Math 172
Assignment 9
Tuesday, April 10, 2018

Problem G, Problem H, 14.6 (2c, 22, 23a, 37)

Problem G Verify that if ω is a complex 3rd root of unity ($\omega^3 = 1$) and $x = u + v$ is a solution to the depressed cubic $x^3 + px + q = 0$, then $x = u\omega + v\omega^2$ and $x = u\omega^2 + v\omega$ are also solutions. Explain why $x = u\omega + v\omega$ is not a solution in the context of the derivation of Cardano's formulas.

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Problem H Use Cardanos formulas to solve $x^3 + x^2 - 2 = 0$. Find all 3 solutions. Then verify that your expressions are equal to the 3 roots: $1, (-1 + i), (-1 - i)$.

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14.6.2c Determine the Galois groups of the following polynomial: $x^3 - x + 1$.

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14.6.22 *(Newton's Formulas)* Let $f(x)$ be a monic polynomial of degree n with roots $\alpha_1, \ldots, \alpha_n$. Let s_i be the elementary symmetric function of degree i in the roots and define $s_i = 0$ for $i > n$. Let $p_i = \alpha_1^i + \cdots + \alpha_n^i$, $i \geq 0$, be the sum of the i^{th} powers of the roots of $f(x)$. Prove Newton's Formulas:

$$\begin{aligned}
&p_1 - s_1 = 0\\
&p_2 - s_1 p_1 + 2s_2 = 0\\
&p_3 - s_1 p_2 + s_2 p_1 - 3s_3 = 0\\
&\quad\vdots\\
&p_i - s_1 p_{i-1} + s_2 p_{i-2} - \cdots + (-1)^{i-1} s_{i-1} p_1 + (-1)^i i s_i = 0.
\end{aligned}$$

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$\boxed{\textbf{14.6.23a} \quad \text{If } x+y+z=1,\, x^2+y^2+z^2=2 \text{ and } x^3+y^3+z^3=3, \text{ determine } x^4+y^4+z^4.}$

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14.6.37 Let $f(x_1, \ldots, x_n)$ be a polynomial which is symmetric in $x_1, \ldots, x_n$. Recall that the degree (sometimes called the *weight*) of the monomial $Ax_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}(a_i \geq 0)$ is $a_1 + a_2 + \cdots + a_n$ and that a polynomial is *homogeneous (of degree m)* if every monomial has the same degree (m).

(a) Show that every polynomial $f(x_1, \ldots, x_n)$ can be written as a sum of homogeneous polynomials. Show that if $f(x_1, \ldots, x_n)$ is symmetric then each of these homogeneous polynomials is also symmetric.

(b) Show that the monomial $Bs_1^{a_1}s_2^{a_2}\ldots s_n^{a_n}$ in the elementary symmetric functions is a homogeneous polynomial in $x_1, x_2, \ldots, x_n$ of degree $a_1 + 2a_2 + \cdots + na_n$.

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