Math 172
Assignment 1
Tuesday, January 23, 2016

7.3 (1, 10, 19, 26, 28) 7.4 (10) and Problem A

7.3.1 Let R be a ring with identity and $1 \neq 0$. Prove that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are isomorphic.

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7.3.10 Decide which of the following ideals are ideals of the ring $\mathbb{Z}[x]$:

(a) the set of all polynomials whose constant term is a multiple of 3

(b) the set of all polynomials whose coefficient of x^2 is a multiple of 3

(c) the set of all polynomials whose constant term, coefficient of x and coefficient of x^2 are zero

(d) $\mathbb{Z}[x^2]$ (i.e., the polynomials in which only even powers of x appear)

(e) the set of polynomials whose coefficients sum to zero

(f) the set of polynomials $p(x)$ such that $p'(0) = 0$, where $p'(x)$ is the usual first derivative of $p(x)$ with respect to x.

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7.3.19 Prove that if $I_1 \subseteq I_2 \subseteq \ldots$ are ideals of R then $\sum_{n=1}^{\infty} I_n$ is an ideal of R.

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7.3.26 The *characteristic* of a ring R is the smallest positive integer n such that

$$\underbrace{1+1+\ldots+1=0}_{n \text{ times}}$$

in R; if no such integer exists the characteristic of R is said to be 0. For example, $\mathbb{Z}/n\mathbb{Z}$ is a ring of characteristic n for each positive integer n and $\mathbb{Z}$ is a ring of characteristic 0.

(a) Prove that the map $\mathbb{Z} \to R$ defined by

$$k \mapsto \begin{cases} 1+1+\ldots+1 \ (k \text{ times}) & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ -1-1-\ldots-1 \ (-k \text{ times}) & \text{if } k < 0 \end{cases}$$

is a ring homomorphism whose kernel is $n\mathbb{Z}$, where n is the characteristic of R. (This explains the use of the terminology "characteristic 0" instead of the archaic phrase "characteristic ∞" for rings in which no sum of 1s is zero.)

(b) Determine the characteristics of the rings $\mathbb{Q}$, $\mathbb{Z}[x]$, $\mathbb{Z}/n\mathbb{Z}[x]$.

(c) Prove that if p is a prime and if R is a commutative ring of characteristic p, then

$$(a+b)^p = a^p + b^p$$

for all $a, b \in R$.

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7.3.28 Prove that an integral domain has characteristic p, where p is either prime or 0 (cf. Exercise 26).

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7.4.10 Assume R is commutative. Prove that if P is a prime ideal of R and P contains no zero divisors then R is an integral domain.

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Problem A

(i) Let R be an integral domain. Prove that the units in $R[x]$ are precisely the constant polynomials $p(x) = u$ where u is a unit in R.

(ii) On the other hand, show that $p(x) = 1 + 2x$ is a unit in $R[x]$, where $R = \mathbb{Z}/4\mathbb{Z}$.

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