Math 321 Fall 2011
Homework 5
Due: October 7, 2011

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books outside of our textbook or other people, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

1. For each of the following groups G and subgroups H, compute the right and left coset decompositions determined by H. (Hint: For all three parts, be as efficient as you can be. The cosets come from an equivalence relation that partitions the set.)
   (a) pg. 108 # 5. Let $G = S_4$ and $H = \{e, (12), (34), (12)(34)\}$.
   (b) pg. 108 # 6. Let $G = S_4$ and $H = \{e, (12)(34), (13)(24), (14)(23)\}$.
   (c) Let $G = D_n$ and $H = (r)$.

2. Let n and a be positive integers. Prove that the equation $ax \equiv 1 \bmod n$ has a solution x if and only if $\gcd(a, n) = 1$.

3. Find a cyclic subgroup of order 4 in $U(40)$ and find a noncylic subgroup of order 4 in $U(40)$. (Don't forget to explain why.)

4. Compute the order of $\overline{13}$ in $U(30)$.

5. Let G be a group with $|G| = pq$ where p and q are prime. Prove that every proper subgroup of G is cyclic.

6. Let G be a group and let H and K be subgroups of G. Let $a \in G$. Show that the two sets $aH \cap aK$ and $a(H \cap K)$ are equal. Thus, the left cosets of the subgroup $H \cap K$ are obtained by intersecting the corresponding left cosets of H and K individually.

7. Use Lagranges Theorem in the multiplicative group $U(n)$ to prove Euler's Theorem: $a^{\phi(n)} \equiv 1 \bmod n$ for every integer a relatively prime to n.

8. Let a and b be nonidentity elements of a group G of order 221, with $o(a) = 13$ and $o(b) = 17$. Prove that the only subgroup of G that contains a and b is G itself.

9. Let p, $k \in \mathbb{Z}^+$ with p prime. Suppose that G is a group with $|G| = p^k$. Show that G has an element of order p.

Challenge

1. Let G be a cyclic group of order n and let k be an integer relatively prime to n.
   (a) Prove that the map $x \to x^k$ is a bijection.
   (b) Prove the same is true for any finite group of order n.