Math 321 Fall 2011
Homework 6
Due: October 14, 2011

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books outside of our textbook or other people, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

1. pg. 112 # 4. Let H and K be normal subgroups of G. Prove that $H \cap K \lhd G$.

2. pg. 112 # 7. Prove that if H is a subgroup of G of index 2, then $H \lhd G$.

3. (a) pg. 116 # 5. Prove that if G is an abelian group and $H < G$ then G/H is abelian.
   (b) Prove that if G is cyclic and $H < G$ then G/H is cyclic.
   (c) If H and G/H are abelian, must G be abelian?

4. If H is a normal subgroup of a group G, prove that the *centralizer of* H *in* G, $C_G(H) = \{g \in G \mid ghg^{-1} = h \text{ for all } h \in H\}$ is a normal subgroup of G.

5. Let G be a group and N a normal subgroup. Prove that the order of the element gN in G/N is n, where n is the smallest positive integer such that $g^n \in N$ (and gN has infinite order if no such positive integer exists). Give an example to show that the order of gN in G/N may be strictly smaller than the order of g in G.

6. Consider the additive quotient group $\mathbb{Q}/\mathbb{Z}$. Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.

7. Let N be a normal subgroup of G and let H be any subgroup of G. Prove that $NH = \{n \cdot h \mid n \in N, h \in H\}$ is a subgroup of G. Give an example to show that NH need not be a subgroup of G if neither N nor H is normal.

8. Let G be a group and H a subgroup of G of index 2. Show that H contains every element of G of odd order.

Challenge

1. Assume both H and K are normal subgroups of G with $H \cap K = 1$. Prove that $xy = yx$ for all $x \in H$ and $y \in K$.

2. Let $H \leq K \leq G$. Prove that $|G : H| = |G : K| \cdot |K : H|$. DO NOT assume G is finite.