Problem Set 5

1. $y_t = \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \varepsilon_t$, $\varepsilon_t \sim WN(0, \Sigma)$, $y_t \in \mathbb{R}^{k \times 1}$

(a) The companion form of the VAR(2) model:

$$\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} = \begin{pmatrix} \Phi_1 & \Phi_2 \\ I_k & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix}$$

where I_k is the $(k \times k)$ identity matrix.

This is a VAR(1) model for Y_t:

$$Y_t = \Phi Y_{t-1} + U_t$$

where $Y_t = (y_t', y_{t-1}')'$, $U_t = (\varepsilon_t', 0)'$ are $(2k \times 1)$ vectors and Φ is a $(2k \times 2k)$ matrix.

(b) y_t is weakly stationary if the eigenvalues of Φ are less than 1 in modulus.

Solve

$$0 = |\Phi - \lambda I_{2k}| = \det\left[\begin{pmatrix} \Phi_1 - \lambda I_k & \Phi_2 \\ I_k & -\lambda I_k \end{pmatrix}\right]$$

$$= \det(-\lambda I_k)\det(\Phi_1 - \lambda I_k - \Phi_2(-\lambda I_k)^{-1} I_k)$$

Simplify the following:

① $-\lambda I_k = \mathrm{diag}(-\lambda \; -\lambda \cdots -\lambda)$

$$\Rightarrow \det(-\lambda I_k) = \prod_{i=1}^{k}(-\lambda) = (-1)^k \lambda^k$$

② $\Phi_2(-\lambda I_k)^{-1} I_k = \Phi_2 (-\lambda)^{-1} I_k^{-1} I_k = \frac{1}{\lambda}\Phi_2$

$$\Rightarrow \det(\Phi_1 - \lambda I_k - \Phi_2(-\lambda I_k)^{-1} I_k)$$

$$= \det(\Phi_1 - \lambda I_k + \frac{1}{\lambda}\Phi_2)$$

$$= \det\left[\frac{1}{\lambda}(\lambda \Phi_1 - \lambda^2 I_k + \Phi_2)\right]$$

$$= \frac{1}{\lambda^k}\det(\Phi_2 + \lambda\Phi_1 - \lambda^2 I_k),$$

$$= \frac{1}{\lambda^k}\det[-(\lambda^2 I_k - \lambda\Phi_1 - \Phi_2)]$$

$$= \frac{(-1)^k}{\lambda^k}\det(\lambda^2 I_k - \lambda\Phi_1 - \Phi_2)$$

by the scalar property of determinants.
The above expressions allow us to simplify

$$
\begin{aligned}
0 = |\Phi - \lambda I_k| &= (-1)^k \lambda_k \frac{(-1)^k}{\lambda^k} \det(\lambda^2 I_k - \lambda\Phi_1 - \Phi_2) \\
&= (-1)^{2k} \det(\lambda^2 I_k - \lambda\Phi_1 - \Phi_2) \\
&= \det(\lambda^2 I_k - \lambda\Phi_1 - \Phi_2) \\
&= |I_k \lambda^2 - \Phi_1 \lambda - \Phi_2|,
\end{aligned}
$$

which brings us to the characteristic polynomial equation for finding the roots λ (i.e. eigenvalues) of y_t. When $|\lambda| < 1$, for all λ, the given VAR(2) model is weakly stationary.

(c) Take

$$
\begin{aligned}
y_t &= \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \varepsilon_t \\
&= \Phi_1 L y_t + \Phi_2 L^2 y_t + \varepsilon_t
\end{aligned}
$$

$$\Rightarrow (I_k - \Phi_1 L - \Phi_2 L^2) y_t = \varepsilon_t$$

The MA coefficients Ψ_i are found by solving

$$(I_k - \Phi_1 L - \Phi_2 L^2)(I_k + \Psi_1 L + \Psi_2 L^2 + \Psi_3 L^3 + \cdots) = I_k$$

$$\Rightarrow \Phi(L)\Psi(L) = I_k$$

$$\Psi(L) = \Phi^{-1}(L)$$

Equating the coefficients of L, L^2, L^3, we get

$$
\begin{aligned}
\Psi_1 &= \Phi_1, \\
\Psi_2 &= \Phi_1 \Psi_1 + \Phi_2, \\
\Psi_3 &= \Phi_1 \Psi_2 + \Phi_2 \Psi_1.
\end{aligned}
$$

2. (a)

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.6 \\ 0.25 & 0.5 \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \begin{pmatrix} 0.1 & 0.4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_{1,t-2} \\ y_{2,t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

$\varepsilon_t \sim WN(0, \Sigma)$

Using our results from part 1 (b),

$$0 = |\Phi - \lambda I_2| = \det\left(I_2 \lambda^2 - \Phi_1 \lambda - \Phi_2\right)$$

$$= \det\begin{pmatrix} \lambda^2 - 0.4\lambda - 0.1 & -0.6\lambda - 0.4 \\ -0.25\lambda & \lambda^2 - 0.5\lambda \end{pmatrix}$$

Check $\lambda = 1$:

$$\det\begin{pmatrix} 1 - 0.4 - 0.1 & -0.6 - 0.4 \\ -0.25 & 1 - 0.5 \end{pmatrix}$$

$$= \det\begin{pmatrix} 0.5 & -1 \\ -0.25 & 0.5 \end{pmatrix}$$

Note $\mathrm{col}(1) = -\frac{1}{2}\mathrm{col}(2)$
$\Rightarrow$ linearly dependent columns $\Rightarrow \det = 0$

$$= (0.5)(0.5) - (-1)(-0.25)$$
$$= 0.25 - 0.25$$
$$= 0$$

$\therefore \lambda = 1$ is a root of the determinant
$\Rightarrow y_t$ is unit root non-stationary

(b) VECM:

$$\Delta y_t + y_{t-1} = \Phi_1 y_{t-1} + \Phi_2 (y_{t-1} - \Delta y_{t-1}) + \varepsilon_t$$

$$\Rightarrow \Delta y_t = (-I_k + \Phi_1 + \Phi_2) y_{t-1} - \Phi_2 \Delta y_{t-1} + \varepsilon_t$$

Let $\Pi = I_k - \Phi_1 - \Phi_2$, $\Gamma_1 = -\Phi_2$.

$$\Rightarrow \Delta y_t = -\Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + \varepsilon_t.$$

$$\therefore \begin{pmatrix} y_{1t} - y_{1,t-1} \\ y_{2t} - y_{2,t-1} \end{pmatrix} = \begin{pmatrix} 0.5 & -1 \\ -0.25 & 0.5 \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \begin{pmatrix} -0.1 & -0.4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_{1,t-1} - y_{1,t-2} \\ y_{2,t-1} - y_{2,t-2} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

(c) Long-Run Relation:

In part (a), we showed the model is a bivariate cointegrated VAR process.

$\Rightarrow \exists$ (2×1) vector $\beta = (\beta_1, \beta_2)'$ such that

$$\beta' y_t = \beta_1 y_{1t} + \beta_2 y_{2t} \sim I(0).$$

Normalize β to $\beta = (I_2, -\tilde{\beta})$. Then

$$\beta' y_t = y_{1t} - \tilde{\beta} y_{2t},$$

which implies

$$y_{1t} = \tilde{\beta} y_{2t} \text{ (long-run equilibrium relation).}$$

Let $\alpha = (\alpha_1, \alpha_2)'$ be a vector of coefficients.

Using matrix Π from part (b), we solve

$$\Pi = \alpha \beta'$$

$$\begin{pmatrix} 0.5 & -1 \\ -0.25 & 0.5 \end{pmatrix} = \begin{pmatrix} \alpha_1 & -\alpha_1 \tilde{\beta} \\ \alpha_2 & -\alpha_2 \tilde{\beta} \end{pmatrix}$$

$$\alpha_1 = 0.5 \Rightarrow -(0.5)\tilde{\beta} = -1 \Rightarrow \tilde{\beta} = 2 \ \checkmark$$

$$\alpha_2 = -0.25 \Rightarrow -(-0.25)\tilde{\beta} = 0.5 \Rightarrow \tilde{\beta} = 2 \ \checkmark$$

$\therefore \ \beta = (I_2, -2)$

$$\Rightarrow y_{1t} = -2 y_{2t}$$

3. Trivariate Process

$$Y_t = \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ -0.16 & 1.08 & 0.08 \\ 0.24 & -0.12 & 0.88 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix}$$

$$\varepsilon_t \sim (\varepsilon_{1t} \ \varepsilon_{2t} \ \varepsilon_{3t})' \sim WN(0, \Sigma), \quad \Sigma \text{ SPD}$$

(a) VECM:

$$\begin{pmatrix} x_t - x_{t-1} \\ y_t - y_{t-1} \\ z_t - z_{t-1} \end{pmatrix} + \begin{pmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ -0.16 & 1.08 & 0.08 \\ 0.24 & -0.12 & 0.88 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix}$$

$$\begin{pmatrix} \Delta x_t \\ \Delta y_t \\ \Delta z_t \end{pmatrix} = \left[\begin{pmatrix} 0.8 & 0.1 & 0.1 \\ -0.16 & 1.08 & 0.08 \\ 0.24 & -0.12 & 0.88 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix}$$

$$= \begin{pmatrix} -0.2 & 0.1 & 0.1 \\ -0.16 & 0.08 & 0.08 \\ 0.24 & -0.12 & -0.92 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix}$$

$$\Rightarrow \Delta Y_t = -\Pi Y_{t-1} + \varepsilon_t$$

where $$\Pi = \begin{pmatrix} 0.2 & -0.1 & -0.1 \\ 0.16 & -0.08 & -0.08 \\ -0.24 & 0.12 & 0.92 \end{pmatrix}$$

(b) Reduced Row-Echelon form:

$$\begin{matrix} R2-0.8R1 \\ R3+1.2R1 \end{matrix} \rightarrow \begin{pmatrix} 0.2 & -0.1 & -0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \text{Rank}(\Pi) = 1 < 3 \Rightarrow m = 1,$$

where m is the number of cointegrating relations

Write $\underset{(3\times3)}{\Pi} = \underset{(3\times1)}{\alpha} \cdot \underset{(1\times3)}{\beta'}$,

where Rank(α) = Rank(β) = 1

The VECM model in part 3(a) becomes

$$\Delta Y_t = -\alpha\beta' Y_{t-1} + \varepsilon_t$$

Let $\beta = (I_1, \beta_1')'$, $\underset{(2\times1)}{\beta_1} = \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \end{pmatrix}$, such that

$$\beta' Y_t = x_t + \tilde{\beta}_1 y_t + \tilde{\beta}_2 z_t \sim I(0)$$

$$\Rightarrow \Pi = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} (1 \ \tilde{\beta}_1 \ \tilde{\beta}_2) = \begin{pmatrix} 0.2 & -0.1 & -0.1 \\ 0.16 & -0.08 & -0.08 \\ -0.24 & 0.12 & 0.92 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha_1 = 0.2 \\ \alpha_1\tilde{\beta}_1 = 0.2\tilde{\beta}_1 = -0.1 \Rightarrow \tilde{\beta}_1 = -0.5 = \tilde{\beta}_2 \end{cases}$$

* Normalized cointegrating vector:
$$\beta = (1 \ -0.5 \ -0.5)'$$
* Cointegrating relation:
$$\beta' Y_t = x_t - 0.5y_t - 0.5z_t$$
* Long-run relation:
$$x_t = 0.5y_t + 0.5z_t$$
* Speed of adjustment coefficients:
$$\alpha = (0.2 \ 0.16 \ -0.24)'$$

(c) Since we assume the standard regularity conditions hold, then OLS estimation of the model will yield consistent and efficient estimates of the VAR coefficients.

→ Consistent Estimates:

The series $\{x_t\}$, $\{y_t\}$, and $\{z_t\}$ are cointegrated such that a stationary ($I(0)$) linear combination exists, so we have consistent estimates of the cointegrating vector and adjustment parameters.

→ Efficient Estimates:

Every equation includes the same set of lagged variables, so we do not need to estimate each equation separately by OLS.