Finite Element Method – Mixed Formulation

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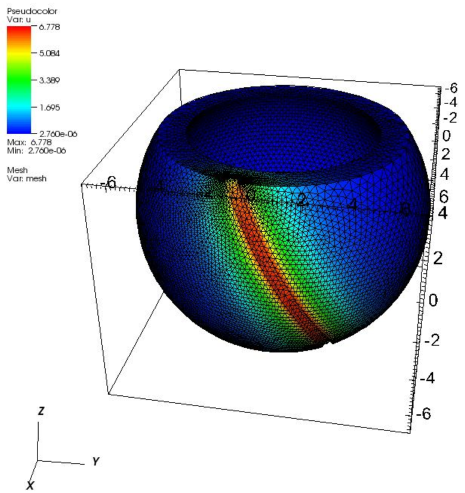
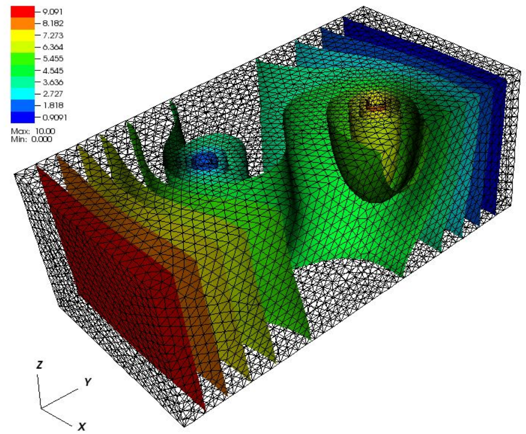
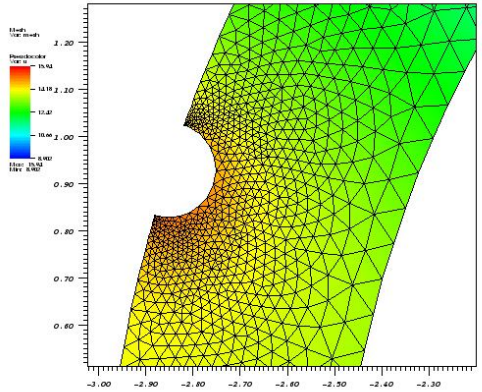


Luty-marzec, 2013



Finite Element Method



- Original problem:

$$\boxed{\begin{cases} -\nabla \cdot K\, \nabla u = f & \text{in} \quad \Omega \\ u = 0 \quad \text{in} \quad \Gamma \end{cases}}$$

- **Weak formulation** (variational problem)

Find $u \in H_g^1(\Omega)$ such that

$$\int_\Omega \nabla u\; K\; \nabla \omega\; d\underline{x} = \int_\Omega f\; \omega\; d\underline{x}, \quad \forall \omega \in H_0^1(\Omega)$$

- Approximated solution $u_h \in H_{gh}^1(\Omega_h)$ is represented as a linear combination of basic functions:

$$u_h = \sum_{i=1}^{n_i} \alpha_i\; \varphi_i(\underline{x}).$$

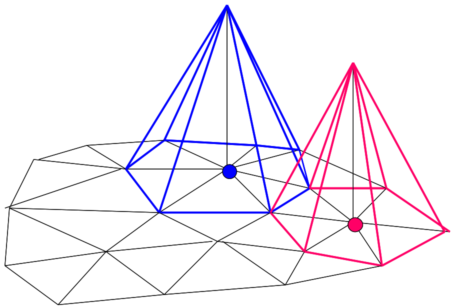
- **Linear** approximation P1 – elementwise linear functions

Any function in $H_h^1(\Omega_h)$ is uniquely defined by its values in nodes of discretization.

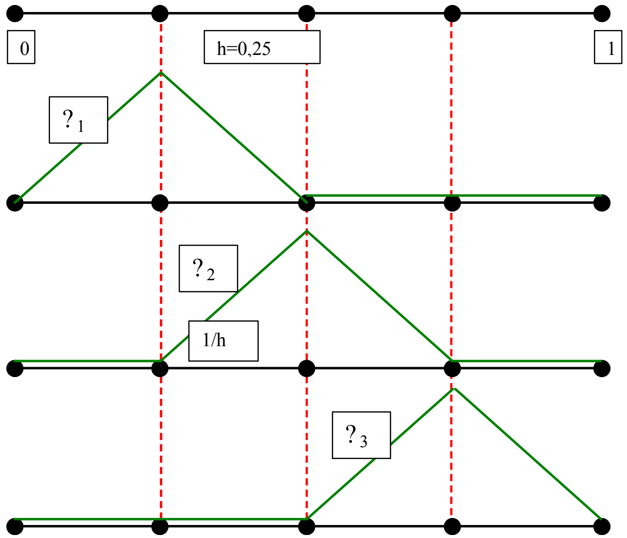
- **Construction of basis in** $H_h^1(\Omega_h)$

Basis function φ_i, connected to a node *i* of a triangulation T_h is:

- continuous function $\Omega_h \to R$
- linear on element
- for any φ_i, $\varphi_i(\underline{x}_j) = \delta_{ij} = \begin{cases} 1 & , \quad i = j \\ 0 & , \quad i \neq j \end{cases}$, $\underline{x}_j$ – node.

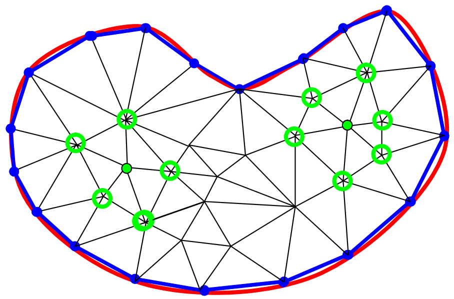


$$\boxed{\sum_{i=1}^{n} \varphi_i = 1 \text{ for } \forall \underline{x} \in \Omega_h}$$



Substituting, we get a system of linear equations $\mathbf{A}\,\boldsymbol{\alpha} = \mathbf{F}$ with unknowns $\alpha_i,\; i = 1,\ldots,n_i$.

Matrix elements:

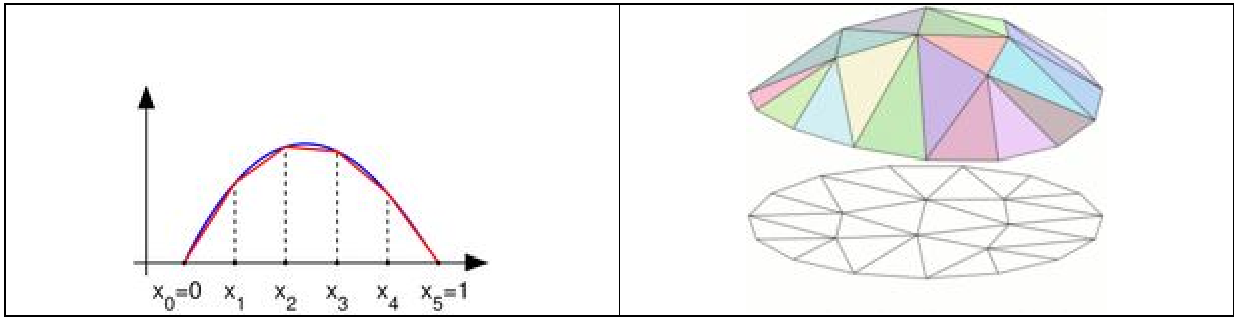


$$\mathbf{A}_{ij} = \int_\Omega \nabla\varphi_i(\underline{x}) \cdot K\, \nabla\varphi_j(\underline{x})\, d\underline{x}\,,\;\; i,\, j = 1,\ldots,n_i$$

RHS vector:

$$\mathbf{F}_i = \int_\Omega f(\underline{x})\,\varphi_i(\underline{x})\,d\underline{x}$$

- A is **sparse**.



Piecewise constant derivatives, what may cause problems while solving transport equation, tracing streamlines, etc., (mass conservation does not hold within an element).

Mixed formulation brings a remedy

Finite elements – Mixed formulation

- Original problem: Find $u : \Omega \to R$ such that:

(1)
$$\begin{cases} -\nabla \cdot K\,\nabla u = f \quad in \quad \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

- Apply an equivalent system of equations:

(2)
$$\begin{cases} \mathbf{q} = -K\,\nabla u \\ \nabla \cdot \mathbf{q} = f \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

(3)
$$\begin{cases} K^{-1}\,\mathbf{q} + \nabla u = 0 \\ \nabla \cdot \mathbf{q} = f \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

Two unknown functions: u (pressure, head), **q** (velocity, volumetric flux)

Standard procedurę -→ towards variational formulation:

First equation of (3) is multiplied by a vector function $\mathbf{v} \in H^1_{0N}(div;\Omega)$, while the second by a scalar function $\varphi \in L^2(\Omega)$ and all is integrated over Ω.

(4) $$\boxed{\begin{cases} \int\limits_\Omega K^{-1}\,\mathbf{q}\cdot\mathbf{v} + \int\limits_\Omega \nabla u \cdot \mathbf{v} = 0 \\ \int\limits_\Omega \nabla\cdot\mathbf{q}\cdot\varphi = \int\limits_\Omega f\cdot\varphi \\ u\big|_{\partial\Omega} = 0 \end{cases}}$$

Green's Theorem is applied to the second term of the first equation:

$$\boxed{\int\limits_\Omega \nabla u\cdot\mathbf{v} = -\int\limits_\Omega u\cdot\nabla\cdot\mathbf{v} + \int\limits_{\partial\Omega} u\cdot\mathbf{n}\cdot\mathbf{v}}$$

Weak formulation in the Mixed form:

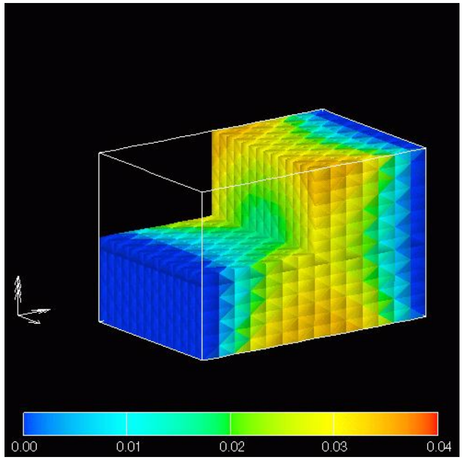
Find $(\mathbf{q}, u) \in H^1_{0N}(div; \Omega) \times L^2(\Omega)$ such that

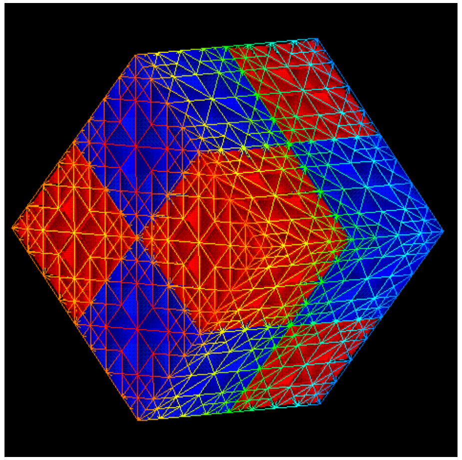
(5)
$$\boxed{\begin{cases} \int\limits_{\Omega} K^{-1}\, \mathbf{q} \cdot \mathbf{v} - \int\limits_{\Omega} u \cdot \nabla \cdot \mathbf{v} = 0 & \forall\, \mathbf{v} \in H^1_{0N}(div; \Omega) \\ -\int\limits_{\Omega} \nabla \cdot \mathbf{q} \cdot \varphi = -\int\limits_{\Omega} f \cdot \varphi & \forall \varphi \in L^2(\Omega) \end{cases}}$$

Under appropriate assumptions, there exists a unique solution of the problem.

Solutions belong to spaces which should be **consistent**.

Condition Ładyżenska - Brezzi – Babuška.





Approximate representations:

(6a) $$\mathbf{q}_h = \sum_{i=1}^{m} \alpha_i \, \boldsymbol{\psi}_i$$

$\mathbf{q}_h$ and $\boldsymbol{\psi}_i$ are vectorial functions, coefficients α_i are scalars.

Degrees of freedom related to $\boldsymbol{\psi}_i$ are linked to edges (2D), or faces (3D) of elements.

(6b) $$u_h = \sum_{i=1}^{n} \beta_i \, \varphi_i$$

Degrees of freedom β_i are linked to elements.

Substituting (6) to (5):

(7) $$\boxed{\begin{cases} \displaystyle\sum_{i=1}^{m} \alpha_i \int_{\Omega} \left(K^{-1} \boldsymbol{\psi}_i\right)^T \cdot \boldsymbol{\psi}_j - \sum_{i=1}^{n} \beta_i \int_{\Omega} \varphi_i \, \nabla \cdot \boldsymbol{\psi}_j = 0 \\ \displaystyle -\sum_{i=1}^{m} \alpha_i \int_{\Omega} \nabla \cdot \boldsymbol{\psi}_i \, \varphi_k = -\int_{\Omega} f \, \varphi_k \end{cases}}$$

j=1,..., m (edges), k=1,...,n (elements)

It is a system of linear equations with *m+n* unknowns:

(8) $$\boxed{\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}}$$, where:

$$\boxed{A = \int_{\Omega} \left(K^{-1} \boldsymbol{\psi}_i\right)^T \cdot \boldsymbol{\psi}_j} , \quad \boxed{B = -\int_{\Omega} \nabla \cdot \boldsymbol{\psi}_i \, \varphi_j}$$

$\boxed{F_1 = 0}$ (here nonzero Dirichlet BC appears)

Remark: Neumann BC is strongly fulfilled in the Mixed formulation
Dirichlet BC is fulfilled in a weak sense (a natural BC).

$$\boxed{F_2 = -\int_{\Omega} f \, \varphi_j}$$

We need spaces now.

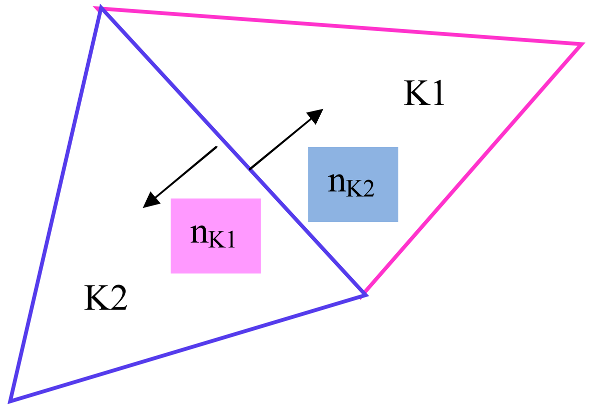
Discretization of Raviart-Thomas (1977).

1) u can be approximated by elementwise constant functions (no continuity across elements is needed):

$V_h = \{v|_K$ constant$\}$

2) Approximation of $H^1_{0N}(div;\Omega)$ is more complicated.

If **q** is smooth on an element $\left(\mathbf{q}|_K \in H^1(K)\right)$ then $\int_\Omega |\nabla \cdot \mathbf{q}|^2 < \infty$ if and only if **n.q** is **continuous across elements.**



Degrees of freedom are normal components of **q** in some points on faces / (edges in 2D).

- **Construction of basis in** $H_h^1(\Omega_h)$

Basis function φ_i, connected to a node *i* of a triangulation T_h is:

- continuous function $\Omega_h \to R$
- linear on element

for any φ_i, $\varphi_i(\underline{x}_j) = \delta_{ij} = \begin{cases} 1 & , \quad i = j \\ 0 & , \quad i \neq j \end{cases}$, $\underline{x}_j$ – node.

Let e_i, $i=1,\ldots,m$ denote numbered faces (edges in 2D), and K_j, $j=1,\ldots,n$ denote numbered elements.

Our vectorial space is spanned by linearly independent vectorial basis functions $\boldsymbol{\Psi}_i$, such that for any $\boldsymbol{\Psi}_i$

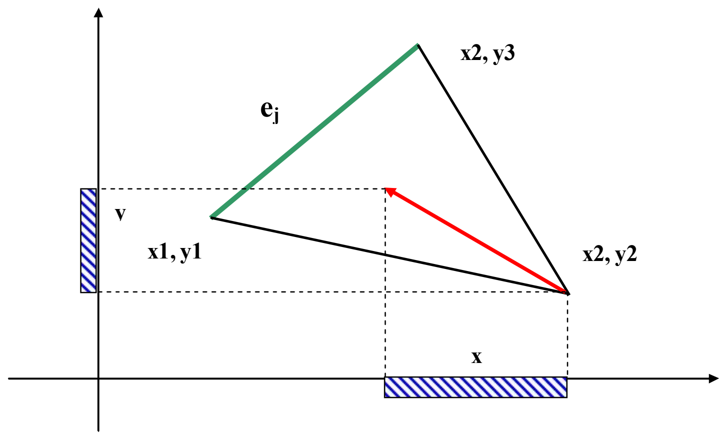
$$\int_{e_j} \mathbf{n}_j \cdot \boldsymbol{\psi}_i \, d\gamma = \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}, \text{ i, j=1,...,M.}$$

Any function **q** belonging to this space has one degree of freedom per face / edge,

$\int_{e_j} \mathbf{n}_i \cdot \mathbf{q} \, d\gamma$ (flux through e_i).

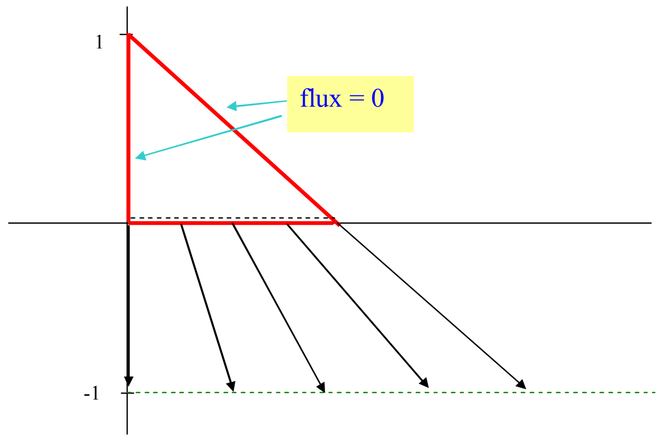
Construction of functions fullfilling:

$$\int_{ej} \mathbf{n}_j \cdot \boldsymbol{\psi}_i \, d\gamma = \delta_{ij} \qquad \int_{ej} \nabla \cdot \boldsymbol{\psi}_i \, d\gamma = 1$$



$$\boxed{\psi_j = \frac{1}{2\Delta_T}\begin{bmatrix} x - x_2 \\ y - y_2 \end{bmatrix}}$$

An "easy-triangle" example:



$$\psi_j = \frac{1}{2\Delta_T}\begin{bmatrix} x \\ y - 1 \end{bmatrix} \qquad n_j = [0,-1]$$

$$2\Delta_T = 1$$

Example in 1D

Not very convincing...

$$-\frac{d^2u}{dx^2} = 10\, w\,(0, 0.5)$$

$$u(0) = 0,\ \left.\frac{du}{dx}\right|_0 = 0$$

Exact solution: $u(x) = -5x^2 + 5x$.

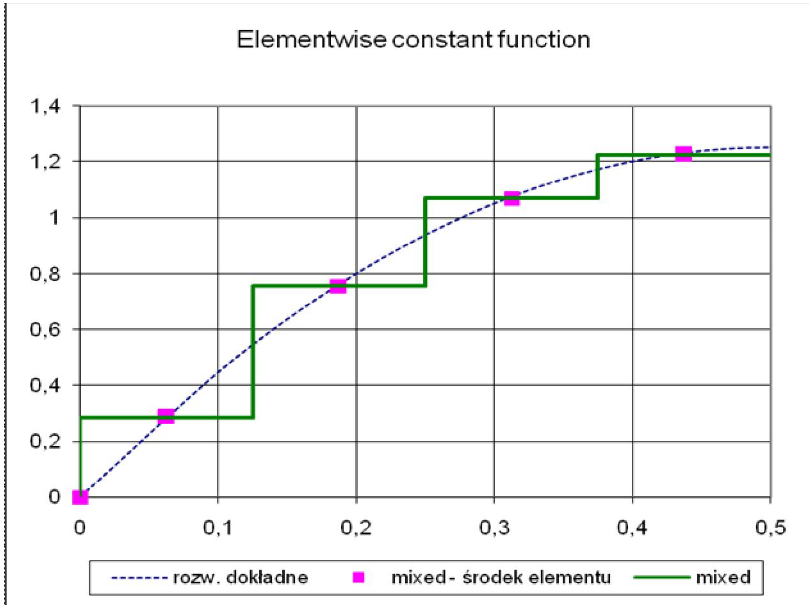
We take:

- elementwise functions to approximate *u* (pressure),
- elementwise linear functions to approximate **q** (velocity).

Finite elements – Mixed formulation in 1D

h= 0,125

FLUX				FUNCTION VALUE					
0,041667	0,020833	0	0	-1	0	0	0		0
0,020833	0,083333	0,020833	0	1	-1	0	0		0
0	0,020833	0,083333	0,020833	0	1	-1	0		0
0	0	0,020833	0,083333	0	0	1	-1		0
-1	1	0	0	0	0	0	0		-1,25
0	-1	1	0	0	0	0	0		-1,25
0	0	-1	1	0	0	0	0		-1,25
0	0	0	-1	0	0	0	0		-1,25



5
3,75
2,5
1,25
0,286458
0,755208
1,067708
1,223958

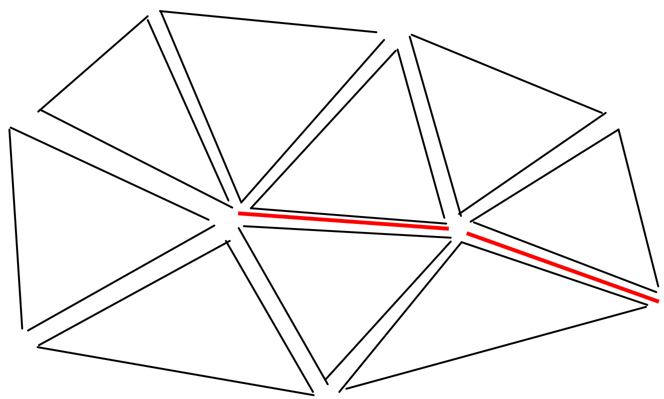
System of linear equations (8) is not good.

- Matrix, though symmetric, is not positive-definied.
- Large size ot the matrix (number of elements + number of faces/edges).

In order to obtain a hybrid version:

- Continuity requirements on faces/edges are relaxed. This way the space of approximate solutions for velocities **q** is enlarged.
- In order to enforce continuity a new variable is introduced acting on faces/edges (Lagrange multipliers).

Each face/edge is 'considered' twice.



What happens:

$$\int_{\Omega} K^{-1}\,\mathbf{q}\cdot\mathbf{v} + \int_{\Omega} \nabla u\cdot\mathbf{v} = 0$$

Green's theorem is applied for each element independently.

$$\int_{\Omega} \nabla u\cdot\mathbf{v} = \sum_{K\in T_h}\int_{K} \nabla u\cdot\mathbf{v} = \sum_{K\in T_h}\Big(-\int_{K} u\,\nabla\cdot\mathbf{v} + \int_{\partial K} u\,\mathbf{n}\cdot\mathbf{v}\Big)$$

In order to enforce continuity of **v**:

$$\sum_{K\in T_h}\int_{\partial K} \mathbf{n}\cdot\mathbf{v}\,\mu = 0 \qquad \forall\mu$$ constant on *e*,

Plus boundary conditions:

$$\sum_{K\in T_h}\int_{\partial K} \mathbf{n}\cdot\mathbf{v}\,\mu\,d\gamma = \int_{\partial\Omega} g_N \mu\,d\gamma \qquad \forall\mu\in ....$$

Mixed-Hybrid formulation:

Find $(\mathbf{q}_h, u_h, \lambda_h) \in .....$ such that:

$\int\limits_{\Omega}(\mathbf{K}^{-1}\mathbf{q}_h)\cdot\mathbf{v}_h - \sum\limits_{K\in T_h}(\int\limits_{K} u_h\, \nabla\cdot\mathbf{v}_h - \int\limits_{\partial K}\lambda_h\, \mathbf{n}\cdot\mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h .....$	$\mathbf{v}_h$ vector function defined before
$-\sum\limits_{K\in T_h}(\int\limits_{K}\nabla\cdot\mathbf{v}_h\, \varphi_h = -\int\limits_{\Omega} f\, \varphi_h \quad \forall \varphi_h .....$	φ_h constant on elements K
$\sum\limits_{K\in T_h}\int\limits_{\partial K}\mathbf{n}_K\cdot\mathbf{v}_h\, \mu_h\, d\gamma = \int\limits_{\partial\Omega} g_N \mu_h\, d\gamma \quad \forall \mu_h \in ....$	μ_h constant on faces/edges, a 'trace' of *u*.

There exists a unique solution of this system. Moreover, this solution are the same as the solution of the problem in mixed firmulation.

Remark: in mixed-hybrid formulation, Dirichlet BCs are strongly fulfilled. Neumann BC are fulfilled in a weak sense.

We arrived to a (hudge!) system of linear equations:

$$\begin{bmatrix} A & B & C \\ B^T & 0 & 0 \\ C^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

Size ot the system:
2x number of faces /edges + number of elements + number of faces/edges
(-- faces/edges with Dirichlet BC)

So what have we gained ????????????????

- It is possible to compute α elementwise:

$$\alpha = A^{-1}(F_1 - B\beta - C\lambda)$$

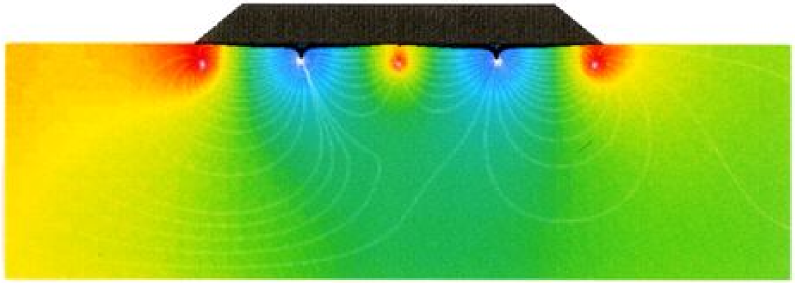
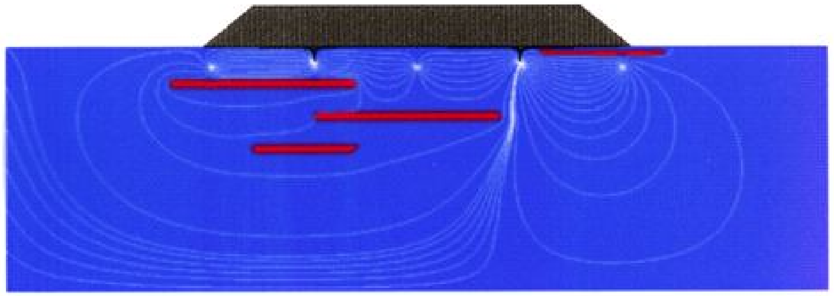
- After substitution and some algebra:

$$\beta = (B^T A^{-1} B)^{-1}(B^T A^{-1}(F_1 - C\lambda) - F_2)$$

- To finally get:

$D\lambda = F$, with D=.... and F=..... D is a matrix 3x3 in 2D (and 4x4 in 3D).
This ‘small’ (local) system of equations is inserted in to a global matrix, using a global numbering of faces/edges.

- The size of the final system of linear equations: number of faces/edges
- Once λ is computed, α and β are computed elementwise.
- **Mass conservation is perfectly fulfilled**

	infiltration from a waste disposal: 0.004 m/d, Regional flow 0.002 m/d Domain 200m x 60 m homogeneous, isotropic
	Conductivity coefficients: 1 m/day (blue), 0.01 m/day (red)