Problem Set 4

1. GARCH(1,1)

$$
\begin{aligned}
h_t &= a_0 + a_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1} \\
&= a_0 + a_1 \varepsilon_{t-1}^2 + \beta_1 (a_0 + a_1 \varepsilon_{t-2}^2 + \beta_1 h_{t-2}) \\
&= a_0 (1+\beta_1) + a_1 (\varepsilon_{t-1}^2 + \beta_1 \varepsilon_{t-2}^2) + \beta_1^2 h_{t-2} \\
&= a_0 (1+\beta_1+\beta_1^2) + a_1 (\varepsilon_{t-1}^2 + \beta_1 \varepsilon_{t-2}^2 + \beta_1^2 \varepsilon_{t-3}^2) + \beta^3 h_{t-3} \\
&\vdots \\
&= \frac{a_0}{1-\beta_1} + a_1 \sum_{i=0}^{\infty} \beta_1^i \varepsilon_{t-1-i}^2
\end{aligned}
$$

Therefore the GARCH(1,1) model can be expressed as a GARCH(0,∞) model, which is equivalent to the ARCH(∞) model.

2. $y_t = \Phi_0 + \Phi_1 y_{t-1} + \varepsilon_t$

$$
\left.
\begin{aligned}
&\varepsilon_t = \sqrt{h_t}\, v_t \,, \quad v_t \sim IID(0,1) \\
&h_t = a_0 + a_1 \varepsilon_{t-1}^2 + a_2 \varepsilon_{t-2}^2
\end{aligned}
\right\} \text{ARCH(2) model}
$$

$|\Phi_t| < 1, \; a_0 > 1, \; a_i \geq 0, \; a_1 + a_2 < 1$

(a) $\Omega_{t-1} = \{y_{t-1}, y_{t-2}, \dots\}$

Conditional mean:

$$
\begin{aligned}
\hat{y}_{t|t-1} = E_{t-1} y_t &= E(\Phi_0 + \Phi_1 y_{t-1} \mid \Omega_{t-1}) \\
&= \Phi_0 + \Phi_1 y_{t-1}
\end{aligned}
$$

Conditional variance:

$$
\begin{aligned}
var(y_t \mid \Omega_{t-1}) &= E_{t-1}(y_t - E_{t-1} y_t)^2 \\
&= E_{t-1}(y_t - \Phi_0 - \Phi_1 y_{t-1})^2 \\
&= E_{t-1}(\varepsilon_t^2) \\
&= h_t \\
&= a_0 + a_1 \varepsilon_{t-1}^2 + a_2 \varepsilon_{t-2}^2
\end{aligned}
$$

(b) Unconditional mean:

$$\mu = E(y_t) = E(\phi_0 + \phi_1 y_{t-1} + \varepsilon_t)$$
$$= \phi_0 + \phi_1 \mu + 0$$
$$\Rightarrow \mu = E(y_t) = \frac{\phi_0}{1-\phi_1}$$

Unconditional variance:

Let $x_t = y_t - \mu$ be the mean adjusted series.

$$\Rightarrow x_t = \cancel{\phi_0} + \phi_1 y_{t-1} + \varepsilon_t - \cancel{\phi_0} - \phi_1 \mu$$
$$= \varepsilon_t + \phi_1 (y_{t-1} - \mu)$$
$$= \varepsilon_t + \phi_1 (\cancel{\phi_0} + \phi_1 y_{t-2} + \varepsilon_{t-1} - \cancel{\phi_0} - \phi_1 \mu)$$
$$= \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 (y_{t-2} - \mu)$$
$$\vdots$$
$$= \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}$$

So
$$\text{var}(x_t) = \text{var}(y_t - \mu)$$
$$= \text{var}(y_t) + \text{var}(\mu) - 2\,\text{cov}(y_t, \mu)$$
$$= \text{var}(y_t)$$

and
$$\text{var}(x_t) = \text{var}\left(\sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}\right)$$
$$= \sum_{i=0}^{\infty} \underbrace{\phi_1^{2i}}\, \underbrace{\text{var}(\varepsilon_{t-i})}$$
$$= \left(\frac{1}{1-\phi_1^2}\right)\left(\frac{a_0}{1-a_1-a_2}\right)$$

$|\phi_1| < 1 \rightarrow$

Since $\text{var}(\varepsilon_t) = \dfrac{a_0}{1-a_1-\cdots-a_q}$ for an ARCH(q) model

3. GARCH(1,2)

$$\varepsilon_t \sim \sqrt{h_t} v_t \,, \quad v_t \sim IID(0,1)$$

$$h_t = a_0 + a_1 \varepsilon_{t-1}^2 + a_2 \varepsilon_{t-2}^2 + \beta_1 h_{t-1}$$

$|\Phi_1| < 1$, $a_0 > 0$, $a_i \geq 0$ for $i = 1,2$, $\beta_1 \geq 0$, $a_1 + a_2 + \beta_1 < 1$

(a)
$$\begin{aligned} E(\varepsilon_t^2 \mid \varepsilon_{t-1}, \varepsilon_{t-2}, \dots) &= E(h_t v_t^2 \mid \varepsilon_{t-1}, \varepsilon_{t-2}, \dots) \\ &= h_t E(v_t^2 \mid \varepsilon_{t-1}, \varepsilon_{t-2}, \dots) \\ &= h_t E(v_t^2) \leftarrow \sigma^2 \text{ (second moment)} \\ &= h_t (1) \\ &= h_t = var(\varepsilon_t \mid \Omega_{t-1}) \end{aligned}$$

(b) Unconditional Variance:

Since $a_1 + a_2 + \beta_1 < 1$, the process is weakly stationary. Therefore $E(h_t) = E(\varepsilon_t^2) = \sigma_t^2$ and we can write

$$\begin{aligned} \sigma_t^2 = var(\varepsilon_t) &= E(var(\varepsilon_t \mid \Omega_{t-1})) + var(\underbrace{E(\varepsilon_t \mid \Omega_{t-1})}_{=0}) \\ &= E(h_t) \\ &= E(a_0 + a_1 \varepsilon_{t-1}^2 + a_2 \varepsilon_{t-2}^2 + \beta_1 h_{t-1}) \\ &= a_0 + a_1 \sigma_t^2 + a_2 \sigma_t^2 + \beta_1 \sigma_t^2 \end{aligned}$$

$$\Rightarrow \sigma_t^2 = var(\varepsilon_t) = \frac{a_0}{1 - a_1 - a_2 - \beta_1}$$

(c) h-step ahead forecasts of h_{t+h} at forecast origin t, for $h \geq 1$

$$h_{t+1} = a_0 + a_1 \varepsilon_t^2 + a_2 \varepsilon_{t-1}^2 + \beta_1 h_t$$

$$\Rightarrow \hat{h}_{t+1|t} = a_0 + a_1 \varepsilon_t^2 + a_2 \varepsilon_{t-1}^2 + \beta_1 h_t$$

$$h_{t+2} = a_0 + a_1 \varepsilon_{t+1}^2 + a_2 \varepsilon_t^2 + \beta_1 h_{t+1}$$

$$\begin{aligned} \Rightarrow \hat{h}_{t+2|t} &= a_0 + a_1 E(\varepsilon_{t+1}^2 \mid \Omega_t) + a_2 \varepsilon_t^2 + \beta_1 E(h_{t+1} \mid \Omega_t) \\ &= a_0 + (a_1 + \beta_1) \hat{h}_{t+1|t} + a_2 \varepsilon_t^2 \end{aligned}$$

$$h_{t+3} = a_0 + a_1 \varepsilon_{t+2}^2 + a_2 \varepsilon_{t+1}^2 + \beta_1 h_{t+2}$$

$$\Rightarrow \hat{h}_{t+3|t} = a_0 + (a_1 + \beta_1) \hat{h}_{t+2|t} + a_2 \hat{h}_{t+1|t}$$

$$\therefore \hat{h}_{t+h|t} = a_0 + (a_1 + \beta_1) \hat{h}_{t+h-1|t} + a_2 \hat{h}_{t+h-2|t}$$

4. $\vec{y}_t = (y_{1t}, y_{2t})^T$ $\quad y_t = \Phi_0 + \Phi y_{t-1} + \varepsilon_t$

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

where $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})^T \sim WN(0, \Sigma)$ with

$$\Sigma = \begin{pmatrix} 1.0 & 0.7 \\ 0.7 & 0.5 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$$

(a) $$|\Phi - \lambda I_2| = \det\left[\begin{pmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \det\begin{pmatrix} 0.2-\lambda & 0.3 \\ -0.6 & 1.1-\lambda \end{pmatrix}$$

$$= (0.2-\lambda)(1.1-\lambda) - (-0.6)(0.3)$$

$$= (0.22 - 1.1\lambda - 0.2\lambda + \lambda^2) + 0.18$$

$$= \lambda^2 - 1.3\lambda + 0.40$$

$$= 0$$

$$\Rightarrow \lambda_{1,2} = \frac{1.3 \pm \sqrt{1.3^2 - 4(0.40)}}{2}$$

$$= \frac{1.3 \pm 0.3}{2}$$

$$\Rightarrow |\lambda_1| = |0.8| < 1, \quad |\lambda_2| = |0.5| < 1$$

$\therefore$ The roots of the equation lie inside the unit circle, meaning $\vec{y}_t$ is weakly stationary.

(b) The mean vector of $\vec{y}_t$ is

$$\vec{\mu} = E(\vec{y}_t) = (I_2 - \Phi)^{-1}\vec{\phi}_0$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{pmatrix}\right]^{-1} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 & -0.3 \\ 0.6 & -0.1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$= \frac{1}{-0.08+0.18} \begin{pmatrix} -0.1 & 0.3 \\ -0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$= \frac{1}{0.10} \begin{pmatrix} 0.2 \\ -0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -8 \end{pmatrix}$$

(c) $T = 100$, $y_{1,100} = 1$, $y_{2,100} = 0.5$

The forecasts of $y_{1,101}$ and $y_{2,101}$ are

$$\hat{y}_{101|100} = \phi_0 + \Phi y_{100}$$

$$= \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 0.35 \\ -0.05 \end{pmatrix}$$

$$= \begin{pmatrix} 4.35 \\ 1.95 \end{pmatrix} = \begin{pmatrix} y_{1,101} \\ y_{2,101} \end{pmatrix}$$

and the associated standard errors are given by the matrix Σ, where

$$\sigma_1 = \sqrt{\sigma_1^2} = \sqrt{1.0} = 1.0$$

$$\sigma_2 = \sqrt{\sigma_2^2} = \sqrt{0.5} \approx 0.71$$

(d) coefficient matrices on...

$$\varepsilon_t \rightarrow \Phi^0 \equiv I_2$$

$$\varepsilon_{t-1} \rightarrow \Phi^1 = \begin{pmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{pmatrix}$$

$$\varepsilon_{t-2} \rightarrow \Phi^2 = \begin{pmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{pmatrix} = \begin{pmatrix} -0.14 & 0.39 \\ -0.58 & 1.03 \end{pmatrix}$$