Math 133: Calculus II

Chapter 17: Key Points *

For most of these results, it is safest to assume the curves C are smooth (no pointy edges) or at least *piecewise-smooth curves* (curves with a finite number of pointy edges, or curves which we can break up into a union of smooth curves). Most of what is discussed will be for the case of functions or vector fields with two variables. The results can be extended with relative ease to functions or vector fields with three variables.

1 Line Integrals

In §17.2 we learned about a special integral: line integrals of a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$. If $\mathbf{F}(x,y)$ is a vector field and $\bullet$ means a dot product, then

$$\int_C \mathbf{F} \bullet \,\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}\left(\mathbf{r}(t)\right) \bullet \mathbf{r}'(t)\,\mathrm{d}t = \int_a^b \mathbf{F}\left(x(t), y(t)\right) \bullet \mathbf{r}'(t)\,\mathrm{d}t.$$

In the special case where $\mathbf{F}(x,y)$ is a force field, this integral has an interpretation as work done by the vector field on a particle moving along the curve C.

2 Special Properties of Gradient Vector Fields

Theorem 1 (Fundamental Theorem of Line Integrals §17.3 $\boxed{2}$)**.** *If C is a curve given by the vector valued function $\mathbf{r}(t)$ for $a \le t \le b$, and f is a differentiable function, then*

$$\int_C \nabla f \bullet d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

What this tells us is that to compute a line integral on a **gradient** vector field we need to only compute the value of f at the terminal point and the initial point, and compute that difference. We will see later that it is very important that we have a gradient vector field, as this result is not true for a general vector field.

WHY is this theorem true? We plug in the values for the gradient (as paritial derivatives) and apply the dot product and then use the Chain Rule.

There are two main consequences of the Fundamental Theorem of Line Integerals. **First**, line integrals of gradient vector fields are *path independent*: if we travel two different ways through the vector field from one point to another, the line integral will be the same, by Theorem $\boxed{2}$.

Second, if C is a *closed path*, one that starts and stops at the same point in the (x, y) plane, then $\int_C \nabla f \cdot d\mathbf{r} = 0$ since the functional value at both the initial and terminal point will be the same.

These are both valuable properties for a line integral to have. They are true for gradient vector fields by Theorem $\boxed{2}$ but are they true for any other vector fields? It turns out that the answer to this question is **No!** These are properties that only hold for line integrals on gradient vector fields. We call these vector fields *conservative vector fields*.

We next show that the two ideas of (1) path independence and (2) an integral evaluating to 0 along all closed paths are identical ideas. Our goal is going to be to show that only conservative vector fields have these properties.

Theorem 2 (§17.3 $\boxed{3}$)**.** $\int_C \mathbf{F}\bullet\ \mathrm{d}\mathbf{r}$ *is independent of path in a region R if and only if* $\int_C \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} = 0$ *for every closed path C in R.*

WHY is this theorem true? First, take any closed path C and suppose that every integral is independent of path in a region R. Break C up into two pieces, one called C_1 from a point A to a point B and the other called C_2 from point B to point A (see Figure 3 pg. 1084). Since we have independence of path, we know travleing along C_1 from A to B or traveling along C_2 from A to B will yield the same integral value. So

$$\int_{C_1} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} = \int_{-C_2} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} \qquad \text{or} \qquad \int_{C_1} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} - \int_{-C_2} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} = 0 \tag{2.1}$$

(where $-C_2$ means travel C_2 backward from A to B). But at the same time we know that $C = C_1 + C_2$ (traveling around C is the same as traveling along C_1 from A to B and then along C_2 from B to A). Hence $\int_C \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} = 0$ by Equation 2.1.

To show the converse, we take any two paths from A to B, call them C_1 and C_2. We need to show that the integrals are independent of path, or $\int_{C_1} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r}$. Similar to above, we know that we can form a closed path C by following C_1 from A to B and then tracing C_2 backward from B to A. Hence

$$0 = \int_C \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} + \int_{-C_2} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} \qquad \text{or} \qquad \int_{C_1} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} = -\int_{-C_2} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F}\bullet\ \mathrm{d}\mathbf{r}.$$

Hence we have independence of path.

Finally we can conclude that path independence in a vector field only occurs when the vector field is conservative.

Theorem 3 (§17.3 $\boxed{4}$)**.** *If* $\int_C \mathbf{F}\bullet\ \mathrm{d}\mathbf{r}$ *is path independent on a region R, then* $\mathbf{F}$ *is a conservative vector field on that region R.*

WHY is this theorem true? The main idea is to create a function f such that $\mathbf{F} = \nabla f$. We define f to be

$$f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F}\bullet \mathrm{d}\mathbf{r}.$$

The rest of this theorem takes a bit of work to prove and we won't go into the specifics here. More details may be found in the book on pages 1084-1085.

For the next two sections, we will assume that $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on the regions we are working over. This condition ensures that we can simply work with the partial derivatives without worrying about whether they exist at every point.

3 Determining a Gradient Vector Field

Section 2 has given us a sense that gradient vector fields are very important. So if we are given some vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$, how do we tell if $\mathbf{F}$ is conservative? What would it mean for $\mathbf{F}$ to be conservative? It would mean that $\mathbf{F} = \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ for some function $f(x, y)$. So $P(x, y)$ would have to equal $\frac{\partial f}{\partial x}$ and $Q(x, y)$ would have to equal $\frac{\partial f}{\partial y}$.

Just messing around for a minute, what would $\frac{\partial P}{\partial y}$ be? Since $P = \frac{\partial f}{\partial x}$, then

$$\frac{\partial P}{\partial y} = \frac{\partial f^2}{\partial y \partial x}.$$

Similarly,

$$\frac{\partial Q}{\partial x} = \frac{\partial f^2}{\partial x \partial y}$$

since $Q = \frac{\partial f}{\partial y}$. This means $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are the two mixed second partials of f. But a result called Clairaut's Theorem tells us these values are the same, or $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

So if $\mathbf{F}$ is a conservative vector field, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. It turns out that the **converse** of this is also true in "nice" regions. The converse says that (assuming the region is "nice") if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ then $\mathbf{F} = \langle P(x, y), Q(x, y) \rangle$ is conservative. To understand why this is true we need a result called Green's Theorem.

4 Green's Theorem

Before we state Green's Theorem, we have to talk about some notation. If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$, then $\int_C \mathbf{F} \bullet \mathrm{d}\mathbf{r}$ can also be written as $\int_C P \ \mathrm{d}x + Q \ \mathrm{d}y$. Also, we say a curve is *positively oriented* curve if we travel the path so that the interior of the region which C is the boundary of is on the left.

Theorem 4 (Green's Theorem, §17.4)**.** *Let C be a positively oriented, piece-wise smooth, simple closed curve in the plane, and let R be the region bounded by C. Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be a vector field. Then*

$$\int_C P \ \mathrm{d}x + Q \ \mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right) \ \mathrm{d}A.$$

What this theorem says is that the line integral of a vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ along a "nice" closed path which is the boundary of a region R is the same as the double integral of the function $f(x, y) = \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}$ over the region R.

Finally, we can now understand why a vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ with the condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ must be a gradient vector field.

Theorem 5 ([§17.3 $\boxed{6}$]). *Let $\mathbf{F} = \langle P, Q\rangle$ be a vector field on an open, simply-connected region R (a "nice" region with no holes or gaps). If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on all of R, then $\mathbf{F}$ is conservative.*

WHY is this theorem true? Since we can assume $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ everywhere, we also know

$$f(x, y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \text{ for all } (x, y).$$

Let C be any closed path and let R be the region which C is the boundary of. The integral of this function f must be 0 also:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \, \mathrm{d}A = 0.$$

But if we plug this function into Green's Theorem, what do we get?

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \, \mathrm{d}A = \int_C P \, \mathrm{d}x + Q \, \mathrm{d}y$$

since C is a closed path. Putting these two pieces of information together gives us $\int_C P \, \mathrm{d}x + Q \, \mathrm{d}y = 0$ for any closed path. But by Theorems $\boxed{3}$ and $\boxed{4}$ this means $\mathbf{F}$ must be conservative.