Problem Set 3

1. $\Delta y_t = \alpha + \beta t + \delta y_{t-1} + \varepsilon_t \,, \quad t = 1, 2, \ldots, T$

$$y_t - y_{t-1} = \alpha + \beta t + \delta y_{t-1} + \varepsilon_t$$

$$\Rightarrow y_t = \alpha + \beta t + (1+\delta) y_{t-1} + \varepsilon_t$$

Suppose we place no restrictions on β and test $H_0 : \delta = 0$ against $H_1 : \delta < 0$.

Under the null hypothesis,

$$y_t = \alpha + \beta t + y_{t-1} + \varepsilon_t$$

Take $y_{t-1} = \alpha + \beta(t-1) + y_{t-2} + \varepsilon_{t-1}$

$$\Rightarrow y_t = 2\alpha + \beta(2t-1) + y_{t-2} + \varepsilon_t + \varepsilon_{t-1}$$

Take $y_{t-2} = \alpha + \beta(t-2) + y_{t-3} + \varepsilon_{t-2}$

$$\Rightarrow y_t = 3\alpha + \beta(3t-3) + y_{t-3} + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}$$

Suppose $t = 3 \Rightarrow y_3 = 3\alpha + 6\beta + y_0 + \varepsilon_3 + \varepsilon_2 + \varepsilon_1$

Continuing this process, y_t becomes

$$y_t = y_0 + \sum_{i=1}^{t} (\alpha + \beta i + \varepsilon_i)$$

$$= y_0 + \alpha t + \underbrace{\beta \frac{t(t+1)}{2}}_{\beta \frac{t^2+t}{2}} + \sum_{i=1}^{t} \varepsilon_i$$

$\therefore$ β accumulates to a quadratic trend

2. AR(P)

$$y_t = \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \cdots + \Phi_p y_{t-p} + \varepsilon_t$$

$$\varepsilon_t \sim IID(0, \sigma^2)$$

(a) $$y_t = \Phi_1 L y_t + \Phi_2 L^2 y_t + \cdots + \Phi_p L^p y_t + \varepsilon_t$$

$$(1 - \Phi_1 L - \Phi_2 L^2 - \cdots - \Phi_p L^p) y_t = \varepsilon_t$$

The roots of the model are

$$1 - \Phi_1 z - \Phi_2 z^2 - \cdots - \Phi_p z^p = 0$$

$$\Rightarrow \quad 1 - \sum_{i=1}^{p} \Phi_i z^i = 0$$

If $z = 1$ is a root, then

$$1 - \sum_{i=1}^{p} \Phi_i = 0$$

$$\Rightarrow \sum_{i=1}^{p} \Phi_i = 1$$

(b) $$\Delta y_t = y_t - y_{t-1}$$

$$= (\Phi_1 - 1) y_{t-1} + \Phi_2 y_{t-2} + \cdots + \Phi_p y_{t-p} + \varepsilon_t$$

Note: $y_i - y_{i-1} = \Delta y_i$

$$\Rightarrow y_{i-1} = y_i - \Delta y_i$$

so $y_{t-2} = y_{t-1} - \Delta y_{t-1}$

$$y_{t-3} = y_{t-2} - \Delta y_{t-2}$$

$$= (y_{t-1} - \Delta y_{t-1}) - \Delta y_{t-2}$$

$$y_{t-4} = y_{t-3} - \Delta y_{t-3}$$

$$= y_{t-1} - \Delta y_{t-1} - \Delta y_{t-2} - \Delta y_{t-3}$$

$\vdots$

$$y_{t-p} = y_{t-1} - \sum_{i=2}^{p} \Delta y_{t-i+1}$$

$$\Longrightarrow \Delta y_t = (\phi_1 - 1) y_{t-1} + \sum_{i=2}^{p} \phi_i \left(y_{t-1} - \sum_{j=2}^{i} \Delta y_{t-j+1} \right) + \varepsilon_t$$

$$= [\phi_1 - 1 + \sum_{i=2}^{p} \phi_i] y_{t-1} - \sum_{i=2}^{p} \phi_i \sum_{j=2}^{i} \Delta y_{t-j+1} + \varepsilon_t$$

$$= [\sum_{i=1}^{p} \phi_i - 1] y_{t-1} - \sum_{j=2}^{p-1} \sum_{i=j}^{p} \phi_i \Delta y_{t-j+1} + \varepsilon_t$$

Let $\phi_1^* = \sum_{i=1}^{p} \phi_i - 1$,

and $\phi_j^* = - \sum_{i=j}^{p} \phi_i$ for $j = 2, 3, \ldots, p$

$$\therefore \Delta y = \phi_1^* y_{t-1} + \phi_2^* \Delta y_{t-1} + \cdots + \phi_p^* \Delta y_{t-p+1} + \varepsilon_t$$

Since $\sum_{i=1}^{p} \phi_i = 1$, (derived in part (a))

testing $\{y_t\}$ is equivalent to testing whether

$$\phi_1^* = 1 - 1 = 0.$$

3. $(1-L) y_t = -1.2 + (1 - 0.6L) \varepsilon_t$

$y_0 = 1$

$\varepsilon_t \sim IID(0,1)$

(a) This is an $I(1)$ process, where $L=1$ is a unit root.

The ARMA(p,d,q) model is given by

$$(1 - \sum_{i=1}^{p} \phi_i L^i)(1-L)^d y_t = \alpha + (1 - \sum_{i=1}^{q} \theta_i L^i) \varepsilon_t.$$

Our model has $p=0$, $d=1$, $q=1$

$\Longrightarrow$ ARMA(0,1,1)

(b) $(1-L)y_t = y_t - y_{t-1} = -1.2 + \varepsilon_t - 0.6\,\varepsilon_{t-1}$

$$\Rightarrow y_t = y_{t-1} - 1.2 + \varepsilon_t - 0.6\,\varepsilon_{t-1}$$

$$\mathbb{E}(y_1) = \mathbb{E}(y_0) - 1.2$$
$$\mathbb{E}(y_2) = \mathbb{E}(y_1) - 1.2 = \mathbb{E}(y_0) - 1.2(2)$$
$$\mathbb{E}(y_3) = \mathbb{E}(y_2) - 1.2 = \mathbb{E}(y_0) - 1.2(3)$$
$$\vdots$$
$$\mathbb{E}(y_t) = \mathbb{E}(y_0) - 1.2t$$
$$= 1 - 1.2t$$

For $t \geq 0$, $\mathbb{E}(y_t) \to -\infty$ as $t \to \infty$

(c) $(1-0.95L)\,x_t = -1.2 + (1-0.6L)\,\varepsilon_t$

$x_0 = 1$

$\varepsilon_t \sim IID(0,1)$

We can rewrite the model as

$$x_t = 0.95\,x_{t-1} - 1.2 + \varepsilon_t - 0.6\,\varepsilon_{t-1}$$

Note that we do not have a unit root, this is an ARMA(1,1) stationary process

$$\mathbb{E}(x_1) = 0.95\,\mathbb{E}(x_0) - 1.2$$
$$= 0.95 - 1.2$$
$$= -0.25$$

$$\mathbb{E}(x_2) = 0.95\,\mathbb{E}(x_1) - 1.2$$
$$= 0.95(-0.25) - 1.2$$

$$\mathbb{E}(x_3) = 0.95\,\mathbb{E}(x_2) - 1.2$$
$$= 0.95(0.95(-0.25) - 1.2) - 1.2$$
$$= 0.95^2(-0.25) - 0.95(1.2) - 1.2$$
$$\mathbb{E}(x_4) = 0.95^3(-0.25) - 0.95^2(1.2) - (0.95)(1.2) - 1.2$$
$$\vdots$$
$$\mathbb{E}(x_t) = 0.95^t(-0.25) - \sum_{i=0}^{t-1} 0.95^i(1.2)$$

Taking the limit,

$$\lim_{t\to\infty}\left[0.95^t(-0.25) - \sum_{i=0}^{t-1} \underbrace{0.95^i(1.2)}_{\text{geometric series}}\right]$$

$$= 0 - \frac{1.2}{1-0.95}$$

$$= -24$$

Since $|\phi_1| = |0.95| < 1$, the process converges to its unconditional mean $\mu = \frac{\phi_0}{1-\phi_1} = \frac{-1.2}{1-0.95} = -24$ as $t \to \infty$.

4. $y_t = (1+0.5L)(1+0.3L^4)\varepsilon_t$, $\varepsilon_t \sim WN(0, \sigma^2)$

(a) Since this is a finite-order MA model, $\{y_t\}$ is weakly stationary.

$$\theta(z) = (1+0.5z)(1+0.3z^4)$$

$$z = -2,\ 0.955 \pm 0.955i$$

$z = -2$ outside the unit circle ✓

$R = \sqrt{0.955^2 + 0.955^2} = 1.35$ outside the unit circle ✓

$\Rightarrow \{y_t\}$ is invertible

(b) $y_t = (1 + 0.5L + 0.3L^4 + 0.15L^5)\varepsilon_t$

$$= \varepsilon_t + 0.5\varepsilon_{t-1} + 0.3\varepsilon_{t-4} + 0.15\varepsilon_{t-5}$$

This is an MA(5) process

Recall for an MA(q) process, $y_t = \sum_{i=0}^{q} \theta_i \varepsilon_{i-1}$

Then our model is given by

$$y_t = \sum_{i=0}^{5} \theta_i \varepsilon_{i-1},$$

where
$$\begin{cases} \theta_0 = 1 \\ \theta_1 = 0.5 \\ \theta_2 = 0 \\ \theta_3 = 0 \\ \theta_4 = 0.3 \\ \theta_5 = 0.15 \end{cases}$$

The autocovariance is $\gamma_h = \sigma^2 \sum_{i=0}^{5-h} \theta_i \theta_{i+|h|}$

So
$$\begin{aligned}
\gamma_0 &= \sigma^2 \sum_{i=0}^{5} \theta_i \theta_i \\
&= \sigma^2 (\theta_0^2 + \theta_1^2 + \underbrace{\cancel{\theta_2^2}}_{=0} + \underbrace{\cancel{\theta_3^2}}_{=0} + \theta_4^2 + \theta_5^2) \\
&= \sigma^2 (1^2 + 0.5^2 + 0.3^2 + 0.15^2) \\
&= 1.3625\,\sigma^2
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= \sigma^2 \sum_{i=0}^{4} \theta_i \theta_{i+1} \\
&= \sigma^2 (\theta_0\theta_1 + \underbrace{\cancel{\theta_1\theta_2}}_{=0} + \underbrace{\cancel{\theta_2\theta_3}}_{=0} + \underbrace{\cancel{\theta_3\theta_4}}_{=0} + \theta_4\theta_5) \\
&= \sigma^2 [(1)(0.5) + (0.3)(0.15)] \\
&= 0.545\,\sigma^2
\end{aligned}$$

$$\begin{aligned}
\gamma_2 &= \sigma^2 \sum_{i=0}^{3} \theta_i \theta_{i+2} \\
&= \sigma^2 (\underbrace{\cancel{\theta_0\theta_2}}_{=0} + \underbrace{\cancel{\theta_1\theta_3}}_{=0} + \underbrace{\cancel{\theta_2\theta_4}}_{=0} + \underbrace{\cancel{\theta_3\theta_5}}_{=0}) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\gamma_3 &= \sigma^2 \sum_{i=0}^{2} \theta_i \theta_{i+3} \\
&= \sigma^2 (\underbrace{\cancel{\theta_0\theta_3}}_{=0} + \theta_1\theta_4 + \underbrace{\cancel{\theta_2\theta_5}}_{=0}) \\
&= \sigma^2 [(0.5)(0.3)] \\
&= 0.15\,\sigma^2
\end{aligned}$$

$$\gamma_4 = \sigma^2 \sum_{i=0}^{1} \theta_i \theta_{i+4}$$
$$= \sigma^2 (\theta_0 \theta_4 + \theta_1 \theta_5)$$
$$= \sigma^2 [(1)(0.3) + (0.5)(0.15)]$$
$$= 0.375\sigma^2$$

$$\gamma_5 = \sigma^2 (\theta_0 \theta_5)$$
$$= \sigma^2 [(1)(0.15)]$$
$$= 0.15\sigma^2$$

$$\gamma_h = 0 \text{ for } h > 5$$

<u>ACF</u>

$$\rho_0 = 1$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{0.545\sigma^2}{1.3625\sigma^2} = 0.4$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = 0$$

$$\rho_3 = \frac{\gamma_3}{\gamma_0} = \frac{0.15\sigma^2}{1.3625\sigma^2} \approx 0.110$$

$$\rho_4 = \frac{\gamma_4}{\gamma_0} = \frac{0.375\sigma^2}{1.3625\sigma^2} \approx 0.275$$

$$\rho_5 = \frac{\gamma_5}{\gamma_0} = \frac{0.15\sigma^2}{1.3625\sigma^2} \approx 0.110$$

$$\rho_h = 0 \text{ for } h > 5$$

(c) Let $\tilde{y}_t = (1 + 0.5L)\varepsilon_t$
$$= \varepsilon_t + 0.5\,\varepsilon_{t-1}$$

$\tilde{y}_t$ is a MA(1) process

$$\begin{cases} \theta_0 = 1 \\ \theta_1 = 0.5 \end{cases}$$

Autocovariance

$$\tilde{\gamma}_0 = \sigma^2(\theta_0^2 + \theta_1^2)$$
$$= \sigma^2(1^2 + 0.5^2)$$
$$= 1.25\sigma^2$$

$$\tilde{\gamma}_1 = \sigma^2(\theta_0\theta_1)$$
$$= 0.5\sigma^2$$

$$\tilde{\gamma}_h = 0 \text{ for } h > 1$$

ACF

$$\tilde{\rho}_0 = 1$$
$$\tilde{\rho}_1 = \frac{\gamma_1}{\gamma_0} = \frac{0.5\sigma^2}{1.25\sigma^2} = 0.4$$

$$\tilde{\rho}_h = 0 \text{ for } h > 1$$

The ACF for $\tilde{y}_t$ becomes 0 after lag 1.
At lag 1, the ACF for $\tilde{y}_t$ is equivalent to the ACF of the original model y_t, i.e. $\rho_1 = \tilde{\rho}_1 = 0.4$.

Let $y_t^* = (1 + 0.3L^4)\varepsilon_t$
$$= \varepsilon_t + 0.3\varepsilon_{t-4}$$
y_t^* is a MA(4) process

$$\begin{cases} \theta_0 = 1 \\ \theta_1 = 0 \\ \theta_2 = 0 \\ \theta_3 = 0 \\ \theta_4 = 0.3 \end{cases}$$

Autocovariance

$$\gamma_0^* = \sigma^2(\theta_0^2 + \cancel{\theta_1^2} + \cancel{\theta_2^2} + \cancel{\theta_3^2} + \theta_4^2)$$
$$= \sigma^2(1^2 + 0.3^2)$$
$$= 1.09\sigma^2$$

$$\gamma_1^* = \sigma^2(\theta_0\cancel{\theta_1} + \theta_1\cancel{\theta_2} + \theta_2\cancel{\theta_3} + \theta_3\cancel{\theta_4})$$
$$= 0$$

$$\gamma_2^* = \sigma^2(\theta_0\cancel{\theta_2} + \theta_1\cancel{\theta_3} + \theta_2\cancel{\theta_4})$$
$$= 0$$

$$\gamma_3^* = \sigma^2(\theta_0\cancel{\theta_3} + \theta_1\cancel{\theta_4})$$
$$= 0$$

$$\gamma_4^* = \sigma^2(\theta_0\theta_4)$$
$$= 0.3\sigma^2$$

$$\gamma_h = 0 \text{ for } h > 4$$

ACF

$$\rho_0 = 1$$
$$\rho_1 = 0$$
$$\rho_2 = 0$$
$$\rho_3 = 0$$
$$\rho_4 = \frac{0.3\sigma^2}{1.09\sigma^2} \approx 0.275$$
$$\rho_h = 0 \text{ for } h > 4$$

The ACF for y_t^* becomes 0 after lag 4.
At lag 4, the ACF for y_t^* is equivalent to the ACF of the original model y_t, i.e. $\rho_4 = \rho_4^* = 0.275$.