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Problem Set 3

Due Date: 9/17 at 11:35 AM

Throughout, assume that $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ is a probability space where Ω is countable (either finite or countably infinite).

(P.1) *Let $g : [0,\infty) \mapsto [0,\infty)$ be strictly increasing. Show that*

$$\mathbb{P}(|X| \geq t) \quad \leq \quad \frac{\mathbb{E}\, g(|X|)}{g(t)},$$

for all $t > 0$.

Proof. Since g is a nonnegative function, we can apply Markov's inequality to the random variable $g(|X|)$ to show

$$\mathbb{P}(g(|X|) \geq g(t)) \quad \leq \quad \frac{\mathbb{E}\,(|X|)}{g(t)}$$

for all $g(t) > 0$. Since g is strictly increasing, then $g(|X|) \geq g(t)$ whenever $|X| \geq t$. Therefore

$$\mathbb{P}(g(|X|) \geq g(t)) \quad = \quad \mathbb{P}(|X| \geq t) \quad \leq \quad \frac{\mathbb{E}\,(|X|)}{g(t)}.$$

□

(P.2) *Show that* $(\mathbb{E}\,X)^2 \leq \mathbb{E}\,X^2$ *always, assuming that both expectations exist.*

Proof. Let X be a real-valued random variable with X^2 in L^1. By Definition 5.2, we define the Variance of X to be

$$\sigma_X^2 = \mathbb{E}(X - \mathbb{E}X)^2.$$

We can expand this definition to show

$$\begin{aligned} \sigma_X^2 &= \mathbb{E}(X - \mathbb{E}X)^2 \\ &= \mathbb{E}(X^2 - 2X\mathbb{E}X + (\mathbb{E}X)^2) \\ &= \mathbb{E}X^2 - \mathbb{E}(2X\mathbb{E}X) + \mathbb{E}((\mathbb{E}X)^2) \end{aligned} \quad . \tag{1}$$

We know that $\mathbb{E}X$ is a constant, i.e. one constant defines the expectation of X. Hence we factor

$$\mathbb{E}(2X\mathbb{E}X) = (2\mathbb{E}X)\mathbb{E}X = 2(\mathbb{E}X)^2$$

and we rewrite

$$\mathbb{E}((\mathbb{E}X)^2) = (\mathbb{E}X)^2.$$

Thus equation (1) becomes

$$\sigma_X^2 = \mathbb{E}X^2 - 2(\mathbb{E}X)^2 + (\mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2.$$

Since the Variance of X is a nonnegative value, we show

$$0 \leq \sigma_X^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2.$$

Therefore

$$\begin{aligned} 0 &\leq \mathbb{E}X^2 - (\mathbb{E}X)^2 \\ (\mathbb{E}X)^2 &\leq \mathbb{E}X^2. \end{aligned}$$

$\square$

(P.3) *Let X be a Geometric(p) random variable, i.e.,*

$$\mathbb{P}(X = x) \quad = \quad (1-p)^x\, p, \qquad x \in \{0, 1, 2, \ldots\},$$

for some $p \in (0,1)$. Show that X is memoryless by showing

$$\mathbb{P}(X > t + s \,|\, X \geq t) \quad = \quad \mathbb{P}(X > s),$$

for all $t, s \in \{0, 1, 2, \ldots\}$.

Proof. We know that $\mathbb{P}(X = x)$ is the probability of all $\omega \in \Omega$ such that $X(\omega) = x$. To find the probability that $\mathbb{P}(X > x)$ we use the cumulative distribution function $\mathbb{P}(X \leq x)$, where

$$\mathbb{P}(X \leq x) = \sum_{i=1}^{x} \mathbb{P}(X = i) = 1 - (1-p)^x.$$

Then we show

$$\mathbb{P}(X > x) = 1 - \mathbb{P}(X \leq x) = 1 - [1 - (1-p)^x] = (1-p)^x. \tag{2}$$

We apply the definition of conditional probability and substitute in the above equation to solve

$$\begin{aligned}
\mathbb{P}(X > t + s \,|\, X \geq t) &= \frac{\mathbb{P}((X > t+s) \cap (X \geq t))}{\mathbb{P}(X \geq t)} \\
&= \frac{\mathbb{P}(X > t+s)}{\mathbb{P}(X \geq t)} \\
&= \frac{(1-p)^{t+s}}{(1-p)^t} \\
&= (1-p)^s \\
&= \mathbb{P}(X > s).
\end{aligned}$$

$\square$

(P.4) *Let* $(A_n)_{n\geq 1}$ *be a sequence of events in* $\mathcal{P}(\Omega)$. *Show that*

$$\mathbb{E}\sum_{n=1}^{\infty} \mathbb{1}_{A_n} \;=\; \sum_{n=1}^{\infty} \mathbb{P}(A_n),$$

where $+\infty$ *is a possible value for each side of the equation.*

Proof. We define the indicator function to be

$$\mathbb{1}_{A_n}(\omega) = \begin{cases} 1 & \text{if } \omega \in A_n \\ 0 & \text{if } x \notin A_n \end{cases}$$

for all $\omega \in \Omega$. Then by definition of expectation we can solve

$$\begin{aligned} \mathbb{E}\sum_{n=1}^{\infty} \mathbb{1}_{A_n} &= \sum_{n=1}^{\infty} (\mathbb{E}\mathbb{1}_{A_n}) \\ &= \sum_{n=1}^{\infty} [1\mathbb{P}(\mathbb{1}_{A_n} = 1) + 0\mathbb{P}(\mathbb{1}_{A_n} = 0)] \\ &= \sum_{n=1}^{\infty} \mathbb{P}(\mathbb{1}_{A_n} = 1) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(A_n). \end{aligned}$$

□

(P.5) *Suppose that X takes all its values in $\mathbb{N} = \{0, 1, 2, \ldots\}$, i.e., $X : \Omega \mapsto \mathbb{N}$. Show that*

$$\mathbb{E}\,X \;=\; \sum_{n=0}^{\infty} \mathbb{P}(X > n).$$

Proof. By definition of expectation of X, we have

$$\mathbb{E}X = \sum_{k=0}^{\infty} k\mathbb{P}(X = k). \tag{3}$$

Since $k \in \mathbb{N}$, we can write $\mathbb{P}(X = k)$ to be

$$\mathbb{P}(X = k) = \mathbb{P}(X \geq k) - \mathbb{P}(X \geq k + 1).$$

Then we can expand equation (3) and solve

$$\begin{aligned}
\mathbb{E}X &= \sum_{k=0}^{\infty} k\mathbb{P}(X = k) \\
&= \sum_{k=0}^{\infty} k\mathbb{P}(X \geq k) - \sum_{k=0}^{\infty} k\mathbb{P}(X \geq k + 1) \\
&= \sum_{k=0}^{\infty} k\mathbb{P}(X \geq k) - \sum_{k=1}^{\infty} (k-1)\mathbb{P}(X \geq k) \\
&= \left[0 + \sum_{k=1}^{\infty} k\mathbb{P}(X \geq k)\right] - \left[0 + \sum_{k=2}^{\infty} (k-1)\mathbb{P}(X \geq k)\right] \\
&= \sum_{k=1}^{\infty} \mathbb{P}(X \geq k) \\
&= \sum_{k=1}^{\infty} \mathbb{P}(X > k - 1).
\end{aligned}$$

Set $n = k - 1$. Then

$$\mathbb{E}X = \sum_{n=0}^{\infty} \mathbb{P}(X > n).$$

$\square$