A Complete Bound on the Chromatic Number and Index of Hypergraphs Through a Generalization of Vizing's Theorem

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Objective

- Investigate generalizations of graph theory through hypergraphs to establish duality between traditional concepts of vertices and edges
- Extend Vizing's Theorem for edge coloring to both hypergraph edge coloring and vertex coloring

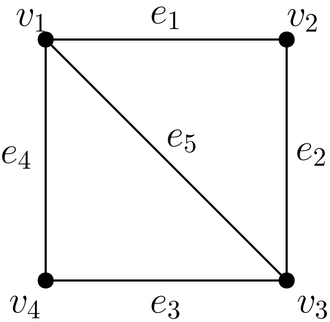
Background

Definition

Two edges are **incident** if they share a common vertex.

Definition

Two vertices are **adjacent** if there exists an edge that contains both vertices.



Background

Definition

A **proper edge coloring** of a graph G is a mapping of colors to the edges such that no two incident edges share the same color.

Definition

The **chromatic index** $\chi\prime(G)$ is the minimum number of colors needed for a proper edge coloring of G.

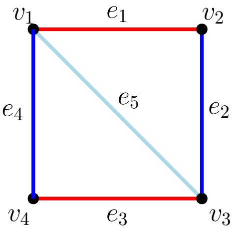


Figure: $\chi\prime(G) = 3$

Vizing's Theorem

Definition

The **degree** of a vertex v is the number of edges of G containing v. The **maximum degree** of a graph across all vertices is denoted by $\Delta(G)$.

Vizing's Theorem

For any simple graph G, the chromatic index satisfies $\chi\prime(G) = \Delta(G) + m$ where $m \in \{0, 1\}$.

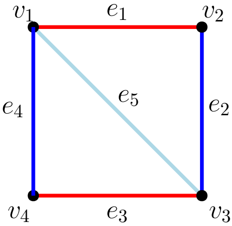
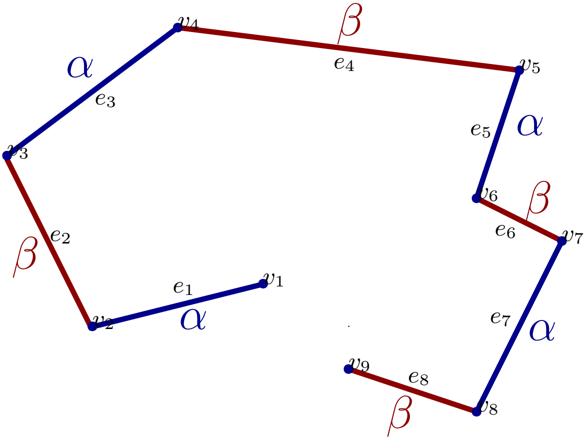


Figure: $\chi\prime(G) = 3$

Definition

A **λ-colored Kempe Chain** of a colored graph G is a maximal path or cycle in G where each edge has a color in the set λ.



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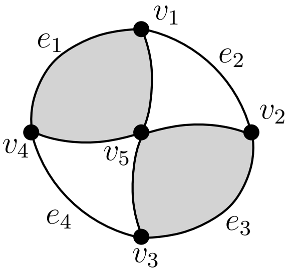
Background

Definition

A **hypergraph** is an ordered pair $G = (V, E)$ consisting of a vertex set V and edge set E, where each edge in E is a subset of V with no restriction on size.



Motivation

Theorem

There exists a logical functor that embeds the category of graphs into the category of hypergraphs.

Background

Definition

A **proper edge coloring** of a hypergraph $G = (V, E)$ is a mapping of colors to E such that no two incident edges share the same color.

- Chromatic Index

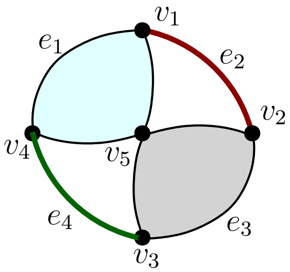


Figure: Proper Edge Coloring

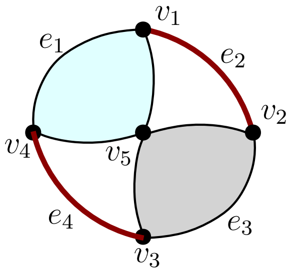


Figure: $\chi\prime(G) = 3$

Background

Definition

A **proper vertex coloring** of a hypergraph $G = (V, E)$ is a mapping of colors to V such that no two adjacent vertices share the same color.

- Chromatic Number

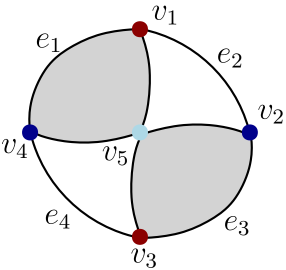
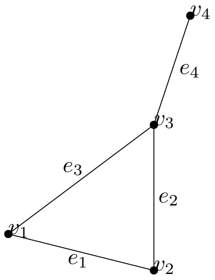


Figure: $\chi(G) = 3$

Motivation

Definition

The **incidence matrix** of a graph/hypergraph $G = (V, E)$ is a binary matrix $I = V \times E$ where entries of 1 represent incidence and entries of 0 represent otherwise.

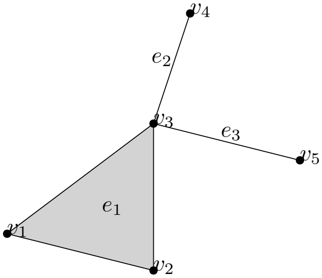


$$I = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Motivation

Definition

The **incidence matrix** of a graph/hypergraph $G = (V, E)$ is a binary matrix $I = V \times E$ where entries of 1 represent incidence and entries of 0 represent otherwise.



$$I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Generalizing Vizing's Theorem

Definition

The **edge size** of an edge e is the number of vertices it contains. The **maximum edge size** of a graph across all edges is denoted by $\nabla(G)$.

Conjecture

For any simple hypergraph G, the chromatic index and chromatic number satisfy:

- $\chi'(G) = \Delta(G) + m_\nabla$ where $m_\nabla \in \{0, 1, ..., \nabla(G) - 1\}$
- $\chi(G) = \nabla(G) + m_\Delta$ where $m_\Delta \in \{0, 1, ..., \Delta(G) - 1\}$.

Definition

A **pseudoartery** is a hypergraph in which every internal vertex v has degree 2 and in which cycles are permitted.

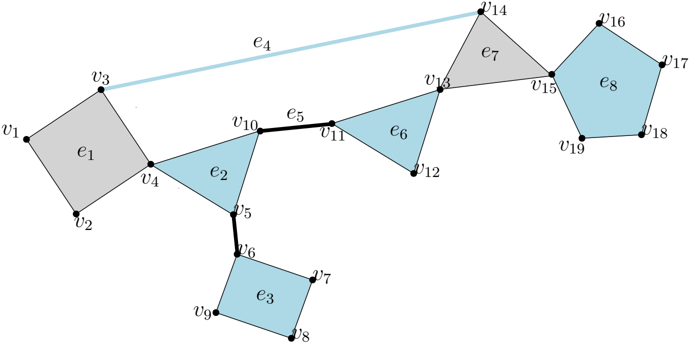


Figure: Bicolored pseudoartery

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Connection to Tutte Polynomials

Definition

The number, $\pi_G(\lambda)$, of proper colorings of G is a polynomial on the set of colors λ and is called the **chromatic polynomial**.

Conjecture

$\pi_G(\lambda) = T(G : 1 - \lambda, 0)$

Connection to Tutte Polynomials

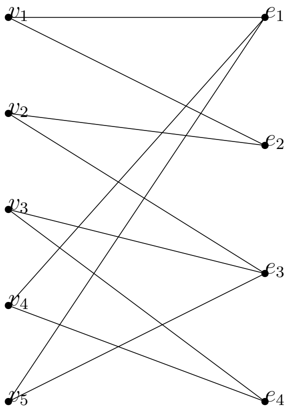

Figure: Bipartite graphs illustrate connection between Vizing's and the Tutte Polynomial.

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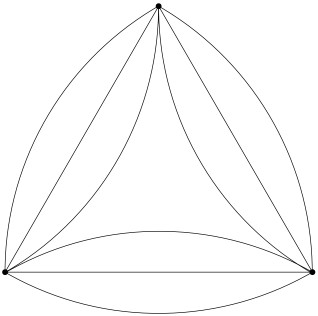
Future Work

- Continue work on our conjectures
- After extending Vizing's to simple hypergraphs, extend to other hypergraphs
  - Extension to dual of Shannon multigraphs



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