Project 6 - Exotic and Path Dependent Options, JumpDiffusion Processes

```
warning('off','all')
warning
```

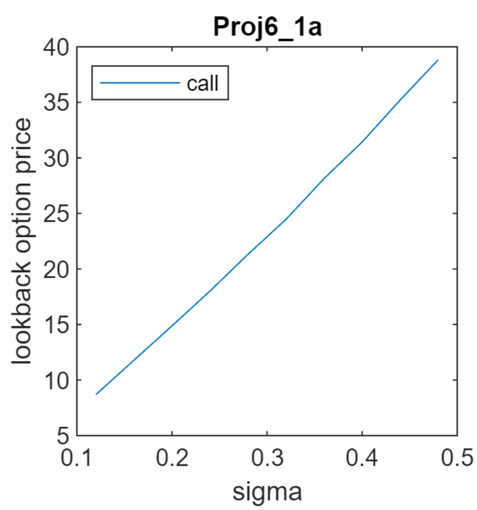
1. Consider a 12-month Fixed Strike Lookback Call and Put options, when the interest rate is 3% per annum, the current stock price is $98 and the strike price is $100. Use the Monte Carlo simulation method to estimate the prices of Call and Put options for the following range of volatilities: from 12% to 48%, in increments of 4%.

The payoff of the Call option is $(S_{\max} - X)^+$, where $S_{\max} = \max\{S_t : t \in [0, T]\}$, and the payoff of the Put option is: $(X - S_{\min})^+$, where $S_{\min}=\min\{S_t : t \in [0, T]\}$.

Inputs: S_0, X, *T*, *r*, , *N*

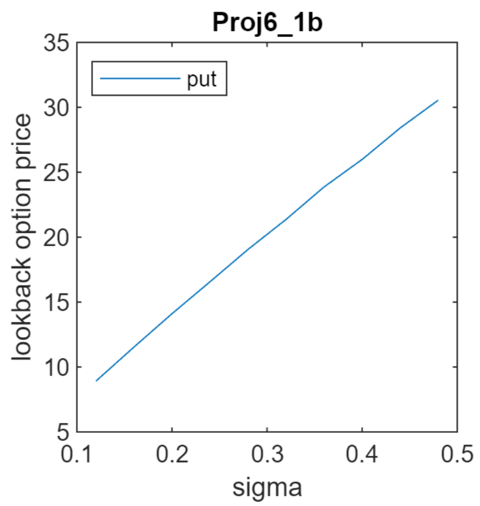
Outputs: Graphs: Call and Put options prices as a function of the volatility. Place the Call graph in Proj6_1a.jpg and the Put graph in Proj6_1b.jpg.

```
r=0.03; S0=98; X=100; sigma_vec = (12:4:48)/100; T=1; N=10000;
dt=0.01;

fast_fun = @(sigma) lookback_option(S0,r,X,sigma,T,N,dt);
[call,put] = arrayfun(@(sigma_i) fast_fun(sigma_i), sigma_vec);

% plot for call option
plot(sigma_vec,call)
set(gcf,'Position',[250 250 250 250])
legend("call",...
    'Location','northwest','NumColumns',1)
ylabel("lookback option price")
xlabel("sigma")
title("Proj6\_1a")
```




```
% plot for put option
```

```
plot(sigma_vec,put)
set(gcf,'Position',[250 250 250 250])
legend("put",...
    'Location','northwest','NumColumns',1)
ylabel("lookback option price")
xlabel("sigma")
title("Proj6\_1b")
```



2. Assume that the value of a collateral follows a jump-diffusion process:

$\frac{dV_t}{V_t^-} = \mu \mathrm{dt} + \sigma dW_t + \gamma dJ_t$, where μ, σ, $\gamma < 0$, and V_0 are given, J is a Poisson process, with intensity λ_1, independent of the Brownian Motion process W.

V_t^- is the value process before jump occurs at time t (if any).

Consider a collateralized loan, with a contract rate per period *r* and maturity *T* on the above-collateral, and assume the outstanding balance of that loan follows this process:

$L_t = a - bc^{12t}$, where a > 0, b > 0, c > 1, and L_0 are given. We have that $L_T = 0$.

Define the following stopping time:

$Q = \min\{t \geq 0 : V_t \in q_t L_t\}$

This stopping time is the first time when the relative value of the collateral (with respect to the outstanding loan balance) crosses a threshold which will be viewed as the "optimal exercise boundary" of the option to default.

Define another stopping time, which is the first time an adverse event occurs:

$S = \min\{t \geq 0 : N_t > 0\}$

Assume that is a Poisson process with intensity of λ_2.

Define $\tau = \min\{Q, S\}$.

We assume the embedded default option will be exercised at time τ, if and only if $\tau < T$.

If the option is exercised at time Q then the payoff to the borrower is: $(L_Q - \epsilon V_Q)^+$.

If the option is exercised at time S then the payoff to the borrower is: abs($L_Q - \epsilon V_Q$), where *abs(.)* is the absolute value function.

Notes:

1. If $\min\{Q, S\} > T$, then there is no default option exercise.
2. ϵ Should be viewed as the recovery rate of the collateral, so (1-ϵ) can be viewed as the legal and administrative expenses.

Assume *J* has intensity λ_1 and *N* has intensity λ_2. *N* is independent of *J* and *W*.

Assume the APR of the loan is $R = r_0 + \delta\lambda_2$, where r_0 is the "risk-free" rate, and δ is a positive parameter to measure the borrower's creditworthiness in determining the contract rate per period: *r.*

We have monthly compounding here, so r = R/12.

Assume that $q_t = \alpha + \beta t$, where β > 0, α < V_0/L_0 and β = $\frac{\varepsilon - \alpha}{T}$.

Use r_0 for discounting cash flows. Use the following base-case parameter values:

V_0 = \$20,000, L_0 = \$22,000, μ= -0.1, σ = 0.2, γ= -0.4, λ_1 = 0.2, T = 5 years, r_0 = 0.02, = 0.25, λ_2 = 0.4,

α = 0.7, ε = 0.95. Notice that, PMT= $\frac{L_0\, r}{\left[1 - \frac{1}{(1+r)^n}\right]}$, where r = R/12, n = T*12, and $a = \frac{\text{PMT}}{r}$, $b = \frac{\text{PMT}}{r(1+r)^n}$,

$c = (1 + r)$. Notice that, q_T= ε.

Write the code as a function Proj6_2func.* that takes λ_1, λ_2 and *T* as parameters, setting defaults if these parameters are not supplied, and outputs the default option price, the default probability and the expected exercise time. Function specification:

function [D, Prob, Et] = Proj6_2func(lambda1, lambda2, T)

(a) Estimate the value of the default option for the following ranges of parameters:

λ_1 from 0.05 to 0.4 in increments of 0.05;

λ_2 from 0.0 to 0.8 in increments of 0.1;

T from 3 to 8 in increments of 1;

(b) Estimate default probability for the following ranges of parameters:.

λ_1 from 0.05 to 0.4 in increments of 0.05;

λ_2 from 0.0 to 0.8 in increments of 0.1;

T from 3 to 8 in increments of 1;

(c) Find the Expected Exercise Time of the default option, conditional on τ < T. That is, estimate $E(\tau \mid < T))$ for the following ranges of parameters:.

λ_1 from 0.05 to 0.4 in increments of 0.05;

λ_2 from 0.0 to 0.8 in increments of 0.1;

T from 3 to 8 in increments of 1;

Inputs: *seed*

Outputs:

i. Values: the default option D, the default probability Prob and the expected exercise time E_t for parts (a), (b) and (c) with λ_1=0.2, λ_2=0.4 and T=5.

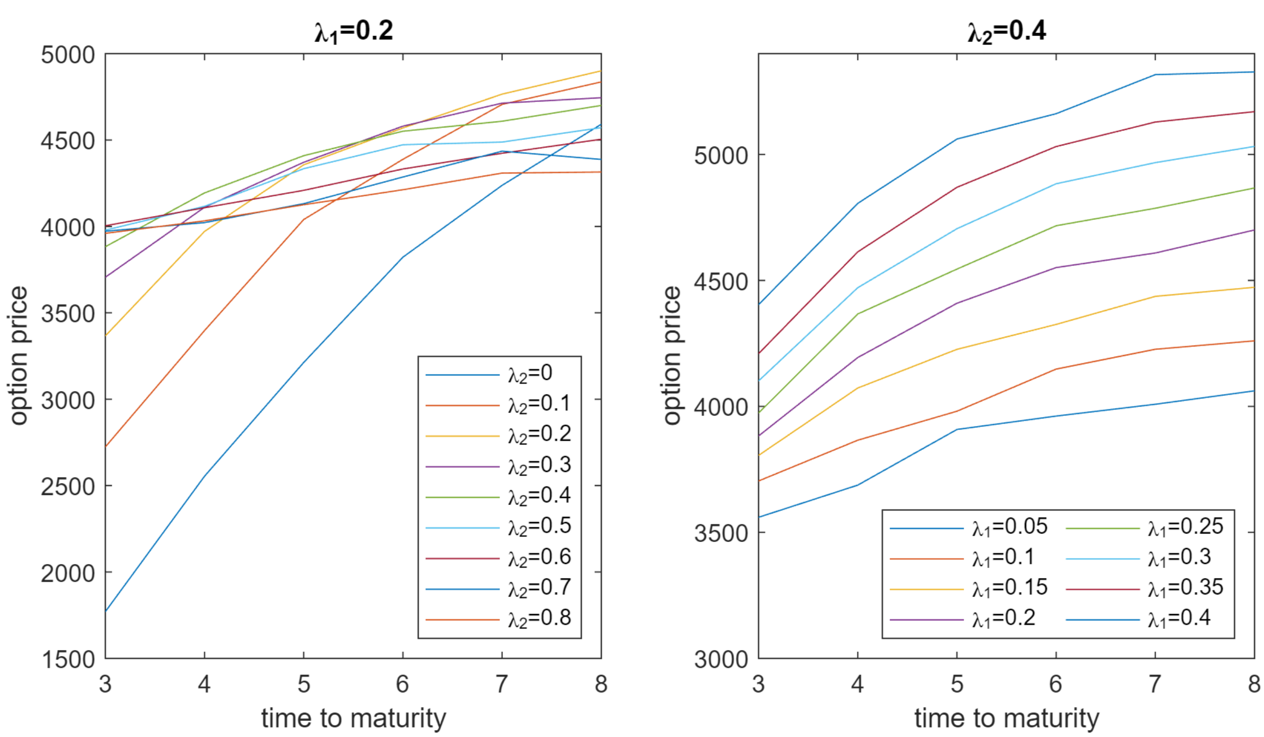
ii. Graphs: For each of (a), (b) and (c) two graphs as a function of T, first with λ_1=0.2 and λ_2 from 0.0 to 0.8 in increments of 0.1, then with λ_2 = 0.4 and λ_1 from 0.05 to 0.4 in increments of 0.05. Put the two graphs in one .png file.

```
% run codes to get option valye, default probability Prob and the expected
% exercise time
lambda1_vec = 0.05:0.05:0.4;
lambda2_vec = 0:0.1:0.8;
T_vec = 3:8;

option_price = zeros(5,8,9);
default_prob = zeros(5,8,9);
exp_ex_time = zeros(5,8,9);
for i = 1:6
    for j = 1:8
        T = T_vec(i);
        l1 = lambda1_vec(j);
        fast_l2 = @(l2) Proj6_2func(l1,l2,T);
        [option_price(i,j,:), default_prob(i,j,:), exp_ex_time(i,j,:)] = arrayfun(@(l2) fast_l2
    end
end
```

(a) Plots

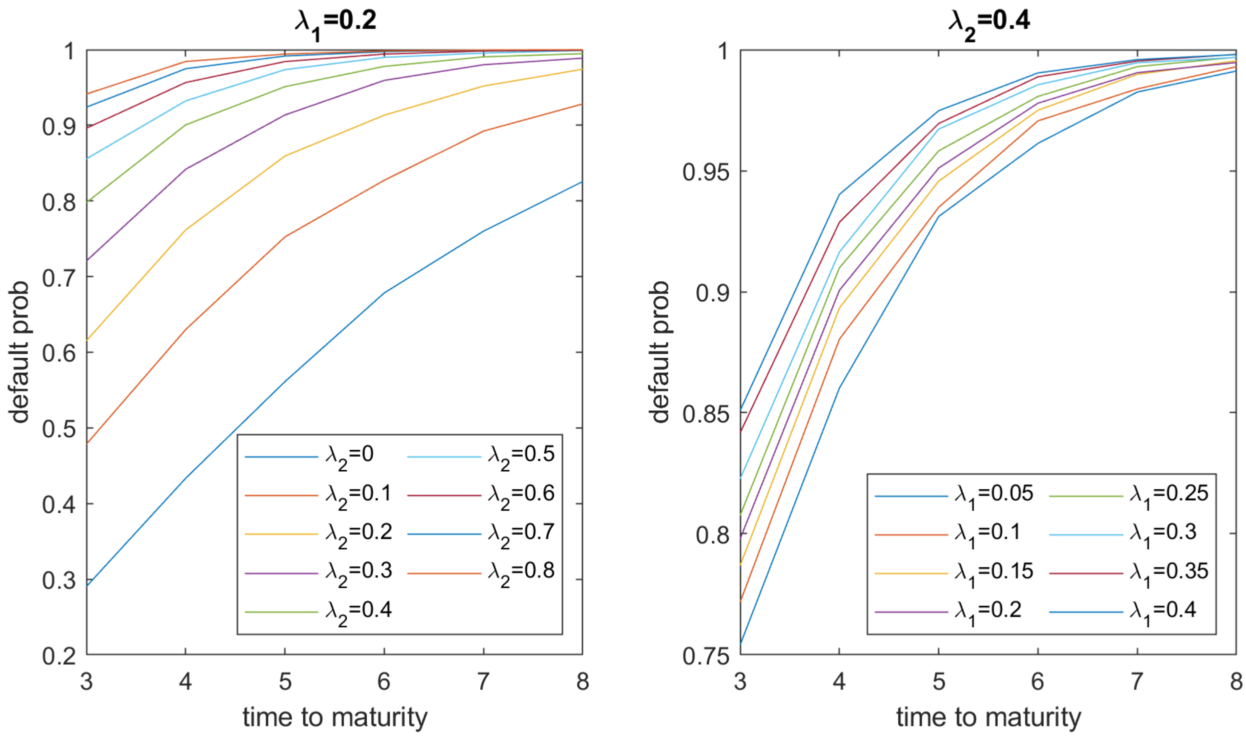
```
legend_vec = strings(9,1);
figure;
subplot(1,2,1);
for k = 1:9
    l2=lambda2_vec(k);
    legend_vec(k) = append("\lambda_2=",string(l2));
```

```
    plot(T_vec,option_price(:,4,k))
    hold on
end
hold off

set(gcf,'Position',[400 400 800 400])
legend(legend_vec,...
    'Location','southeast','NumColumns',1)
ylabel("option price")
xlabel("time to maturity")
title("\lambda_1=0.2")

subplot(1,2,2);
legend_vec2 = strings(8,1);
for k = 1:8
    l1=lambda1_vec(k);
    legend_vec2(k) = append("\lambda_1=",string(l1));
    plot(T_vec,option_price(:,k,5))
    hold on
end
hold off

set(gcf,'Position',[400 400 800 400])
legend(legend_vec2,...
    'Location','southeast','NumColumns',2)
ylabel("option price")
ylim([3000 5400])
xlabel("time to maturity")
title("\lambda_2=0.4")
```



(b) Plots

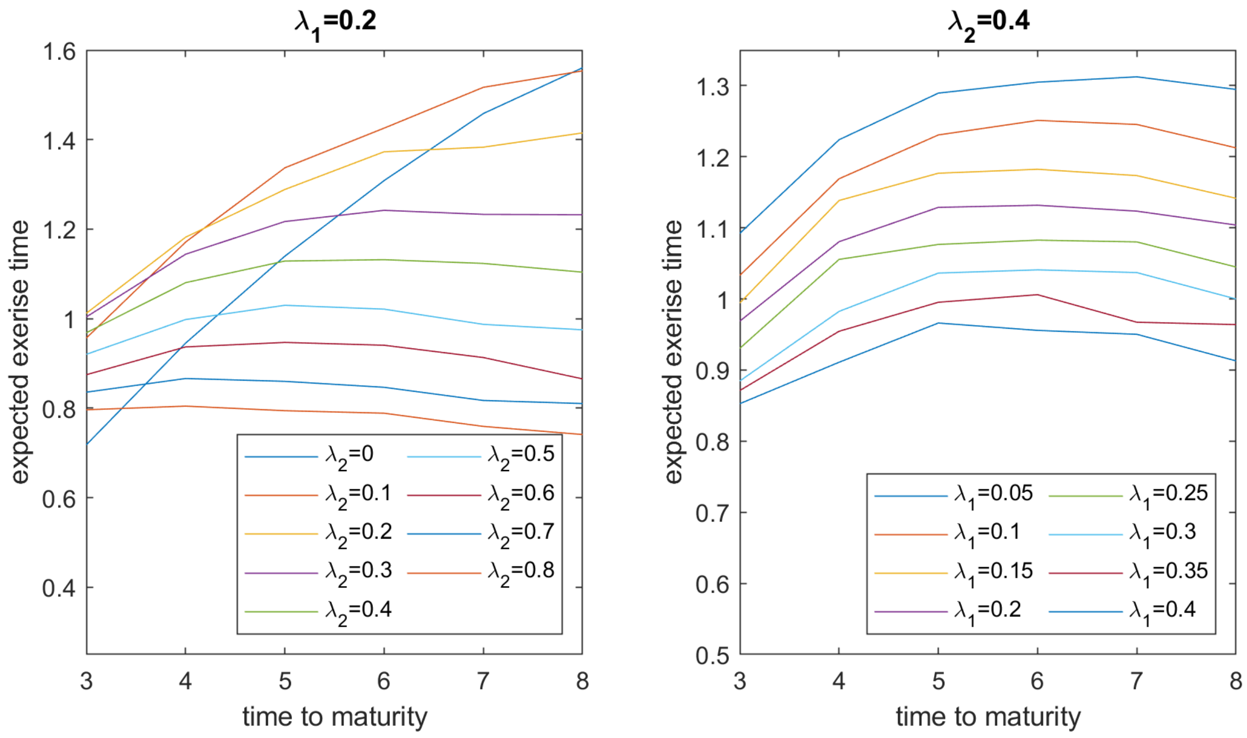
```
figure;
subplot(1,2,1);
legend_vec = strings(9,1);
for k = 1:9
    l2=lambda2_vec(k);
    legend_vec(k) = append("\lambda_2=",string(l2));
    plot(T_vec,default_prob(:,4,k))
    hold on
end
hold off

set(gcf,'Position',[400 400 800 400])
legend(legend_vec,...
    'Location','southeast','NumColumns',2)
ylabel("default prob")
xlabel("time to maturity")
title("\lambda_1=0.2")

subplot(1,2,2);
legend_vec2 = strings(8,1);
for k = 1:8
    l1=lambda1_vec(k);
    legend_vec2(k) = append("\lambda_1=",string(l1));
    plot(T_vec,default_prob(:,k,5))
    hold on
end
hold off

set(gcf,'Position',[400 400 800 400])
legend(legend_vec2,...
    'Location','southeast','NumColumns',2)
ylabel("default prob")
xlabel("time to maturity")
title("\lambda_2=0.4")
```



(c) Plots

```
figure;
subplot(1,2,1);
legend_vec = strings(9,1);
for k = 1:9
    l2=lambda2_vec(k);
    legend_vec(k) = append("\lambda_2=",string(l2));
    plot(T_vec,exp_ex_time(:,4,k))
    hold on
end
hold off

set(gcf,'Position',[400 400 800 400])
legend(legend_vec,...
    'Location','southeast','NumColumns',2)
ylabel("expected exerise time")
ylim([0.25 1.6])
xlabel("time to maturity")
title("\lambda_1=0.2")

subplot(1,2,2);
legend_vec2 = strings(8,1);
for k = 1:8
    l1=lambda1_vec(k);
    legend_vec2(k) = append("\lambda_1=",string(l1));
    plot(T_vec,exp_ex_time(:,k,5))
    hold on
end
```

```
hold off

set(gcf,'Position',[400 400 800 400])
legend(legend_vec2,...
    'Location','southeast','NumColumns',2)
ylabel("expected exerise time")
ylim([0.5 1.35])
xlabel("time to maturity")
title("\lambda_2=0.4")
```




```
function [call,put] = lookback_option(S0,r,X,sigma,T,N,dt)
    step = ceil(T/dt);
    St = zeros(N,step+1); St(:,1)=S0;
    for i = 2:(step+1)
        Wt = normrnd(0,sqrt(dt),[1,N/2]);
        St(:,i) = St(:,i-1) + r* St(:,i-1)*dt + sigma*St(:,i-1).*[Wt, -Wt]';
    end
    call_payoff = max(max(St,[],2)-X,0);
    put_payoff = max(X-min(St,[],2),0);
    call = mean(exp(-r*T)*call_payoff,1);
    put = mean(exp(-r*T)*put_payoff,1);
end

function result = find_0(vec,axis)
    result = find(vec,axis);
    [row,col] = size(result);
```

```
    if row*col==0
        % assign Inf if the set is empty
        result = Inf;
    end
end

function value = get_Vt(Vt, tao, ttime)
% helper function
    if isinf(tao(ttime))
        value = nan;
    else
        value = Vt(ttime,tao(ttime)+1); % +1 is to modify for the position
    end
end

function [D, Prob, Et] = Proj6_2func(lambda1, lambda2, T)
    V0 = 20000;L0=22000; N=10000;
    mu=-0.1;sigma=0.2;gamma=-0.4;
    r0=0.02; delta = 0.25; alpha=0.7;
    epsilon=0.95; % recovery rate of the collateral
    R = r0+delta*lambda2;
    r=R/12; beta = (epsilon-alpha)/T;
    % q_T = epsilon;
    n=T*12; PMT = L0*r/(1-1/((1+r)^n));
    a = PMT/r; b=PMT/(r*(1+r)^n); c=1+r;

    dt = 0.01;
    step = T/dt;

    Nt = poissrnd(lambda2*dt,N,step);
    fast_find = @(try_time) find_0(Nt(try_time,:)>0,1);
    S = arrayfun(@(try_time) fast_find(try_time),1:N);


    Vt = zeros(N,step+1); Vt(:,1) = V0;
    q0 = alpha;
    qLt = zeros(1,step+1); qLt(1) = q0*L0;
    J = poissrnd(lambda1*dt,N,step);
    for i = 2:(step+1)
        Wt = normrnd(0,sqrt(dt),[1,N/2]);
        t = dt*i-dt;
        Lt = a-b*c^(12*t);
        q_t = alpha+beta*t; % beta>0, alpha<V0/L0
        qLt(i) = q_t*Lt;
        Vt(:,i) = Vt(:,i-1) + Vt(:,i-1)*mu*dt + sigma*Vt(:,i-1).*[Wt, -Wt]' + gamma * J(:,i-1).
    end

    fast_find2 = @(try_time) find_0(qLt>=Vt(try_time,:),1)-1;
    Q = arrayfun(@(ttime) fast_find2(ttime),1:N);
    tao = min(Q,S);
```

```
    %Prob = mean(mean((qLt(2:(step+1))>=Vt(2:(step+1)))|(Nt>0),2),1);
    Prob = mean(~isinf(tao),2);

    if isinf(tao)
        D = 0;
        Et = Inf; % or inf?
    else
        inf_flag = isinf(tao);
        type_flag = (tao==Q); % 1 if type is Q; o.w. type is S or inf
        Lt = a-b*c.^(12.*(tao.*dt)); Lt(inf_flag) = nan;

        fast_get = @(ttime) get_Vt(Vt, tao, ttime);
        V_qs = arrayfun(@(ttime) fast_get(ttime),1:N);
        Q_value = (type_flag & (~inf_flag)).*max(Lt-epsilon.*V_qs,0);
        Q_value(isnan(Q_value))=0;
        S_value = ((~type_flag) & (~inf_flag)).*abs(Lt-epsilon.*V_qs);
        S_value(isnan(S_value))=0;
        option = Q_value + S_value;
        option(inf_flag) = 0;
        D = mean(option,2)*exp(-r0*T);
        Et = mean(tao(~inf_flag).*dt,2);
    end
end
```