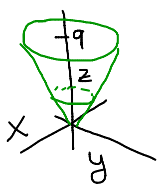
Math 133: Calculus II

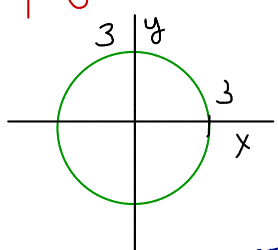
Final Exam Review Solutions

1. Evaluate $\iiint_R z$ where R is the region between $x^2 + y^2 = z$ and $z = 9$.

Cone is $x^2+y^2=z$



cylindrical coordinates
project onto xy



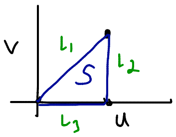
$r=\sqrt{x^2+y^2}$ so $r^2=x^2+y^2$

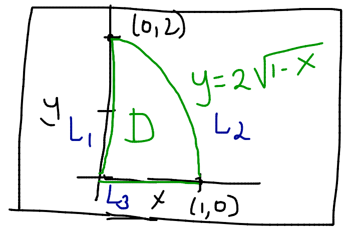
$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=3}\int_{z=r^2}^{z=9} z\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 \left[\frac{z^2}{2}r\right]_{z=r^2}^{z=9} dr\,d\theta = \int_0^{2\pi}\int_0^3 \frac{81}{2}r - \frac{r^5}{2}\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{81}{4}r^2 - \frac{r^6}{12}\right]_0^3 d\theta = \int_0^{2\pi} \frac{3^6}{4} - \underbrace{\frac{\cancel{3^6}}{\cancel{12}}}_{3^5/4}\,d\theta = \int_0^{2\pi}\frac{3^5\cdot\cancel{2}}{\cancel{4}\,2}\,d\theta = \frac{243}{2}\theta\Big]_0^{2\pi}$$

$$=\boxed{243\pi}$$

2. (a) Find the image of the region S defined as the triangle with coordinates $(0,0)$, $(1,0)$, $(1,1)$ under the transformation $T(u,v) = (u^2 - v^2, 2uv)$.





L_1: $u = v$, $0 \le u \le 1$

So $u^2 - v^2 = 0$
$y = 2uv = 2u^2$

$x = 0$
$0 \le y \le 2$

L_2: $u = 1$

So $\begin{cases} x = 1 - v^2 \\ y = 2v \end{cases}$

$x = 1 - \left(\frac{y}{2}\right)^2$
$y = 2\sqrt{1-x}$

L_3: $v = 0$, $0 \le u \le 1$

So $x = u^2$
$y = 0$
$0 \le x \le 1$

(b) Let D be the region found in (a). Compute $\iint\limits_D \sqrt{x^2 + y^2}\, dx\, dy$ using the transformation from (a).

Jacobian is $\begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$

$$\iint_S f(u^2 - v^2, 2uv)\,|4u^2 + 4v^2|\, du\, dv = \int_{v=0}^{v=1}\int_{u=v}^{u=1} \sqrt{(u^2 - v^2)^2 + (2uv)^2}\, |4u^2 + 4v^2|\, du\, dv$$

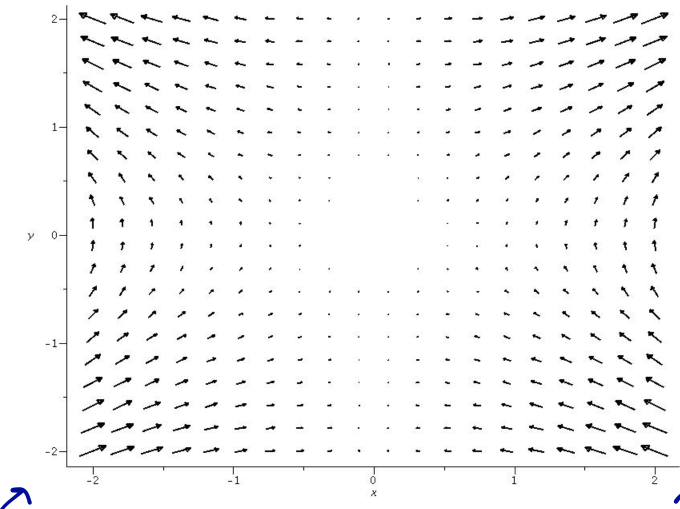
$$= \int_0^1 \int_{u=v}^{u=1} \sqrt{\underbrace{u^4 - 2u^2v^2 + v^4 + 4u^2v^2}_{= (u^2+v^2)^2}} \cdot (4u^2 + 4v^2)\, du\, dv = \int_0^1 \int_{u=v}^{u=1} 4(u^2+v^2)(u^2+v^2)\, du\, dv$$

$$= \int_0^1 \int_{u=v}^{u=1} 4(u^4 + 2u^2v^2 + v^4)\, du\, dv = \int_0^1 4\left(\frac{u^5}{5} + \frac{2u^3v^2}{3} + uv^4\right)\Bigg]_{u=v}^{u=1} dv =$$

$$\int_0^1 4\left(\frac{1}{5} + \frac{2}{3}v^2 + v^4\right) - 4\underbrace{\left(\frac{v^5}{5} + \frac{2v^5}{3} + v^5\right)}_{= \frac{3v^5 + 10v^5 + 15v^5}{15} = \frac{28v^5}{15}} dv \quad = 4\left(\frac{1}{5}v + \frac{2}{9}v^3 + \frac{v^5}{5} - \frac{\overset{14}{\cancel{28}}}{15 \cdot \underset{3}{\cancel{6}}} v^6\right)\Bigg]_0^1$$

$$= \boxed{4\left(\frac{1}{5} + \frac{2}{9} + \frac{1}{5} - \frac{14}{45}\right)} - 0$$

3. Find the gradient vector field ∇f of $f(x, y) = x^2y$ and sketch it.



Above is the vector field.

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \boxed{\langle 2xy, x^2 \rangle}$$

4. Evaluate the line integral $\int_C \mathbf{F} \bullet \, d\mathbf{r}$ where $\mathbf{F}$ is the vector field $\langle y^2, x^2 \rangle$ and C is the curve $y = x^{-1}$ for $1 \leq x \leq 2$.

We parametrize the curve C as $x=t$, $y=1/t$ for $1 \leq t \leq 2$

Then $$\int_C \vec{F} \cdot d\vec{r} = \int_1^2 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_1^2 \underbrace{\left\langle \left(\frac{1}{t}\right)^2, t^2 \right\rangle}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{\left\langle 1, -\frac{1}{t^2} \right\rangle}_{\vec{r}\,'(t)} dt =$$

$$= \int_1^2 \frac{1}{t^2} - 1\,dt = -\frac{1}{t} - t \Big]_1^2 dt = -\frac{1}{2} - 2 + 1 + 1 = \boxed{-1/2}$$

5. Find the work done by the force field $\mathbf{F}(x, y) = \langle x^2, y - x\rangle$ in moving an object along the curve given by $\mathbf{r}(t) = \langle (1 + \sin t), (t - \cos \cancel{t})\rangle$ from $t = 0$ to $t = \pi$.

typo. Should be t.

We are asking for $\int_C \vec{F}\cdot d\vec{r}$ where C is defined by $\vec{r}(t)$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} \overbrace{\langle (1+\sin t)^2, t-\cos t -1-\sin t\rangle}^{\vec{F}(\vec{r}(t))} \bullet \overbrace{\langle \cos t, 1+\sin t\rangle}^{\vec{r}\,'(t)}\, dt$$

$$= \int_0^{\pi} \underbrace{(1+2\sin t+\sin^2 t)\cos t}_{\cancel{\cos t}+2\sin t\cancel{\cos t}+\sin^2 t\cos t} + t - \cancel{\cos t} - 1 - \cancel{\sin t} + t\sin t - \cos t\cancel{\sin t} - \cancel{\sin t} - \overbrace{\sin^2 t}^{=\left(\frac{1-\cos 2t}{2}\right)}\, dt$$

$$= \int_0^{\pi} \sin t\cos t + \sin^2 t\cos t + t - 1 - 2\sin t + t\sin t - \sin^2 t\, dt$$

$u = \sin t \quad du = \cos t\, dt$

parts $u = t \quad v = -\cos t$
$du = 1\,dt \quad dv = \sin t\, dt$

$$= \frac{\sin^2 t}{2} + \frac{\sin^3 t}{3} + \frac{t^2}{2} - t + 2\cos t - t\cos t + \underbrace{\int_0^{t} \cos t\, dt}_{=\sin t} - \frac{t}{2} + \frac{\sin 2t}{4}\Bigg]_0^{\pi}$$

$$= \underset{0}{\cancel{\frac{\sin^2 \pi}{2}}} + \underset{0}{\cancel{\frac{\sin^3 \pi}{3}}} + \frac{\pi^2}{2} - \pi + 2\underset{-1}{\cancel{\cos\pi}} - \pi\underset{-1}{\cancel{\cos\pi}} + \underset{0}{\cancel{\sin\pi}} - \pi/2 + \frac{\overset{0}{\cancel{\sin 2\pi}}}{4} -$$

$$\left(\underset{0}{\cancel{\frac{\sin^2 0}{2}}} + \underset{0}{\cancel{\frac{\sin^3 0}{3}}} + \frac{0}{2} - 0 + 2\underset{1}{\cancel{\cos}}0 - 0\cos 0 + \underset{0}{\cancel{\sin 0}} - \frac{0}{2} + \underset{0}{\cancel{\frac{\sin 0}{4}}}\right) = \frac{\pi^2}{2} - \cancel{\pi} + \cancel{2} + \cancel{\pi} - \pi/2 - \cancel{2}$$

$$= \boxed{\pi^2/2 - \pi/2}$$

6. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f so that $\mathbf{F} = \nabla f$ and compute $\int_C \mathbf{F} \bullet\ d\mathbf{r}$ where C is the parametrization $x = t^2$ and $y = 1/t$ for $1 \le t \le 2$.

(a) $\mathbf{F}(x, y) = \langle \frac{x}{y^2+1}, \frac{y}{x^2+1}\rangle$

We first check the partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$

$$\frac{\partial P}{\partial y} = -\frac{2yx}{(y^2+1)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{-2xy}{(x^2+1)^2}$$

Since these are not equal, this is <u>not</u> conservative by Theorem $\boxed{5}$

(b) $\mathbf{F}(x, y) = \langle 2xy + y^3, x^2 + 3xy^2 + 2y \rangle$

$$\frac{\partial P}{\partial y} = 2x + 3y^2, \quad \frac{\partial Q}{\partial x} = 2x + 3y^2 \quad \text{plus continuous everywhere}$$

Since these partials are the same, this is conservative by Theorem $\boxed{6}$.

To find f we consider $f = \int f_x\,dx = \int 2xy + y^3\,dx = x^2y + y^3x + g_1(y)$

and $f = \int f_y\,dy = \int x^2 + 3xy^2 + 2y\,dy = x^2y + xy^3 + y^2 + g_2(x)$

Putting these together gives us $f = x^2y + xy^3 + y^2 + K$

Finally we compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{r}(t) = \langle t^2, 1/t \rangle$, $1 \le t \le 2$

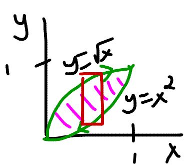
So $\int_C \vec{F} \cdot d\vec{r} = \int_1^2 \langle \underbrace{2\cancel{t^2}\tfrac{1}{\cancel{t}}}_{2t} + \tfrac{1}{t^3}, t^4 + \underbrace{3\cancel{t^2}\cdot\tfrac{1}{\cancel{t^2}}}_{3} + 2\cdot\tfrac{1}{t} \rangle \cdot \langle 2t, -\tfrac{1}{t^2} \rangle\,dt =$

$$\int_1^2 \underbrace{\cancel{4t^2} + \cancel{\tfrac{2}{t^2}} - \cancel{t^2}}_{3t^2} - \cancel{\tfrac{3}{t^2}} - \tfrac{2}{t^3}\,dt = t^3 + \tfrac{1}{t} + \tfrac{1}{t^2}\Big]_1^2 = 8 + \tfrac{1}{2} + \tfrac{1}{4} - 1 - 1 - 1 = \boxed{5\tfrac{3}{4}}$$

7. Use Green's Theorem to evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle x + y, x^2 - y \rangle$ and C is the boundary of the region defined by $y = x^2$ and $y = \sqrt{x}$, negatively oriented from $0 \le x \le 1$.

C green

D pink



$$\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$$ (this gives us a positively oriented curve)

By Green's Theorem

$$\int_{-C} \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \int_{x=0}^{x=1}\int_{y=x^2}^{y=\sqrt{x}} 2x - 1\,dy\,dx =$$

$$\int_{x=0}^{x=1} 2xy - y\Big]_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 \underbrace{2\cancel{x\sqrt{x}}}_{x^{3/2}} - \sqrt{x} - 2x^3 + x^2\,dx$$

$$= \frac{2x^{5/2}}{5/2} - \frac{x^{3/2}}{3/2} - \frac{2x^4}{4} + \frac{x^3}{3}\Big]_0^1 = \frac{4}{5} - \frac{2}{3} - \frac{1}{2} + \frac{1}{3} - 0$$

$$= \frac{4}{5} - \frac{1}{3} - \frac{1}{2} = \frac{24}{30} - \frac{10}{30} - \frac{15}{30} = -\frac{1}{30}$$

So $\int_C \vec{F} \cdot d\vec{r} = -(-1/30) = \boxed{1/30}$