Project 7- Finite Difference Methods for Option Pricing

1. Consider the following information on the stock of company XYZ: The volatility of the stock price is σ = 20% per annum. Assume the prevailing risk-free rate is r = 4% per annum. Use the X = ln(S) transformation of the Black-Scholes PDE, and Δt = 0.002, with $\Delta X = \sigma\sqrt{\Delta t}$; then with $\Delta X = \sigma\sqrt{3\Delta t}$; then with $\Delta X = \sigma\sqrt{4\Delta t}$, and a uniform grid (on X) to price a European Put option with strike price of K = \$10, expiration of 0.5-years, and current stock prices ranging from \$4 to \$16; using the specified methods below:

(a) Explicit Finite-Difference method,

```
dX_time = [1,3,4];
fast_EFD = @(dX_t) EFD(dX_t);
option_price_EFD = arrayfun(@(dX_t) fast_EFD(dX_t),dX_time,'uniformoutput',false);
option_price_EFD = cell2mat(option_price_EFD);
option_price_EFD' % show the put option price
```

```
ans = 3×13
    0.0003    0.0008    0.0039    0.0158    0.0597    0.1956    0.5256    1.0697 ···
    0.0004    0.0008    0.0044    0.0147    0.0674    0.1869    0.4849    1.0746
    0.0004    0.0010    0.0033    0.0186    0.0597    0.1954    0.4863    1.0696
```

```
% Black-Scholes Pricing Formula
S0_vec = 16:-1:4;
fast_BSM = @(S0) BSM(S0,"put");
option_price_BMS = arrayfun(@(S0) fast_BSM(S0),S0_vec)% Black-Scholes Pricing for put option
```

```
option_price_BMS = 1×13
    0.0001    0.0006    0.0031    0.0137    0.0525    0.1715    0.4647    1.0244 ···
```

```
% relative error
fast_err = @(i) option_price_EFD(:,i)-option_price_BMS';
relative_err1 = arrayfun(@(i) fast_err(i),1:3,'uniformoutput',false);
relative_err1 = cell2mat(relative_err1);
relative_err1'% show the relative error compared with Black-Scholes Pricing
```

```
ans = 3×13
    0.0002    0.0002    0.0008    0.0021    0.0072    0.0240    0.0610    0.0453 ···
    0.0003    0.0002    0.0013    0.0010    0.0149    0.0153    0.0203    0.0502
    0.0003    0.0004    0.0001    0.0050    0.0073    0.0239    0.0216    0.0452
```

(b) Implicit Finite-Difference method,

```
dX_time = [1,3,4];
fast_IFD = @(dX_t) IFD(dX_t);
option_price_IDF = arrayfun(@(dX_t) fast_IFD(dX_t),dX_time,'uniformoutput',false);
option_price_IDF = cell2mat(option_price_IDF);
option_price_IDF' % show the put option price
```

```
ans = 3×13
    0.0003    0.0007    0.0032    0.0133    0.0514    0.1734    0.4814    1.0059 ···
    0.0003    0.0007    0.0037    0.0123    0.0581    0.1655    0.4428    1.0108
    0.0004    0.0009    0.0027    0.0157    0.0514    0.1733    0.4441    1.0059
```

```
% relative error
fast_err = @(i) option_price_IDF(:,i)-option_price_BMS';
relative_err2 = arrayfun(@(i) fast_err(i),1:3,'uniformoutput',false);
relative_err2 = cell2mat(relative_err2);
relative_err2'% show the relative error compared with Black-Scholes Pricing
```

```
ans = 3×13
    0.0002    0.0000    0.0001   -0.0004   -0.0011    0.0019    0.0167   -0.0185 ···
    0.0002    0.0001    0.0005   -0.0013    0.0057   -0.0060   -0.0219   -0.0136
    0.0002    0.0002   -0.0004    0.0021   -0.0010    0.0017   -0.0206   -0.0185
```

(c) Crank-Nicolson Finite-Difference method.

```
dX_time = [1,3,4];
fast_CNFD = @(dX_t) CNFD(dX_t);
option_price_CNDF = arrayfun(@(dX_t) fast_CNFD(dX_t),dX_time,'uniformoutput',false);
option_price_CNDF = cell2mat(option_price_CNDF);
option_price_CNDF' % show the put option price
```

```
ans = 3×13
    0.0003    0.0006    0.0032    0.0132    0.0513    0.1736    0.4817    1.0060 ···
    0.0003    0.0007    0.0036    0.0123    0.0581    0.1656    0.4431    1.0109
    0.0003    0.0008    0.0027    0.0157    0.0514    0.1734    0.4444    1.0060
```

```
% relative error
fast_err = @(i) option_price_CNDF(:,i)-option_price_BMS';
relative_err3 = arrayfun(@(i) fast_err(i),1:3,'uniformoutput',false);
relative_err3 = cell2mat(relative_err3);
relative_err3' % show the relative error compared with Black-Scholes Pricing
```

```
ans = 3×13
    0.0002    0.0000    0.0001   -0.0004   -0.0011    0.0020    0.0170   -0.0184 ···
    0.0002    0.0000    0.0005   -0.0014    0.0057   -0.0059   -0.0216   -0.0135
    0.0002    0.0002   -0.0004    0.0020   -0.0011    0.0019   -0.0203   -0.0184
```

Inputs: K, σ, T, Δt

Outputs: i. Values: Pa, Pb and Pc for the European Put option using each of the methods (a), (b) and (c). ii. Writeup: compare the three methods from (a), (b) and (c) and comment. To compare, calculate the relative error with respect to the prices derived from Black-Scholes-Merton formula. Do this for current stock prices of $4 to $16 in $1 increments and put them in a table. Put the table and your comments in a .pdf file.

```
% plot the relative error
figure;
legend_vec = ["\DeltaX = \sigma\surd{\Deltat}","\DeltaX = \sigma\surd{3\Deltat}",...
    "\DeltaX = \sigma\surd{4\Deltat}"];
subplot(1,3,1);
for k = 1:3
    plot(S0_vec, relative_err1(:,k))
    hold on
end
hold off
```

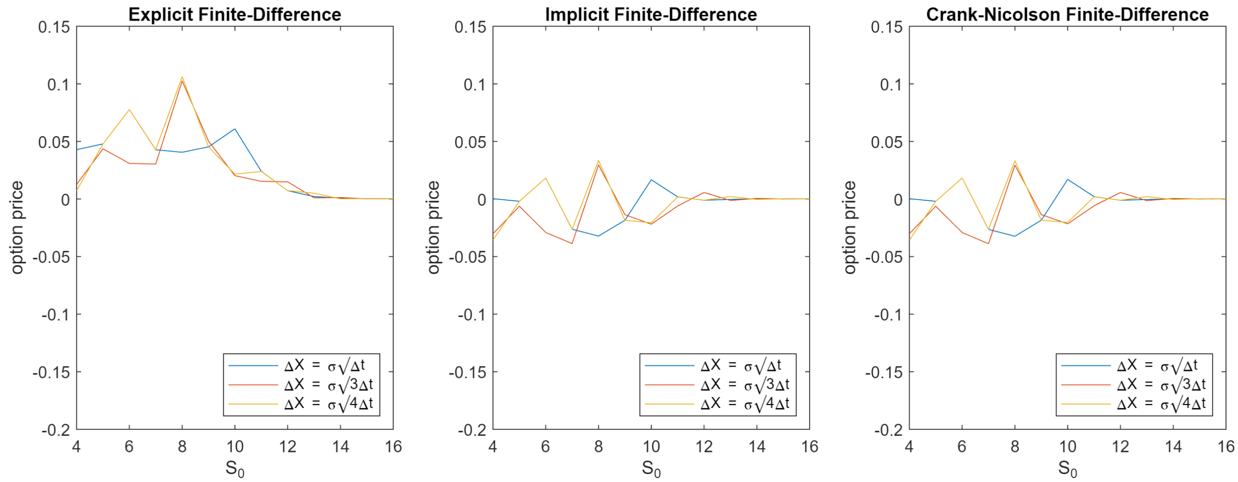
```
set(gcf,'Position',[400 400 1200 400])
legend(legend_vec,...
    'Location','southeast','NumColumns',1)
ylabel("option price")
ylim([-0.2,0.15])
xlabel("S_0")
title("Explicit Finite-Difference")

subplot(1,3,2);
for k = 1:3
    plot(S0_vec, relative_err2(:,k))
    hold on
end
hold off

set(gcf,'Position',[400 400 1200 400])
legend(legend_vec,...
    'Location','southeast','NumColumns',1)
ylabel("option price")
ylim([-0.2,0.15])
xlabel("S_0")
title("Implicit Finite-Difference")

subplot(1,3,3);
for k = 1:3
    plot(S0_vec, relative_err3(:,k))
    hold on
end
hold off

set(gcf,'Position',[400 400 1200 400])
legend(legend_vec,...
    'Location','southeast','NumColumns',1)
ylabel("option price")
ylim([-0.2,0.15])
xlabel("S_0")
title("Crank-Nicolson Finite-Difference")
```




```
mean(relative_err1.^2,1)
```

```
ans = 1×3
    0.0015    0.0014    0.0019
```

```
mean(relative_err2.^2,1)
```

```
ans = 1×3
10^-3 ×
    0.2068    0.3772    0.3220
```

```
mean(relative_err3.^2,1)
```

```
ans = 1×3
10^-3 ×
    0.2086    0.3758    0.3204
```

Comments:

Among the three methods (Explicit Finite-Difference, Implicit Finite-Difference and Crank-Nicolson Finite-Difference Methods), Implicit Finite-Difference Method and Crank-Nicolson Finite-Difference Methods have similar good performance on relative error. However, compared with these two methods, the Explicit Finite-Difference Method has larger relative errors.

In the selection of ΔX, $\Delta X = \sigma\sqrt{3\Delta t}$ and $\Delta X = \sigma\sqrt{4\Delta t}$ have similar errors, while $\Delta X = \sigma\sqrt{\Delta t}$ always produces different error, especially when $S_0 < \$11$. Since the algorithm is proved to be stable and converging when $\Delta X \geq \sigma\sqrt{3\Delta t}$, I believe that the results are better prediction when $\Delta X = \sigma\sqrt{3\Delta t}$ or $\Delta X = \sigma\sqrt{4\Delta t}$.

2. Consider the following information on the stock of company XYZ: The volatility of the stock price is σ = 20% per annum. Assume the prevailing risk-free rate is r = 4% per annum. Use the Black-Scholes PDE (for S) to price American Call and American Put options with strike prices of K = \$10, expiration of 0.5-years, and current stock prices for a range from \$4 to \$16; using the specified methods below:

(a) Explicit Finite-Difference method,

```
dX_time = [0.5, 1];
```

```
fast_EFD = @(dX_t) EFD_am(dX_t);
[option_price_EFDam_call,option_price_EFDam_put] = arrayfun(@(dX_t) fast_EFD(dX_t),dX_time,'uni
option_price_EFDam_call = cell2mat(option_price_EFDam_call);% transfer from cell to matrix
option_price_EFDam_call'% show the call option price
```

```
ans = 2×13
    6.1987    5.1989    4.2013    3.2116    2.2488    1.3642    0.6540    0.2167 ⋯
    6.1995    5.1994    4.2019    3.2116    2.2450    1.3481    0.6230    0.2010
```

```
option_price_EFDam_put = cell2mat(option_price_EFDam_put);% transfer from cell to matrix
option_price_EFDam_put'% show the put option price
```

```
ans = 2×13
    0.0005    0.0008    0.0033    0.0138    0.0519    0.1707    0.4733    1.0757 ⋯
    0.0013    0.0013    0.0039    0.0139    0.0482    0.1545    0.4400    1.0516
```

(b) Implicit Finite-Difference method,

```
dX_time = [0.5, 1];
fast_IFD = @(dX_t) IFD_am(dX_t);
[option_price_IFDam_call,option_price_IFDam_put] = arrayfun(@(dX_t) fast_IFD(dX_t),dX_time,'uni
option_price_IFDam_call = cell2mat(option_price_IFDam_call);% transfer from cell to matrix
option_price_IFDam_call'% show the call option price
```

```
ans = 2×13
    6.1985    5.1988    4.2014    3.2117    2.2489    1.3638    0.6534    0.2166 ⋯
    6.1993    5.1993    4.2019    3.2117    2.2451    1.3479    0.6223    0.2011
```

```
option_price_IFDam_put = cell2mat(option_price_IFDam_put); % transfer from cell to matrix
option_price_IFDam_put'% show the put option price
```

```
ans = 2×13
    0.0005    0.0009    0.0034    0.0139    0.0519    0.1703    0.4725    1.0752 ⋯
    0.0013    0.0013    0.0040    0.0140    0.0483    0.1542    0.4391    1.0513
```

(c) Crank-Nicolson Finite-Difference method.

```
dX_time = [0.5, 1];
fast_CNFD = @(dX_t) CNFD_am(dX_t);
[option_price_CNFDam_call,option_price_CNFDam_put] = arrayfun(@(dX_t) fast_CNFD(dX_t),dX_time,'
option_price_CNFDam_call = cell2mat(option_price_CNFDam_call);% transfer from cell to matrix
option_price_CNFDam_call'% show the call option price
```

```
ans = 2×13
    6.1985    5.1988    4.2013    3.2116    2.2489    1.3640    0.6537    0.2166 ⋯
    6.1993    5.1993    4.2018    3.2117    2.2450    1.3480    0.6227    0.2010
```

```
option_price_CNFDam_put = cell2mat(option_price_CNFDam_put);% transfer from cell to matrix
option_price_CNFDam_put'% show the put option price
```

```
ans = 2×13
    0.0005    0.0008    0.0034    0.0138    0.0519    0.1705    0.4729    1.0755 ⋯
    0.0013    0.0013    0.0039    0.0140    0.0482    0.1543    0.4395    1.0514
```

Choose Δt = 0.002, with ΔS = 0.5, or with ΔS = 1.

Inputs: K, σ, T, Δt

Outputs: i. Values: Ca, Cb, Cc, Pa, Pb and Pc and for the American call and put options using each of the methods (a), (b) and (c). ii. Graphs: Plot the American Call option price as a function of the current stock price from \$4 to \$16 in \$1 increments for methods (a), (b) and (c) on the same graph. Use a color legend or linestyles to differentiate the plots. Do the same for the American Put option in another graph. Place the two graphs in a .pdf file.

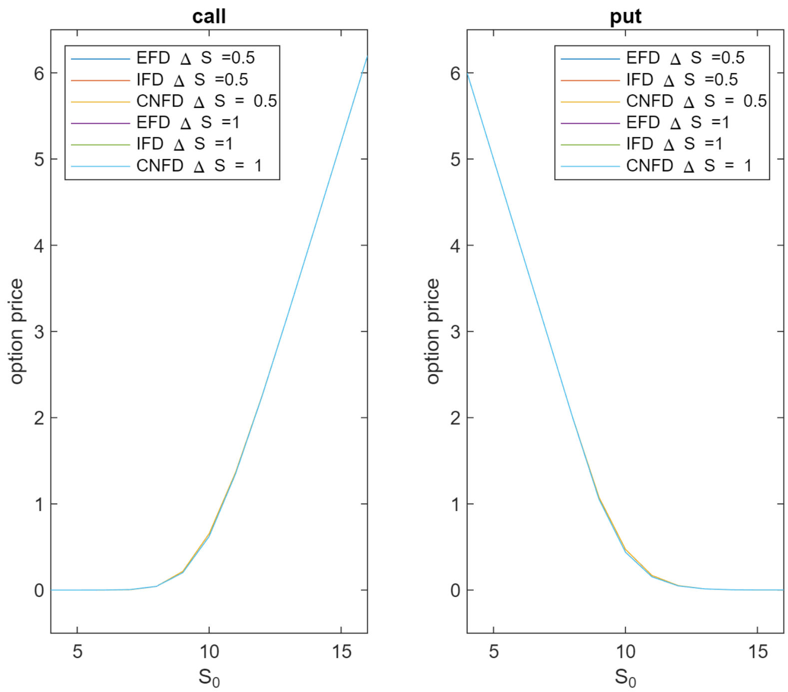
```
% plot
figure;
legend_vec = ["EFD \Delta S =0.5","IFD \Delta S =0.5","CNFD \Delta S = 0.5",...
    "EFD \Delta S =1","IFD \Delta S =1","CNFD \Delta S = 1"];
subplot(1,3,1);
for k = 1:2
    plot(S0_vec, option_price_EFDam_call(:,k))
    hold on
    plot(S0_vec, option_price_IFDam_call(:,k))
    hold on
    plot(S0_vec, option_price_CNFDam_call(:,k))
    hold on
end
hold off

set(gcf,'Position',[400 400 1000 500])
legend(legend_vec,...
    'Location','northwest','NumColumns',1)
ylabel("option price")
ylim([-0.5,6.5])
xlabel("S_0")
title("call")

subplot(1,3,2);
for k = 1:2
    plot(S0_vec, option_price_EFDam_put(:,k))
    hold on
    plot(S0_vec, option_price_IFDam_put(:,k))
    hold on
    plot(S0_vec, option_price_CNFDam_put(:,k))
    hold on
end
hold off

set(gcf,'Position',[400 400 1000 500])
legend(legend_vec,...
    'Location','northeast','NumColumns',1)
ylabel("option price")
ylim([-0.5,6.5])
xlabel("S_0")
```

```
title("put")
```



Functions used in the previous codes:

```
function option_price = BSM(S0, type)
    sigma=0.2; r=0.04; K=10; T=0.5;
    d1 = (log(S0/K) + (r+sigma^2/2)*T)/(sigma*sqrt(T));
    d2 = d1-sigma*sqrt(T);
    if type=="call"
        option_price = normcdf(d1)*S0-normcdf(d2)*K*exp(-r*T);
    elseif type == "put"
        option_price = normcdf(-d2)*K*exp(-r*T)-normcdf(-d1)*S0;
    else
        % other input: return empty vector
        option_price = [];
    end
end

function option_price = EFD(dX_times)
% Explicit Finite-Difference method for put option
    Smin = 4; Smax = 16; Xmin = log(Smin); Xmax = log(Smax);
    sigma = 0.2; r = 0.04; dt = 0.002;
    K = 10; T = 0.5;
    M = ceil(T/dt);
    dX = sigma*sqrt(dX_times*dt);
    N = ceil(((Xmax-Xmin)/dX)/2); % build a grid with shape (2N+1, M)

    % Pu, Pm, Pd
```

```
    P_u = dt*(sigma^2/(2*(dX^2))+(r-sigma^2)/(2*dX));
    P_m = 1-dt*(sigma/dX)^2-r*dt;
    P_d = dt*(sigma^2/(2*(dX^2))-(r-sigma^2)/(2*dX));

    % matrix A
    A_mat = zeros(2*N+1);
    for i=1:2*N+1
        if i == 1  % max
            A_mat(i,1:3) = [P_u,P_m,P_d];
        elseif i == 2*N+1  % min
            A_mat(i,i-2:i) = [P_u,P_m,P_d];
        else
            A_mat(i,i-1:i+1) = [P_u,P_m,P_d];
        end
    end

    % the stock price at the grid made up of X
    S_vec = 1:(2*N+1); S_vec(1) = Smax;
    % the current stock price whose the option price will be returned
    S_vec_return = 16:-1:4;j=1;
    % find the index that the price should be returned
    S_idx = 4:16; min0 = 1;
    for i = 1:(2*N+1)
        if i ~= 1
        % -delta S = S_end * (exp(-delta X)-1), where X=lnS
            S_vec(i) = S_vec(i-1)*exp(-dX);
        end
        diff = abs(S_vec(i) - S_vec_return(j));
        if diff <= min0
            min0 = diff;
            S_idx(j) = i;
        else% diff > min0
            min0 = 1;
            j = j+1;
        end
    end

    F_vec = max(K-S_vec',0);
    for i=1:M
        B_vec = zeros(2*N+1,1); B_vec(2*N+1,1) = S_vec(2*N)-S_vec(2*N+1);
        F_vec = A_mat*F_vec + B_vec; % F_t = A*F_(t+1)+B
    end
    option_price = F_vec(S_idx);
end

function [option_price_call,option_price_put] = EFD_am(dS)
% Explicit Finite-Difference method for American call and put option
    Smin=4; Smax=16;
    sigma=0.2; r=0.04; dt=0.002;
    K=10; T=0.5;
```

```matlab
    % build a grid with shape (2N+1, M)
    % unit: # of deltas in the $1 gap of Stock Price
    unit = ceil(1/dS); N = unit * ceil((Smax-Smin)/2);
    S0 = Smin+N*dS; M = ceil(T/dt);

    % Pu, Pm, Pd
    j_vec = (16*unit):-1:(4*unit);
    P_u = dt*((sigma*j_vec).^2+r*j_vec)/2;
    P_m = 1-dt*((sigma*j_vec).^2+r);
    P_d = dt*((sigma*j_vec).^2-r*j_vec)/2;

    % matrix A
    A_mat = zeros(2*N+1);
    for i=1:2*N+1
        if i == 1  % max
            A_mat(i,1:3) = [P_u(i),P_m(i),P_d(i)];
        elseif i == 2*N+1  % min
            A_mat(i,i-2:i) = [P_u(i),P_m(i),P_d(i)];
        else
            A_mat(i,i-1:i+1) = [P_u(i),P_m(i),P_d(i)];
        end
    end

    S_vec = (dS*N:-dS:-dS*N)+S0;

    % call option price
    F_vec = max(S_vec' - K,0);
    standard = F_vec;
    for i=1:M
        B_vec = zeros(2*N+1,1); B_vec(1,1) = dS;
        F_vec = A_mat*F_vec + B_vec;
        idx = standard>F_vec; % update the early exercise
        F_vec(idx) = standard(idx);
    end
    S_vec0_idx = 1:unit:(2*N+1);
    option_price_call = F_vec(S_vec0_idx);

    % put option price
    F_vec = max(K-S_vec',0);
    standard = F_vec;
    for i=1:M
        B_vec = zeros(2*N+1,1); B_vec(2*N+1,1) = dS;
        F_vec = A_mat*F_vec + B_vec; % F_t = A*F_(t+1)+B
        idx = standard>F_vec; % update the early exercise
        F_vec(idx) = standard(idx);
    end
    option_price_put = F_vec(S_vec0_idx);
end

function option_price = IFD(dX_times)
```

```
% Implicit Finite-Difference method for put option
    Smin = 4; Smax = 16; Xmin = log(Smin); Xmax = log(Smax);
    sigma = 0.2; r = 0.04; dt = 0.002;
    K = 10; T = 0.5;
    % build a grid with shape (2N+1, M)
    % unit: # of deltas in the $1 gap of Stock Price
    M = ceil(T/dt);
    dX = sigma*sqrt(dX_times*dt);
    N = ceil(((Xmax-Xmin)/dX)/2);

    % Pu, Pm, Pd
    P_u = -1/2*dt*(sigma^2/(dX^2)+(r-sigma^2/2)/dX);
    P_m = 1+dt*(sigma/dX)^2+r*dt;
    P_d = -1/2*dt*(sigma^2/(dX^2)-(r-sigma^2/2)/dX);

    A_mat = zeros(2*N+1);
    for i=1:2*N+1
        if i == 1  % max
            A_mat(i,1:2) = [1,-1];
        elseif i == 2*N+1  % min
            A_mat(i,i-1:i) = [1,-1];
        else
            A_mat(i,i-1:i+1) = [P_u,P_m,P_d];
        end
    end

    % the stock price at the grid made up of X
    S_vec = 1:(2*N+1); S_vec(1) = Smax;
    % the current stock price whose the option price will be returned
    S_vec_return = 16:-1:4;j=1;
    % find the index that the price should be returned
    S_idx = 4:16; min0 = 1;
    for i = 1:(2*N+1)
        if i ~= 1
            S_vec(i) = S_vec(i-1)*exp(-dX);
        end
        diff = abs(S_vec(i) - S_vec_return(j));
        if diff <= min0
            min0 = diff;
            S_idx(j) = i;
        else% diff > min0
            min0 = 1;
            j = j+1;
        end
    end

    F_vec = max(K-S_vec',0);
    for i=1:M
        F_vec(2*N+1,1) = -S_vec(2*N)+S_vec(2*N+1); F_vec(1,1) = 0;
        F_vec = A_mat\F_vec; % A * F_t = F_(t+1) => F_t = inv(A)*F_(t+1)
```

```
    end

    option_price = F_vec(S_idx);
end

function [option_price_call,option_price_put] = IFD_am(dS)
% Implicit Finite-Difference method for American call and put option
    Smin=4;Smax=16;
    sigma=0.2; r=0.04; dt=0.002;
    K=10; T=0.5;
    % build a grid with shape (2N+1, M)
    % unit: # of deltas in the $1 gap of Stock Price
    M = ceil(T/dt);
    unit = ceil(1/dS); N = unit * ceil((Smax-Smin)/2);
    S0 = Smin+N*dS;

    j_vec = (16*unit):-1:(4*unit);
    P_u = -dt*((sigma*j_vec).^2+r*j_vec)/2;
    P_m = 1+dt*((sigma*j_vec).^2+r);
    P_d = -dt*((sigma*j_vec).^2-r*j_vec)/2;

    A_mat = zeros(2*N+1);
    for i=1:2*N+1
        if i == 1  % max
            A_mat(i,1:2) = [1,-1];
        elseif i == 2*N+1  % min
            A_mat(i,i-1:i) = [1,-1];
        else
            A_mat(i,i-1:i+1) = [P_u(i),P_m(i),P_d(i)];
        end
    end

    S_vec = (dS*N:-dS:-dS*N)+S0;

    % call option price
    F_vec = max(S_vec'-K,0);
    standard = F_vec;
    for i=1:M
        F_vec(1,1) = dS; F_vec(2*N+1,1) = 0;
        F_vec = A_mat\F_vec;
        idx = standard>F_vec; % update the early exercise
        F_vec(idx) = standard(idx);
    end
    S_vec0_idx = 1:unit:(2*N+1);
    option_price_call = F_vec(S_vec0_idx);

    % put option price
    F_vec = max(K-S_vec',0);
    standard = F_vec;
    for i=1:M
```

```
        F_vec(2*N+1,1) = -dS; F_vec(1,1) = 0;
        F_vec = A_mat\F_vec;% A * F_t = F_(t+1) => F_t = inv(A)*F_(t+1)
        idx = standard>F_vec; % update the early exercise
        F_vec(idx) = standard(idx);
    end
    option_price_put = F_vec(S_vec0_idx);
end



function option_price = CNFD(dX_times)
% Crank-Nicolson Finite-Difference method for put option
    Smin = 4; Smax = 16; Xmin = log(Smin); Xmax = log(Smax);
    sigma = 0.2; r = 0.04; dt = 0.002;
    K = 10; T = 0.5;
    % build a grid with shape (2N+1, M)
    % unit: # of deltas in the $1 gap of Stock Price
    M = ceil(T/dt);
    dX = sigma*sqrt(dX_times*dt);
    N = ceil(((Xmax-Xmin)/dX)/2);

    P_u = -1/4*dt*(sigma^2/(dX^2)+(r-sigma^2/2)/dX);
    P_m = 1 + dt/2*(sigma/dX)^2+r*dt/2;
    P_d = -1/4*dt*(sigma^2/(dX^2)-(r-sigma^2/2)/dX);
    % matrix A
    A_mat = zeros(2*N+1);
    for i=1:2*N+1
        if i == 1  % max
            A_mat(i,1:2) = [1,-1];
        elseif i == 2*N+1  % min
            A_mat(i,i-1:i) = [1,-1];
        else
            A_mat(i,i-1:i+1) = [P_u,P_m,P_d];
        end
    end
    % matrix B
    B_mat = A_mat;
    for i=1:2*N+1
        if (i ~= 1) && (i ~= 2*N+1)
            B_mat(i,i-1:i+1) = [-P_u,-P_m+2,-P_d];
        end
    end
    % the stock price at the grid made up of X
    S_vec = 1:(2*N+1); S_vec(1) = Smax;
    % the current stock price whose the option price will be returned
    S_vec_return = 16:-1:4;j=1;
    % find the index that the price should be returned
    S_idx = 4:16; min0 = 1;
    for i = 1:(2*N+1)
        if i ~= 1
```

```
            S_vec(i) = S_vec(i-1)*exp(-dX);
        end
        diff = abs(S_vec(i) - S_vec_return(j));
        if diff <= min0
            min0 = diff;
            S_idx(j) = i;
        else % diff > min0
            min0 = 1;
            j = j+1;
        end
    end

    F_vec = max(K-S_vec',0);
    for i=1:M
        % build z vector
        z_vec = B_mat*F_vec;
        z_vec(2*N+1,1) = -S_vec(2*N)+S_vec(2*N+1); z_vec(1,1) = 0;
        F_vec = A_mat\z_vec;% A * F = Z => F = inv(A)*Z
    end
    option_price = F_vec(S_idx);
end


function [option_price_call,option_price_put] = CNFD_am(dS)
% Crank-Nicolson Finite-Difference method for American call and put option
    Smin=4;Smax=16;
    sigma=0.2; r=0.04; dt=0.002;
    K=10; T=0.5;
    % build a grid with shape (2N+1, M)
    % unit: # of deltas in the $1 gap of Stock Price
    M = ceil(T/dt);
    unit = ceil(1/dS); N = unit * ceil((Smax-Smin)/2);
    S0 = Smin+N*dS;

    % Pu, Pm, Pd
    j_vec = (16*unit):-1:(4*unit);
    P_u = -dt*((sigma*j_vec).^2+r*j_vec)/4;
    P_m = 1+dt*((sigma*j_vec).^2+r)/2;
    P_d = -dt*((sigma*j_vec).^2-r*j_vec)/4;


    A_mat = zeros(2*N+1);
    for i=1:2*N+1
        if i == 1  % max
            A_mat(i,1:2) = [1,-1];
        elseif i == 2*N+1  % min
            A_mat(i,i-1:i) = [1,-1];
        else
            A_mat(i,i-1:i+1) = [P_u(i),P_m(i),P_d(i)];
        end
```

```
    end

    B_mat = A_mat;
    for i=1:2*N+1
        if (i ~= 1) && (i ~= 2*N+1)
            B_mat(i,i-1:i+1) = [-P_u(i),-P_m(i)+2,-P_d(i)];
        end
    end

    S_vec = (dS*N:-dS:-dS*N)+S0;

    % call option price
    F_vec = max(S_vec'-K,0);
    standard = F_vec;
    for i=1:M
        z_vec = B_mat*F_vec;
        z_vec(1,1) = dS; z_vec(2*N+1,1) = 0;
        F_vec = A_mat\z_vec; % A * F = Z => F = inv(A)*Z
        idx = standard>F_vec; % update the early exercise
        F_vec(idx) = standard(idx);
    end
    S_vec0_idx = 1:unit:(2*N+1);
    option_price_call = F_vec(S_vec0_idx);

    % put option price
    F_vec = max(K-S_vec',0);
    standard = F_vec;
    for i=1:M
        z_vec = B_mat*F_vec;
        z_vec(2*N+1,1) = -dS; z_vec(1,1) = 0;
        F_vec = A_mat\z_vec;
        idx = standard>F_vec; % update the early exercise
        F_vec(idx) = standard(idx);
    end
    option_price_put = F_vec(S_vec0_idx);
end
```