{P}(A)$.
Therefore

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cup B)} < \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

Thus

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cup B)} \leq \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

and so

$$\mathbb{P}(A \cap B \mid A \cup B) \leq \mathbb{P}(A \cap B \mid A).$$

□

(P.4) *Let* $A, B, C \in \mathcal{A}$ *be independent events and assume that* $\mathbb{P}(A \cap B) > 0$. *Show that* $\mathbb{P}(C \mid A \cap B) = \mathbb{P}(C)$.

Proof. By definition of mutual independence, we note that

$$\mathbb{P}(C \cap (A \cap B)) = \mathbb{P}(C)\mathbb{P}(A \cap B) = \mathbb{P}(C)\mathbb{P}(A)\mathbb{P}(B). \tag{1}$$

Then by definition of conditional probability, we use (1) to solve

$$\begin{aligned} \mathbb{P}(C|A \cap B) &= \frac{\mathbb{P}(C \cap (A \cap B))}{\mathbb{P}(A \cap B)} \\ &= \frac{\mathbb{P}(C)\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(A)\mathbb{P}(B)} \\ &= \mathbb{P}(C). \end{aligned}$$

$\square$

(P.5) *Consider two independent coin tosses, which we can model with the probability space* $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$*, where* $\Omega = \{(H,H),(T,T),(H,T),(T,H)\}$ *with the definition*

$$\mathbb{P}(\{(T_1,T_2)\}) \quad = \quad \frac{1}{4}, \qquad (T_1,T_2) \in \Omega.$$

Consider the following two sub-σ*-algebras of* $\mathcal{P}(\Omega)$*:*

$$\mathcal{G}_1 \quad = \quad \{\emptyset, \Omega, \{(H,H),(H,T)\},\ \{(T,H),(T,T)\}\}$$

and

$$\mathcal{G}_2 \quad = \quad \{\emptyset, \Omega, \{(H,H),(T,H)\}, \{(H,T),(T,T)\}\}.$$

Show that $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are independent* σ*-algebras.*

Proof. Let $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$. By definition, we can show events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Case 1: $A = \emptyset$.
Then $\mathbb{P}(A) = 0$ and

$$\mathbb{P}(A \cap B) = \mathbb{P}(\emptyset \cap B) = \mathbb{P}(\emptyset) = 0 = \mathbb{P}(A)\mathbb{P}(B)$$

for all $B \in \mathcal{G}_2$. By symmetry, if we take $B = \emptyset$, then the result is true for any $A \in \mathcal{G}_1$.

Case 2: $A = \Omega$.
Then $\mathbb{P}(A) = 1$ and

$$\mathbb{P}(A \cap B) = \mathbb{P}(\Omega \cap B) = \mathbb{P}(B) = (1)\mathbb{P}(B) = \mathbb{P}(A)\mathbb{P}(B)$$

for all $B \in \mathcal{G}_2$. By symmetry, if we take $B = \Omega$, then the result is true for any $A \in \mathcal{G}_1$.

Case 3: Suppose $A = \{(H,H),(H,T)\}$ and $B = \{(H,H),(T,H)\}$. Then

$$\mathbb{P}(A) = \mathbb{P}(\{(H,H),(H,T)\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

and similarly

$$\mathbb{P}(B) = \mathbb{P}(\{(H,H),(T,H)\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Since $A \cap B = \{(H,H)\}$, we solve to show

$$\mathbb{P}(A \cap B) = \mathbb{P}(\{(H,H)\}) = \frac{1}{4}$$

$=$

$$\mathbb{P}(A)\mathbb{P}(B) = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}.$$

Thus A and B are independent events. Then by Theorem 3.1 we know that so also are A and B^c, A^c and B, and A^c and B^c. Therefore $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent for any combination of events $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$, and so $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent σ-algebras.

$\square$

(P.6) *Consider the previous problem. How would you extend this to an arbitrary, but finite, number of coin flips? An arbitrary and possibly countable infinite number of coin flips?*

Proof. We extend the previous problem to consider an arbitrary, but finite, number of independent coin flips. Let $\{\mathcal{G}_n\}_{n\geq 1}$ be an arbitrary sequence of sub-σ-algebras of $\mathcal{P}(\Omega)$ in the probability space $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$, where $\Omega = \{H, T\}^n$ for n coin flips. Define events $A_{i,t_i} \in \mathcal{G}_i$ for $i = 1, \ldots, n$, $t_i \in \{H, T\}$, where

$$\mathcal{G}_i = \{\emptyset, \Omega, A_{i,H}, A_{i,T}\},$$

Case 1: Let $B_i \in \mathcal{G}_i$ and consider a $B_j \in \mathcal{G}_j$ where $B_j = \emptyset$ for some arbitrary $j \in \{1, \ldots, n\}$. Then $\mathbb{P}(B_j) = 0$ and for all $B_i \in \mathcal{G}_i$,

$$\mathbb{P}(B_j \bigcap_{i=1}^{n} B_i) = \mathbb{P}(\emptyset) = 0 = \mathbb{P}(B_j) \prod_{i=1}^{n} \mathbb{P}(B_i).$$

Case 2: Now let $B_i \in \mathcal{G}_i$ and consider a $B_k \in \mathcal{G}_k$ where $B_k = \Omega$ for some arbitrary $k \in \{1, \ldots, n\}$. Then $\mathbb{P}(B_k) = 1$ and for all $B_i \in \mathcal{G}_i$,

$$\mathbb{P}(B_k \bigcap_{i=1}^{n} B_i) = \mathbb{P}(\bigcap_{i=1}^{n} B_i) = \prod_{i=1}^{n} \mathbb{P}(B_i).$$

Case 3: Lastly we consider $A_{i,t_i} \in \mathcal{G}_i$. Since each coin toss is independent, and we have two outcomes $A_{i,H}$ or $A_{i,T}$, then

$$\mathbb{P}(A_{i,t_i}) = \frac{1}{2}$$

for all i. Therefore for n independent coin tosses we have

$$\prod_{i=1}^{n} \mathbb{P}(A_{i,t_i}) = (\frac{1}{2})^n = \frac{1}{2^n}.$$

We also know $|\bigcap_{i=1}^{n} A_{i,t_i}| = 1$, since only one sequence corresponds to these outcomes, and the total number of possible outcomes is $|\Omega| = 2^n$, so

$$\mathbb{P}(\bigcap_{i=1}^{n} A_{i,t_i}) = \frac{1}{|\Omega|} = \frac{1}{2^n}.$$

Therefore

$$\mathbb{P}(\bigcap_{i=1}^{n} A_{i,t_i}) = \frac{1}{2^n} = \prod_{i=1}^{n} \mathbb{P}(A_{i,t_i})$$

and by definition we have shown that for n extended coin tosses, the collection of events A_{i,t_i} is an independent collection for any $i = 1, \ldots, n$. Thus $\{\mathcal{G}_n\}_{n\geq 1}$ are an independent sequence of sub-σ-algebras.

We can also extend to an arbitrary and possibly countable infinite number of coin flips, where $\Omega = \{H, T\}^{\infty}$. For infinite, independent coin tosses, we can take any finite sequence of sub-σ-algebras and show the above is true.

$\square$