Math 321 Fall 2011
Homework 8 version 2.0
Due: November 4, 2011

Insert the usual blurb here about working together and writing distinct solutions.

1. pg. 150 # 2. In a ring R with unity element, e show that $(-e)\cdot(-e) = e$. In the familiar arithmetic systems, this boils down to $(-1)\cdot(-1) = 1$.

2. Let $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ (this should look familiar to those of you who took Combinatorics). This value is always an integer.

   (a) pg. 161 #4. Prove that if n is a prime and $0 < k < n$ then $\binom{n}{k}$ is divisible by n.

   (b)pg. 161 #5. Let R be a ring with unity element having prime characteristc $n > 0$. Prove that $(a+b)^n = a^n + b^n$. (Hint: Use pg. 160 #2, but you don't need to prove it.)

3. Let m and n be positive integers and let k be the least commmon multiple of m and n. Show that $m\mathbb{Z} \cap n\mathbb{Z} = k\mathbb{Z}$.

4. Let $S = \{a + bi \mid a, b \in \mathbb{Z}, b \text{ even}\}$. Show that S is a subring of $\mathbb{Z}[i]$ but not an ideal of $\mathbb{Z}[i]$.

5. Let R be a commutative ring and let A be any subset of R. Show that the *annihilator* of A, $\operatorname{Ann}(A) = \{r \in R \mid ra = 0 \text{ for all } a \in A\}$ is an ideal.

6. pg. 166 # 1. Prove that the isomorphic image of an integral domain is an integral domain.

7. Let $S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$. Show that $\eta : \mathbb{C} \to S$ given by

   $$\eta(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

   is a ring isomorphism.

8. Let R be a ring. An element $a \in R$ is called an idempotent if $a^2 = a$. Notice that 0 and e are idempotents in every ring R.

   (a) Show that if $a \in R$ is both a unit and an idempotent, then $a = e$.

   (b) Show that if R is an integral domain, then 0 and 1 are the only idempotents of R.

   (c) Find all idempotents in $\mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/18\mathbb{Z}$.

9. Recall that $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. Define a function $N : \mathbb{Z}[i] \to \mathbb{N}$ by letting $N(a + bi) = a^2 + b^2$. The function N is called the norm on $\mathbb{Z}[i]$.

   (a). Show that $N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta)$ for all α and $\beta \in \mathbb{Z}[i]$.

   (b) Show that if α is a unit in $\mathbb{Z}[i]$, then $N(\alpha) = 1$.

   (c) Find all units in $\mathbb{Z}[i]$.