Project 3 - Low Discrepancy Sequences and Simulation of Stochastic Processes

1. Evaluate the following expected values and probabilities:

$\mathrm{p1} = P(Y_2 > 5)$ $\quad \mathrm{e1} = E\left(X_2^{\frac{1}{3}}\right),\ \mathrm{e2} = E(Y_3), \mathrm{e3} = E(X_2 Y_2\ 1(X_2 > 1))$,

where the Ito's processes X and Y evolve according to the following SDEs:

$$dX_t = \left(\frac{1}{5} - \frac{1}{2}X_t\right)\mathrm{dt} + \frac{2}{3}dW_t,\ X_0 = 1; \quad dY_t = \left(\left(\frac{2}{1+t}\right)Y_t - \frac{1+t^3}{3}\right)\mathrm{dt} + \frac{1+t^3}{3}dZ_t,\ Y_0 = \frac{3}{4},$$

and W, Z are independent Wiener Processes.

Inputs: $X_0,\ Y_0$

Outputs: Values: p1, e1, e2, e3.

```
rng(100);
n=100000; dt=0.001; T=3; % note: dt is delta time
% X0, Y0
X0 = 1;
Y0 = 3/4;
% number of steps
num_steps = T/dt;
% note: Xt_vector(1) = X0; Yt_vector(1) = Y0;
Xt_vector = ones(num_steps+1,n); Xt_vector(1,:) = ones(1,n) * X0;
Yt_vector = ones(num_steps+1,n); Yt_vector(1,:) = ones(1,n) * Y0;
for t = 1:num_steps
    % Wiener Processes
    Wt = normrnd(0,sqrt(dt),[1,n]);
    Zt = normrnd(0,sqrt(dt),[1,n]);
    % dX = (1/5 -1/2 Xt)dt + 2/3 dWt
    Xt_vector(t+1,:) = Xt_vector(t,:) + (1/5 -1/2 * Xt_vector(t,:))*dt + 2/3 * Wt;
    % dY = (2/(1+t) Yt + (1+t^3)/3)dt + (1+t^3)/3 dZt
    Yt_vector(t+1,:) = Yt_vector(t,:) + (2/(1+(t-1)*dt) * Yt_vector(t,:) + (1+((t-1)*dt)^3)/3)
end

% output p1, e1, e2, e3
% p1 = P(Y2>5)
Y2 = Yt_vector(2/dt+1,:);
p1 = sum(Y2>5)/length(Y2)
```

```
p1 = 0.9766
```

```
% e1 = E(X2^(1/3))
X2 = Xt_vector(2/dt+1,:);
e1 = mean(nthroot(X2,3),2)
```

```
e1 = 0.6396
```

```
% e2 = E(Y3)
Y3 = Yt_vector(3/dt+1,:);
e2 = mean(Y3,2)
```

```
e2 = 26.1443
```

```
% e3 = E(X2Y2 1(X2>1))
e3 = mean(X2.*Y2.*(X2>1),2)
```

```
e3 = 3.9217
```

2. Estimate the following expected values:

$$\text{e1} = E(1+X_2)^{1/3},\ \text{e2} = E(X_1 Y_1),$$

where $dX_t = \frac{1}{4}X_t\,\text{d}t + \frac{1}{3}X_t\,dW_t - \frac{3}{4}X_t\,dZ_t,\ X_0 = 1;\quad Y_t = e^{-0.08t+\frac{1}{3}W_t+\frac{3}{4}Z_t},$

and W, Z are independent Wiener Processes.

Inputs: X_0

Outputs: Values: e*1*, *e2*

```
% set seed
rng(9001);
n=100000; dt =0.001; T=3; % note: dt is delta time
% X0, Y0
X0 = 1;
Y0 = 1; % set Y0=1, which has no influence on Yi where i>=1

% number of steps
num_steps = T/dt;
% note: Xt_vector(1) = X0; Yt_vector(1) = Y0;
Xt_vector = ones(num_steps+1,n); Xt_vector(1,:) = ones(1,n) * X0;
Yt_vector = ones(num_steps+1,n); Yt_vector(1,:) = ones(1,n) * Y0;
for t = 1:num_steps
    % Wiener Processes
    delta_Wt = normrnd(0,sqrt(dt),[1,n]);
    delta_Zt = normrnd(0,sqrt(dt),[1,n]);
    if t == 1
        Wt = delta_Wt;
        Zt = delta_Zt;
    else
        Wt = Wt + delta_Wt;
        Zt = Zt + delta_Zt;
    end
    % dX = 1/4 Xt dt + 1/3 Xt dWt - 3/4 Xt dZt
    Xt_vector(t+1,:) = Xt_vector(t,:) + 1/4 * Xt_vector(t,:) * dt + 1/3 * Xt_vector(t,:) .* de
    % dY = exp(-0.08t + 1/3 Wt + 3/4 Zt)
    Yt_vector(t+1,:) = exp(-0.08*(t-1)*dt + 1/3 * Wt + 3/4 * Zt);
```

```
end

% e1 = E(1+X3)^(1/3)
X3 = Xt_vector(3/dt+1,:);
e1 = mean(nthroot(1+X3,3),2)
```

```
e1 = 1.3381
```

```
% e2 = E(X1 Y1)
X1 = Xt_vector(1/dt+1,:);
Y1 = Yt_vector(1/dt+1,:);
e2 = mean(X1.*Y1,2)
```

```
e2 = 1.0574
```

3.

(a) Write a code to compute prices of European Call options via Monte Carlo simulation. Use variance reduction techniques (e.g. Antithetic Variates) in your estimation. The code should be generic: for any input of the 5 model parameters - S_0, T, X, r, σ - the output is the corresponding price of the European call option.

```
% parameters
S0=88; T=0.5; X=90;
r=0.04; sigma=0.2; n=100000;

% set seed
%rng(21);
% Antithetic Variates
%Wt
WT = normrnd(0,sqrt(T),[1,n]);
% St
ST_pos = S0 * exp(sigma*WT + (r-sigma^2/2)*T);
ST_neg = S0 * exp(sigma*(-WT) + (r-sigma^2/2)*T);
% call price
C1_pos = exp(-r*T)*max(ST_pos-X, 0);
C1_neg = exp(-r*T)*max(ST_neg-X, 0);
C1 = mean((C1_pos+C1_neg)/2,2)
```

```
C1 = 4.8398
```

(b) Write code to compute the prices of European Call options by using the Black-Scholes formula. Use the approximation of (·) described in Chapter 3. The code should be generic: for any input values of the 5 parameters - S_0, T, X, r, σ - the output is the corresponding price of the European call option.

```
% parameters
S0=88; T=0.5; X=90;
r=0.04; sigma=0.2;
% use a function called Nx_generate for N(.)
% N(x) = 1-1/2*(1+d1*x+d2*x^2+d3*x^3+d4*x^4+d5*x^5+d6*x^6)^(-16) if x >= 0
% N(x) = 1-N(-x) if x < 0
```

```
d1 = (log(S0/X)+(r+power(sigma,2)/2)*T)/(sigma*sqrt(T));
d2 = d1 - sigma*sqrt(T);
% call price
C2 = S0*Nx_generate(d1)-Nx_generate(d2)*X*exp(-r*T)
```

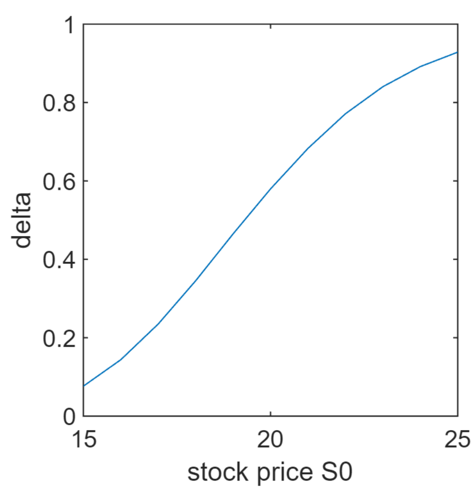
```
C2 = 4.8587
```

(c) Estimate the European call option's greeks - delta, gamma, theta, and vega - and graph them as functions of the initial stock price

S_0. Use X = 20, σ = 0.25, r = 0.04 and T = 0.5 in your estimations. Use the range [15, 25] for S_0, with a step size of 1. You will have 4 different graphs for each of the 4 greeks.

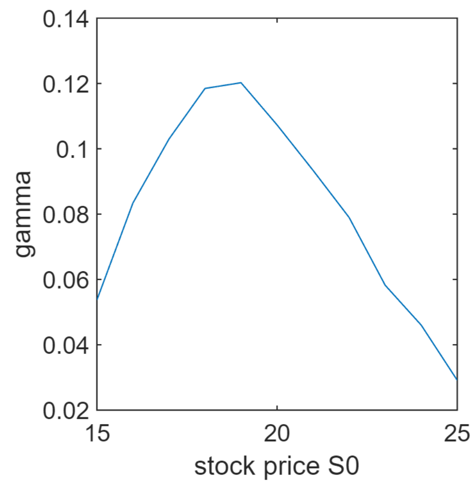
In all cases, dt (time-step) should be user-defined. Use dt=0.004 (a day) as a default value.

```
T=0.5; X=20; r=0.04; sigma=0.25; dt=0.004; delta = 0.01;
% delta
delta_vector = 1:10;
for S0 = 15:25
    delta_vector(S0-14) = call_price_MC(S0+delta, T, X, r, sigma, dt) - call_price_MC(S0-delta,
    delta_vector(S0-14) = delta_vector(S0-14)/(2*delta);
end

x_axis_vector = 15:25;

plot(x_axis_vector,delta_vector)
xlabel('stock price S0')
ylabel('delta')
```

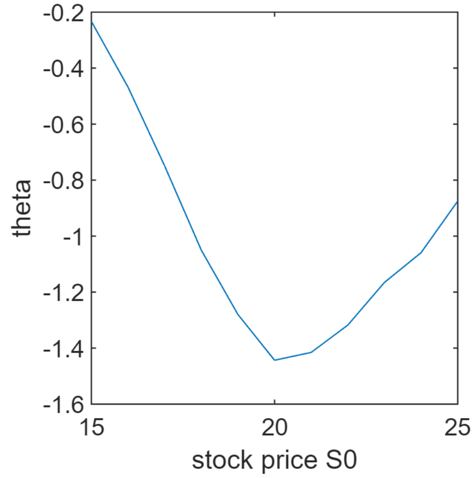



```
% gamma
gamma_vector = 1:10;
for S0 = 15:25
    % delta of S0+0.01
    delta1 = call_price_MC(S0+delta*2, T, X, r, sigma, dt) - call_price_MC(S0, T, X, r, sigma,
    delta1 = delta1/(2*delta);
    % delta of S0-0.01
    delta2 = call_price_MC(S0, T, X, r, sigma, dt) - call_price_MC(S0-delta*2, T, X, r, sigma,
```

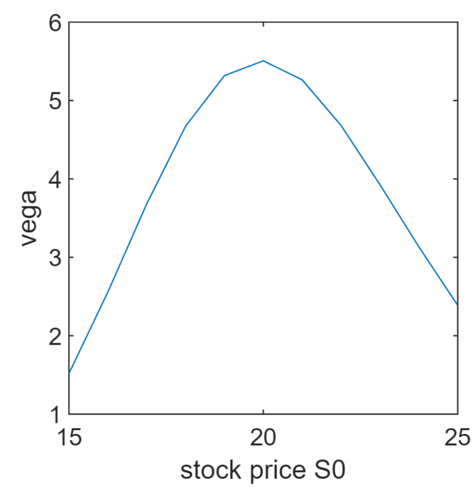
```
    delta2 = delta2/(2*delta);
    gamma_vector(S0-14) = (delta1-delta2)/(2*delta);
end

x_axis_vector = 15:25;

plot(x_axis_vector,gamma_vector)
xlabel('stock price S0')
ylabel('gamma')
```




```
% theta
theta_vector = 1:10;
for S0 = 15:25
    % as time increase, the time to maturity decreases
    theta_vector(S0-14) = call_price_MC(S0, T-dt, X, r, sigma, dt) - call_price_MC(S0, T, X, r,
    theta_vector(S0-14) = theta_vector(S0-14)/dt;
end

x_axis_vector = 15:25;

plot(x_axis_vector,theta_vector)
xlabel('stock price S0')
ylabel('theta')
```




```
% vega
vega_vector = 1:10;
for S0 = 15:25
    vega_vector(S0-14) = call_price_MC(S0, T, X, r, sigma+delta, dt) - call_price_MC(S0, T, X,
    vega_vector(S0-14) = vega_vector(S0-14)/(2*delta);
end

x_axis_vector = 15:25;

plot(x_axis_vector,vega_vector)
xlabel('stock price S0')
ylabel('vega')
```



4. Consider the following 2-factor model for stock prices with stochastic volatility:

$$\begin{cases} dS_t &= rS_t\mathrm{dt} + \sqrt{V_t}\,S_t\,dW_t^1 \\ dV_t &= \alpha(\beta - V_t)\mathrm{dt} + \sigma\,\sqrt{V_t}\,dW_t^2 \end{cases}$$

where the Brownian Motion processes above are correlated: $dW_t^1 dW_t^2 = \rho$dt, where the correlation ρ is a constant in [-1,1].

Estimate the price of a European Call option (via Monte Carlo simulation) that has a strike price of K and matures in T years.

Use the following default parameters of the model: ρ = -0.6, r = 0.03, S_0 = $48, V_0 = 0.05, =0.42, α = 5.8, β = 0.0625.

Use the Full Truncation, Partial Truncation, and Reflection methods, and provide 3 price estimates by using the tree methods.

```
rho = -0.6; r = 0.03; S0 = 48; V0 = 0.05; sigma = 0.42; alpha = 5.8; beta = 0.0625;
K = 50; T = 0.5; dt=0.004; n=100000;

% Wt
L = chol([1,rho;rho,1],'lower'); % cholesky decomposition

% set seed
%rng(170);
% Full Truncation
num_steps = T/dt;
St_vector = ones(num_steps+1,n); St_vector(1,:) = ones(1,n) * S0;
Vt_vector = ones(num_steps+1,n); Vt_vector(1,:) = ones(1,n) * V0;
for t = 1:num_steps
    Wt = normrnd(0,1,[2,n]);
    Wt = L*Wt;
    Wt1 = Wt(1,:);
    Wt2 = Wt(2,:);
    % dSt = rSt dt + sqrt(Vt)St dWt1
    St_vector(t+1,:) = St_vector(t,:) + r*St_vector(t,:) * dt + sqrt(max(Vt_vector(t,:),0)) .*
    % dVt = alpha(beta-Vt) dt + sigma sqrt(Vt) dWt2
    Vt_vector(t+1,:) = Vt_vector(t,:) + alpha * (beta-max(Vt_vector(t,:),0)) * dt + sigma * sq
end
mean(St_vector(num_steps+1,:))
```

```
ans = 48.7135
```

```
C1 = exp(-r*T) * mean(max(St_vector(num_steps+1,:)-K, 0),2)
```

```
C1 = 2.6087
```

```
% Partial Truncation
St_vector = ones(num_steps+1,n); St_vector(1,:) = ones(1,n) * S0;
Vt_vector = ones(num_steps+1,n); Vt_vector(1,:) = ones(1,n) * V0;
for t = 1:num_steps
    Wt = normrnd(0,1,[2,n]);
    Wt = L*Wt;
    Wt1 = Wt(1,:);
    Wt2 = Wt(2,:);
    % dSt = rSt dt + sqrt(Vt)St dWt1
```

```
    St_vector(t+1,:) = St_vector(t,:) + r*St_vector(t,:) * dt + sqrt(max(Vt_vector(t,:),0)) .*
    % dVt = alpha(beta-Vt) dt + sigma sqrt(Vt) dWt2
    Vt_vector(t+1,:) = Vt_vector(t,:) + alpha * (beta-Vt_vector(t,:)) * dt + sigma * sqrt(max(V
end

C2 = exp(-r*T) * mean(max(St_vector(num_steps+1,:)-K, 0),2)
```

```
C2 = 2.6236
```

```
% Reflection methods
St_vector = ones(num_steps+1,n); St_vector(1,:) = ones(1,n) * S0;
Vt_vector = ones(num_steps+1,n); Vt_vector(1,:) = ones(1,n) * V0;
for t = 1:num_steps
    Wt = normrnd(0,1,[2,n]);
    Wt = L*Wt;
    Wt1 = Wt(1,:);
    Wt2 = Wt(2,:);
    % dSt = rSt dt + sqrt(Vt)St dWt1
    St_vector(t+1,:) = St_vector(t,:) + r*St_vector(t,:) * dt + sqrt(abs(Vt_vector(t,:))) .* St
    % dVt = alpha(beta-Vt) dt + sigma sqrt(Vt) dWt2
    Vt_vector(t+1,:) = abs(Vt_vector(t,:)) + alpha * (beta-abs(Vt_vector(t,:))) * dt + sigma *
end

C3 = exp(-r*T) * mean(max(St_vector(num_steps+1,:)-K, 0),2)
```

```
C3 = 2.5903
```

5. The objective of this exercise is to compare a sample of Pseudo-Random numbers with a sample of Quasi-Monte Carlo numbers of Uniform [0,1]x[0,1]:

a) Generate 100 2-dimensional Uniform [0,1]x[0,1] vectors by using any one of the algorithms for random number generation.

```
U_vector = zeros(100,2);
U_vector(:,1) = generate_U(100,101).'; % generate one-demensional 100 uniformly random numbers
U_vector(:,2) = generate_U(100,7001).'; % generate one-demensional 100 uniformly random numbers
U_vector
```

```
U_vector = 100×2
    0.0008    0.0548
    0.2901    0.2248
    0.9053    0.5368
    0.6098    0.2287
    0.4725    0.5310
    0.9067    0.2285
    0.2740    0.6935
    0.3845    0.5940
    0.7148    0.6146
    0.2195    0.9995
      ⋮
```

b) Generate 100 points of the 2-dimensional Halton sequences, using bases 2 and 7.

```
H_vector = zeros(100,2);
H_vector(:,1) = GetHalton(100, 2); % generate one-demensional Halton sequence with 100 numbers
H_vector(:,2) = GetHalton(100, 7); % generate one-demensional Halton sequence with 100 numbers
H_vector
```

```
H_vector = 100×2
    0.5000    0.1429
    0.2500    0.2857
    0.7500    0.4286
    0.1250    0.5714
    0.6250    0.7143
    0.3750    0.8571
    0.8750    0.0204
    0.0625    0.1633
    0.5625    0.3061
    0.3125    0.4490
      ⋮
```

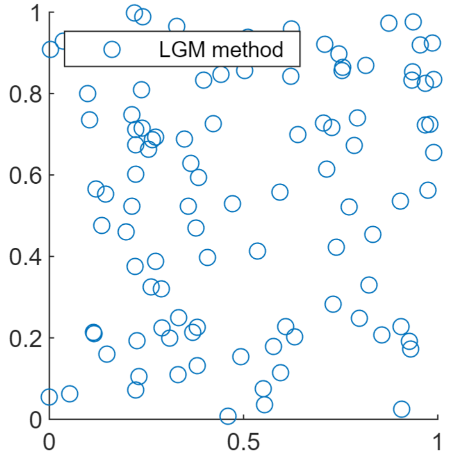
c) Generate 100 points of the 2-dimensional Halton sequences, using bases 2 and 4. (Note: 4 is a nonprime number!).

```
H2_vector = zeros(100,2);
H2_vector(:,1) = GetHalton(100, 2); % generate one-demensional Halton sequence with 100 numbers
H2_vector(:,2) = GetHalton(100, 4); % generate one-demensional Halton sequence with 100 numbers
H2_vector
```

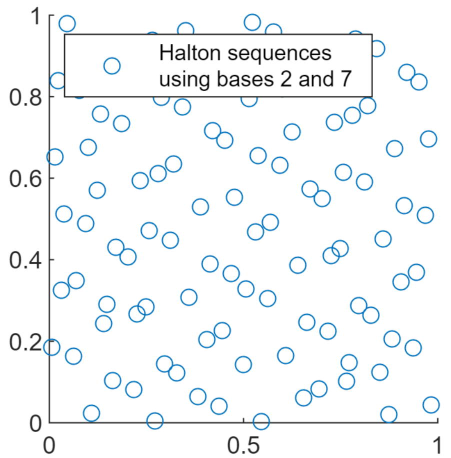
```
H2_vector = 100×2
    0.5000    0.2500
    0.2500    0.5000
    0.7500    0.7500
    0.1250    0.0625
    0.6250    0.3125
    0.3750    0.5625
    0.8750    0.8125
    0.0625    0.1250
    0.5625    0.3750
    0.3125    0.6250
      ⋮
```

d) Draw all 3 sequences of random numbers on separate graphs. Are there differences in the three sets (visual test only)? Comment on your observations.

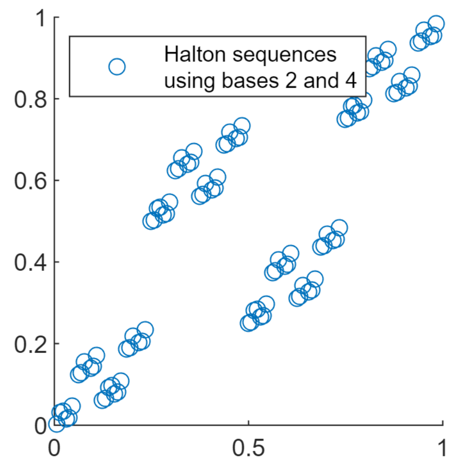
```
% 100 2-dimensional Uniform [0,1]x[0,1] vectors by using LGM method
scatter(U_vector(:,1),U_vector(:,2))
set(gcf,'Position',[250 250 250 250])
legend("LGM method",...
    'Location','northwest','NumColumns',1)
```




```
% 100 points of the 2-dimensional Halton sequences by using bases 2 and 7
scatter(H_vector(:,1),H_vector(:,2))
set(gcf,'Position',[250 250 250 250])
legend("Halton sequences " + newline + "using bases 2 and 7",...
    'Location','northwest','NumColumns',1)
```




```
% 100 points of the 2-dimensional Halton sequences by using bases 2 and 4
scatter(H2_vector(:,1),H2_vector(:,2))
set(gcf,'Position',[250 250 250 250])
legend({"Halton sequences " + newline + "using bases 2 and 4"},...
    'Location','northwest','NumColumns',1)
```



Halton sequence by using two different prime numbers looks more "uniform random" than those generated from LGM method. But, in terms of the Halton sequence by using non-prime numbers, the points look highly correlated. So, generally, among these three methods, Halton sequence by using two different prime numbers have the best performance.

e) Use 2-dimensional Halton sequences to compute the following integral:

$$I = \int_0^1 \int_0^1 e^{-xy}\left(\sin(6\pi x) + \cos^{\frac{1}{3}}(2\pi y)\right)\mathrm{d}x\mathrm{d}y$$

Use N=10,000 in your simulations. Try different couples for bases: (2,4), (2,7), (5,7).

```
% bases are (2,4)
x = GetHalton(10000, 2); % generate one-demensional Halton sequence with 10000 numbers base 2
y = GetHalton(10000, 4); % generate one-demensional Halton sequence with 10000 numbers base 4
sim = exp(-x.*y).*(sin(6*pi*x)+nthroot(cos(2*pi*y),3));
I = mean(sim,1)
```

```
I = -0.0049
```

```
% bases are (2,7)
x = GetHalton(10000, 2); % generate one-demensional Halton sequence with 10000 numbers base 2
y = GetHalton(10000, 7); % generate one-demensional Halton sequence with 10000 numbers base 7
sim = exp(-x.*y).*(sin(6*pi*x)+nthroot(cos(2*pi*y),3));
I = mean(sim,1)
```

```
I = 0.0261
```

```
% bases are (5,7)
x = GetHalton(10000, 5); % generate one-demensional Halton sequence with 10000 numbers base 5
y = GetHalton(10000, 7); % generate one-demensional Halton sequence with 10000 numbers base 7
sim = exp(-x.*y).*(sin(6*pi*x)+nthroot(cos(2*pi*y),3));
I = mean(sim,1)
```

```
I = 0.0262
```

```
function call_price = call_price_MC(S0, T, X, r, sigma, dt)
    % compute call price with Monte Carlo Simultion
    n=100000;
    % set seed
    rng(21);
    % Antithetic Variates
    %Wt
    WT = sum(normrnd(0,sqrt(dt),[T/dt,n]),1);
    % St
    ST_pos = S0 * exp(sigma*WT + (r-sigma^2/2)*T);
    ST_neg = S0 * exp(sigma*(-WT) + (r-sigma^2/2)*T);
    % call price
    C1_pos = exp(-r*T)*max(ST_pos-X, 0);
    C1_neg = exp(-r*T)*max(ST_neg-X, 0);
    call_price = mean((C1_pos+C1_neg)/2,2);
end

function N_x=Nx_generate(x)
    % generate N(.) with numerical methods
    % N(x) = 1-N(-x) if x < 0
    % N(x) = 1-1/2*(1+d1*x+d2*x^2+d3*x^3+d4*x^4+d5*x^5+d6*x^6)^(-16) if x >= 0
    N_x=(x<0)+sign(x)*Nx_pos_generate(sign(x)*x);
end

function N_x=Nx_pos_generate(x)
    % generate N(.) with numerical methods under positive x condition
    % N(x) = 1-1/2*(1+d1*x+d2*x^2+d3*x^3+d4*x^4+d5*x^5+d6*x^6)^(-16) if x >= 0
    d1 = 0.0498673470; d2 = 0.0211410061; d3 = 0.0032776263;
    d4 = 0.0000380036; d5 = 0.0000488906; d6 = 0.0000053830;
    N_x = 1 - 1/2 * (1+d1*x+d2*(x.^2)+d3*(x.^3)+d4*(x.^4)+d5*(x.^5)+d6*(x.^6)).^(-16);
end

function U_vector = generate_U(num, seed)
    % generate num uniformly distributed numbers in [0,1]
    m = pow2(31)-1;
    a = power(7,5)+3;
    X_vector = 1:num;
    for i = 1:num
        if i == 1
            % use 1 as X_0
            X_vector(i) = mod(a*seed,m);
        else
            % generate next X based on current X
            X_vector(i) = mod(a*X_vector(i-1),m);
        end
    end
    % divide X by m
    U_vector = X_vector/m;
```

```
end

function Seq = GetHalton(HowMany, Base)
    % generate one demensional Helton Sequence
    % note: adopted from Brandimartets "Numerical methods in finance and economics: a MATLAB-ba
    Seq = zeros(HowMany,1);
    % NumBits = m+1
    % m^r <= HowMany < m^(r+1) So, r <= log(HowMany)/log(Base) < r+1
    NumBits = 1+ceil(log(HowMany)/log(Base));
    VetBase = Base.^(-(1:NumBits));
    WorkVet = zeros(1,NumBits);
    for i=1:HowMany
        j=1;
        ok = 0; % ok is a flag
        while ok == 0
            WorkVet(j) = WorkVet(j)+1;
            if WorkVet(j) < Base
                ok = 1;
            else
                WorkVet(j) = 0;
                j = j+1;
            end
        end
        Seq(i) = dot(WorkVet,VetBase);
    end
end
```