Math 172
Assignment 6
Tuesday, February 27, 2018

13.4.4, Problems B, C, D

13.4.4 Determine the splitting field and its degree over $\mathbb{Q}$ for $x^6 - 4$.

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B. Let p be a prime number and F the field of integers mod p, and let $p(x)$ and $q(x)$ be any two irreducible polynomials of degree 2 over F. Show that the fields $F[x]/(p(x))$ and $F[x]/(q(x))$ are isomorphic by constructing an explicit isomorphism.

(a) Show that if $p = 2$, then the statement is true (what are the irreducibles?), so you may as well assume for the remainder of the problem that p is not 2.

(b) Show that if you construct a ring homomorphism $\phi : F[x] \to F[y]$ so that the ideal $(p(x))$ gets sent into the ideal $(q(y))$, then you will have a well-defined homomorphism γ on their quotients:

$$\gamma : F[x]/(p(y)) \to F[x]/(q(y)).$$

(c) Since $\{[1], [x]\}$ are a basis for $F[x]/(p(x))$, then it suffices to specify where 1 and x go under ϕ. As a ring homomorphism, $\phi(1) = 1$. Now you'll need to set

$$\phi(x) = ax + b$$

for some a and b, but choose a and b such a way that $\phi(p(x))$ is sent to a multiple of $q(x)$.

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C. Find a real number u such that $Q(\sqrt{3}, \sqrt{5}) = Q(u)$.

$\blacksquare$

D. Suppose K is an extension of F, and $\phi : K \to K$ is an isomorphism that leaves every element of F fixed. Show that any polynomial in $F[x]$ that has a root r in K also has $\phi(r)$ as a root.

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