Math 321 Fall 2016
Homework 10
Due: November 23, 2016

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books outside of our textbook or other people, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

1. An ideal A of a commutative ring R with unity is said to be finitely generated if there exist elements $a_1, a_2, \ldots a_n$ of A such that $A = (a_1, a_2, \ldots, a_n)$. An integral domain R is said to satisfy the ascending chain condition if every strictly increasing chain of ideals $I_1 \subset I_2 \cdots$ must be finite in length. Show that an integral domain R satisfies the ascending chain condition if and only if every ideal of R is finitely generated.
2. 19.10
3. 19.12 We used this in the proof of Theorem 19.3.
4. 19.13 We used this in the proof of mod p irreducibility.
5. (a) How many roots does $X^2 + \overline{3}X + \overline{2}$ have in $\mathbb{Z}/6\mathbb{Z}$?

   (b) Find, with proof, all the irreducible polynomials of degree 2 or 3 in $\mathbb{Z}/2\mathbb{Z}[X]$.

   (c) Show that the polynomial $\overline{2}X + \overline{1}$ in $\mathbb{Z}/4\mathbb{Z}[X]$ has a multiplicative inverse in $\mathbb{Z}/4\mathbb{Z}[X]$.
6. (a) If I is an ideal of a ring R, prove that $I[X]$ is an ideal of $R[X]$.

   (b) Let R be a commutative ring with unity. If I is a prime ideal of R, prove that $I[X]$ is a prime ideal of $R[X]$.
7. Determine (with explanation) whether the following polynomials are irreducible in the rings indicated.

   (a) $X^4 + X + \overline{1} \in \mathbb{Z}/2\mathbb{Z}[X]$

   (b) $X^2 + X + \overline{4} \in \mathbb{Z}/11\mathbb{Z}[X]$

   (c) $X^6 + 30X^5 - 15X^3 + 6X - 120 \in \mathbb{Z}[X]$

   (d) $X^2 + X + 4 \in \mathbb{Z}[X]$

   (e) $\frac{3}{7}X^4 - \frac{2}{7}X^2 + \frac{9}{35}X + \frac{3}{5} \in \mathbb{Q}[X]$ (Hint: Part (a) might come in handy).