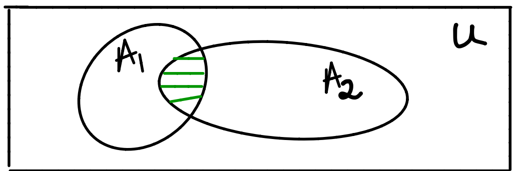
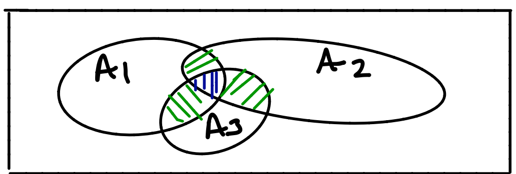
Inclusion-Exclusion[1]

Question: Let A_1 and A_2 be sets. How many objects are in $A_1 \cup A_2$?



From this Venn Diagram, we conclude $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$. as elements in the intersection (the green shaded part) count twice, once in $|A_1|$ and once in $|A_2|$.

Question: What about 3 sets?



From this Venn Diagram, we conclude $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$ as again, elements in the green intersections contribute twice in the sum $|A_1| + |A_2| + |A_3|$ but when we subtract them off, we remove the blue shaded area too many times.

We think of the A_i as sets of elements of some big set U (the universe) with some property.

Ex: A used car dealer has 18 cars: 9 with automatic transmission (AT), 12 with power steering (PS), 8 power brakes (PB). But 7 have both AT and PS, 4 have AT and PB, 5 have PS and PB, 3 have PS and PB and AT. How many cars have only AT? How many of them have none of these?

We let A_1 be the set of cars with AT, P_2 the set of car with PS, and P_3 be the set of cars with PB, so $|A_1| = 9$ (number of cars with AT), $|A_2| = 12$, and $|A_3| = 8$. Notice $|A_1|$ is too big to answer the first question. We must subtract out those cars with AT and one other property. But once we do that, we've removed all cars that have all three properties too many times. So to answer the first question we must compute $|A_1| - |A_1 \cap A_2| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3| = 9 - 7 - 4 + 3 = 1$.

The second question is really asking how many elements are NOT in the sets (or with none of the properties) so take $|U|$ minus what we said above. (This is the subtraction principle.) So here we compute $|U| - |A_1 \cup A_2 \cup A_3| = |U| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3| = 18 - 9 - 12 - 8 + 7 + 4 + 5 - 3 = 2$.

[1](modified from a lecture by A. Uzzell)

Theorem 1 (Inclusion/Exclusion). *Let A_i be finite sets. Then* $|A_1 \cup A_2 \cup \cdots \cup A_m| =$

$$\sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{m+1} |A_1 \cap A_2 \cap \cdots \cap A_m|. \quad (0.1)$$

The Theorem counts the number of objects with at least one of the properties described in the A_i. From this theorem we can also state the number of elements which are in *none* of the A_i. This value is $|U| - |A_1 \cup A_2 \cup \cdots \cup A_m|$ which is equal to

$$|U| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^m |A_1 \cap A_2 \cap \cdots \cap A_m|.$$

Proof: We will prove this using a combinatorial argument. We will show both sides count the same value, which means they must be equal. The left side counts those objects with at least 1 property. We show the right side also counts those objects with at least 1 property, but in a different way. Take an object that has exactly 1 property. Then it is in exactly one subset on the right and so contributes 1 to the sum. Next, if an object has 2 properties, on the right it is counted twice, once for each individual set it is in, and then subtracted out in the intersection of those two sets, so again it adds a total of 1 to the right side. If an object has 3 properties it is counted 3 times in each single set, then it is subtracted $\binom{3}{2} = 3$ times for each of the pairs of intersections, and finally added back one time for triple intersection. This means it contributes a total of $3 - 3 + 1 = 1$ to the right side. In general, if an element shows up in n of the A_i, then it contributes to $\binom{n}{1} = n$ of the single sets, it is subtracted from $\binom{n}{2}$ of the intersections of two of the A_i's, added back $\binom{n}{3}$ times for the intersection of three of the A_i's, and so on. Thus to determine how much this object contributes to the right side, we must evaluate $\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k}$, which you proved equals 1 on a homework assignment (using the Binomial Theorem).

Now, since every object counted on the left side shows up exactly once on the right side, these values must be the same.$\square$

Ex: How many integers up to 300 are divisible by 2, 3, and 5? We define A_1 to be the set of numbers up to 300 which are divisible by 2, we define A_2 to be the set of numbers up to 300 divisible by 3, and we let A_3 be the set of numbers divisible by 5. This problem is asking us to find the size of $A_1 \cup A_2 \cup A_3$, which by Inclusion-Exclusion we know is

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

By prior work in the class, we know $|A_1| = 300/2 = 150$, $|A_2| = 300/3 = 100$, and $|A_3| = 300/5 = 60$. Similarly $A_1 \cap A_2$ is the number of divisors of 6, $A_1 \cap A_3$ is the number of divisors of 10, and $A_2 \cap A_3$ is the number of divisors of 15. This gives sizes of $300/6 = 50$, $300/10 = 30$, and $300/15 = 20$, respectively. Finally $A_1 \cap A_2 \cap A_3$ will be the divisors of 30, which are of size $300/30 = 10$. Plugging these values into (0.1) gives $150 + 100 + 60 - 50 - 30 - 20 + 10 = 220$.

Formula for $\phi(n)$

We use Inclusion-Exclusion to prove a formula for $\phi(n)$ (which we already derived another way using the definition of a multiplicative function in class).

Suppose $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is in standard form. Then, for some $m \in Z_n$ to contribute to $\phi(n)$, we must have $(m, n) = 1$. But by George (Theorem 1.6.2), this means that $(m, p_i^{a_i}) = 1$ for all $1 \leq i \leq k$ and this implies $(m, p_i) = 1$ since $p_i \nmid m$.

Instead of counting which m satisfy this (are relatively prime to all the p_i), we will instead count the complement. We will use Inclusion-Exclusion to count how many elements in Z_n are divisible by at least one of the p_i. Then the value for $\phi(n)$ will be $|Z_n| = n$ minus this value (so Z_n plays the role of the universe U here).

We let A_i be the set of elements of Z_n which p_i divides. As we saw in the example above, the number of elements in Z_n which p_i divides is n/p_i which means $|A_i| = n/p_i$. Similarly $A_i \cap A_j$ will be the elements in Z_n which both p_1 and p_2 divide. So $|A_i \cap A_j| = n/p_i p_2$. Applying Inclusion-Exclusion (or the complement of it) we get

$$\phi(n) = n - \sum_i \frac{n}{p_i} + \sum_{i<j} \frac{n}{p_i p_j} + \cdots + (-1)^k \frac{n}{p_1 \cdots p_k}.$$

We factor out an n from each term which gives

$$\phi(n) = n \left(1 - \sum_i \frac{1}{p_i} + \sum_{i<j} \frac{1}{p_i p_j} + \cdots + (-1)^k \frac{1}{p_1 \cdots p_k} \right).$$

Finally, we apply the technique of grouping to factor the terms inside the parentheses above. Split those terms into those with a $\frac{1}{p_1}$ and those without. By symmetry these will match if we factor out the $\frac{1}{p_1}$ from those terms. Grouping, and then repeating for each of the remaining p_i gives us

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

Notice this equation matches Corollary 2.4.3. The last step may be easier to initially see by working backwards. Take the final answer and "FOIL" it out to get the previous line. Or it is perhaps best seen by an example.

Ex: Suppose $n = p_1^{a_1} p_2^{a_2} p_3^{a_3}$. Then this is saying

$$\phi(n) = n \left(1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} + \frac{1}{p_1 p_2} + \frac{1}{p_1 p_3} + \frac{1}{p_2 p_3} - \frac{1}{p_1 p_2 p_3} \right).$$

If we separate the terms with a $\frac{1}{p_1}$ and those without we get

$$\phi(n) = n \left(1 - \frac{1}{p_2} - \frac{1}{p_3} + \frac{1}{p_2 p_3} - \frac{1}{p_1} \left(1 - \frac{1}{p_2} - \frac{1}{p_3} + \frac{1}{p_2 p_3} \right) \right) = \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2} - \frac{1}{p_3} + \frac{1}{p_2 p_3} \right).$$

Repeat with p_2 now to get

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2} - \frac{1}{p_2} \left(1 - \frac{1}{p_3}\right) \right) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \left(1 - \frac{1}{p_3}\right).$$