Problem Set 1

1. $y_t = 1 + \varepsilon_t + .6\,\varepsilon_{t-1} + .2\,\varepsilon_{t-2}$

   $\varepsilon_t \sim WN(0,1)$

   We know:
   - $\mathbb{E}(\varepsilon_t) = 0$, $var(\varepsilon_t) = 1$ for all t
   - $var(c) = 0$ for a constant c
   - $\mathbb{E}(\varepsilon_t \varepsilon_s) = 0$ for all $t \neq s$
   - $\rho_h = 0$ for $h > 2$

$$\gamma_0 = var(y_t)$$
$$= \overset{=0}{\cancel{var}(1)} + \overset{=1}{var(\varepsilon_t)} + \overset{=.6^2(1)}{.6^2\, var(\varepsilon_{t-1})} + \overset{=.2^2(1)}{.2^2\, var(\varepsilon_{t-2})}$$
$$+ .6 \underbrace{cov(\cancel{\varepsilon_t}, \varepsilon_{t-1})}_{=0} + .2 \underbrace{cov(\cancel{\varepsilon_t}, \varepsilon_{t-2})}_{=0} + (.6)(.2) \underbrace{\cancel{cov}(\varepsilon_{t-1}, \varepsilon_{t-2})}_{=0}$$
$$= 1 + .36(1) + .04(1)$$
$$= 1.4$$

$$\gamma_1 = cov(y_t, y_{t-1})$$
$$= cov(1 + \cancel{\varepsilon_t} + .6\,\varepsilon_{t-1} + .2\,\varepsilon_{t-2},\ 1 + \varepsilon_{t-1} + .6\,\varepsilon_{t-2} + .2\,\cancel{\varepsilon_{t-3}})$$
$$= .6(1) + (.2)(.6)(1)$$
$$= .72$$

$$\gamma_2 = cov(y_t, y_{t-2})$$
$$= cov(1 + \cancel{\varepsilon_t} + .6\,\cancel{\varepsilon_{t-1}} + .2\,\varepsilon_{t-2},\ 1 + \varepsilon_{t-2} + .6\,\cancel{\varepsilon_{t-3}} + .2\,\cancel{\varepsilon_{t-4}})$$
$$= .2$$

$$\gamma_3 = cov(y_t, y_{t-2})$$
$$= cov(1 + \cancel{\varepsilon_t} + .6\,\cancel{\varepsilon_{t-1}} + .2\,\cancel{\varepsilon_{t-2}},\ 1 + \cancel{\varepsilon_{t-3}} + .6\,\cancel{\varepsilon_{t-4}} + .2\,\cancel{\varepsilon_{t-5}})$$
$$= 0$$

Autocorrelation Function:

$$\rho_0 = \frac{\gamma_0}{\gamma_0} = \frac{1.4}{1.4} = 1$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{.72}{1.4} = .51$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{.2}{1.4} = .14$$

$$\rho_h = 0 \quad (h > 2)$$

2. $y_t = \beta_0 + \beta_1 t + \varepsilon_t + \theta \varepsilon_{t-1}$

$\varepsilon_t \sim WN(0, \sigma^2)$

A time series is stationary if it has constant mean ($\mathbb{E}(y_t) = \mu$) and the covariance function depends only on $|t_i - t_j|$ ($\gamma(t_i, t_j) = \text{cov}(y_{t_i}, y_{t_j})$).

(a) $\mathbb{E}(y_t) = \mathbb{E}(\beta_0) + \mathbb{E}(\beta_1 t) + \underbrace{\mathbb{E}(\cancel{\varepsilon_t})}_{=0} + \underbrace{\mathbb{E}(\theta \cancel{\varepsilon_{t-1}})}_{=0}$

$= \beta_0 + \beta_1 t$

The series mean varies with time, since y_t is dependent on t, so this series is <u>not</u> weakly stationary.

(b) $x_t = y_t - y_{t-1}$

$= \cancel{\beta_0} + \cancel{\beta_1} t + \varepsilon_t + \theta \varepsilon_{t-1} - \cancel{\beta_0} - \beta_1(\cancel{t} - 1) - \varepsilon_{t-1} - \theta \varepsilon_{t-2}$

$= \beta_1 + \varepsilon_t + (\theta - 1)\varepsilon_{t-1} - \theta \varepsilon_{t-2}$

$\mathbb{E}(x_t) = \mathbb{E}(\beta_1) + \underbrace{\mathbb{E}(\cancel{\varepsilon_t})}_{=0} + (\theta - 1)\underbrace{\mathbb{E}(\varepsilon_{\cancel{t-1}})}_{=0} + \theta \underbrace{\mathbb{E}(\varepsilon_{\cancel{t-2}})}_{=0}$

$= \beta_1$

$\gamma_0 = \text{var}(x_t)$

$= \underbrace{\text{var}(\cancel{\beta_1})}_{=0} + \text{var}(\varepsilon_t) + (\theta - 1)^2 \text{var}(\varepsilon_{t-1}) + \theta^2 \text{var}(\varepsilon_{t-2})$

$= \sigma^2 + (\theta^2 - 2\theta + 1)\sigma^2 + \theta^2 \sigma^2$

$= 2\sigma^2(\theta^2 - \theta + 1)$

$\gamma_1 = \text{cov}(x_t, x_{t-1})$

$= \text{cov}(\beta_1 + \cancel{\varepsilon_t} + (\theta - 1)\varepsilon_{t-1} - \theta \varepsilon_{t-2}, \beta_1 + \varepsilon_{t-1} + (\theta - 1)\varepsilon_{t-2} - \theta \cancel{\varepsilon_{t-3}})$

$= (\theta - 1)\text{var}(\varepsilon_{t-1}) - (\theta)(\theta - 1)\text{var}(\varepsilon_{t-2})$

$= (\theta - 1)\sigma^2 - (\theta^2 - \theta)\sigma^2$

$= \sigma^2(-\theta^2 + 2\theta - 1)$

$\gamma_2 = \text{cov}(x_t, x_{t-2})$

$= \text{cov}(\beta_1 + \cancel{\varepsilon_t} + (\theta - \cancel{1})\varepsilon_{t-1} - \theta \varepsilon_{t-2}, \beta_1 + \varepsilon_{t-2} + (\theta - \cancel{1})\varepsilon_{t-3} - \theta \cancel{\varepsilon_{t-4}})$

$= -\theta \,\text{var}(\varepsilon_{t-2})$

$= -\theta \sigma^2$

$\gamma_3 = \text{cov}(x_t, x_{t-3})$

$= 0$

The series mean is constant over time and the covariance function depends only only on the time between observations. The series x_t <u>is</u> weakly stationary.

3. $y_t = \varepsilon_t + \theta \varepsilon_{t-1}$

$\varepsilon_t \sim WN(0, \sigma^2)$

(a) Suppose $|\theta| < 1$

Let $\varepsilon_t = y_t - \theta \varepsilon_{t-1}$

$$= y_t - \theta(y_{t-1} - \theta \varepsilon_{t-2}) = y_t - \theta y_{t-1} + \theta^2 \varepsilon_{t-2}$$

$$= y_t - \theta(y_{t-1} - \theta(y_{t-2} - \theta \varepsilon_{t-3}))$$

$$= y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 \varepsilon_{t-3}$$

$$\vdots$$

$$= \sum_{i=0}^{\infty} (-\theta)^i y_{t-i}$$

In terms of y_s, where $s \le t$, we can set $s = t - i$ and write

$$\varepsilon_t = \sum_{t-s=0}^{\infty} (-\theta)^{t-s} y_s$$

More recent observations have less weight than observations from the distant past.

(b) Suppose $|\theta| > 1$

Let $\varepsilon_t = \theta^{-1} \varepsilon_{t+1} + \theta^{-1} y_{t+1}$

$$= \theta^{-1}(\theta^{-1} \varepsilon_{t+2} + \theta^{-1} y_{t+2}) + \theta^{-1} y_{t+1}$$

$$= \theta^{-2} \varepsilon_{t+2} + \theta^{-2} y_{t+2} + \theta^{-1} y_{t+1}$$

$$= \theta^{-2}(\theta^{-1} \varepsilon_{t+3} + \theta^{-1} y_{t+3}) + \theta^{-2} y_{t+2} + \theta^{-1} y_{t+1}$$

$$= \theta^{-3} \varepsilon_{t+3} + \theta^{-3} y_{t+3} + \theta^{-2} y_{t+2} + \theta^{-1} y_{t+1}$$

$$\vdots$$

$$= \sum_{i=1}^{\infty} \theta^{-i} y_{t+1}$$

In terms of y_s, where $s \ge t$, we can set $s = t + i$ and write

$$\varepsilon_t = \sum_{s-t=0}^{\infty} \theta^{t-s} y_s$$

4. $x_t = \varepsilon_t - .4\,\varepsilon_{t-1}\,,\quad \varepsilon_t \sim WN(0,4)$
$y_t = u_t - .7\,u_{t-1}\,,\quad \varepsilon_t \sim WN(0,1)$
$\Big\}$ ε_t, u_t independent

Let $z_t = x_t + y_t$.

$$\begin{aligned}
\gamma_0 &= var(z_t) \\
&= var(\varepsilon_t) + .4^2 var(\varepsilon_{t-1}) + var(u_t) + .7^2 var(u_{t-1}) \\
&= 4 + (.16)(4) + 1 + (.49)(1) \\
&= 6.13
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= cov(z_t, z_{t-1}) \\
&= cov(x_t + y_t, x_{t-1} + y_{t-1}) \\
&= cov(\cancel{\varepsilon_t} - .4\varepsilon_{t-1} + \cancel{u_t} - .7u_{t-1}, \\
&\qquad\quad \varepsilon_{t-1} - 4\cancel{\varepsilon_{t-2}} + u_{t-1} - .7\cancel{u_{t-2}}) \\
&= (-.4)\,var(\varepsilon_{t-1}) + (-.7)\,var(u_{t-1}) \\
&= (-.4)(4) + (-.7)(1) \\
&= -2.3
\end{aligned}$$

(a) $\rho_0 = 1$

$\rho_1 = \dfrac{-2.3}{6.13} = -.375$

(b) Suppose $z_t = e_t + \theta e_{t-1}\,,\quad e_t \sim WN(0, \sigma_e^2)$

$$\begin{aligned}
var(z_t) &= var(e_t) + \theta^2 var(e_{t-1}) \\
&= \sigma_e^2 + \theta^2 \sigma_e^2 \\
&= (1+\theta^2)\,\sigma_e^2 \\
&= \gamma_0
\end{aligned}$$

$$\begin{aligned}
cov(z_t, z_{t-1}) &= cov(\cancel{e_t} + \theta e_{t-1}, e_{t-1} + \theta \cancel{e_{t-2}}) \\
&= \theta\, cov(e_{t-1}) \\
&= \theta\,\sigma_e^2 \\
&= \gamma_1
\end{aligned}$$

$$\Rightarrow \rho_1 = \frac{\theta\sigma_e^2}{(1+\theta^2)\sigma_e^2} = \frac{\theta}{1+\theta^2} = -.375$$

$\Rightarrow .375\theta^2 + \theta + .375 = 0$
$\theta = -.45\,,\ -\cancel{2}.2$
$\Big\}$ we want z_t to be invertible, i.e. $|\theta| < 1$, so we chose $\theta = -.45$.

$$\Rightarrow -2.3 = \theta\sigma_e^2 \Rightarrow \sigma_e^2 = \frac{-2.3}{-.45} = 5.1$$

$\therefore\ z_t = e_t - .45\,e_{t-1}\,,\quad e_t \sim WN(0, 5.1)$

5. $y_t = y_{t-1} - .24\, y_{t-2} + \varepsilon_t$, $\varepsilon_t \sim WN(0, \sigma^2)$

(a) $y_t = L y_t - .24 L^2 y_t + \varepsilon_t$

$\Rightarrow (1 - L + .24 L^2)\, y_t = \varepsilon_t$

$\Rightarrow y_t = (1 - L + .24 L^2)^{-1} \varepsilon_t$

(b) $\lambda^2 - \lambda + .24 = 0$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4(.24)}}{2} = .5 \pm .1$$

$\Rightarrow |\lambda_1|, |\lambda_2| < 1 \Rightarrow$ this series is weakly stationary

y_t does not have complex roots, hence it does not exhibit stochastic business cycles

(c) MA(∞) representation of an AR(2) process

$$\begin{cases} a + b = 1 \\ \lambda_2 a + \lambda_1 b = 0 \end{cases}$$

$\lambda_1 = .4$, $\lambda_2 = .6$

$$\Rightarrow a = \frac{\lambda_1}{\lambda_1 - \lambda_2} = \frac{.4}{.4 - .6} = \frac{.4}{-.2} = -2$$

$$b = \frac{\lambda_2}{\lambda_2 - \lambda_1} = \frac{.6}{.6 - .4} = \frac{.6}{.2} = 3$$

$$\Rightarrow y_t = \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$$

$$= \sum_{i=0}^{\infty} (a \lambda_1^i + b \lambda_2^i)\, \varepsilon_{t-i}$$

$\Rightarrow \{\psi_i\} \equiv \{-2(.4)^i + 3(.6)^i\}$, $i = 0, 1, \ldots$

(d) $x_t = 2 + x_{t-1} - .24 x_{t-2} + \varepsilon_t$

$$\mathbb{E}(x_t) = \mu$$
$$= \mathbb{E}(2) + \mathbb{E}(x_{t-1}) - .24\,\mathbb{E}(x_{t-2}) + \mathbb{E}(\varepsilon_t)$$
$$= 2 + \mu - .24\mu + 0$$
$$= 2 + .76\mu$$
$$\Rightarrow (1 - .76)\mu = 2 \rightarrow (1-\phi_1)\mu = \phi_0$$
$$.24\mu = 2$$
$$\mu = 8.3$$

MA(∞) representation of an AR(2) process

$$x_t - \mu = \phi_1(x_{t-1} - \mu) + \phi_1^2(x_{t-1} - \mu) + \varepsilon_t$$
$$\Rightarrow x_t - 8.3 = .76(x_{t-1} - 8.3) + .76^2(x_{t-1} - 8.3) + \varepsilon_t$$

Let $\tilde{x}_t = x_t - 8.3$.

Since $|.76| < 1$, we can write

$$\tilde{x}_t = \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i} = \sum_{i=0}^{\infty} .76^i \varepsilon_{t-i} = x_t - 8.3$$
$$\Rightarrow x_t = 8.3 + \sum_{i=0}^{\infty} .76^i \varepsilon_{t-i}$$

(e) $z_t = z_{t-1} - .3 z_{t-2} + \varepsilon_t$

$$= L z_t - .3 L^2 z_t + \varepsilon_t$$
$$\Rightarrow (1 - L + .3L^2) z_t = \varepsilon_t$$
$$\Rightarrow z_t = (1 - L + .3L^2)^{-1} \varepsilon_t$$

$$\lambda^2 - \lambda + .3 = 0$$
$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4(.3)}}{2} = \frac{1}{2} \pm i\frac{\sqrt{.2}}{2}$$
$$\Rightarrow \left.\begin{array}{l} \lambda_1 = 1/2 + i\sqrt{.2}/2 \\ \lambda_2 = 1/2 - i\sqrt{.2}/2 \end{array}\right\} a = \frac{1}{2},\ b = \frac{\sqrt{.2}}{2}$$
$$R = \sqrt{(1/2)^2 + (\sqrt{.2}/2)^2}$$
$$= .55 < 1$$

$\Rightarrow$ The series <u>is</u> stationary

Complex roots give rise to business cycles

$\Rightarrow$ average length of the business cycles:

$$k = \frac{2\pi}{\cos^{-1}\left(\frac{a}{R}\right)} = \frac{2\pi}{\cos^{-1}\left(\frac{.5}{.55}\right)} = 14.94$$