Maximize $f(x,y,z) = x^2+y^2+z^2$ relative to the constraint $\underbrace{x^4+y^4+z^4}_{g(x,y,z)}=1$ → level curve of $g(x,y,z)$

We need to find all points where $\nabla f = \overset{\text{scalar}}{\lambda} \nabla g$

$$\nabla f = \langle 2x, 2y, 2z \rangle \qquad \lambda \nabla g = \langle 4\lambda x^3, 4\lambda y^3, 4\lambda z^3 \rangle$$

We have 4 equations to use:

⓪ $x^4+y^4+z^4=1$ ← constraint above

① $2x = 4\lambda x^3$

② $2y = 4\lambda y^3$

③ $2z = 4\lambda z^3$

①–③: From equation $\nabla f = \lambda \nabla g$

From ①, ②, and ③ we get $x=0$ or $x^2 = 1/2\lambda$ and $y=0$ or $y^2 = 1/2\lambda$ and $z=0$ or $z^2 = 1/2\lambda$

There are 8 options here. We combine them into 4 cases.

Case I: $x=y=z=0$ This case fails because it does not satisfy ⓪. ✗

Case II: $x=y=0$, $z^2 = 1/2\lambda$ By ⓪ $z^4=1 \Rightarrow z=\pm 1$ (similarly if $x=z=0$ or $y=z=0$).

This gives us 6 potential points: $(0,0,\pm1)$, $(0,\pm1,0)$, $(\pm1,0,0)$ * f is 1 on all these points

Case III: $x=0$, $y^2=z^2=1/2\lambda$ * By ⓪ $y^4+z^4=1$

(other 2 cases give same f value)

$$(1/2\lambda)^2 + (1/2\lambda)^2 = 1$$

$$\lambda = \pm 1/\sqrt{2}$$

plug into * $z^2=y^2=\sqrt{2}/2$

f is $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=\sqrt{2}$ on these points *

Case IV: $x^2=y^2=z^2=1/2\lambda$ ** Once more by ⓪ $x^4+y^4+z^4=1$

$$3\,(1/2\lambda)^2=1$$

$$4\lambda^2=3$$

so $\lambda^2 = 3/4$

$$\lambda = \pm\sqrt{3}/2$$

plug into ** $x^2=y^2=z^2=1/\sqrt{3}$

f is $\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}=\frac{3}{\sqrt{3}}=\sqrt{3}$ * on this point.

Finally compare all possible f values at these points (* values) to see the max is $\sqrt{3}$ and the min is 1.

What is the point on the plane $z = x + y + 1$ closest to the point $(1,0,0)$?

We are optimizing $f(x,y,z) = \underbrace{(x-1)^2 + y^2 + z^2}_{\text{distance formula}}$ relative to the constraint

$$\underbrace{\overbrace{z-x-y}^{g(x,y,z)} = 1}_{\text{level curve on } g(x,y,z).}$$

We need to find all points where $\nabla f = \overset{\text{scalar}}{\lambda} \nabla g$

$$\nabla f = \langle 2(x-1), 2y, 2z \rangle \qquad \lambda \nabla g = \langle -\lambda, -\lambda, \lambda \rangle$$

We have 4 equations to use:

⓪ $z - x - y = 1$ ← constraint above

① $2(x-1) = -\lambda$

② $2y = -\lambda$

③ $2z = \lambda$

From equation $\nabla f = \lambda \nabla g$

Directly from ①, ②, and ③ we get

$$x = \underbrace{\frac{-\lambda + 2}{2}}_{\text{"} 1 - \lambda/2}, \quad y = -\lambda/2, \quad z = \lambda/2 \quad *$$

Plugging these into ⓪ we get

$$\lambda/2 - (1 - \lambda/2) - (-\lambda/2) = 1$$

$$3\lambda/2 = 2$$

$$\lambda = 4/3$$

(Technically, to confirm min, we need to note that no other values are closer. We could do that by testing a few very close points to see it is a min.)

Plugging λ into the equations labeled * gives us the final answer:

$$\boxed{x = 1/3, \quad y = -2/3, \quad z = 2/3}$$

$$f(1/3, -2/3, 2/3) = (-2/3)^2 + (-2/3)^2 + (2/3)^2$$

$$= 12/9 = 4/3$$

What is the dimension of the box with largest volume and total surface area of 64 cm^2?

We are maximizing $V = \underbrace{xyz}_{f(x,y,z)}$ with the constraint $\cancel{64} = \cancel{2}xy + \cancel{2}yz + \cancel{2}xz$

level curve of $g(x,y,z)$ $\boxed{32 = \underbrace{xy + yz + xz}_{g(x,y,z)}}$

We need to find all points where $\nabla f = \lambda \nabla g$ scalar

$$\nabla f = \langle yz, xz, xy \rangle \qquad \lambda \nabla g = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y) \rangle$$

We have 4 equations to use:

⓪ $32 = xy + yz + xz$ ← constraint above

① $yz = \lambda(y+z)$

② $xz = \lambda(x+z)$

③ $xy = \lambda(x+y)$

From equation $\nabla f = \lambda \nabla g$

We use a different technique and subtract: ① − ②

$$yz - xz = \lambda y + \cancel{\lambda z} - \lambda x - \cancel{\lambda z}$$
$$(y-x)z = (y-x)\lambda$$

So either $y = x$ or $\lambda = z$.

If $\lambda = z$ then ① becomes $y\cancel{z} = \cancel{yz} + z^2$.
But then $z = 0$ which does not maximize volume!
So this option gives a min with, for example
$x = 32, y = 1, z = 0$.

In the case $y = x$, now consider ② − ③.

$$x(z-y) = \cancel{\lambda x} + \lambda z - \cancel{\lambda x} - \lambda y$$
$$x(z-y) = \lambda(z-y)$$

So either $z = y$ or $x = \lambda$ (same issue)

This will be a max instead of min since we found a min above and practically, there must be a max.

Together we get $x = y = z$ so from ⓪ $3x^2 = 32$ $\boxed{x = y = z = \sqrt{32/3}}$