MATH 215 Thurs, Sept. 8th, 2016

Systems of Linear Algebraic Equations

- Linear Systems (L.S.) appear in solving some ODE's and PDE's.

Definition: A linear system in m equations and n variables $(x_1, x_2 \cdots, x_n)$ has the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m \end{aligned}$$

- In $a_{ij}x_j$, i (j) represent the equation (j variables) number
- m and n are finite
- a_{ij}'s and c_j's are real numbers for $i = 1, 2, \cdots m$; $j = 1, 2, \cdots, n$

Definition: If all c_j's are zero, a system is said to be a homogeneous (homog) linear system.
If at least one c_j's is non-zero, the system is non-homogenous

Definition: If at least one of the variables (x_j's) appears non-linearly, in any equation, the system is considered non-linear.

Definition: If a system has at least one solution, it is said to be consistent. If the system has no solutions, it is considered to be inconsistent.

Example. $\left.\begin{aligned} x_1 - \sqrt{2}x_2 &= 6 \\ 3x_1 + e^{-1}x_2 &= -\tfrac{1}{2} \end{aligned}\right\}①$ $\left.\begin{aligned} 2x_1 + x_2 &= 0 \\ x_1 - \sin(x_2) + x_2 &= 1 \end{aligned}\right\}②$

- Note the system is non-linear due to $\sin(x_2)$

Definition: The set $\{s_1, s_2 \cdots, s_n\}$ is a solution of a system if the m equations of the said system are satisfied when we substitute $x_1 = s_1$, $x_2 = s_2 \cdots$, $x_n = s_n$ into system.

Example: Show that $\{1, 2\}$ is a solution of the L.S.

$$\begin{cases} 2x_1 - x_2 = 0 \\ x_1 + 2x_2 = 5 \end{cases}$$

Check: $2(1) - (2) = 0$ ✓
$(1) + 2(2) = 5$ ✓

Solving Linear Systems

- Method 1: Gauss Elimination / Back Substitution
  - Use Gauss Elimination with the Elementary Equation Operations (EEO):
    - → Multiply by non-zero constant $(E_j \to \alpha E_j)$
    - → Interchange two equations $(E_j \leftrightarrow E_k)$
    - → Add a multiple of one equation to another
  - Note that in elimination, each EEO must be reversible, and must leave the solution set unchanged.
  - Every system produced during elimination procedure is equivalent to the original system.
  - Equivalent systems: If one L.S. is obtained from another system by a finite number of EEO's, then the two are equivalent.

Example: Determine if the L.S. has a solution:

$$\begin{cases} ① & x_1 + x_2 - x_3 = 2 \\ ② & 2x_1 - x_2 + x_3 = 7 \\ ③ & x_1 + 2x_2 + 3x_3 = 9 \end{cases}$$

$E_2 \to E_2 + (-2)E_1$

$$\begin{array}{rrrrl} & 2x_1 & - x_2 & + x_3 & = 7 \\ + & -2x_1 & -2x_2 & +2x_3 & = -4 \\ \hline ②' & & -3x_2 & +3x_3 & = 3 \end{array}$$

$E_3 \to E_3 + (-1)E_1$

$$\begin{array}{rrrrl} & x_1 & + 2x_2 & + 3x_3 & = 9 \\ + & -x_1 & - x_2 & + x_3 & = -2 \\ \hline ③' & & x_2 & + 4x_3 & = 7 \end{array}$$

$E_{2'} \to E_{2'} + 3E_{3'}$

$$\begin{array}{rrrl} & -3x_2 & + 3x_3 & = 3 \\ + & 3x_2 & + 12x_3 & = 21 \\ \hline & & 15x_3 & = 24 \\ & & x_3 & = \frac{8}{5} \end{array}$$

③': $x_2 + 4\left(\frac{8}{5}\right) = \frac{35}{5}$

$x_2 = \frac{3}{5}$

①: $x_1 + \frac{3}{5} - \left(\frac{8}{5}\right) = 2$

$x_1 = 3$

$$\boxed{x_1 = 3,\ x_2 = \frac{3}{5},\ x_3 = \frac{8}{5}}$$

∴ The system has a unique solution.

MATH 215 Tues, Sept. 13th, 2016

SOLVING LINEAR SYSTEMS CONT'D

Example: Consider the system

$$\begin{cases} x_1 + x_2 - x_3 + 3x_4 = -2 & (1) \\ x_1 + x_2 + 2x_3 + 2x_4 = 3 & (2) \\ x_1 + x_2 + 3x_3 - x_4 = 4 & (3) \end{cases}$$

$E_2 \to E_2 + (-1)E_1$, $E_3 \to E_3 + (-1)E_1$

$$\begin{cases} x_1 + x_2 - x_3 + 3x_4 = -2 & (1) \\ 3x_3 - x_4 = 5 & (2)' \\ 4x_3 - 4x_4 = 6 & (3)' \end{cases}$$

$E_3 \to E_3 + (-4)E_2$, $-8x_3 = -14$, $x_3 = \frac{7}{4}$

$(2)'$: $3(\frac{7}{4}) - 5 = x_4 = \frac{1}{4}$

$E_3 \to x_3 = \frac{7}{4}$, $x_4 = \frac{1}{4}$, $\because E_1$ $x_1 + x_2 = -1$

$\therefore$ Solution set is $\{\alpha, -\alpha-1, \frac{7}{4}, \frac{1}{4}\}$

We also say that the L.S. has a 1-parameter family of solutions.

Example: Consider the system

$$\begin{cases} 2x_1 + 4x_2 = 1 & (1) \\ x_1 + 2x_2 = 3 & (2) \end{cases}$$

$E_1 \to E_1 + (-2)E_2$

$(1)'$: $0 = -5$ ← Impossible!

$\therefore$ The L.S. is inconsistent.

EXISTENCE / UNIQUENESS FOR L.S.

- If m (# equations) $< n$ (# variables), the system can be both consistent or inconsistent. If it is consistent, it cannot have a unique solution. It will have a p-parameter family of solutions, where $n-m \leq p \leq n$.
- If $m \geq n$, system can be consistent or inconsistent.
- If it's consistent, it can have a unique solution or a p-parameter family of solutions, where $1 \leq p \leq n$.
- If $m < n$, then there is an infinity of solutions in addition to the trivial set.

Example ① $x_1 + x_2 + 2x_3 - 2x_4 + 2x_5 = 0$
② $3x_1 + 3x_2 + 5x_3 + 2x_5 = 0$ $\Big\}\ m < n$

$E_2 \rightarrow E_2 + (-3)E_1$:

$$\begin{array}{rl} & -3x_1 - 3x_2 - 6x_3 + 6x_4 - 6x_5 = 0 \\ + & \underline{3x_1 + 3x_2 + 5x_3 - 0x_4 + 2x_5 = 0} \\ & -x_3 + 6x_4 - 4x_5 = 0 \end{array}$$

Let $x_4 = \alpha_1$, $x_5 = \alpha_2$. $\longrightarrow$ $x_3 = 6\alpha_1 - 4\alpha_2$

Plug into equation ①:

$x_1 = -x_2 - 2(6\alpha_1 - 4\alpha_2) + 2\alpha_1 - 2\alpha_2$, choose $x_2 = \alpha_3$

$x_1 = -\alpha_3 - 10\alpha_1 + 6\alpha_2$

$\therefore$ The 3-parameter family of solutions is:

$\{-\alpha_3 - 10\alpha_1 + 6\alpha_2,\ \alpha_3,\ 6\alpha_1 - 4\alpha_2,\ \alpha_1,\ \alpha_2\}$

MATRIX NOTATION

In the L.S:

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = c_n \end{array}$$

- Omitting the x_j's leads to the rectangular array, called augmented matrix.

$$[A \,\vdots\, \vec{C}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & c_2 \\ a_{m1} & a_{m2} & \cdots & a_{mn} & c_m \end{array}\right]$$

- Note that there are m rows and n columns in $[A]$.
- To find the solution, we use elementary row operations (E.R.O.'s):
  - $\rightarrow$ Addition of a multiple of one row to another row $(R_j \rightarrow R_j + \alpha R_k)$
  - $\rightarrow$ multiplying a row by a non-zero constant $(R_j \rightarrow \alpha R_j)$
  - $\rightarrow$ Interchange two rows $(R_j \leftrightarrow R_k)$
- In the Gauss-Jordan reduction method, the final result, after applying the E.R.O's to $[A \,\vdots\, \vec{C}]$ is a new augmented matrix.

MATRICES CONT'D

- The new augmented matrix, $[A' : \vec{c}]$, is in reduced row-echelon form
- Reduced Row-Echelon Form (RREF): A matrix m is said to be in RREF if it satisfies the following:
  (1) In each row not made up entirely of zeros, the first non-zero element is a 1. This is called a leading 1.
  (2) In any two consecutive rows not made entirely of zeros, the leading 1 of the lower row is to the right to the leading 1 of the upper row.
  (3) Any column contains a leading 1, every other element in that column is a zero.
  (4) All rows made up entirely of zeros are grouped together at the bottom of the matrix.

Example:

$$M_1 = \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \quad \text{RREF}$$

$$M_2 = \left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \quad \text{Not RREF}$$

- The process of using ERO's to convert a matrix to RREF is called the Gauss-Jordan reduction method.

Example: Use the Gauss-Jordan reduction to solve the LS:

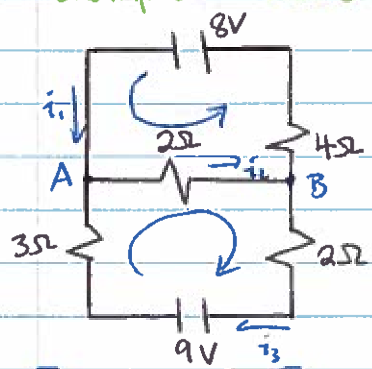
$$-x_1 + x_2 - x_3 + 3x_4 = 0$$
$$3x_1 + x_2 - x_3 - x_4 = 0$$
$$2x_1 - x_2 - 2x_3 - x_4 = 1$$

$$[A : \vec{c}] = \left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 + (-2)R_1]{R_2 \to R_2 + (-3)R_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{4}R_2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 5 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 + (-1)R_2]{R_1 \to R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + R_3']{R_3 \to -\frac{1}{3}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & -\frac{1}{3} \end{array}\right] = [A' : \vec{C}]$$

Example: Solve for i_1, i_2, i_3

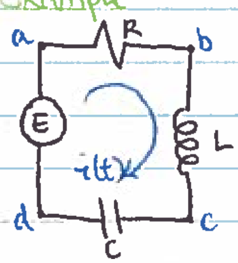


$$\begin{aligned} i_1 - i_2 - i_3 &= 0 \\ -i_1 + i_2 - i_3 &= 0 \\ 4i_1 + 2i_2 \quad &= 8 \\ 2i_2 + 5i_3 &= 9 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -1 & 1 & -1 & 0 \\ 4 & 2 & 0 & 8 \\ 0 & 2 & 5 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 4 & 8 \\ 0 & 2 & 5 & 9 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ & & & \\ 0 & 0 & -3 & -4 \\ & & & \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore i_1 = 1, \; i_2 = 2, \; i_3 = 3$$

Example:



$$V_{ab} = Ri(t) \qquad V_{bc} = L\frac{di(t)}{dt} \qquad V_{cd} = \frac{1}{C}\left[q(t_0) + \int_{t_0}^{t} i(s)\,ds\right]$$

$$V_{ab} + V_{bc} + V_{cd} + V_{da} = 0$$

$$\downarrow$$

$$⊛ \quad Li''(t) + Ri'(t) + \frac{1}{C}i(t) = E'(t)$$

$$① \quad i'' + ai' + bi = f(t)$$

where $a = \frac{R}{L}$, $b = \frac{1}{CL}$, $f(t) = \frac{E'(t)}{L}$

General solution of DE:

$$i(t) = i_h(t) + i_p(t)$$

where (in this example) $i_h(t)$ is the solution of the homogeneous DE, $i'' + ai' + bi = 0$ and $i_p(t)$ is the particular solution of the DE ①.

VECTOR SPACES

Example: In the RLC, let $L = 20H$, $R = 80\Omega$, $C = 10^{-2}F$, $e(t) = 50\sin(2t)$, Find $i(t)$.

$$① \to 20i'' + 80i' + \frac{1}{10^2}i = 100\cos(2t)$$
$$② \quad i'' + 4i' + 5i = 5\cos(2t)$$

General Solution is: $i(t) = i_h(t) + i_p(t)$,

here: $i_h(t) = C_1 e^{-2t}\cos t + C_2 e^{-2t}\sin t$

where C_1 and C_2 are constants to be found and

$$i_p(t) = A\cos(2t) + B\sin(2t)$$

where A and B are constants to be found.

Since $i_p(t)$ is a solution of ②,

$$i_p'' + 4i_p' + 5i_p = 5\cos(2t)$$
$$-4A\cos(2t) - 4B\sin(2t) + 4(-2A\sin(2t) + 2B\cos(2t)) + 5(A\cos(2t) + B\sin(2t)) = 5\cos(2t)$$

Coefficients of $\cos(2t)$: $-4A + 8B + 5A = 5$

Coefficients of $\sin(2t)$: $-4B - 8A + 5B = 0$

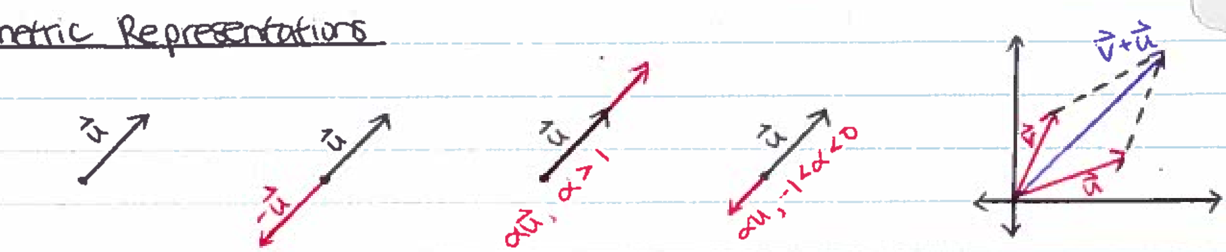
$$A = 5 - 8B, \quad B - 8A = 0 \to B = \frac{8}{13}, \ A = \frac{1}{13}$$
$$i_p(t) = \frac{1}{13}\cos(2t) + \frac{8}{13}\sin(2t)$$

N-SPACE

$$\mathbb{R}^n = \left\{ \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \text{ such that } u_1, u_2 \cdots u_n \in \mathbb{R} \right\}$$

- $u_1, u_2 \cdots, u_n$ are called the components of $\vec{u}$
- For Simplicity, we write the vector $\vec{u}$ as follows:
  $\vec{u} = (u_1, u_2 \cdots, u_n)$
- If $n=2$, $\mathbb{R}^2 = \{\vec{u} = (u_1, u_2) ; u_1, u_2 \in \mathbb{R}\}$
- If $\vec{u}, \vec{v} \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$, we define $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$
  - Addition: $\vec{u} + \vec{v} = (u_1, u_2) + (v_1, v_2)$
    $= (u_1 + v_1, u_2 + v_2) \in \mathbb{R}^2$
  - Multiplication: $\alpha\vec{u} = \alpha(u_1, u_2)$
    $= (\alpha u_1, \alpha u_2) \in \mathbb{R}^2$
  - Subtraction: $\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v} = (u_1, u_2) + (-v_1, v_2)$
    $= (u_1 - v_1, u_2 - v_2) \in \mathbb{R}^2$

Geometric Representations



Triangular Addition

- Properties
  - $\rightarrow \vec{u}+\vec{v}=\vec{v}+\vec{u}$
  - $\rightarrow (\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
  - $\rightarrow \vec{u}+\vec{0}=\vec{u}$
  - $\rightarrow \vec{u}+(-\vec{u})=\vec{0}$
  - $\rightarrow 1\cdot\vec{u}=\vec{u}$
  - $\rightarrow 0\cdot\vec{u}=\vec{0}$
  - $\rightarrow \alpha(\beta\vec{u})=(\alpha\beta)\vec{u}$
  - $\rightarrow \alpha(\vec{u}+\vec{v})=\alpha\vec{u}+\alpha\vec{v}$
  - $\rightarrow (\alpha+\beta)\vec{u}=\alpha\vec{u}+\beta\vec{u}$
  - $\rightarrow (-1)\cdot\vec{u}=-\vec{u}$
  - $\rightarrow \alpha\vec{0}=\vec{0}$

Dot product, Norm, and Angles for n-space

- Norm: If $\vec{u}\in\mathbb{R}^n$, the norm of $\vec{u}$ is defined by
$$\|\vec{u}\|=\sqrt{u_1^2+u_2^2\cdots+u_n^2}=\sqrt{\sum_{j=1}^{n}u_j^2}$$
- Norm is also called length or magnitude of a vector
- Dot product: The dot product of non-zero vectors $\vec{u}$ and $\vec{v}$ is defined as:
$$\vec{u}\cdot\vec{v}=\|\vec{u}\|\cdot\|\vec{v}\|\cos(\theta)$$
- Note that dot product is a scalar!
- The dot product of $\vec{u},\vec{v}\in\mathbb{R}^2$ in terms of the vector components is
$$\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2$$
- If $\vec{u},\vec{v}\in\mathbb{R}^n$, then $\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2\cdots+u_nv_n=\sum_{j=1}^{n}u_jv_j$
- $\|u\|^2=\vec{u}\cdot\vec{u}=\sum_{j=1}^{n}u_j^2$
- $\|\vec{u}\|=\sqrt{\vec{u}\cdot\vec{u}}$
- $\theta=\cos^{-1}\left(\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}\right)$

PROPERTIES OF DOT PRODUCTS

- If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, then
  - $\rightarrow \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (Commutative)
  - $\rightarrow \vec{u} \cdot \vec{u} > 0 \quad \forall \ \vec{u} \neq \vec{0}$ (Non-negative)
  - $\rightarrow (\alpha \vec{u} + \beta \vec{v}) \cdot \vec{w} = \alpha(\vec{u} \cdot \vec{w}) + \beta(\vec{v} \cdot \vec{w})$ (Linearity)

Ex: Find the angle between $(3,4)$ and $(-1,7)$

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right) = \frac{(3)(-1) + (4)(7)}{(5) \cdot \sqrt{50}} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \boxed{\frac{\pi}{4}}$$

Ex: Expand the dot product $(4\vec{t} + \vec{u}) \cdot (2\vec{v} - \vec{w})$

$$(4\vec{t} + \vec{u}) \cdot (2\vec{v} - \vec{w}) = 4\vec{t} \cdot [2\vec{v} - \vec{w}] + \vec{u} \cdot [2\vec{v} - \vec{w}]$$
$$= 8(\vec{t} \cdot \vec{v}) - 4(\vec{t} \cdot \vec{w}) + 2(\vec{u} \cdot \vec{v}) - (\vec{u} \cdot \vec{w})$$

Schwartz Inequality:

$$\boxed{|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|}$$

Properties of the Norm

- If $\vec{u} \in \mathbb{R}^n$, $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{\sum u_j^2}$, then
  - $\rightarrow \|\alpha \vec{u}\| = |\alpha| \, \|\vec{u}\|$ (Scaling)
  - $\rightarrow \|\vec{u}\| > 0 \quad \forall \ \vec{u} \neq 0$ (Non-negative)
  - $\rightarrow \|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ (Triangle Inequality)

Orthogonality

- The vectors $\vec{u}$ and $\vec{v}$ are said to be orthogonal if $\vec{u} \cdot \vec{v} = 0$
- The set of vectors $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n\}$ is said to be orthogonal if
  $$\vec{u}_j \cdot \vec{v}_i = 0 \quad \forall \ i \neq j, \quad i,j = 1,2 \cdots, n$$
- The zero vector is orthogonal to all vectors including itself
- If $\vec{u} \neq 0$ and $\vec{v} \neq 0$ such that $\vec{u} \cdot \vec{v} = 0$, then
  $$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right) = \cos^{-1}(0) = \frac{\pi}{2}$$

and we say $\vec{u}$ and $\vec{v}$ are perpendicular.

Normalization

- If $\vec{u} \neq 0$, then the normalized vector of $\vec{u}$ is:

$$\vec{a} = \frac{1}{\|\vec{u}\|} \vec{u}$$

- The length of $\vec{a}$ is always 1.
- The vector of unit length is called the unit vector.
- A set of vectors is said to be orthonormal if it is orthogonal and each vector has length 1.

Example: $\vec{u}_1 = (0, 0, 1)$ $\vec{u}_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$ $\vec{u}_3 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$

$\vec{u}_i \cdot \vec{u}_j = 0 \quad \forall \; i, j = 1, 2, 3$

$\|\vec{u}_i\| = 1 \quad \forall \; i = 1, 2, 3$

Applications of the dot product

- If a constant vector force $\vec{F}$ acts to move an object on a straight surface in the $\overrightarrow{PQ}$ direction, then the work done to move this object is: $W = \overrightarrow{PQ} \cdot \vec{F}$
- $\overrightarrow{PQ}$ is the displacement vector: $\overrightarrow{PQ} = (q_1 - p_1, q_2 - p_2)$
- If we have the magnitude of the force $\|\vec{F}\|$ in moving an object on a plane with angle θ, then the vector force is:

$$\vec{F} = (\|\vec{F}\| \cos\theta, \|\vec{F}\| \sin\theta)$$

Vector Space

- A non-empty set S of object (often called vectors) is said to be a vector space if the following properties are satisfied:
  - A. (i) S is closed under the addition operation: if $\vec{u}, \vec{v} \in S$, then $\vec{u} + \vec{v} \in S$

    (ii) $\vec{u} + \vec{v} = \vec{v} + \vec{u} \quad \forall \; \vec{u}, \vec{v} \in S$

    (iii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \quad \forall \; \vec{u}, \vec{v}, \vec{w} \in S$

    (iv) S contains a unique zero such that $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u} \quad \forall \; \vec{u} \in S$

    (v) For each $\vec{u} \in S$, there is a unique negative inverse vector $-\vec{u} \in S$ s/t $\vec{u} + (-\vec{u}) = \vec{0}$

VECTOR SPACE FORMATION

- B (i) S is closed under the scalar multiplication operation:
  $\alpha\vec{u} \in S \quad \forall \; \vec{u} \in S, \alpha \in \mathbb{R}$
  (ii) $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u} + \alpha\vec{v} \quad \forall \; \vec{u}, \vec{v} \in S$
  (iii) $(\alpha+\beta)\vec{u} = \alpha\vec{u} + \beta\vec{u} \quad \forall \; \vec{u} \in S, \alpha, \beta \in \mathbb{R}$
  (iv) $\alpha(\beta\vec{u}) = (\alpha\beta)\vec{u} \quad \forall \; \vec{u} \in S, \alpha, \beta \in \mathbb{R}$
  (v) $1\vec{u} = \vec{u} \quad \forall \; \vec{u} \in S$

- $S = \mathbb{R}^n$ with classical addition and scalar multiplication form a vector space called Euclidean (vector) space

Example: Let S be a set of 1st-order polynomials. That is, if $p(x) \in S$, then $p(x) = a_0 + a_1 x$.
i.e. $S = \{p(x) = a_0 + a_1 x, \; a_0, a_1 \in \mathbb{R}\}$.
Show that S with classical addition and scalar multiplication forms a vector space called first-order polyspace.

(i) If $p(x), q(x) \in S$, then

$p(x) = a_0 + a_1 x \longrightarrow p(x)+q(x) = (a_0+b_0) + (a_1 x + b_1 x) \in S$
$q(x) = b_0 + b_1 x \qquad \therefore$ S is closed under this addition.

(ii) $p(x)+q(x) = (a_0+b_0) + (a_1+b_1)x$
$= (b_0+a_0) + (b_1+a_1)x$
$= (b_0 + b_1 x) + (a_0 + a_1 x)$
$= q(x) + p(x)$

(iii) $[p(x)+q(x)] + r(x) = [(a_0+a_1 x) + (b_0 + b_1 x)] + (c_0 + c_1 x)$
$= a_0 + (b_0 + c_0) + [a + (b_1 + c_1)]x$
$= (a_0 + a_1 x) + [(b_0 + c_0) + (b_1 + c_1)x]$
$= p(x) + [q(x) + r(x)]$

(iv) If $0 \in S$, then $0(x) = 0 + 0x$
If $p(x) \in S$, then $p(x) + 0(x) = (a_0 + a_1 x) + (0 + 0x)$
$= (a_0 + 0) + (a_1 + 0)x$
$= a_0 + a_1 x = p(x)$

(v) $p(x) + (-p(x)) = 0 = a_0 + a_1 x - (a_0 + a_1 x)$
If $p(x) \in S$ and $\alpha \in \mathbb{R}$, then $\alpha p(x) = \alpha(a_0 + a_1 x) = q(x)$
$q(x) = \alpha a_0 + \alpha a_1 x \quad , \alpha a_0, \alpha a_1 \in \mathbb{R}. \quad q(x) \in S$
$\therefore$ S is closed under scalar multiplication.

Example: Let $S = \mathbb{R}^2$ and the two operations of addition and scalar multiplication be defined as follows:

(i) If $\vec{u}, \vec{v} \in S$, then $\vec{u}+\vec{v} = (u_1+v_1, u_2+1)$

(ii) If $\vec{u} \in S$, and $\alpha \in \mathbb{R}$, then $\alpha\vec{u} = (\alpha u_1, \alpha u_2)$.

Determine if S equipped with the above operation forms a vector space.

(A.i) Let $\vec{u}, \vec{v} \in S$. Then $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$

and $\vec{u}+\vec{v} = (u_1+v_1, u_2+v_2) \in S = \mathbb{R}^2$

(A.ii) $\vec{u}+\vec{v} = (u_1+v_1, u_2+1) \longrightarrow \vec{u}+\vec{v} \neq \vec{v}+\vec{u}$

$\vec{v}+\vec{u} = (v_1+u_1, v_2+1)$

$\therefore$ S does not form a vector space.

- Given a vector space S and an inner product $\vec{u}\cdot\vec{v}$, for $\vec{u}, \vec{v} \in S$, if the given dot product satisfies the properties of the dot product, then S is called inner product space.
- Similarly, if the vector space S is equipped with a norm $\|\vec{u}\|$, for $\vec{u} \in S$, and this norm satisfies the properties of the norm, then S is called normed vector space.
- If the vector space S is equipped with $\vec{u}\cdot\vec{v}$ & $\|\vec{u}\|$, for $\vec{u}, \vec{v} \in S$, and these vectors satisfy the properties of dot product and norm, the S is called normed inner product space.

TYPES OF DOT PRODUCTS

$$\vec{u}\cdot\vec{v} = \sum_{j=1}^{n} u_j v_j \qquad \vec{u}\cdot\vec{v} = \sum_{j=1}^{n} w_j u_j v_j$$

- Where W_j are fixed constants called weights.
- For an integrable vector space, we may define:

$$u(x)\cdot v(x) = \langle u(x), v(x)\rangle = \int_0^1 w(x)u(x)v(x)\,dx \quad \text{or} \quad \int_0^1 u(x)v(x)\,dx$$

TYPES OF NORMS

$$\|\vec{u}\| = \sqrt{\sum_{j=1}^{n} u_j^2} \qquad \|\vec{u}\| = \sqrt{\sum_{j=1}^{n} w_j u_j^2}$$

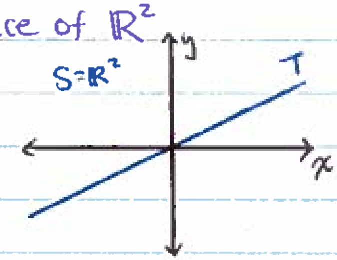
$$\|\vec{u}\| = |u_1| + |u_2| + \cdots + \|u_n\|$$

SPAN AND SUBSPACE

SUBSPACES

- Let $(S, +, *)$ denote a vector space S with addition $(+)$ and scalar multiple $(*)$. A non-empty set T (we write $T \subseteq S$) is said to be a subspace of S, if it contains the zero vector of S and:
- for all $\vec{u}, \vec{v} \in T$ and $\alpha \in \mathbb{R}$, then
  - (i) T is closed under addition operation
    $\vec{u} + \vec{v} \in T$
  - (ii) T is closed under scalar multiplication
    $\alpha \vec{u} \in T$

Example: Show that $T = \{(x,y) \mid 2x - 3y = 0\}$ is a subspace of $\mathbb{R}^2$



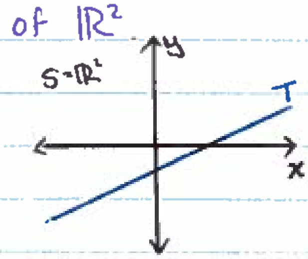
- T is a line passing through the origin $(0,0)$
  $2(0) - 3(0) = 0 \rightarrow (0,0) \in T$
- We know that T is also not empty.
- Let $\vec{x}_1 = (x_1, y_1) \in T \rightarrow 2x_1 - 3y_1 = 0$
  $\vec{x}_2 = (x_2, y_2) \in T \rightarrow 2x_2 - 3y_2 = 0$

(i) $\vec{x}_1 + \vec{x}_2 = 2(x_1 + x_2) - 3(y_1 + y_2) = 0$
$= 2(x_1) - 3(y_1) + 2(x_2) - 3(y_2)$
$= (2x_1 - 3y_1) + (2x_2 - 3y_2) = 0 + 0 = 0$
$\vec{x}_1 + \vec{x}_2 \in T$

(ii) $\alpha \vec{x}_1 = (\alpha x_1, \alpha y_1) = 2(\alpha x_1) - 3(\alpha y_1) = 0$
$= \alpha(2x_1 - 3y_1) = 0\alpha = 0$
$\alpha \vec{x}_1 \in T$

$\therefore$ T is a subspace of $\mathbb{R}^2$.

Example: Determine if $\{(x,y) \mid 2x - 3y = 1\}$ is a subspace of $\mathbb{R}^2$



- T does not pass through $(0,0)$. It is not a subspace of $\mathbb{R}^2$
- Choose $\vec{x}_1, \vec{x}_2 \in T$: $\rightarrow 2x_1 - 3y_1 = 1$
  $\rightarrow 2x_2 - 3y_2 = 1$

$\vec{x}_1 + \vec{x}_2 = (x_1 + x_2, y_1 + y_2)$
$\rightarrow 2(x_1 + x_2) - 3(y_1 + y_2) = (2x_1 - 3y_1) + (2x_2 - 3y_2) = 2$
$\rightarrow 2 \neq 1, \quad \vec{x}_1 + \vec{x}_2 \notin T$

Remark:

(i) A vector space S (or subspace T) must have at least one vector.
For example, for $\alpha = 0 \in \mathbb{R}$, $x \in S$ or $x \in T$, we have
$0\vec{x} = \vec{0} \in S$ (or $\in T$)

(ii) $T = \{\vec{0}\}$ is a subspace of $\mathbb{R}^n$. This is called the trivial / zero subspace

(iii) $\mathbb{R}^n$ is a subspace of itself.

SPAN

- Let $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$. We say that $\vec{v}$ is a linear combination of the vectors of B if there exist scalar numbers $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$ such that:

$$\vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_k\vec{v}_k$$

- Where $\vec{v}$ is a system with k unknowns, $\alpha_1, \alpha_2 \cdots, \alpha_k$.

Example: Show that $\vec{\omega} = (3,5)$ is a linear combination of $\vec{u} = (1,2)$ and $\vec{v} = (1,1)$

$$(3,5) = \alpha_1(1,2) + \alpha_2(1,1) \rightarrow \alpha_1 + \alpha_2 = 3 \qquad 2\alpha_1 + \alpha_2 = 5$$

$$\alpha_1 = 2 \qquad \alpha_1 = 1$$

$$\boxed{\vec{\omega} = 2\vec{u} + \vec{v}}$$

- If $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ are vectors in a vector space S, then the set of all possible linear combinations of these vectors in B is called the span of B.
- Span of B can be denoted by:

$$\text{Span } B \quad \text{or} \quad \text{span}\{\vec{v}_1, \vec{v}_2 \cdots . \vec{v}_k\}$$

- The set B, as defined, is called the spanning / generating set of span B.

THEOREM FOR SPAN & SUBSPACE:

If $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ are vectors in a vector space S, then span$\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is itself a vector space (or subspace) of S.

SPANNING PROBLEMS

Example: Determine the span of $\{\vec{u}_1 = (1,2)\}$ in $\mathbb{R}^2$

$$\vec{u} = \alpha_1 \vec{u}_1 \rightarrow (u_1, u_2) = \alpha_1(1,2) = (\alpha_1, 2\alpha_1)$$

$$\alpha_1 = u_1 \qquad \alpha_1 = \tfrac{1}{2}u_2$$

- In this example, $\vec{u}$ spans a subspace (L) in a $\mathbb{R}^2$
- The spanning (or generating) set of span$\{u_1\} = L$ is $\vec{u}_1$.

Example: $\vec{u}_1 = (1,2)$ $\vec{u}_2 = (3,1)$ in $\mathbb{R}^2$

$$\vec{u} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2$$

$$(u_1, u_2) = \alpha_1(1,2) + \alpha_2(3,1)$$

$$= (\alpha_1, 2\alpha_1) + (3\alpha_2, \alpha_2)$$

$$= (\alpha_1 + 3\alpha_2, 2\alpha_1 + \alpha_2)$$

$$\alpha_1 + 3\alpha_2 = u_1 \qquad 2\alpha_1 + \alpha_2 = u_2$$

$$\alpha_1 = \tfrac{1}{13}u_1 + \tfrac{3}{5}u_2 \qquad \alpha_2 = \tfrac{1}{5}(6u_1 - u_2)$$

- For any $\vec{u} = (u_1, u_2) \in \mathbb{R}^2$, there exist α_1 and α_2.

Example: Determine the span of $\{\vec{v}_1 = (1,0), \vec{v}_2 = (1,2), \vec{v}_3 = (1,1)\}$ in $\mathbb{R}^2$

$$\vec{u} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3$$

$$(u_1, u_2) = \alpha_1(1,0) + \alpha_2(1,2) + \alpha_3(1,1)$$

$$u_1 = \alpha_1 + \alpha_2 + \alpha_3 \qquad u_2 = 2\alpha_2 + \alpha_3$$

$$\alpha_3 = u_2 - 2\alpha_2 \qquad \alpha_1 = u_1 - \alpha_2 - (u_2 - 2\alpha_2)$$

Choose $\alpha_2 = \alpha^*$, $\alpha^* \in \mathbb{R}^2$

System is consistent with infinite solutions

In this example, span$\{v_1, v_2, v_3\} = \mathbb{R}^2$
Notice: $\vec{v}_1 = 2\vec{v}_3 - \vec{v}_2$: In this case, we write
span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ = span$\{\vec{v}_2, \vec{v}_3\} = \mathbb{R}^2$

Example: Find the span $\{1, 2x, -x^2\}$ in $P_2(x)$ (polynomial 2^{nd} order)

Let $p(x) = a + bx + cx^2 \in P_2(x)$

$$p(x) = \alpha_1(1) + \alpha_2(2x) + \alpha_3(-x^2)$$

$$\begin{cases} \alpha_1 = a \\ 2\alpha_2 = b \\ -\alpha_3 = c \end{cases}$$

$$\alpha_1 = a \qquad \alpha_2 = \tfrac{1}{2}b \qquad \alpha_3 = -c$$

$\therefore$ span$\{1, 2x, -x^2\} = P_2(x)$ and it spans the entire space

Mon, Oct. 3rd, 2016

Linear Dependence / Independence

Definition: A set of vectors $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is said to be linearly dependent if there exists scalars $\alpha_1, \cdots, \alpha_k$ (not all zeroes) such that:

$$\vec{u} = \alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k$$

Definition: A set of vectors $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is said to be linearly independent if the only solution of $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 \cdots + \alpha_k\vec{v}_k = \vec{0}$ is the trivial solution $\alpha_1, \cdots, \alpha_k = 0$

Test for dependence:

- We have $\{\vec{v}_1, \cdots, \vec{v}_k\}$
- Equation (*): $\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k = \vec{0}$
- Solve * for $\alpha_1, \cdots, \alpha_k$

(i) If * has a unique solution ($\alpha_1 = \alpha_k = 0$) then the set is linearly independent.

(ii) If * has ∞ solutions, which include the trivial solution, the set is linearly dependent.

Example: Determine if the set $\{\vec{v}_1 = (2,3), \vec{v}_2 = (1,6)\}$ is LI or LD.

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 = \vec{0}$$

$$\alpha_1(2,3) + \alpha_2(1,6) = \vec{0}$$

$$2\alpha_1 + \alpha_2 = 0 \qquad 3\alpha_1 + 6\alpha_2 = 0$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 3 & 6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \qquad \alpha_1 = \alpha_2 = 0; \text{ LI}$$

Example: Determine if $\{2x, -5, 10-4x\}$ is LI or LD

$$\alpha_1 P_1(x) + \alpha_2 P_2(x) + \alpha_3 P_3(x) = 0$$

$$\alpha_1 2x - 5\alpha_2 + \alpha_3(10-4x) = 0 + 0x$$

$$-5\alpha_2 + 10\alpha_3 = 0 \qquad 2\alpha_1 - 4\alpha_3 = 0$$

$$\alpha_2 = 2\alpha_3 \qquad \alpha_1 = 2\alpha_3$$

$\therefore$ There are ∞ solutions, LD.

THEOREMS CONT'D

Theorem: A set containing a zero vector is linearly dependent.

Proof: Consider the set $\{\vec{v}_1, \cdots, \vec{v}_k\}$

Let's assume $\vec{v}_k = \vec{0}$

$\vec{v}_k = \vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_{k-1}$

$\alpha_1 = \alpha_2 = \alpha_{k-1} = 0$

α_k can be any number

$\therefore$ The set is linearly dependent.

Theorem: Every finite orthogonal set of non-zero vectors is linearly independent

Proof: Consider the set $\{\vec{v}_1, \cdots, \vec{v}_k\}$

Let $\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k = \vec{0}$

Dot $\vec{v}_1$ into both sides of equation

$\vec{v}_1 \cdot (\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k) = \vec{v}_1 \cdot \vec{0}$

$\alpha_1(\vec{v}_1 \cdot \vec{v}_1) + \cdots \alpha_k(\vec{v}_k \cdot \vec{v}_1) = \vec{0}$ (by linearity of dot product)

$\alpha_1 \|\vec{v}_1\|^2 + 0 \cdots + 0 = \vec{0}$

Since $\|\vec{v}_1\|^2 \neq 0$ as $\vec{v}_1 \neq 0$, $\alpha_1 = 0$

Dot $\vec{v}_2$ into both sides of equation, repeat, until $\alpha_k = 0$

$\therefore$ The homogeneous system has only trivial solution $\alpha_1 = \alpha_2 \cdots = \alpha_k = 0$.

Thus, the set is also linearly independent.

BASES, EXPANSIONS, DIMENSIONS

Definition: The set $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is said to be a basis for a vector space S if

(i) $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is a spanning set for S.

(ii) $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is linearly independent.

Example: Let $\{\vec{v}_1 = (1,-3,0), \vec{v}_2 = (3,0,4), \vec{v}_3 = (11,-6,2)\}$

Let T be a subspace of $S = \mathbb{R}^3$ given by $T = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Find a basis for T.

$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = 0$

$\alpha_1(1,-3,0) + \alpha_2(3,0,4) + \alpha_3(11,-6,2) = 0$

$$\begin{aligned} \alpha_1 + 3\alpha_2 + 11\alpha_3 &= 0 \\ -3\alpha_1 \qquad - 6\alpha_3 &= 0 \\ 4\alpha_2 + 2\alpha_3 &= 0 \end{aligned} \qquad \begin{aligned} \alpha_1 &= -2 \\ \alpha_2 &= -3 \\ \alpha_3 &= 1 \end{aligned}$$

The set is linearly dependent. In fact, $\vec{v}_3 = \alpha_2\vec{v}_2 + \alpha_1\vec{v}_1$

$\therefore \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\{\vec{v}_2, \vec{v}_1\}$

Moreover, $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent because

$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 = \vec{0}$ has unique solution $\alpha_1 = \alpha_2 = 0$.

Hence $\{\vec{v}_1, \vec{v}_2\}$ is $\ell.i$ and spans T. Thus, it's a basis for T.

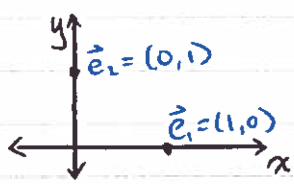
> **Exercise:** Given $\{\vec{v}_1 = (1,-1), \vec{v}_2 = (-3,-2), \vec{v}_3 = (0,2)\}$, find a basis for $\mathbb{R}^2$.

Standard Basis for $\mathbb{R}^n$

Definition: The standard basis for $\mathbb{R}^n$ is:

$$\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_j, \cdots, \vec{e}_n\}$$

where $\vec{e}_j = (0, 0, \cdots \underbrace{1}_{j^{th} \text{ component}}, 0, 0 \cdots, 0) \in \mathbb{R}^n$



In $\mathbb{R}^2$, the standard basis $\{\vec{e}_1, \vec{e}_2\} = \{(1,0), (0,1)\}$

- Every vector $\vec{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n$ can be written as a linear combination of the standard basis of $\mathbb{R}^n$

  $\vec{v} = (v_1, v_2 \cdots, v_n) = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$

- Standard basis for $\mathbb{P}_n(x)$ is:

  $\{1, x, x^2, \cdots, x^n\}$

DIMENSIONS

Definition: If a vector space S has a basis with n vector, then we say that the dimension of S is n and we write

$$\boxed{\dim S = n}$$

Remarks: (i) If a vector space S has has a basis with an arbitrarily large number of vectors, we say that S is "infinite-dimensional".

(ii) The dimension of the trivial vector space $S = \{0\}$ is zero because the corresponding basis is empty.

Theorem: $\dim \mathbb{R}^n = n$

Theorem: The dimension of span $\{u_1, u_2, \cdots, u_k\}$, where the $\vec{u}_j$'s are not all zero, is equal to the largest number of linearly independent vectors within the spanning set.

Proof: Let $U = \{\vec{u}_1, \cdots, \vec{u}_k\}$. Let the largest number of linearly independent vectors in U be N, where $1 \leq N \leq k$. Without loss of generality, assume that the L.I. N vectors have been numbered so that $\vec{u}_1, \cdots, \vec{u}_N$. Then, the remaining vectors of U, $U_{N+1}, \cdots U_k$, can be written as a linear combination of $\vec{u}_1, \cdots, \vec{u}_N$. Then, each vector in span U = span $\{\vec{u}_1, \cdots, \vec{u}_N\}$ can be written as a linear combination of $\vec{u}_1, \cdots, \vec{u}_N$. Thus, $\{\vec{u}_1, \cdots, \vec{u}_N\}$ is L.I. and spans span U
$\therefore$ By previous theorem, $\dim\{\text{span } U\} = N$

ORTHOGONAL BASES

- Suppose that $\{\vec{v}_1, \vec{v}_2 \cdots, \vec{v}_k\}$ is an orthogonal basis for a vector space S. That is, it is both a basis for S, and orthogonal
- Suppose that we want to expand a vector $\vec{w} \in S$ in terms of $\vec{v}_j$s:
$$\vec{w} = \alpha_1 \vec{v}_1 \cdots \alpha_k \vec{v}_k \qquad (*)$$
- How can we find α_j?

Dot (*) with $\vec{v}_1 + \vec{v}_2 \cdots + \vec{v}_K$:

$$\vec{w} \cdot \vec{v}_1 = \alpha_1 (\vec{v}_1 \cdot \vec{v}_1) + \alpha_2 (\vec{v}_2 \cdot \vec{v}_1) \cdots + \alpha_K (\vec{v}_K \cdot \vec{v}_1)$$
$$\vec{w} \cdot \vec{v}_2 = \alpha_1 (\vec{v}_1 \cdot \vec{v}_2) + \alpha_2 (\vec{v}_2 \cdot \vec{v}_2) \cdots + \alpha_K (\vec{v}_K \cdot \vec{v}_2)$$
$$\vdots$$
$$\vec{w} \cdot \vec{v}_K = \alpha_1 (\vec{v}_1 \cdot \vec{v}_K) + \alpha_2 (\vec{v}_2 \cdot \vec{v}_K) \cdots + \alpha_K (\vec{v}_K \cdot \vec{v}_K)$$

- System is uncoupled in b equations and b unknowns, $\alpha_1, \cdots, \alpha_K$.
- The solution of the system is:

$$\alpha_1 = \frac{\vec{w} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}, \quad \alpha_2 = \frac{\vec{w} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}, \quad \cdots, \quad \alpha_K = \frac{\vec{w} \cdot \vec{v}_K}{\vec{v}_K \cdot \vec{v}_K}$$

- Thus, if $\{\vec{v}_1, \vec{v}_2 \cdots, \vec{v}_K\}$ is an orthogonal basis, then

$$\boxed{\vec{w} = \left(\frac{\vec{w} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{w} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 + \cdots + \left(\frac{\vec{w} \cdot \vec{v}_K}{\vec{v}_K \cdot \vec{v}_K}\right)\vec{v}_K}$$

- If, moreover, $\|\vec{v}_j\|^2 = (\vec{v}_j \cdot \vec{v}_j) = 1$ (ie. St is orthonormal), then

$$\vec{w} = (\vec{w} \cdot \vec{v}_1)\vec{v}_1 + (\vec{w} \cdot \vec{v}_2)\vec{v}_2 + \cdots + (\vec{w} \cdot \vec{v}_K)\vec{v}_K$$

Example: Expand $\vec{w} = (-1, 5)$ in terms of $\{\vec{v}_1 = (2,0), \vec{v}_2 = (0,3)\}$

The set is orthogonal basis

$$\vec{w} = \left(\frac{\vec{w} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{w} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2$$
$$= \left(\frac{-2}{4}\right)\vec{v}_1 + \left(\frac{15}{9}\right)\vec{v}_2 = \boxed{\frac{5}{3}\vec{v}_2 - \frac{1}{2}\vec{v}_1}$$

BEST APPROXIMATION

- Let S be a normed inner product vector space (ie. a v. space with norm and inner / dot product defined)
- Let the norm be the natural norm (ie $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$)
- Let dim $S = N$
- Let $\{\vec{u}_1, \vec{u}_2 \cdots, \vec{u}_N\}$ be an orthonormal basis for S and $\vec{w} \in S$, then

$$\vec{w} = \sum_{j=1}^{N} \alpha_j \vec{u}_j = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_N \vec{u}_N, \text{ where}$$

$$\alpha_j = \vec{w} \cdot \vec{u}_j, \quad \text{for } j = 1, 2, \cdots, N.$$

APPROXIMATIONS CONT'D

- If $\dim S > N$, what is the best approximation of $\vec{w}$ in terms of the vectors of the set $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_N\}$

Theorem: Let $\vec{w}$ be any vector in a normed inner product vector space S with natural norm and let $\{\vec{u}_1, \vec{u}_2 \cdots, \vec{u}_N\}$ be an orthonormal set in S, then the best approximation

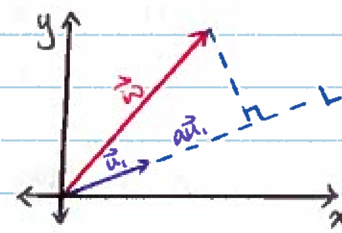
$$\vec{w} \approx c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_N\vec{u}_N = \sum_{j=1}^{N} c_j\vec{u}_j$$

is obtained when $c_j = \vec{w}\cdot\vec{u}_j$, $j = 1, 2, 3, \cdots, N$

Example: $S = \mathbb{R}^2$, $\vec{w} = (1,1)$, $\{\vec{u}_1 = \frac{1}{13}(12,5)\}$

$N = 1$, $\dim S = 2$

$$\begin{aligned}\vec{w} &\approx (\vec{w}\cdot\vec{u}_1)\vec{u}_1 \\ &= \left(\tfrac{12}{13} + \tfrac{5}{13}\right)\left(\tfrac{12}{13}, \tfrac{5}{13}\right) \\ &= \tfrac{1}{169}(204, 85) \\ &\approx (1.2, 0.5)\end{aligned}$$



MATRICES & L.I.'s

MATRICES & MATRIX ALGEBRA

Definition: An $m \times n$ matrix is a rectangular array of numbers

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \\ \begin{array}{cccc} \uparrow & & \uparrow & \\ 1^{st}\text{-column} & & j^{th}\text{-column} & \end{array} \end{array} \begin{array}{l} \leftarrow 1^{st}\text{-row} \\ \\ \\ \leftarrow i^{th}\text{-row} \\ \\ \\ \end{array}$$

- m = # of rows ; n = # of columns
- $m \times n$ is the dimension of the matrix A.

$$A = [a_{ij}]_{m\times n} \quad \text{or} \quad A = (a_{ij})_{m\times n}$$

- $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$

Definition: Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are equal iff $a_{ij} = b_{ij}$ $\forall\ i, j$.

Definition: If $m = n$, then $A_{n\times n}$ is a square matrix

Definition: A square matrix $U = [u_{ij}]_{n\times n}$ is said to be upper triangular if all entries below the diagonal are all zero : $u_{ij} = 0$ $\forall\ i > j$

Definition: A square matrix $L = [l_{ij}]_{n\times n}$ is said to be lower triangular if all entries above the diagonal are all zero : $l_{ij} = 0$ $\forall\ i < j$

Definition: A matrix $D = [d_{ij}]_{n\times n}$ is said to be diagonal if it is both upper and lower triangular. $d_{ij} = 0$ $\forall\ i \neq j$

- Denoted by $D = diag(d_{11}, d_{22}, \cdots, d_{nn})$

$$\begin{bmatrix} 10 & -2 & 0 \\ 0 & 0 & 19 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

upper triangular matrix   lower triangular matrix   diagonal matrix

MATRIX ADDITION, SCALAR MULTIPLICATIONS

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, $\alpha \in \mathbb{R}$

Addition: $A \cdot B = [a_{ij}]_{m\times n} \times [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$

Multiplication: $\alpha A = \alpha [a_{ij}]_{m\times n} = [\alpha a_{ij}]_{m\times n}$

PROPERTIES

- If A, B, and C are $m\times n$ matrices, $0 = [0_{ij}]_{m\times n}$ (zero matrix) and $\alpha, \beta \in \mathbb{R}$, then:

$$A + B = B + A \qquad \alpha(\beta A) = (\alpha\beta)A$$

$$A + (B + C) = (A + B) + C \qquad (\alpha + \beta)A = \alpha A + \beta A$$

$$A + 0 = A \qquad \alpha(A + B) = \alpha A + \alpha B$$

$$A + (-A) = 0 \qquad 1 \cdot A = A$$

- Any non-empty set of all $m\times n$ matrices is a vector space called matrix space:

$$S_{m\times n} = M_{m\times n} = \{A : A \text{ is } m\times n \text{ matrix}\}$$

- Example of a matrix subspace:

$$T_{n\times n} = \{A : A \text{ is an upper/lower tri. matrix}\} \subseteq S_{n\times n}$$

MATRIX MULTIPLICATION

- Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$
- We define the matrix $C = AB$ whose ij entry is defined by:

$$\boxed{C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}} \quad , \quad \begin{aligned} i &= 1, 2, \cdots, m \\ j &= 1, 2, \cdots, p \end{aligned}$$

Example: Evaluate AB where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}_{2\times 3} \qquad B = \begin{bmatrix} 4 & 1 & 0 \\ 0 & -4 & 0 \\ -2 & 0 & 3 \end{bmatrix}_{3\times 3}$$

$$AB = \begin{bmatrix} (2)(4) + 0 + 0 & (2)(1) + (1)(-4) + 0 & 0 + 0 + 0 \\ (-1)(4) + 0 + (1)(-2) & (-1)(1) + (3)(-4) + 0 & 0 + 0 + 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 & -2 & 0 \\ -6 & -13 & 3 \end{bmatrix}$$

Can we evaluate BA?

$\rightarrow$ No! The matrices are not conformable for multiplication

Example: Evaluate AB and BA where:

$$A = [2 \;\; 3]_{1\times 2} \quad , \quad B = \begin{bmatrix} -1 \\ 5 \end{bmatrix}_{2\times 1}$$

$$AB = [-2 + 15]_{1\times 1} = [13]_{1\times 1} \qquad BA = \begin{bmatrix} -2 & -3 \\ 10 & 15 \end{bmatrix}_{2\times 2}$$

- In general, $AB \neq BA$
- In $A_{m\times n} \times B_{m\times n}$: (i) $AB = (m\times n)(n\times m) = (m\times m)$

  (ii) $BA = (n\times m)(m\times n) = (n\times n)$
- In BA: (i) A is pre-multiplied by B

  (ii) B is post-multiplied by A.

- The algebraic linear system in m equations and n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m \end{aligned}$$

- This system can be written in a vector (or matrix) form as:

$$A\vec{x} = \vec{c} \quad , \text{ where}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \qquad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}_{m\times 1}$$

coefficient matrix — variable vector — constant vector

- Here, $A\vec{x} = \vec{c}$: $(m\times n)(n\times 1) = (m\times 1)$
- System of 1st-order DE's:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 \\ x_2' &= a_{21}x_1 + a_{22}x_2 \end{aligned} \longrightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- If A is an $m\times n$ matrix, then:
  (i) $A\cdot A\cdot \cdots A = A^p$, where p is a positive integer
  (ii) $A^p \cdot A^q = A^{p+q}$, where p, q are positive integers

- If D is an $n\times n$ diagonal matrix, then:
  (i) $D^p = \text{diag}(d_{11}^p, d_{22}^p, \cdots, d_{nn}^p)$

- The identity matrix of dimension $n\times n$ has the form:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

$I_n = [\delta_{ij}]_{n\times n}$ where δ_{ij} is the Kronker delta symbol defined as:

$$\delta_{ij} = \begin{cases} 1 & , \text{ if } i = j \\ 0 & , \text{ if } i \neq j \end{cases}$$

THE IDENTITY MATRIX

Properties of the Identity Matrix

- $A_{n\times n} I_n = I_n A_{n\times n} = A_{n\times n}$
- $A_{n\times n}^{\,0} = I_n$
- $0A = 0$; $A0 = 0$
- $0A = A0 = 0_{m\times n}$

Theorem: Even if $A \neq 0$, $AB = AC$ does not imply that $B = C$
(ii) $AB = 0$ does not imply $A = 0$ and/or $B = 0$
(iii) $A^2 = I$ does not imply $A = I$ or $A = -I$
(iv) $AB \neq BA$

- These are the differences between the multiplication of real numbers and the multiplication of matrices.

Properties of Matrix Multiplication:

- Let A, B, and C be (suitably) conformable matrices, $\alpha, \beta \in \mathbb{R}$
  (i) $(\alpha A)B = A(\alpha B) = \alpha(AB)$ (associative)
  (ii) $A(BC) = (AB)C$
  (iii) $(A+B)C = AC + BC$ (distributive)
  (iv) $C(A+B) = CA + CB$
  (v) $A(\alpha B + \beta C) = \alpha AB + \beta AC$ (linearity)

Proof (ii):
Dimensions: Let $A_{m\times n}$, $B_{n\times p}$, $C_{p\times q}$

Let $AB = U = [u_{ij}]_{m\times p} = \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]_{m\times p}$

Let $BC = V = [v_{ij}]_{n\times q} = \left[\sum_{k=1}^{p} b_{ik}c_{kj}\right]_{n\times q}$

$A(BC) = F = [f_{ij}]_{m\times q} = \left[\sum_{k=1}^{n} a_{ik}v_{kj}\right]_{m\times q}$

$(AB)C = G = [g_{ij}]_{m\times q} = \left[\sum_{k=1}^{p} u_{ik}c_{kj}\right]_{m\times q}$

$$[f_{ij}] = \left[\sum_{k=1}^{n} a_{ik}v_{kj}\right] = \left[\sum_{k=1}^{n} a_{ik}\left(\sum_{\ell=1}^{p} b_{k\ell}c_{\ell j}\right)\right] = \left[\sum_{\ell=1}^{p}\left(\sum_{k=1}^{n} a_{ik}b_{k\ell}\right)c_{\ell j}\right]$$
$$= \left[\sum_{\ell=1}^{p} u_{i\ell}c_{\ell j}\right] = [g_{ij}]$$

$[f_{ij}] = [g_{ij}] \rightarrow (AB)C = A(BC)$

Partitioning

Let $A = \left[\begin{array}{cc|c} 7 & -1 & 2 \\ \hline 0 & 1 & -3 \\ 12 & 11 & 4 \end{array}\right] \longrightarrow \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

- The blocks or submatrices are defined by:

$$A_{11} = [7 \;\; -1], \quad A_{12} = [2], \quad A_{21} = \begin{bmatrix} 0 & 1 \\ 12 & 11 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Addition:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}_{m \times n}, \quad B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} \end{bmatrix}_{m \times n}, \quad A+B = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & \cdots & A_{1n}+B_{1n} \\ \vdots & \vdots & & \vdots \\ A_{m1}+B_{m1} & A_{m2}+B_{m2} & \cdots & A_{mn}+B_{mn} \end{bmatrix}$$

- A_{ij} and B_{ij} are conformable for addition here.

Multiplication:

- If $m=n$ and we denote $AB = C$, then

$$C_{ij} = \left[\sum_{k=1}^{n} A_{ik} B_{kj}\right] \quad \text{or} \quad C = \begin{bmatrix} \sum_{k=1}^{n} A_{1k}B_{k1} & \cdots & \sum_{k=1}^{n} A_{1k}B_{kn} \\ \vdots & & \vdots \\ \sum_{k=1}^{n} A_{nk}B_{k1} & \cdots & \sum_{k=1}^{n} A_{nk}B_{kn} \end{bmatrix}$$

- A_{ij} and B_{ij} must be conformable for multiplication

Example: Calculate AB if:

$$A = \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 2 & 1 & 0 \\ \hline 0 & 5 & 3 \end{array}\right] \qquad B = \left[\begin{array}{c|c} 4 & 1 \\ -1 & 0 \\ \hline -2 & 5 \end{array}\right]$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad AB = \begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

$$A_{11}B_{11}+A_{12}B_{21} = \begin{bmatrix} -1 \\ 7 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, \quad A_{11}B_{12}+A_{12}B_{22} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$A_{21}B_{11}+A_{22}B_{21} = [-5] + [-6] = [-11], \quad A_{21}B_{12}+A_{22}B_{22} = [0] + [15] = [15]$$

$$\boxed{AB = \begin{bmatrix} 1 & -5 \\ 7 & 2 \\ -11 & 15 \end{bmatrix}}$$

THE TRANSPOSE MATRIX

Definition: Given any $m \times n$ matrix, $A = [a_{ij}]_{m \times n}$, the transpose of A, denoted by A^T or A-transpose, is defined by:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}_{n \times m}$$

- The $n \times m$ matrix is obtained by interchanging the rows and columns of A: the ij-entry becomes the ji-entry

$$\boxed{a_{ij}{}^T = a_{ji}}$$

- A^T is NOT the T^{th} power of A!

Example: Find A^T

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 1 & \frac{1}{2} & 7 \end{bmatrix}_{2 \times 3} \qquad A^T = \begin{bmatrix} -2 & 1 \\ 3 & \frac{1}{2} \\ 0 & 7 \end{bmatrix}_{3 \times 2}$$

__Properties of the Transpose__

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(\alpha A)^T = \alpha A^T$

(iv) $(AB)^T = B^T A^T$

Proof (iv): Let $AB = C$, $C = [c_{ij}]$. Then

$$c_{ij}{}^T = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki} = \sum_{k=1}^{n} b_{ki} a_{jk}$$

$$= \sum_{k=1}^{n} b_{ik}{}^T a_{kj}{}^T$$

$$(AB)^T = B^T \cdot A^T$$

- It also follows from (iv) that $(ABC)^T = C^T B^T A^T$

DOT PRODUCT IN A MATRIX FORM

- Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ $\quad \vec{x}\cdot\vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

- $\vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 \cdots + x_ny_n = \sum_{j=1}^{n} x_jy_j$
- In matrix form:

$$\vec{x}\cdot\vec{y} = \vec{x}^T\cdot\vec{y} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + \cdots x_ny_n = \sum_{j=1}^{n} x_jy_j$$

Definition: A square matrix is said to be symmetric if $A^T = A$.
→ It is also said to be skew-symmetric or anti-symmetric if $A^T = -A$.

DETERMINANTS

- The determinant of a $n\times n$ matrix $A = [a_{ij}]$ is defined by the cofactor expansion

$$\det(A) = \sum_{1}^{n} a_{jk} A_{jk}$$

where the summation is carried out on j for any fixed value of k $(1 \le k \le n)$ or on k for any fixed value of j $(1 \le j \le n)$.

- A_{jk} is called the cofactor of the a_{jk} element and is defined by:

$$\boxed{A_{jk} = (-1)^{j+k} M_{jk}}$$

where M_{jk} is called the minor of a_{jk}, namely, the determinant of the $(n-1)\times(n-1)$ matrix, when the row and column containing a_{jk} (the j^{th} row and k^{th} column) are struck out.

- Another notation for the determinant of A is $|A|$. This is not the absolute val!

Examples:

(1) $A = [-5]$, $|A| = -5$

DETERMINANTS CONT'D

(ii) $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\det(A) = |A| = \sum_{k=1}^{2} a_{1k} A_{1k}$ (fixing $j=1$)

$$\begin{aligned} |A| &= a_{11}A_{11} + a_{12}A_{12} \\ &= a_{11}[(-1)^{1+1} M_{11}] + a_{12}[(-1)^{1+2} M_{12}] \\ &= a_{11}(1)|a_{22}| + a_{12}(-1)|a_{21}| \\ &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

Example

$A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ $\det(A) = \sum_{k=1}^{3} a_{1k} A_{1k}$

$$\begin{aligned} \det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= a_{11}[(-1)^{1+1}M_{11}] + a_{12}[(-1)^{1+2}M_{12}] + a_{13}[(-1)^{1+3}M_{13}] \end{aligned}$$

$M_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$ $M_{12} = \begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix}$ $M_{13} = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$

$$= 2(6-8) + (-1)(5)(18-10) + 4(12-5)$$

$$\boxed{= -16}$$

- If $A = [a_{ij}]$ is an $n \times n$ upper/lower triangular matrix, then

$$\boxed{|A| = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}}$$

Properties of Determinants

(i) If $A \xrightarrow[c_j = c_j + \alpha c_k]{r_j = r_j + \alpha r_k} B$, r = row, c = column, $|A| = |B|$

(ii) If $A \xrightarrow[c_j \leftrightarrow c_k]{r_j \leftrightarrow r_k} B$, $|A| = -|B|$

(iii) If $A_{n\times n}$ is a triangular matrix, then
$|A| = \prod_{i=1}^{n} a_{ii}$

(iv) If all entries in any row or any column are zero, then
$|A| = 0$.

(v) If $A \xrightarrow[c_i = \alpha c_i]{r_i = \alpha r_i} B$, then $|A| = \alpha|B|$

(vi) $|\alpha A| = \alpha^n |A|$, where A is $n \times n$

(vii) If, in a matrix A, $r_i = \alpha r_k$ or $c_i = \alpha c_k$, $|A| = 0$.

Example: Evaluate $|A|$ if $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{bmatrix}$

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} \quad \begin{matrix} r_2 = r_2 - 2r_1 \\ r_3 = r_3 - 3r_1 \end{matrix} \longrightarrow \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} \quad r_2 \leftrightarrow r_3 \longrightarrow \begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix}$$

$$|A| = (2)(-6)(-5) = \boxed{60}$$

- If, in A, any row (or column) is a linear combination of other rows (or columns) then $|A| = 0$.
- Symbolically, in A, if

$$\left\{ \begin{matrix} r_i = \alpha r_j + \beta r_k \\ c_i = \alpha c_j + \beta c_k \end{matrix} \right\} \text{ then } |A| = 0$$

- If any one row or column of A is written as $a = b + c$, then

$$|A|_a = |A|_b + |A|_c$$

Example: Evaluate $|A|$ if $A = \begin{bmatrix} 3 & -8 & 2 \\ 1 & 2 & 4 \\ -2 & 0 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1+2 & -11+3 & 1+1 \\ 1 & 2 & 4 \\ -2 & 0 & 0 \end{bmatrix} \qquad |A| = \begin{vmatrix} 1 & -11 & 1 \\ 1 & 2 & 4 \\ -2 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & 0 & 0 \end{vmatrix}$$

(x) $|A^T| = |A|$

(xi) $|AB| = |A||B|$

(xii) $|A+B| \neq |A| + |B|$

(xiii) $|\alpha A + \beta B| \neq \alpha |A| + \beta |B|$

RANK

Definition: The maximum number of linearly independent rows in a matrix A is called the row rank of A.

Definition: The maximum number of linearly independent columns in a matrix A is called the column rank of A.

- If A is an $n \times m$ zero matrix, then the row rank = column rank = 0

Theorem: For any matrix A, the number of linearly independent row vectors is equal to the number of linearly independent column vectors, and these, in turn, equal the rank of A.

Notation: rank of A = rank(A) or r(A)

Theorem: Row (or column) equivalent matrices have the same rank. That is, elementary row (or column) operation do not alter the rank of a matrix.

- To evaluate rank(A), convert A into row echelon form (REF) or column echelon form (CEF). Then, the number of non-zero rows (or columns) is rank(A)

Example: Find rank(A) where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & 1 & 5 \end{bmatrix} \begin{matrix} r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - 3r_1 \end{matrix} \longrightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 7 & -4 \end{bmatrix} \quad \text{rank}(A) = 2$$

- In this example, the 2nd row = (-2) 1st row

Row and column vector space

- Let A be an $m \times n$ matrix with rows denoted by $r_1, r_2, \cdots, r_m$ and columns denoted by $c_1, c_2, \cdots, c_n$.
- The row vector space (or row space) of A is defined by
$$\text{span}\{r_1, r_2, \cdots, r_m\}$$
- The column vector space (or column space) of A is defined by
$$\text{span}\{c_1, c_2, \cdots, c_n\}$$
- The dimension of the row (and column) space of A is equal to the number of linearly independent rows (and columns) in A.

Example: How many L.I. vectors are in the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, where

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 7 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 \to r_2 - 2r_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 5 & 5 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 \to r_2 - 5r_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

- Since $\text{rank}(A) = 2$, there are 2 linear independent vectors in this set. Thus, $\dim\{\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}\} = 2$.
- The dimension of the row space of A is 2. The basis for that row space of A is given.

Application of Rank to Systems

Theorem: Consider the system $A\vec{x} = \vec{c}$, where A is an $m \times n$ matrix.

There is (i) no solution iff $r(A|\vec{c}) \neq r(A)$

(ii) a unique solution iff $r(A|\vec{c}) = r(A) = n$

(iii) an $(n-r)$-parameter family of solutions iff $r(A|\vec{c}) = r(A) < n$.

Example: Consider the system $\begin{cases} 2x - y = 3 \\ x + y + 2z = 1 \end{cases}$

$$[A|\vec{c}] = \left[\begin{array}{ccc|c} 2 & -1 & 0 & 3 \\ 1 & 1 & 2 & 1 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & -1 & 0 & 3 \end{array}\right] \xrightarrow{r_2 \to r_2 - 2r_1}$$

$$[A|\vec{c}] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -3 & -4 & 1 \end{array}\right] \qquad r(A|\vec{c}) = r(A) = 2, \quad 2 < n = 3$$

Thus, the system has a 1-parameter family of solutions $\{\alpha,\ 2\alpha - 3,\ 2 - \frac{3}{2}\alpha\}$

Example: Consider the system

$$\begin{cases} x + y + 3z = 2 \\ 3x + y = 0 \\ 4x + 2y + 3z = 1 \end{cases} \qquad [A|\vec{c}] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 3 & 1 & 0 & 1 \\ 4 & 2 & 3 & 1 \end{array}\right] \begin{array}{l} r_2 = r_2 - 3r_1 \\ r_3 = r_3 - 4r_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -2 & -9 & -3 \\ 0 & -2 & -9 & -5 \end{array}\right] \xrightarrow{r_3 \to r_3 - r_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -2 & -9 & -3 \\ 0 & 0 & 0 & -2 \end{array}\right] \qquad \begin{array}{l} r(A) = 2 \\ r(A|\vec{c}) = 3 \end{array}$$

- Since $r(A|\vec{c}) \neq r(A)$, the system is inconsistent!

Theorem: If A is $m \times n$, then

$$A\vec{x} = \vec{0}$$

(i) A is consistent

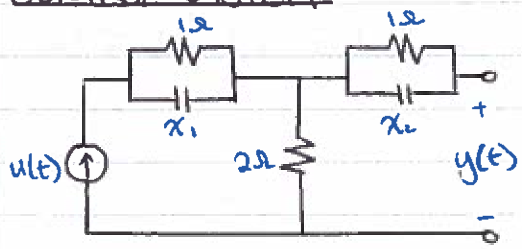
(ii) A admits the trivial solution $\vec{x} = \vec{0}$

(iii) A admits the unique solution $\vec{x} = \vec{0}$ iff $r(A) = n$

(iv) A admits an $(n-r)$-family of solutions (non-trivial), in addition to the trivial solution, iff $r(A) < n$

APPLICATIONS TO RANK OF A MATRIX

CONTROL SYSTEM



→ u(t) is the current source «input»
→ y(t) is the measured voltage «output»
→ $C_1 = C_2 = 1F$

- For $i = 1, 2$, x_i is the voltage across the capacitor with capacitance C_i.
- Does u(t) have an effect on x_2 or can u(t) control x_2?
  → Controllability provides an answer
- Can the initial voltage x_1 be observed from y?
  → Observability provides an answer.
- Let $\vec{x}(0) = \vec{x}_0$ be the initial voltage vector.
- Let $\vec{x}(t)$ be the voltage vector at time t.
- Here, $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

Controllability: In order to be able to do whatever you want with given initial voltage of the system (or network) under the control input, the system must be controllable.

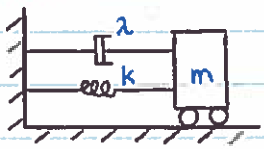
- Consider the n-dimensional, p-input system (or state equation)
$$\vec{x}\,' = A\vec{x} + B\vec{u}$$
- $\vec{x} \in \mathbb{R}^n$, where $\vec{x} = [x_1\ x_2 \cdots x_n]'$
- A is an $n \times n$ matrix ; B is an $n \times p$ matrix
- $\vec{u} \in \mathbb{R}^p$, where $\vec{u} = [u_1\ u_2 \cdots u_p]'$

- Special case: $p = 1$ (single input)
  → B is an $n \times 1$ matrix ; u(t) is scalar.
- We can construct the controllability matrix

$$\boxed{\tilde{C}(A,B) = \tilde{C} = [\underbrace{\underset{n\times1}{B}\quad \underset{\substack{(n\times n)(n\times1)\\=(n\times1)}}{AB}\quad \underset{n\times1}{A^2B}\ \cdots\ \underset{n\times1}{A^{n-1}B}}_{n \text{ columns}}]_{n\times n}}$$

- A system, or matrix pair (A, B) is controllable iff
$$\text{rank}(\tilde{C}) = n \qquad [\text{full row rank}]$$

Example: Consider the mass-spring-damper system:



m = mass of body
λ = damping coefficient
k = stiffness coefficient
x = position of mass

- By Newton's Second Law, the motion of the mass is:
$$mx'' + \lambda x' + kx = f(t)$$
$$x'' + \frac{\lambda}{m}x' + \frac{k}{m}x = u(t)$$

Is the system controllable?

$\vec{x} = ?$ $\quad A = ?$ $\quad B = ?$ $\quad u(t) = ?$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{array}{l} x_1' = x_2 \\ x_2' = -\frac{k}{m}x_1 - \frac{\lambda}{m}x_2 + u(t) \end{array}$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\lambda}{m} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\vec{x}\,' = A\vec{x} + B\vec{u} \longrightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\lambda}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Construct the controllability matrix
$$\tilde{C} = [B \;\; AB] = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{\lambda}{m} \end{bmatrix}_{2\times 2} \qquad \text{rank}(\tilde{C}) = 2$$

INVERSE MATRIX

- Motivation: If we have the scalar equation $ax = b$, then $x = \frac{1}{a} \cdot b$, where $a \neq 0$
- Here, $\frac{1}{a}$ is called the multiplication inverse of a.
- To solve $A\vec{x} = \vec{b}$, we could find the multiplication inverse of the matrix A.

Definition: A $n \times n$ matrix A is said to be non-singular or invertible if there exists a matrix B such that:

$$\boxed{BA = AB = I_n}$$

- The matrix $B_{n \times n}$ is said to be the multiplication inverse of A.
- If B and C are both multiplication inverses of A, then

$$\begin{aligned} B &= B(I) = B(AC) \\ &= (BA)C = IC \\ &= C \end{aligned}$$

- If a matrix A is non-singular, then it has at most one inverse
- If A is a non-singular matrix, its inverse is denoted by A^{-1}, and we call it A-inverse
- If A^{-1} is the inverse of A, then A is the inverse of A^{-1}.
- That is, $\boxed{(A^{-1})^{-1} = A}$

- For example: $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$

are inverses of each other because

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

INVERSE MATRIX CONT'D

Definition: An $n \times n$ matrix is said to be singular if it does not have a multiplication inverse.

Theorem: If A and B are non-singular $n \times n$ matrices, then AB is also non-singular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof: $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$
$= B^{-1}(I)B = B^{-1}B = I$
$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$
$= A(I)A^{-1} = AA^{-1} = I$

- It follows that if $A_1, A_2, \cdots, A_k$ are all non-singular, then $(A_1 A_2 \cdots A_k) = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$
- If $A^{-1}A = I$, then $\det(A^{-1}A) = \det(I) = 1$
- However, $\det(A^{-1}A) = \det(A^{-1})\det(A) = 1$

INVERSE AND THE DETERMINANT

- If a matrix A has an inverse A^{-1}, then it is necessary that $\det(A) \neq 0$.
- Moreover, $\det(A) = \dfrac{1}{\det(A^{-1})}$
- A is singular iff $\det(A) = 0$

FINDING LEMMA

- To find A^{-1} of a non-singular matrix A, we need to find lemma.

Lemma: Let A be an $n \times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for $1 \leq k \leq n$, then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A), & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

ADJOINT OF MATRIX

Definition: Let $A = [a_{ij}]$ be an $n \times n$ matrix. We define a new matrix called the adjoint of A by

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} =: [A_{jk}]_{n \times n}$$

- Where A_{jk} is the cofactor of the a_{jk} element in A.
- One can show that $A(\text{adj } A) = \det(A) I_n$
- If A is non-singular, then $\det(A) \neq 0$. We may write

$$A\left(\frac{1}{\det(A)} \text{adj}(A)\right) = I_n$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} \text{adj}(A)} \quad , \quad \text{when } \det(A) \neq 0$$

- Example: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \text{adj}(A)$

→ A is transformed into its cofactor matrix, where $A_{ij} = (-1)^{i+j} M_{ij}$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

→ Thus, if A is invertible, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- Remember this only works for $\det(A) \neq 0$!

MATH 215

Example: Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, Find $|A|$, adj A, A^{-1}

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -7 \\ 0 & -3 & -4 \end{bmatrix} \xrightarrow{\frac{-4}{3}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -7 \\ 0 & 0 & \frac{-6}{3} \end{bmatrix}$$

$$\det(A) = \tfrac{3}{4}(-4)(-\tfrac{5}{3})$$
$$|A| = 5$$

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} -3 & -1 \\ -2 & -3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & -2 \\ -2 & -3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} -2 & -2 \\ -3 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$\text{adj}(A) = \begin{bmatrix} 2 & -7 & 4 \\ 1 & 4 & -3 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{5}\begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

Theorem: If A is an $n \times n$ matrix, and $\det(A) \neq 0$, then $A\vec{x} = \vec{c}$ admits the unique Solution
$$\vec{x} = A^{-1}\vec{c}$$

Proof: $A^{-1}(A\vec{x}) = A^{-1}\vec{c}$
$I\vec{x} = A^{-1}\vec{c}$
$\vec{x} = A^{-1}\vec{c}$

Example: Solve the linear system

$$2x + y + 2z = 1$$
$$3x + 2y + 2z = 0$$
$$x + 2y + 3z = 4$$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Since $\det(A) \neq 0$, thus $\vec{x} = A^{-1}\vec{c}$

$$\vec{x} = \frac{1}{5}\begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 8 \end{bmatrix}$$

PROPERTIES OF THE INVERSE

(i) If A and B are of the same dimensions and both non-singular, then AB is also non-singular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(ii) If A is non-singular, then

$$(A^T)^{-1} = (A^{-1})^T$$

(iii) If A is non-singular, then

$$(A^{-1})^{-1} = A \quad \text{and} \quad (A^m)^n = A^{mn}$$

(iv) If A is non-singular, then

$$AB = AC \text{ implies } B = C$$
$$AB = 0 \text{ implies } B = 0$$

CRAMER'S RULE

Theorem: Let A be an $n \times n$ non-singular matrix and $\vec{b} \in \mathbb{R}^n$.
Let A_i be the matrix obtained by replacing the i^{th}-column of A by $\vec{b}$.
If $\vec{x}$ is the unique solution of $A\vec{x} = \vec{b}$, then

$$\boxed{x_i = \frac{|A_i|}{|A|}}, \text{ for } 1 \le i \le n$$

Proof:

- Since $\vec{x} = A^{-1}\vec{b} = \left(\frac{1}{|A|}\operatorname{adj}(A)\right)\vec{b}$
- It follows that

$$x_i = \frac{1}{|A|}(b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni})$$

$$x_i = \frac{1}{|A|}|A_i|$$

Example: Use Cramer's Rule to solve the system

$$x + 2y + z = 5$$
$$2x + 2y + z = 6$$
$$x + 2y + 3z = 9$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & +2 \end{vmatrix} = -4 \qquad |A_1| = \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = -4$$

$$|A_2| = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = -4 \qquad |A_3| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = -8$$

$$x_1 = \frac{|A_1|}{|A|} = 1, \quad x_2 = \frac{|A_2|}{|A|} = 1, \quad x_3 = \frac{|A_3|}{|A|} = 2$$

EVALUATION OF A^{-1} BY E.R.O.'s

- Consider the system

$$A\vec{x} = \vec{c} \quad \text{or} \quad A\vec{x} = I_n\vec{c}\,, \qquad A \in \mathbb{R}^{n\times n}$$

- Using a sequence of E.R.O's leads to

$$I_n\vec{x} = A^{-1}\vec{b} \quad \text{or} \quad \boxed{\vec{x} = A^{-1}\vec{b}}$$

Example: Use the E.R.O's to find the inverse of:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 4 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \quad \text{[augmented matrix]}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 5 & -2 & -3 & 1 & 0 \end{array}\right] \xrightarrow[r_3 \to r_3 - 5r_2]{r_1 \to r_1 + r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 8 & -3 & 1 & -5 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -\frac{6}{8} & \frac{2}{8} & \frac{-2}{8} \\ 0 & 0 & 1 & \frac{-3}{8} & \frac{1}{8} & \frac{-5}{8} \end{array}\right] \qquad \text{Thus, } A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ \frac{-3}{4} & \frac{1}{4} & \frac{-1}{4} \\ \frac{-3}{8} & \frac{1}{8} & \frac{-5}{8} \end{bmatrix}$$

- Remark: If an ERO is applied to I, then the new equivalent matrix to I is called the elementary matrix and denoted by E. Here, if an E.R.O. # i is applied to I, we get E_i.

$$I = E_6 E_5 \cdots E_2 E_1 A$$

$$A = E_6^{-1} E_5^{-1} \cdots E_2^{-1} E_1^{-1} I$$

$$A^{-1} = E_6 E_5 \cdots E_2 E_1 I$$

PROPERTIES OF THE INVERSE CONT'D

Proof (ii):

Method 1:

- $(AA^{-1})^T = I^T = I$
- $(AA^{-1})^T = (A^{-1})^T (A)^T = I^T = I$

$\rightarrow$ $(A^{-1})^T$ is the inverse of A^T, that is:

$$(A^{-1})^T = (A^T)^{-1}$$

Method 2:

- $A^{-1}A = AA^{-1} = I$
- $A^T(A^{-1})^T = (A^{-1})^T A^T = I^T = I$

$\rightarrow$ Again, $(A^{-1})^T$ is the inverse of A^T.

Proof (iii):

$$A^{-1}(AB) = A^{-1}(AC)$$
$$(A^{-1}A)B = (A^{-1}A)C$$
$$IB = IC$$
$$B = C$$

LU FACTORIZATION / DECOMPOSITION

- This is an alternative method to solve
$$A\vec{x} = \vec{c} \quad , \text{ where } A \in \mathbb{R}^{n \times n}$$
- We can decompose the matrix $A_{3\times3}$ as follows:
$$A = L \cdot U = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$
- In Doolittle's method, we can set l_{ii} to 1 for all i.
- The system then becomes
$$(LU)\vec{x} = \vec{c} \rightarrow L(U\vec{x}) = \vec{c}$$
- Let $U\vec{x} = \vec{y}$, then
$$L\vec{y} = \vec{c}$$
- To solve for $\vec{x}$,

  (i) Determine L and U

  (ii) Solve $L\vec{y} = \vec{c}$ for $\vec{y}$

  (iii) Solve $U\vec{x} = \vec{y}$ for $\vec{x}$

Example: Use Doolittle's method to solve

$$2x_1 + x_2 = 4$$
$$8x_1 + 2x_2 = 1$$

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{bmatrix}$$

$u_{11} = 2$, $u_{12} = 1$ $\quad l_{21}(2) = 8 \quad l_{21} = 4 \quad u_{22} + (4)(1) = 2 \quad u_{22} = -2$

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$$

* Check $LU = A$

$$L\vec{y} = \vec{c} \longrightarrow \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \longrightarrow \begin{matrix} y_1 = 4 \\ y_2 = -15 \end{matrix} \quad y = \begin{bmatrix} 4 \\ -15 \end{bmatrix}$$

$$U\vec{x} = \vec{y} \longrightarrow \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -15 \end{bmatrix} \quad \begin{matrix} 2x_1 + x_2 = 4 \\ 2x_2 = 15 \end{matrix} \quad \boxed{x = \begin{bmatrix} \frac{3}{4} \\ \frac{15}{2} \end{bmatrix}}$$

EIGENVALUES

Definition: Suppose that A is an $n \times n$ matrix. A non-zero vector $\vec{v} \in \mathbb{R}^n$ satisfying $A\vec{v} = \lambda\vec{v}$ is called an eigenvector of A and the scalar λ is called an eigenvalue of A. The pair $\lambda, \vec{v}$ is called the eigenpair of A.

FINDING EIGENVALUES AND EIGENVECTORS OF MATRICES

- Suppose that λ is an eigenvalue of $A_{n \times n}$. Then, there exists a non-zero eigenvector $\vec{v} \in \mathbb{R}^n$ such that
$$A\vec{v} = \lambda\vec{v} = \lambda I_n \vec{v}$$
- Also, note that
$$(A - \lambda I_n)\vec{v} = (\lambda I_n - A)\vec{v} = 0$$
- Hence, we have a homogenous system of n equations in the unknowns $v_1, v_2, \dots, v_n$. We learned that this homogeneous system has a non-trivial solution $\vec{v}$ iff
$$\det(A - \lambda I_n) = \det(\lambda I_n - A) = 0$$

- The equation solves for the Eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A_{n\times n}$.
- This equation is known as the characteristic equation of A.
- The left-hand side ($\det(A-\lambda I_n)$) is called the characteristic polynomial and is denoted by:

$$p(\lambda) = \det(A - \lambda I_n)$$

- If λ^* is an eigenvalue of A, then all non-zero solutions of the homogeneous system

$$(A - \lambda^* I)\vec{v} = (\lambda^* I - A)\vec{v} = 0$$

are eigenvalues of A corresponding to λ^*

Definition: Let λ be an eigenvalue of $A_{n\times n}$. Then, the set containing the zero vector and all eigenvectors of A corresponding to λ is called the eigenspace of λ.

- Remark: The eigenspace of any eigenvalue λ must contain at least one non-zero vector. Hence, the dimensions of the eigenspace must be at least one.

Example: Find the eigenvalues of A and the corresponding eigenvectors and eigenspaces, where

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A - \lambda I_2 = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = p(\lambda) = (5-\lambda)(4-\lambda) - 6$$
$$= \lambda^2 - 9\lambda + 14$$
$$= (\lambda-2)(\lambda-7) = 0$$

$\lambda_1 = 2, \quad \lambda_2 = 7$ are eigenvalues of A.

Cont'd : Solve $(A-\lambda I)\vec{v}=0$ for $\vec{v}\neq\vec{0}$

(i) Take $\lambda=2$: $[A-2I]=\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \rightarrow \begin{cases} 3v_1+2v_2=0 \\ 3v_1+2v_2=0 \end{cases}$

choose $v_2=\alpha$, $\alpha\in\mathbb{R}$, $\alpha\neq 0$ $\qquad v_1=\frac{-2}{3}\alpha$

If $\alpha=2$, $\vec{v}_1=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\alpha\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$

- There is only one linearly independent eigenvector
- The corresponding eigenspace is

$$\text{span}\left\{\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right\} \text{ or } \left\{\alpha\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}, \alpha\in\mathbb{R}\right\}$$

- Clearly, the dim of this eigenspace is 1 ; as there is one l.i. e-vector.
- A basis for the eigenspace is $\{[-\frac{2}{3} \;\; 1]'\}$

(ii) Take $\lambda=7$:

$(A-7I)\vec{v}=\vec{0} \longrightarrow \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad v_1-v_2=0$, $v_1=v_2$

- We choose $v_2=\alpha$, $\alpha\in\mathbb{R}$, $\alpha\neq 0$.
- The corresponding eigenvector is:

$$\vec{v}_2=\alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha\neq 0$$

- The corresponding eigenspace is

$$\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \text{ or } \left\{\alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \alpha\in\mathbb{R}\right\}$$

- There is one linearly independent eigenvector. Dim of eigenspace is 1
- A basis for the eigenspace is $\{[1 \;\; 1]'\}$

Note: the two eigenvectors $[-\frac{2}{3} \;\; 1]'$ and $[1 \;\; 1]'$ are linearly independent!

Example: Find the eigenvalues of A and the corresponding eigenvectors and eigenspaces, where

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$A - \lambda I_3 = \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_3) = -\lambda^3 + 6\lambda^2 + 15\lambda + 8$$
$$= -(\lambda+1)^2(\lambda-8) = 0$$

$$\lambda_1 = -1 \quad \lambda_2 = -1 \quad \lambda_3 = 8$$

- Here, the algebraic multiplicity of -1 is 2 ; that of 8 is 1

(i) $\lambda = -1$ $\qquad (A-(-1)I)\vec{v} = 0$ for $\vec{v} \neq 0$

$$A + I = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow v_1 + \tfrac{1}{2}v_2 + v_3 = 0$$

Taking $v_2 = \alpha$, $v_3 = \beta$, $\vec{v} \neq 0$

$$v_1 = -\tfrac{1}{2}\alpha - \beta$$

$$\vec{v} = \begin{bmatrix} -\frac{1}{2}\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Choose $\alpha = 1$ and $\beta = 0$; also $\alpha = 0$ and $\beta = 1$ to get:

$$\vec{v}_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Corresponding eigenspace to $\lambda_1 = \lambda_2 = -1$:

$$\operatorname{span}\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ or } \left\{ \alpha \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \alpha, \beta \in \mathbb{R} \right\}$$

- Dim of this vectorspace is 2.

Definition: The trace of $A = [a_{ij}]_{n\times n}$ is defined by:

$$tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

- For an $n\times n$ matrix A, the characteristic polynomial $p(\lambda) = (A - \lambda I)$ is of degree n.
- $|A - \lambda I| = |(-1)(\lambda I - A)| = (-1)^n |\lambda I - A|$
- The total number of e. values of $A_{n\times n}$ is n. The eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_n$. They may be repeated.
- The constant term in $p(\lambda) = |A - \lambda I|$ is $|A| = \prod_{i=1}^{n} \lambda_i$
- The eigenvalues of A^k are $\lambda^k{}_1, \lambda^k{}_2, \cdots \lambda^k{}_n$ (i)
- The matrix A is invertible iff all eigenvalues are non-zero.
- If λ is an eigenvalue of A, then λ is an eigenvalue of A^T (ii)
- If A is invertible, then the eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}$

Proof:

(i) Let λ be an eigenvalue of A

$$A\vec{v} = \lambda\vec{v}$$
$$A(A\vec{v}) = A\lambda\vec{v}$$
$$A^2\vec{v} = A\lambda\vec{v}$$
$$A^2\vec{v} = \lambda(A\vec{v})$$
$$A^2\vec{v} = \lambda(\lambda\vec{v})$$
$$A^2\vec{v} = \lambda^2\vec{v}$$

(ii) $|A - \lambda I| = |(A - \lambda I)^T|$

$$= |A^T - (\lambda I)^T|$$
$$= |A^T - \lambda I|$$

OR

$$A^T\vec{v} = \lambda(I\vec{v})^T$$
$$= \lambda I\vec{v}$$
$$= A\vec{v}$$

Theorem: Suppose that $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalues of $A_{n\times n}$ ($\lambda_i \neq \lambda_j$ for all $i \neq j$) with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$.
Hence, the set of $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is linearly independent.

APPLICATIONS TO ELEMENTARY SINGULARITIES

- Consider the system of 1st-order differential equations:

$$x' = ax + by \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \vec{x}\,' = A\vec{x}$$
$$y' = cx + dy$$

Definition: The equilibrium solution is a solution that does not change with time.
- Hence, an equilibrium solution is a constant solution,
- i.e. $\vec{x}_* = \begin{bmatrix} x_* \\ y_* \end{bmatrix}$

- An equilibrium point is written as an ordered pair (x_*, y_*)
- Equilibrium point is also called a singular point.
- Note that $x' = 0$ and $y' = 0$

- We are interested in investigating how the solution to the system behaves near the singular point when time changes.
- We need to find the eigenvalue of A.
- The solution of the system is given by:

$$\boxed{\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}}$$

→ λ_1, λ_2 are the eigenvalues of A
→ $\vec{v}_1, \vec{v}_2$ are the eigenvectors of A
→ c_1, c_2 are constants determined by IC's.

STABILITY OF SINGULAR POINTS

- If the solution of the system starts close to the singular point, it will stay close to that point, or go to that point.
- We can determine the stability of singular points based on the signs of λ_1 and λ_2.
- There are three cases:

(i) λ's are the same sign
→ if $\lambda < 0$, point is stable
→ if $\lambda > 0$, point is unstable

(ii) λ's are real and opposite signs

→ Singular point is stable

(iii) λ's are complex

→ $\alpha < 0$, point is stable focus (spirial)

→ $\alpha > 0$, point is unstable focus (spirial)

→ $\alpha = 0$, point is centre.

Example: Find the singular point of the following system and its stability:

$$x' = x + 3y$$
$$y' = 5x + 3y$$

Singular point (set x', y' to 0):

$$x + 3y = 0$$
$$5x + 3y = 0$$

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \qquad |A| \neq 0$$

→ As $\det(A) \neq 0$, system has unique trivial solution

Finding eigenvalues, $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{vmatrix} \qquad (1-\lambda)(3-\lambda) - 15 = 0$$
$$\lambda^2 - 4\lambda - 12 = 0$$

→ The eigenvalues of A are: $\lambda_1 = -2$, $\lambda_2 = 6$

Find eigenvectors for λ_1, λ_2

$\lambda_1 = -2$ $\qquad (A - (-2)I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix} \vec{v} = \vec{0} \qquad \vec{v}_1 = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_2 = 6$ $\qquad (A - (6)I)\vec{v} = \vec{0}$

$$\begin{bmatrix} -5 & 3 \\ 5 & -3 \end{bmatrix} \vec{v} = 0 \qquad \vec{v}_2 = \alpha \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

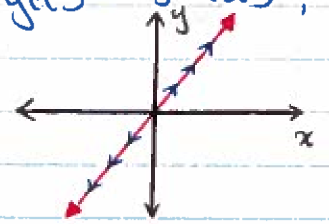
$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

$$\boxed{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t}}$$

- Note that if $C_1 = 0$, then

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3C_2 e^{6t} \\ 5C_2 e^{6t} \end{bmatrix} \qquad \begin{aligned} x(t) &= 3C_2 e^{6t} \\ y(t) &= 5C_2 e^{6t} \end{aligned}$$

- We have $y(t) = \frac{5}{3} x(t)$, a line in $\mathbb{R}^2$

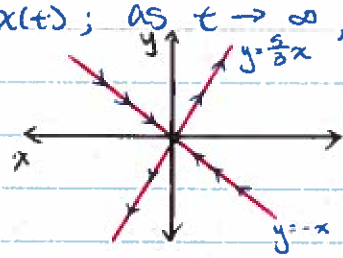


phase plane or phase portrait of $y = \frac{5}{3}x$, $C_1 = 0$

- If $C_2 = 0$, then

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} C_1 e^{-2t} \\ -C_1 e^{-2t} \end{bmatrix} \qquad \begin{aligned} x(t) &= C_1 e^{-2t} \\ y(t) &= -C_1 e^{-2t} \end{aligned}$$

- $y(t) = -x(t)$; as $t \to \infty$, $x(t), y(t) \to 0$



Singular point at $(0,0)$

SYMMETRIC MATRICES

- Recall that a matrix A is symmetric if $A_{n \times n}^T = A$

Theorem: If A is symmetric, then all of its eigenvalues are real.

Theorem: If an eigenvalue of an symmetric matrix A is of multiplicity k (ie $\lambda_1, \lambda_2, \cdots, \lambda_n = \alpha$), then the eigenspace corresponding to the eigenvalue λ is of dimension k.

Theorem: If A is symmetric, then the corresponding eigenvectors to distinct eigenvalues are are orthogonal.

Theorem: If A is symmetric, then the corresponding eigenvector to distinct eigenvectors are orthogonal.

Proof: Let $\vec{x}_j$ and $\vec{x}_k$ be eigenvectors corresponding to distinct eigenvalues λ_j and λ_k, respectively.

$$A\vec{x}_j = \lambda_j \vec{x}_j \qquad A\vec{x}_k = \lambda_k \vec{x}_k$$
$$\vec{x}_k A\vec{x}_j = \vec{x}_k \cdot \lambda_j \vec{x}_j \qquad A\vec{x}_k \vec{x}_j = \lambda_k \vec{x}_k \cdot \vec{x}_j$$
$$\vec{x}_k^T (A\vec{x}_j) = \lambda_j (\vec{x}_k^T \vec{x}_j) \qquad (A\vec{x}_k)^T \vec{x}_j = \lambda_k \vec{x}_k^T \vec{x}_j$$
$$\vec{x}_k A\vec{x}_j = \lambda_j (\vec{x}_k^T \vec{x}_j) \qquad \vec{x}_k^T A^T \vec{x}_j = \lambda_k \vec{x}_k^T \vec{x}_j$$
$$\vec{x}_k^T A \vec{x}_j = \lambda_k \vec{x}_k^T \vec{x}_j$$
$$\lambda_j (\vec{x}_k^T \vec{x}_j) = \lambda_k (\vec{x}_k^T \vec{x}_j)$$
$$(\lambda_j - \lambda_k)(\vec{x}_k^T \vec{x}_j) = 0$$
$$\lambda_j - \lambda_k \neq 0, \therefore$$
$$\vec{x}_k^T \vec{x}_j = \vec{x}_k \vec{x}_j = 0 \quad \square$$

- Using the properities $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$ and $(AB)^T = B^T A^T$
- Also, in a symmetric matrix, $A^T = A$.

- Consider the matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Here, $A^T = A$

$$-(\lambda - 4)(\lambda - 1)^2 = 0$$
$$\lambda_1 = 4, \quad \lambda_2 = \lambda_3 = 1$$

Eigenvectors: $\vec{x}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{x}_2 = \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

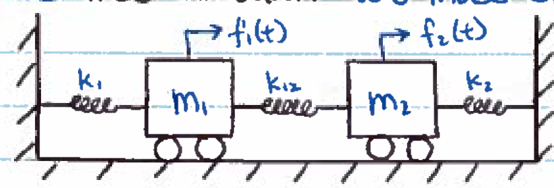
- In accordance with theorems of symmetric matrices
  - (i) λ's are real
  - (ii) $\lambda_1 = 4$ is of multiplicity 1. Its eigenspace has dim 1; $\lambda_{2,3} = 1$ is of multiplicity 2. Its eigenspace has dim 2.
  - (iii) $\vec{x}_1 \cdot \vec{x}_2 = 0$ for all choices of α, β, γ. The eigenvectors are orthogonal.

APPLICATIONS OF EIGENVALUES

- Notice that the eigenvectors $[-1, 0, 1]^T$ and $[-1, 1, 0]^T$ are linearly independent.
- However, the basis for the eigenspace P_1 is not orthogonal.
- This does not violate the theorem as they come from the same eigenvalue.
- We can make the space orthogonal by choosing different values of β and γ.

APPLICATIONS

Consider the free vibration two mass-spring system



- Here,
  - → m_1, m_2 are masses
  - → k_1, k_{12}, k_2 are the stiffness of the springs
  - → f_1 and f_2 are external (input) forces
  - → x_1, x_2 are the positions of the masses

- By Newton's Second Law

$$m_1 x_1'' + (k_1 + k_{12})x_1 - k_{12}x_2 = f_1(t)$$
$$m_2 x_2'' - k_{12}x_1 + (k_2 + k_{12})x_2 = f_2(t)$$

→ If we let $m_1 = m_2 = k_1 = k_{12} = k_2 = 1$ and $f_1(t) = f_2(t) = 0 \ \forall t$,

$$x_1'' + 2x_1 - x_2 = 0$$
$$x_2'' - x_1 + 2x_2 = 0$$

- Assume that we are interested in the oscillatory behaviour of the system. That is, mathematically, we assume

$$x_1(t) = q_1 \sin(\omega t + \phi) \qquad \longrightarrow \qquad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \sin(\omega t + \phi)$$
$$x_2(t) = q_2 \sin(\omega t + \phi)$$

- where q_i's are the amplitudes, ω is the frequency, and ϕ the phase angle.

$$-w^2 q_1 + 2q_1 - q_2 = 0$$
$$-w^2 q_2 - q_1 + 2q_2 = 0$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = w^2 \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \longrightarrow A\vec{q} = w^2\vec{q}$$

- $\lambda = w^2$ is the eigenvalue of A and $\vec{q}$ its corresponding eigenvector.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} \qquad (2-\lambda)^2 - 1 = 0 \qquad \lambda_1 = 1$$
$$\lambda^2 - 4\lambda + 3 = 0 \qquad \lambda_2 = 3$$

- $w = \sqrt{\lambda}$, $w_1 = 1$, $w_2 = \sqrt{3}$. These are frequencies

Eigenvectors:

$$\lambda_1 = 1 \rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha \neq 0$$

$$\vec{x}_1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(t + \phi_1)$$

$$\lambda_2 = 3 \longrightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \vec{v}_2 = \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \beta \neq 0$$

$$\vec{x}_2(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin(\sqrt{3}\,t + \phi_2)$$

$$\vec{x}(t) = \vec{x}_1(t) + \vec{x}_2(t)$$

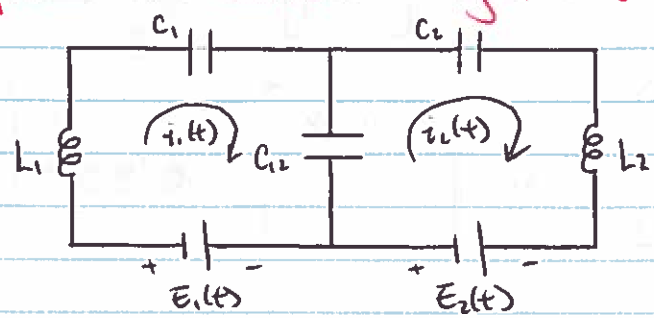
$$\boxed{\vec{x}(t) = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(t + \phi_1) + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin(\sqrt{3}\,t + \phi_2)}$$

Remarks:

- Each eigen pair defines a vibration mode
- The eigenvalues give the vibration frequencies ($w = \sqrt{\lambda}$)
- The eigenvectors give the mode shape or configuration.
- $w_1 = 1$ is the low mode; $w_2 = \sqrt{3}$ is the high mode.
- The frequencies are called natural frequences (external force, $f_1(t) = f_2(t) = 0$)
- $\vec{v}_1 \cdot \vec{v}_2$ are orthogonal - $\vec{v}_1 \cdot \vec{v}_2 = 0$

APPLICATIONS OF EIGENVALUES CONT'D

Example: Consider the following circuit.



mesh 1: $L_1 i_1' + \frac{1}{C_1}\int i_1\,dt + \frac{1}{C_{12}}(i_1 - i_2)\,dt = E_1(t)$

mesh 2: $L_2 i_2' + \frac{1}{C_2}\int i_2\,dt + \frac{1}{C_{12}}(i_2 - i_1)\,dt = E_2(t)$

$$\begin{cases} L_1 i_1'' + \left(\frac{1}{C_1} + \frac{1}{C_{12}}\right) i_1 - \frac{1}{C_{12}} i_2 = E_1'(t) \\ L_2 i_2'' + \left(\frac{1}{C_2} + \frac{1}{C_{12}}\right) i_2 - \frac{1}{C_{12}} i_1 = E_2'(t) \end{cases}$$

- Taking $L_1 = L_2 = C_1 = C_2 = C_{12} = 1\,H / 1\,F$ and $E_1(t) = E_2(t) = 0$

$$\begin{aligned} i_1'' + 2i_1 - i_2 &= 0 \\ i_2'' - i_1 + 2i_2 &= 0 \end{aligned} \longrightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \omega^2 \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

SIMILAR MATRICES

- Suppose that we have $A\vec{x} = \vec{y}$, where $A \in \mathbb{R}^{n\times n}$
- Set $\vec{x} = Q\vec{\tilde{x}}$ and $\vec{y} = Q\vec{\tilde{y}}$, where Q is invertible $n\times n$ matrix.

$$A(Q\vec{\tilde{x}}) = Q\vec{\tilde{y}}$$
$$Q^{-1}AQ\vec{\tilde{x}} = Q^{-1}Q\vec{\tilde{y}}$$
$$\tilde{A}\vec{\tilde{x}} = \vec{\tilde{y}}, \quad \tilde{A} = Q^{-1}AQ$$

Definition: Given any invertible matrix Q, matrices A and $\tilde{A} = Q^{-1}AQ$ are said to be similar.

$$\boxed{\text{Exercise: Show that if } A \text{ and } \tilde{A} \text{ are similar, then they have the same characteristics and eigenvalues.}}$$

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad Q^{-1}AQ = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \qquad Q = \begin{bmatrix} -\frac{1}{2} & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad Q^{-1}BQ = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

DIAGONALIZATION

- Consider the system:

$$\vec{x}\,' = A\vec{x} \quad , \quad \vec{x} \in \mathbb{R}^n \quad , \quad A \in \mathbb{R}^{n \times n}$$

- If A is not diagonal, then the system is coupled.
- Our aim is to convert the system into an uncoupled system by diagonalizing A, if possible.
- Let $\vec{x} = Q\vec{y}$, where Q is an $n \times n$ invertible constant matrix

$$\vec{x}\,' = (Q\vec{y})' = Q\vec{y}\,'$$
$$Q\vec{y}\,' = AQ\vec{y}$$
$$\vec{y}\,' = Q^{-1}AQ\vec{y}$$

- Thus, given a matrix A, the aim is to find Q so that

$$Q^{-1}AQ = D$$

- Here, D is diagonal ; hence, the system becomes uncoupled.

Definition: Let A be an $n \times n$ matrix. If there exists an invertible matrix Q and diagonal matrix D such that $Q^{-1}AQ = D$, then we say that A is diagonalizable and that Q diagonalizes A.

- The matrix Q is called the modal matrix of A.

VANDERMONDE DETERMINANT

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} \prod_{j<k\le n} (\lambda_j - \lambda_k)$$

- If $n=2$, $\begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} = \lambda_1 - \lambda_2$
- If $n=3$, $\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} = (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)$

Theorem: Let A be an $n\times n$ matrix.

(i) A is diagonalizable iff it has n linearly independent eigenvectors.

(ii) If A has n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n$, and we make them columns of Q so that $Q = [\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n]$ then $Q^{-1}AQ = D$ is diagonal and the j^{th}-diagonal element of D is the j^{th}-eigenvalue of A.

Theorem: If an $n\times n$ matrix A has distinct eigenvalues $\lambda_1, \lambda_2 \cdots, \lambda_n$, then the corresponding eigenvectors are linearly independent

Proof: We need to show that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0 \qquad (1)$$

This holds only if $c_1 = c_2 = \cdots = c_n = 0$.

Multiply (1) by A and noting that $A\vec{v}_j = \lambda_j\vec{v}_j$

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_n\lambda_n\vec{v}_n = 0$$

$$c_1\lambda_1^2\vec{v}_1 + c_2\lambda_2^2\vec{v}_2 + \cdots + c_n\lambda_n^2\vec{v}_n = 0$$

$$\vdots$$

$$c_1\lambda_1^{n-1}\vec{v}_1 + c_2\lambda_2^{n-1}\vec{v}_2 + \cdots + c_n\lambda_n^{n-1}\vec{v}_n = 0$$

We can convert this equation to its matrix form

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1\vec{v}_1 \\ c_2\vec{v}_2 \\ \vdots \\ c_{n-1}\vec{v}_{n-1} \\ c_n\vec{v}_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

- The determinant of the coefficient is a Vandermonde determinant which is non-zero if the λ_j's are distinct
- The system has a unique trivial solution
  $$c_1\vec{v}_1 = \vec{0}, \quad c_2\vec{v}_2 = \vec{0} \quad \cdots, \quad c_n\vec{v}_n = \vec{0}$$
- Since $\vec{v}_j$'s are eigenvectors ($\neq \vec{0}$ by definition), it follows that $c_1 = c_2 = \cdots = c_n = 0$
- Thus, $\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n$ are linearly independent $\square$.

Theorem: If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

- Note: the converse of this theorem is not necessarily true.

Example: Consider the system

$$x' = y$$
$$y' = -16x - 10y$$

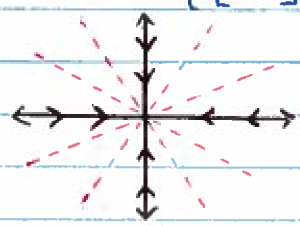
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -16 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- $\lambda(A)$: $\lambda_1 = -8$, $\lambda_2 = -2$; $\vec{v}_1 = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
- $Q = [\vec{v}_1, \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ -8 & -2 \end{bmatrix}$; $\tilde{A} = Q^{-1}AQ = \begin{bmatrix} -8 & 0 \\ 0 & -2 \end{bmatrix}$ $\lambda(\tilde{A}) = -8,$

$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \end{bmatrix} = \begin{bmatrix} -8 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \quad \text{or} \quad \begin{aligned} \tilde{x}' &= -8\tilde{x} \\ \tilde{y}' &= -2\tilde{y} \end{aligned} \rightarrow \begin{aligned} \tilde{x} &= A_1 e^{-8t} \\ \tilde{y} &= A_2 e^{-2t} \end{aligned}$$

- Eigenvectors of $\tilde{A}$ are:
  $$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Basis of A: $\left\{ \begin{bmatrix} 1 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ (1)
- Basis of $\hat{A}$: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ [canonical basis] (2)



(1)

(2)

Theorem: Each symmetric matrix is diagonalizable

Proof: Previous theorems state that A is diagonalizable iff it has n linearly independent eigenvectors (11.4.1) and every symmetric matrix has n orthogonal eigenvectors (11.3.4)

→ Suppose that for a symmetric matrix A we use the normalized eigenvectors of A to form its modal matrix Q of A so that

$$Q = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$$

$$Q^T Q = \begin{bmatrix} \vec{v}_1^{\,T} \\ \vec{v}_2^{\,T} \\ \vdots \\ \vec{v}_n^{\,T} \end{bmatrix} [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n] = \begin{bmatrix} \vec{v}_1^{\,T}\vec{v}_1 & \vec{v}_1^{\,T}\vec{v}_2 & \cdots & \vec{v}_1^{\,T}\vec{v}_n \\ \vec{v}_2^{\,T}\vec{v}_1 & \vec{v}_2^{\,T}\vec{v}_2 & \cdots & \vec{v}_2^{\,T}\vec{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_n^{\,T}\vec{v}_1 & \vec{v}_n^{\,T}\vec{v}_2 & \cdots & \vec{v}_n^{\,T}\vec{v}_n \end{bmatrix} = I_n$$

→ $Q^T Q = I_n$, hence,

$$\boxed{Q^{-1} = Q^T}$$

→ This is guarenteed by normalized eigenvectors

- The above relation is uncorrelated with $\vec{v}_j$'s being eigenvectors.
- It is true for any square matrix Q having columns that are orthonormal
- In this case, Q is said to be an orthogonal matrix.
- If A is symmetric, then $Q^{-1} A Q = D$ is diagonal whether or not Q has columns that are normalized.

JORDAN NORMAL FORM

- If $A_{n\times n}$ is not diagonalizable, it can be triangularized.
- In this case, a "generalized modal matrix" P can be found for A so that
$$P^{-1}AP = J$$
where J is triangular.
- J is called Jordan form; it is an upper triangular matrix with main diagonal containing the eigenvalues of A
- J contains all zeros above the main diagonal except for 1's immediately above one or more diagonal elements.

Example: Consider a 6×6 matrix A which has eigenvalues λ's and eigenvectors as follows:

(i) $\lambda_1, \lambda_1, \lambda_1$ and only one linearly independent eigenvector $\vec{u}_1$.

(ii) λ_2 is single with an eigenvector $\vec{v}_1$

(iii) λ_3, λ_3 and only one linearly independent e.vector.

- In this example, there are only 3 linearly independent eigenvectors.
- To form P, we need 3 more linearly independent eigenvectors

(1) We have $\lambda_1, \lambda_2, \lambda_3$ and $\vec{u}_1$. Find $\vec{u}_2$ and $\vec{u}_3$

$(A-\lambda_1 I)\vec{u}_1 = 0$

$(A-\lambda_1 I)\vec{u}_2 = \vec{u}_1$ [solve for $\vec{u}_2$]

$(A-\lambda_1 I)\vec{u}_3 = \vec{u}_2$ [solve for $\vec{u}_3$]

→ $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly independent vectors.

→ $\vec{u}_2$ and $\vec{u}_3$ are called generalized eigenvectors

(2) Repeat with $\lambda_2, \vec{v}_1$

(3) Repeat with $\lambda_3, \lambda_3, \vec{w}_1$:

$(A-\lambda_3 I)\vec{w}_1 = 0$

$(A-\lambda_3 I)\vec{w}_2 = \vec{w}_1$ [solve for $\vec{w}_2$]

→ $\vec{w}_1$ and $\vec{w}_2$ are linearly independent

→ $\vec{w}_2$ is the generalized eigenvector

- We are now able to form the generalized modal matrix P

$$P = [\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{v}_1, \vec{w}_1, \vec{w}_2]_{6\times 6}$$

$$P^{-1}AP = J = \left[\begin{array}{ccc|c|cc} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{array}\right]$$

Example: Find the generalized modal matrix P of

$$A = \begin{bmatrix} 2 & -1 & 2 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 5 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3 = 2$

$\vec{u}_1 = [1, 0, 0, 0]^T$

$\lambda_4 = 5$

$\vec{v}_1 = [0, 0, 0, 1]^T$

- Generalized eigenvectors are

$(A - \lambda_1 I)\vec{u}_2 = \vec{u}_1$ $\leftarrow$ Solve for $\vec{u}_2$

$\vec{u}_2 = [\alpha, 1, 1, \frac{2}{3}]^T$, $\alpha \in \mathbb{R}$

$(A - \lambda_1 I)\vec{u}_3 = \vec{u}_2$ $\leftarrow$ Solve for $\vec{u}_3$

$\vec{u}_3 = [\beta, \alpha + 2, \alpha + 1, \frac{2}{3}\alpha + \frac{5}{9}]$

$$P = [\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{v}_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & \frac{2}{3} & \frac{5}{9} & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

<u>APPLICATIONS</u>

- $\vec{x}' = A\vec{x}$
- Let $x = P\tilde{x}$, P is the generalized modal matrix of A.
- Then,

$$\vec{\tilde{x}}' = P^{-1}AP\vec{\tilde{x}} = J\vec{\tilde{x}}.$$

MATH 215 FINAL EXAM REVIEW

- Solving systems of equations
  - Gauss elimination / back substitution
  - Gauss Jordan (RREF)

  Used in finding eigenvectors. Use RREF to find rank of matrix.
  Applications of systems, ie. circuits.

- Vector Spaces
  - Norm, inner / dot products and their properties
  - Schwartz and triangle inequality
  - Orthogonality and orthonormality.
  - Properties of the subspace $\{\vec{0}$, addition, multiplication$\}$
  - Spanning sets and linear dependence
  - Basis of sets using L.I. / L.D. and spanning set
  - Best approximation problems
  - Applications, ie. circuits and mass-spring.

- Matrices:
  - Arithemetic operations
  - All theorems and proofs
  - Partitions
  - Second-order DE to system $\vec{x}' = A\vec{x}$
  - Transpose
  - Determinant (2 methods) - Know the properties
  - Inverse (adjoint and REF methods)
  - Using inverse and determinants to solve systems
  - Rank and the controllability matrix, $\tilde{C}$
  - Higher order DE $\rightarrow$ system $\vec{x}' = A\vec{x} + \vec{b}u$
  - Cramer's rule and LU decomposition (Dolittle)

- Eigenvalues
  - Properties
  - Theorem and proofs.
  - Applications, ie. Singularities, sketching, stability.
  - Similar matrices and their diagonalization / Jordan form.
  - Symmetric matrices and orthogonal eigenvectors.
  - Orthogonal matrix $Q^{-1} = Q^T$