Problem Set 1

Due Date: 9/5 at 11:35 AM

Throughout, assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space.

(P.1) *Let Ω be a finite set. Show that the set of all subsets of Ω, the power set $\mathcal{P}(\Omega)$, is also finite and that it is a σ-algebra.*

Proof. By definition of a subset, we have $\Omega \subseteq \Omega$. Then $\Omega \in \mathbb{P}(\Omega)$. By the same definition, we also have $\emptyset \subseteq \Omega$. Then $\emptyset \in \mathbb{P}(\Omega)$.

Let $A \in \mathbb{P}(\Omega)$. Since $A \subseteq \Omega$, then $\Omega \backslash A = A^c \subseteq \Omega$. Therefore $A^c \in \mathcal{P}(\Omega)$, because $A^c \subseteq \Omega$.

Let $A_1, A_2, A_3, \ldots$ be a countable sequence of events in $\mathcal{P}(\Omega)$. Then $A_n \subseteq \Omega$ for each n, and we show

$$A_1 \cup A_2 \cup A_3 \cdots = \bigcup_{n=1}^{\infty} A_n \subseteq \Omega.$$

Thus

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{P}(\Omega).$$

Therefore $\mathcal{P}(\Omega)$ is a σ-algebra.

Let $n = |\Omega|$ for finite set Ω. Then $\mathbb{P}(\Omega)$ is all k-combinations of elements in Ω for $k \in \{0, 1, \ldots, n\}$. Then

$$\begin{aligned} |\mathbb{P}(\Omega)| &= \sum_{n=0}^{n} \binom{n}{k} < \infty \\ &= 2^n. \end{aligned}$$

Therefore $\mathbb{P}(\Omega)$ is finite. $\square$

(P.2) *Let $\{\mathcal{G}_s\}_{s\in S}$ be an arbitrary family of σ-algebras defined on an abstract space Ω. Show*

$$\mathcal{H} \quad = \quad \bigcap_{s\in S} \mathcal{G}_s$$

is also a σ-algebra.

Proof. By definition of σ-algebra, $\emptyset \in \mathcal{G}_s$ for all $s \in S$. Then

$$\emptyset \in \bigcap_{s\in S} \mathcal{G}_s = \mathcal{H}.$$

Similarly, $\Omega \in \mathcal{G}_s$ for all $s \in S$, so

$$\Omega \in \bigcap_{s\in S} \mathcal{G}_s = \mathcal{H}.$$

Let $A \in \mathcal{H}$. Then $A \in \mathcal{G}_s$ for all $s \in S$ by definition of $\mathcal{H}$. Since each $\{\mathcal{G}_s\}_{s\in S}$ is a σ-algebra and therefore closed under complements, then $A^c \in \mathcal{G}_s$ for each $s \in S$. Therefore

$$A^c \in \bigcap_{s\in S} \mathcal{G}_s = \mathcal{H}.$$

Let $(A_n)_{n\geq 1}$ be a countable sequence of events in $\mathcal{H}$. Then for each n, $A_n \in \mathcal{H}$ implies $A_n \in \mathcal{G}_s$, for all $s \in S$. By definition of σ-algebra, each $\{\mathcal{G}_s\}_{s\in S}$ is closed under countable unions. Therefore

$$\bigcup_{n=1}^{\infty} A_n \in \bigcap_{s\in S} \mathcal{G}_s = \mathcal{H}.$$

Thus $\mathcal{H}$ is a σ-algebra. $\square$

(P.3) *Let $\mathcal{A}$ be a σ-algebra and $(A_n)_{n\geq 1}$ a sequence of events in $\mathcal{A}$ and define*

$$\liminf_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m \qquad \textit{and} \qquad \limsup_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Show the following:

(a) $\liminf\limits_{n\to\infty} A_n \in \mathcal{A}$.

(b) $\limsup\limits_{n\to\infty} A_n \in \mathcal{A}$.

(c) $\liminf\limits_{n\to\infty} A_n \subseteq \limsup\limits_{n\to\infty} A_n$, *and show it is possible for* $\liminf\limits_{n\to\infty} A_n \subset \limsup\limits_{n\to\infty} A_n$.

(a)

Proof. Since $A_m \in \mathcal{A}$ for all $m \geq n$, $n \in \{1, 2, \dots\}$, then

$$\bigcap_{m=n}^{\infty} A_m \in \mathcal{A}$$

for all n by proposition of intersection of countable sets for σ-algebra. Then

$$\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m \in \mathcal{A}$$

by definition of σ-algebra, because $\mathcal{A}$ is closed under countable unions.
Therefore $\liminf\limits_{n\to\infty} A_n \in \mathcal{A}$. $\square$

(b)

Proof. Since $A_m \in \mathcal{A}$ for all $m \geq n$, $n \in \{1, 2, \dots\}$, then

$$\bigcup_{m=n}^{\infty} A_m \in \mathcal{A}$$

for all n by definition of σ-algebra, because $\mathcal{A}$ is closed under countable unions. Then

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \in \mathcal{A}$$

by proposition of σ-algebra.
Therefore $\limsup\limits_{n\to\infty} A_n \in \mathcal{A}$. $\square$

(c)

Proof. Let $\omega \in \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$. Then $\omega \in \bigcap_{m=n_0}^{\infty} A_m$ for some n_0. Thus $\omega \in A_m$ for all $n \geq n_0$ and it follows that $\omega \in \bigcup_{m=n}^{\infty} A_m$ for all $n \geq 1$. This implies $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_n$. Therefore $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.

Furthermore, consider set $A = \{0, 1\}$ with sequence of events

$$(A_n) = (\{0\}, \{1\}, \{0\}, \{1\}, \{0\}, \{1\}, \dots).$$

Let $x \in \bigcup_{m=n}^{\infty} A_m$. Since $\bigcup_{m=n}^{\infty} A_m = \{0, 1\}$ for all $m \geq n$, then

$$x \in \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_n = \{0, 1\}.$$

We also have $\bigcap_{m=n}^{\infty} A_n = \emptyset$ for all $m \geq n$.
Therefore

$$\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_n = \emptyset.$$

Then there exists an $x \in \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_n$ such that $x \notin \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_n$.
Therefore it is possible $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$.

$\square$

(P.4) *Let $\mathcal{A}$ be a σ-algebra of subsets of Ω and let $B \in \mathcal{A}$. Show that*

$$\mathcal{F} \quad = \quad \{A \cap B \,:\, A \in \mathcal{A}\}$$

is a σ-algebra of subsets of B.

Proof. By definition of σ-algebra, $\Omega \in \mathcal{A}$. Then $\Omega \cap B = B \in \mathcal{F}$. Similarly, $\emptyset \in \mathcal{A}$ by definition of σ-algebra. Thus $\emptyset \cap B = \emptyset \in \mathcal{F}$.

Let $C \in \mathcal{F}$, such that $C = A \cap B$ for some $A \in \mathcal{A}$. We write

$$\begin{aligned} C^c = B \backslash C &= B \cap C^c \\ &= B \cap (A \cap B)^c \\ &= B \cap (A^c \cup B^c) \end{aligned}$$

using DeMorgan's Law. Since $B \in \mathcal{A}$ and $\mathcal{A}$ is closed under complements by definition of σ-algebra, then $B^c \in \mathcal{A}$. By the same definition, $A^c \in \mathcal{A}$ because $A \in \mathcal{A}$. By definition of σ-algebra, $\mathcal{A}$ is closed under countable unions, therefore $A^c \cup B^c \in \mathcal{A}$.

Let $(C_i)_{i \geq 1}$ be a countable sequence of events in $\mathcal{F}$ such that $C_i = A_i \cap B$ for some $A_i \in \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra,

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}.$$

By the distributive property of sets, we show

$$\begin{aligned} \bigcup_{i=1}^{\infty} C_i &= \bigcup_{i=1}^{\infty} (A_i \cap B) \\ &= B \cap \bigcup_{i=1}^{\infty} A_i \end{aligned}$$

Therefore $\bigcup_{i=1}^{\infty} C_i \in \mathcal{F}$ for all i.
Thus $\mathcal{F}$ is a σ-algebra.

$\square$

(P.5) *Let f be a function mapping Ω to another space E with a σ-algebra $\mathcal{E}$. Let*

$$\mathcal{A} \quad = \quad \{A \subset \Omega : \textit{exists } B \in \mathcal{E} \textit{ such that } A = f^{-1}(B)\}.$$

Show that $\mathcal{A}$ is a σ-algebra on Ω.

Proof. By definition of σ-algebra, $\emptyset \in \mathcal{E}$. Then $f\{\emptyset\} = \emptyset$, so that $A = f^{-1}(\emptyset) = \emptyset$. Since $\emptyset \subset \Omega$ then $\emptyset \in \mathcal{A}$. Because $\mathcal{E}$ is a σ-algebra on space E, then $E \in \mathcal{E}$. Then $f\{\Omega\} = E$. Thus $A = f^{-1}(E) = \Omega$. Therefore $\Omega \in \mathcal{A}$.

Let $A \in \mathcal{A}$. Then by construction there exists $B^c \in \mathcal{E}$ such that $A = f^{-1}(B)$. Then

$$A^c = (f^{-1}(B))^c = f^{-1}(B^c).$$

Since $\mathcal{E}$ is a σ-algebra, then there exists $B^c \in \mathcal{E}$ such that $A^c = f^{-1}(B^c)$. Thus $A^c \in \mathcal{A}$.

Let $(A_n)_{n\geq 1}$ be a sequence of countable events in $\mathcal{A}$. For each n, there exists a $B_n \in \mathcal{E}$ such that $A_n = f^{-1}(B_n)$. Given that $\mathcal{E}$ is a σ-algebra, we know $\mathcal{E}$ is closed under countable unions. Thus there exists $\bigcup_{n=1}^{\infty} B_n \in \mathcal{E}$ such that

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} f^{-1}(B_n) = f^{-1}(\bigcup_{n=1}^{\infty} B_n).$$

Therefore $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Thus $\mathcal{A}$ is a σ-algebra on Ω. $\square$

(P.6) *Show the following for $A, B \in \mathcal{A}$:*

(a) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
(b) $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$.
(c) $\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$.

(a)

Proof. Suppose $A \cap B \neq \emptyset$. We show

$$\begin{aligned} \mathbb{P}(B) &= \mathbb{P}((B \backslash A) \cup (A \cap B)) \\ &= \mathbb{P}(B \backslash A) + \mathbb{P}(A \cap B). \end{aligned} \tag{1}$$

We can rewrite (1) as

$$\mathbb{P}(B \backslash A) = \mathbb{P}(B) - \mathbb{P}(A \cap B). \tag{2}$$

Then

$$\begin{aligned} \mathbb{P}(A \cup B) &= \mathbb{P}(A \cup (B \backslash A)) \\ &= \mathbb{P}(A) + \mathbb{P}(B \backslash A) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B). \end{aligned}$$

□

(b)

Proof. Since $A \in \mathcal{A}$, then $A \subseteq \Omega$. By definition of complement,

$$A^c = \Omega \backslash A \subseteq \Omega. \tag{3}$$

Then

$$A \cup A^c = A \cup (\Omega \backslash A) = \Omega \subseteq \Omega. \tag{4}$$

Since $\mathcal{A}$ is a σ-algebra, then $\Omega \in \mathcal{A}$ and it follows that

$$\begin{aligned} \mathbb{P}(A) + \mathbb{P}(A^c) &= \mathbb{P}(A \cup A^c) \\ &= \mathbb{P}(A \cup (\Omega \backslash A)) \\ &= \mathbb{P}(\Omega) \\ &= 1. \end{aligned} \tag{5}$$

Therefore $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$.

□

(c)

Proof. By definition of set difference, $A \backslash B = A \cap B^c$. Thus we show

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}((A \backslash B) \cup (A \cap B)) \\ &= \mathbb{P}((A \cap B^c) \cup (A \cap B)) \\ &= \mathbb{P}(A \cap B^c) + \mathbb{P}(A \cap B). \end{aligned} \tag{6}$$

Therefore $\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$.

□