Problem Set 2

1. ARMA(p,q) Model:

$$y_t = \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{i=0}^{q} \theta_i \varepsilon_{t-i} \ , \ \theta_0 = 1$$

Stationary if roots of

$$1 - \phi_1 z - \dots - \phi_p z^p = 0$$

lie outside the unit circle.

Invertible if roots of

$$1 + \theta_1 z + \dots + \theta_q z^q = 0$$

lie outside the unit circle

(a) $y_t = y_{t-1} + \varepsilon_t + \frac{1}{2}\varepsilon_{t-1} + \frac{1}{2}\varepsilon_{t-2}$

$\Rightarrow$ ARMA (1, 2)

$$1 - \phi_1 z = 1 - 1 = 0$$

$\Rightarrow$ not stationary

$$1 + \theta_1 z + \theta_2 z^2 = 1 + \frac{1}{2} z + \frac{1}{2} z^2 = 0$$

$$z^2 + z + 2 = 0$$

$$z = \frac{-1 \pm \sqrt{1 - 4(1)(2)}}{2}$$

$$= \frac{-1 \pm i\sqrt{7}}{2}$$

$$R = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{7}{4}} = \sqrt{2} > 1$$

$\Rightarrow$ invertible

(b) $y_t = 2y_{t-1} - 2y_{t-2} + \varepsilon_t - \frac{3}{4}\varepsilon_{t-1}$

$\Rightarrow$ ARMA(2,1)

$1 - \phi_1 z - \phi_2 z^2 = 1 - 2z + 2z^2 = 0$

$2z^2 - 2z + 1 = 0$

$z = \frac{2 \pm \sqrt{4 - 4(2)(1)}}{4}$

$= \frac{2 \pm 2i}{4}$

$R = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} < 1$

$\Rightarrow$ not stationary

$1 + \theta_1 z = 1 - \frac{3}{4}z = 0$

$z = \frac{4}{3} > 1$

$\Rightarrow$ invertible

(c) $y_t = \frac{5}{4}y_{t-1} + \frac{3}{2}y_{t-2} + \varepsilon_t - 2\varepsilon_{t-1}$

$\Rightarrow$ ARMA(2,1)

$1 - \phi_1 z - \phi_2 z^2 = 1 - \frac{5}{4}z - \frac{3}{2}z^2 = 0$

$6z^2 + 5z - 4 = 0$

$z = \frac{-5 \pm \sqrt{25 - 4(6)(-4)}}{12}$

$= \frac{-5 \pm 4\sqrt{6}}{12}$

$z_1 = -1.233 < 1$

$z_2 = 0.3998 < 1$

$\Rightarrow$ not stationary

$1 + \phi_1 z = 1 - 2z = 0$

$z = \frac{1}{2} < 1$

$\Rightarrow$ not invertible

2. $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$, $\varepsilon_t \sim WN(0, \sigma^2)$

Recall $E(\varepsilon_t^2) = \sigma^2$ and $\gamma_h = E[y_t y_{t-h}]$

$E(\varepsilon_t \varepsilon_{t-1}) = 0$ $\gamma_0 = E[y_t^2]$

$E(\varepsilon_t y_t) = \sigma^2$

$$\gamma_0 = E(y_t^2)$$
$$= \phi_1 E(y_t y_{t-1}) + \phi_2 E(y_t y_{t-2}) + E(y_t \varepsilon_t) + \theta_1 E(y_t \varepsilon_{t-1})$$

$$E(y_t \varepsilon_{t-1}) = \phi_1 E(y_{t-1} \varepsilon_{t-1}) + \theta_1 E(\varepsilon_{t-1}^2)$$
$$= \phi_1 \sigma^2 + \theta_1 \sigma^2$$

$$= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2 + \theta_1 (\phi_1 \sigma^2 + \theta_1 \sigma^2)$$
$$= \phi_1 \gamma_1 + \phi_2 \gamma_2 + (\theta_1 \phi_1 + \theta_1^2 + 1)\sigma^2$$

$$\gamma_1 = E(y_t y_{t-1})$$
$$= \phi_1 E(y_{t-1}^2) + \phi_2 E(y_{t-1} y_{t-2}) + \underbrace{E(\varepsilon_t y_{t-1})}_{=0} + \theta_1 E(\varepsilon_{t-1} y_{t-1})$$
$$= \phi_1 \gamma_0 + \phi_2 \gamma_1 + \theta_1 \sigma^2$$

$$\gamma_2 = E(y_t y_{t-2})$$
$$= \phi_1 E(y_{t-1} y_{t-2}) + \phi_2 E(y_{t-2}^2) + \underbrace{E(\varepsilon_t y_{t-2})}_{=0} + \theta_1 \underbrace{E(\varepsilon_{t-1} y_{t-2})}_{=0}$$
$$= \phi_1 \gamma_1 + \phi_2 \gamma_0$$

$E(\varepsilon_t y_{t-h}) = 0$ for $h \geq 1$

$E(\varepsilon_{t-1} y_{t-h}) = 0$ for $h \geq 2$

$\Rightarrow \gamma_h = \phi_1 \gamma_{h-1} + \phi_2 \gamma_{h-2}$ for $h \geq 2$

3. AR(2)

$$y_t = 3 + 0.3 y_{t-1} + 0.04 y_{t-2} + \varepsilon_t \quad , \quad \varepsilon_t \sim WN(0, .25)$$

$$y_{99} = 4$$

$$y_{100} = 3.5$$

(a) One-step forecast:

$$\hat{y}_{101|100} = 3 + 0.3\,E[y_{100} | \Omega_{100}] + .04\,E[y_{99} | \Omega_{100}] + \underbrace{\cancel{E[\varepsilon_{101} | \Omega_{100}]}}_{=0}$$

$$= 3 + 0.3 y_{100} + 0.04 y_{99}$$

$$= 3 + 0.3(3.5) + 0.04(4)$$

$$= 4.21$$

$$var(e_{101}) = \sigma^2 = 0.25$$

$$\sigma = \sqrt{0.25} = 0.5$$

$$CI = \hat{y}_{101|100} \pm 1.96\,\sigma \left(\sum_{i=0}^{0} \phi_1^{2i}\right)^{1/2}$$

$$= 4.21 \pm 1.96(0.5)(1)$$

$$= 4.21 \pm 0.98$$

Two-Step Forecast:

$$\hat{y}_{102|100} = 3 + 0.3\,E[y_{101} | \Omega_{100}] + 0.4\,E[y_{100} | \Omega_{100}]$$

$$= 3 + 0.3\,\hat{y}_{101|100} + 0.04\,y_{100}$$

$$= 3 + 0.3(4.21) + 0.04(3.5)$$

$$= 4.403$$

$$var(e_{102}) = \sigma^2(1 + \phi_1^2) = 0.25(1 + 0.3^2) = 0.2725$$

$$\tilde{\sigma} = \sqrt{0.2725} = 0.522$$

$$CI = \hat{y}_{102|100} \pm 1.96\,\tilde{\sigma} \left(\sum_{i=0}^{1} \phi_1^{2i}\right)^{1/2}$$

$$= 4.403 \pm 1.96(0.522)(1 + 0.3^2)$$

$$= 4.403 \pm 1.115$$

(b) 100-step forecast:

Since $|\phi_1|, |\phi_2| < 1$, we know

$$\hat{y}_{200|100} \rightarrow \mu = \frac{\phi_0}{1-\phi_1-\phi_2} = \frac{3}{1-0.3-0.04} = \boxed{4.545},$$

the unconditional mean.

Additionally,

$$var(e_{200}) \rightarrow \frac{\sigma^2}{1-\phi_1^2-\phi_2^2} = \frac{0.25}{1-0.09-0.0016} = \boxed{0.275}$$

4. ARMA (1,1)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

(a) One-Step Forecast:

$$\begin{aligned}
\hat{y}_{T+1|T} &= E[y_{T+1} | \Omega_T] \\
&= \phi_0 + \phi_1 E[y_T | \Omega_T] \\
&\qquad + E[\varepsilon_{T+1} \cancel{|} \Omega_T] + \theta_1 E[\varepsilon_T | \Omega_T] \\
&= \boxed{\phi_0 + \phi_1 y_T + \theta_1 \varepsilon_T}
\end{aligned}$$

$$\begin{aligned}
e_{T+1} &= y_{T+1} - \hat{y}_{T+1|T} \\
&= (\cancel{\phi_0} + \phi_1 \cancel{y_T} + \varepsilon_{T+1} + \theta_1 \cancel{\varepsilon_T}) \\
&\qquad - (\cancel{\phi_0} + \phi_1 \cancel{y_T} + \theta_1 \cancel{\varepsilon_T}) \\
&= \boxed{\varepsilon_{T+1}}
\end{aligned}$$

(b) $\hat{y}_{T+2|T} = E[y_{T+2} | \Omega_T]$

$$= \phi_0 + \phi_1 E[y_{T+1} | \Omega_T] + E[\cancel{\varepsilon_{T+2} | \Omega_T}] + \theta_1 E[\cancel{\varepsilon_{T+1} | \Omega_T}]$$

$$= \phi_0 + \phi_1 \hat{y}_{T+1}$$

$$= \phi_0 + \phi_1 \phi_0 + \phi_1^2 y_T + \phi_1 \theta_1 \varepsilon_T$$

$$\hat{y}_{T+3|T} = E[y_{T+3} | \Omega_T]$$

$$= \phi_0 + \phi_1 \hat{y}_{T+2}$$

$$= \underbrace{\phi_0 + \phi_1 \phi_0 + \phi_1^2 \phi_0}_{\text{geometric series}} + \phi_1^{\underbrace{3}_{=h}} y_T + \phi_1^{\underbrace{2}_{=h-1}} \theta_1 \varepsilon_T$$

At $h \geq 2$, the expected error terms are 0.

By forward iteration,

$$\hat{y}_{T+h|T} = E[y_{T+h} | \Omega_T] = \mu$$

$$= \phi_0 \left(\frac{1 - \phi_1^h}{1 - \phi_1} \right) + \phi_1^h y_T + \phi_1^{h-1} \theta_1 \varepsilon_T$$

If $|\phi_1| < 1$, then $\phi_1^h \to 0$ as $h \to \infty$.

Therefore

$$\lim_{h \to \infty} \hat{y}_{T+h|T} = \phi_0 \left(\frac{1}{1 - \phi_1} \right) = \mu,$$

meaning the h-step forecast decays exponentially to the unconditional mean,

$$\mu = \frac{\phi_0}{1 - \phi_1}.$$

(c)

$$
\begin{aligned}
e_{T+1} &= y_{T+1} - \hat{y}_{T+1|T} \\
&= (\cancel{\phi_0} + \cancel{\phi_1 y_T} + \varepsilon_{T+1} + \cancel{\theta_1 \varepsilon_T}) \\
&\quad - (\cancel{\phi_0} + \cancel{\phi_1 y_T} + \cancel{\theta_1 \varepsilon_T}) \\
&= \varepsilon_{T+1}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{var}(e_{T+1}) &= \mathrm{var}(\varepsilon_{T+1}) \\
&= \sigma^2
\end{aligned}
$$

$$
\begin{aligned}
e_{T+2} &= y_{T+2} - \hat{y}_{T+2|T} \\
&= (\cancel{\phi_0} + \phi_1 y_{T+1} + \varepsilon_{T+2} + \theta_1 \varepsilon_{T+1}) \\
&\quad - (\cancel{\phi_0} + \phi_1 \phi_0 + \phi_1^2 y_T + \phi_1 \theta_1 \varepsilon_T) \\
&= (\cancel{\phi_1 \phi_0} + \cancel{\phi_1^2 y_T} + \phi_1 \varepsilon_{T+1} + \cancel{\phi_1 \theta_1} \varepsilon_T \\
&\quad + \varepsilon_{T+2} + \theta_1 \varepsilon_{T+1}) - (\cancel{\phi_1 \phi_0} + \cancel{\phi_1^2 y_T} + \cancel{\phi_1 \theta_1} \varepsilon_T) \\
&= \varepsilon_{T+2} + (\theta_1 + \phi_1)\varepsilon_{T+1}
\end{aligned}
$$

$$
\mathrm{var}(e_{T+2}) = (1 + \theta_1 + \phi_1)\sigma^2
$$

$$
\begin{aligned}
e_{T+3} &= y_{T+3} - \hat{y}_{T+3|T} \\
&= (\cancel{\phi_0} + \phi_1 y_{T+2} + \varepsilon_{T+3} + \theta_1 \varepsilon_{T+2}) \\
&\quad - (\cancel{\phi_0} + \phi_1 \phi_0 + \phi_1^2 \phi_0 + \phi_1^3 y_T + \phi_1^2 \theta_1 \varepsilon_T) \\
&= (\cancel{\phi_1 \phi_0} + \phi_1^2 y_{T+1} + \phi_1 \varepsilon_{T+2} + \phi_1 \theta_1 \varepsilon_{T+1} \\
&\quad + \varepsilon_{T+3} + \theta_1 \varepsilon_{T+2}) \\
&\quad - (\cancel{\phi_1 \phi_0} + \phi_1^2 \phi_0 + \phi_1^3 y_T + \phi_1^2 \theta_1 \varepsilon_T) \\
&= \cancel{\phi_1^2 \phi_0} + \cancel{\phi_1^3 y_T} + \phi_1^2 \varepsilon_{T+1} + \cancel{\phi_1^2 \theta_1} \varepsilon_T \\
&\quad + \phi_1 \varepsilon_{T+2} + \phi_1 \theta_1 \varepsilon_{T+1} + \varepsilon_{T+3} + \theta_1 \varepsilon_{T+2}) \\
&\quad - (\cancel{\phi_1^2 \phi_0} + \cancel{\phi_1^3 y_T} + \cancel{\phi_1^2 \theta_1} \varepsilon_T) \\
&= \varepsilon_{T+3} + (\phi_1 + \theta_1)\varepsilon_{T+2} + \phi_1(\phi_1 + \theta_1)\varepsilon_{T+1}
\end{aligned}
$$

$$var(e_{T+3}) = (1 + \Phi_1 + \Phi_1 + \Phi_1\theta_1 + \Phi_1^2)\sigma^2$$

The coefficient of σ^2 increases with each forecast. Hence the variance grows as $h \to \infty$.