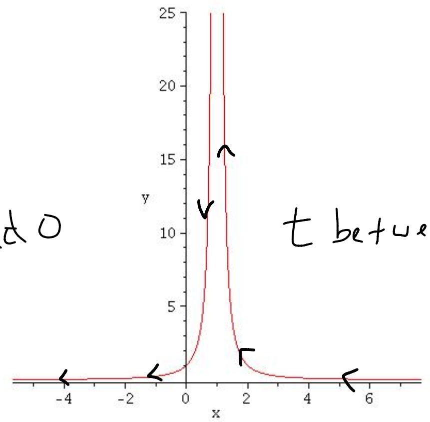
Math 133
Exam 1 Review Solutions

1. (a) Sketch the curve defined by the parametric equations $x = 1 + t^{-1}$ and $y = t^2$. Indicate with an arrow the direction which the curve is traced as t increases.



(b) Eliminate the parameter in the equations from (a) to find a Cartesian equation of the curve.

Solve for t to get $t^{-1} = x-1$ or $t = \frac{1}{x-1}$. Plugging in to the equation for y gives $\boxed{y = \frac{1}{(x-1)^2}}$

2. Given the vectors $\mathbf{u} = \langle 1, -3, 2\rangle$ and $\mathbf{v} = \langle -2, 1, 5\rangle$ and $\mathbf{w} = \langle 3, 2, 2\rangle$, compute

(a) $\mathbf{u} + \mathbf{v}$

$\langle 1,-3,2\rangle + \langle -2,1,5\rangle = \boxed{\langle -1,-2,7\rangle}$

(b) $\mathbf{u} \bullet \mathbf{v}$

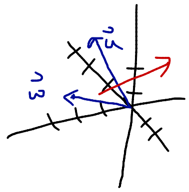
$\langle 1,-3,2\rangle \bullet \langle -2,1,5\rangle = 1\cdot(-2) + (-3)\cdot 1 + 2\cdot 5 = -2 - 3 + 10$
$= \boxed{5}$

(c) $\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 2 \\ 3 & 2 & 2 \end{vmatrix} = \vec{i}\begin{vmatrix} -3 & 2 \\ 2 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -3 \\ 3 & 2 \end{vmatrix}$

$= \vec{i}(-6-4) - \vec{j}(2-6) + \vec{k}(2-(-4))$

$$\boxed{\begin{aligned} &= -10\vec{i} + 4\vec{j} + 11\vec{k} \\ &= \langle -10, 4, 11\rangle \end{aligned}}$$

(d) Which direction is the vector $\mathbf{u} \times \mathbf{w}$ pointing?



$\vec{u} = \langle 1, -3, 2 \rangle$
$\vec{w} = \langle 3, 2, 2 \rangle$

To curl our fingers from $\vec{u}$ to $\vec{w}$ in the direction of the shortest angle, $\boxed{\text{our thumb must face into the page or roughly in the negative } x \text{ axis direction}}$

3. (a) Find a vector in the direction of $\mathbf{u} = \langle 4, 0, -3 \rangle$ but with magnitude 7.

We first find the unit vector in the directions of $\vec{u}$. This is

$$\frac{\vec{u}}{|\vec{u}|} = \langle 4, 0, -3 \rangle \cdot \frac{1}{5} = \langle \tfrac{4}{5}, 0, \tfrac{-3}{5} \rangle.$$

$|\vec{u}| = \sqrt{16+0+9} = \sqrt{25} = 5$

Then to get a vector of magnitude 7 in that direction, we multiply the unit vector by 7 to get $\langle \frac{4}{5}, 0, \frac{-3}{5} \rangle \cdot 7 = \boxed{\langle \frac{28}{5}, 0, \frac{-21}{5} \rangle}$

(b) Find a vector which is orthogonal to $\mathbf{u}$.

We need to find a vector $\langle a, b, c \rangle$ so that $\vec{u} \cdot \langle a, b, c \rangle = 0$ or

$\langle 4, 0, -3 \rangle \cdot \langle a, b, c \rangle = 4a + 0 \cdot b - 3 \cdot c = 0$

or $4a - 3c = 0$.

Let $a = 3$, $b = 1$, $c = 4$ ← or any other value of b too.

So the vector $\boxed{\langle 3, 1, 4 \rangle}$

4. (a) Find vector and scalar equations of the plane through the point $(0, 1, 4)$ and with normal vector $\langle 4, -3, -5 \rangle$.

We know $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$ is the vector equation. To this example we get

$$\boxed{\langle 4, -3, -5 \rangle \cdot \langle x, y, z \rangle = \langle 4, -3, -5 \rangle \cdot \langle 0, 1, 4 \rangle}$$

To find the scalar equation, we multiply the vector equation out to get

$$4x - 3y - 5z = 0 - 3 - 20$$

$$\boxed{4x - 3y - 5z = -23}$$ ← this is the linear equation

or

$$\boxed{4x - 3(y-1) - 5(z-4) = 0}$$

(b) Find vector and scalar equations of the plane through the points $(-3, 1, 1)$, $(5, 2, -1)$, and $(1, 7, -2)$.

We need to find the normal vector. To do this, we need to identify two vectors on the plane and then take their cross product.

The vectors $\langle -3,1,1\rangle - \langle 5,2,-1\rangle = \langle -8,-1,2\rangle = \vec{u}$ and

$\langle -3,1,1\rangle - \langle 1,7,-2\rangle = \langle -4,-6,3\rangle = \vec{v}$ are both on the plane

Then $\vec{u}\times\vec{v} = \langle -8,-1,2\rangle \times \langle -4,-6,3\rangle =$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -8 & -1 & 2 \\ -4 & -6 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 2 \\ -6 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} -8 & 2 \\ -4 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} -8 & -1 \\ -4 & -6 \end{vmatrix}$$

$$= \langle -3+12, -(-24+8) + 48-4\rangle = \langle 9, 16, 44\rangle = \vec{n}$$

So the vector equation for this plane is

$$\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$$

$$\boxed{\langle 9,16,44\rangle\cdot\langle x,y,z\rangle = \langle 9,16,44\rangle\cdot\langle 1,7,-2\rangle}$$

Answers may vary depending on which points you pick.

While the scalar equation for this plane is

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\boxed{9(x-1) + 16(y-7) + 44(z+2) = 0}$$

5. Find parametric equations for the tangent line to the curve $\mathbf{r}(t) = \left\langle 4 - t^3, e^{-t}, \frac{1}{t+1}\right\rangle$ at $t = 3$.

First we find $\vec{r}\,'(t) = \langle -3t^2, -e^{-t}, \frac{-1}{(t+1)^2}\rangle$.

Second, notice that $\vec{r}(3) = \langle 4-27, e^{-3}, \frac{1}{4}\rangle = \langle -23, e^{-3}, \frac{1}{4}\rangle$

and $\vec{r}\,'(3) = \langle -27, -e^{-3}, \frac{-1}{16}\rangle$

So we use the vector equation of a line to get

$$\langle -23, e^{-3}, \tfrac{1}{4}\rangle + t\cdot\langle -27, -e^{-3}, \tfrac{-1}{16}\rangle$$

$$= \langle -23-27t, e^{-3} - te^{-3}, \tfrac{1}{4} - \tfrac{1}{16}t\rangle$$

$$= \langle -23-27t, e^{-3}(1-t), \tfrac{1}{4}(1-\tfrac{t}{4})\rangle$$

This gives the parametric equations

$$\boxed{x = -23-27t,\ y = e^{-3}(1-t),\ z = \tfrac{1}{4}(1-\tfrac{t}{4})}$$

6. Two particles travel along the space curves $\mathbf{r}_1(t) = \langle 3t-1, 4t+2, t-2\rangle$ and $\mathbf{r}_2(t) = \langle t-2, 4t-4, -t\rangle$.

(a) Do the particles collide? If so, when?

We need to find if both curves hit the same point at the same time.

Is there a t so that

$$\left.\begin{aligned} 3t-1 &= t-2 \\ 4t+2 &= 4t-4 \\ t-2 &= -t \end{aligned}\right\}\text{ simultaneously?}$$

No. Notice that the second equation is impossible for all t.

there are other reasons too.

(b) Do their paths intersect? If so, where?

We need to determine if there are times t_1 and t_2 where the first particle is at a particular point at time t_1 and the 2nd particle is at the same point at time t_2.

To do this, we try to find t_1 and t_2 values so that

$$\left.\begin{aligned} &(1)\ 3t_1-1 = t_2-2 \\ &(2)\ 4t_1+2 = 4t_2-4 \\ &(3)\ t_1-2 = -t_2 \end{aligned}\right\}$$

by (3) we know $t_2 = 2-t_1$

plug into (1) to get

$$3t_1 - 1 = (2-t_1)-2$$
$$4t_1 = 1$$
$$t_1 = 1/4$$

plug $t_1 = 1/4$ into (3) to get $t_2 = 2 - \frac{1}{4} = 7/4$

Check (1) & (2) to confirm:

$$3\left(\tfrac{1}{4}\right) - 1 = -\tfrac{1}{4} = \tfrac{7}{4} - 2 \checkmark$$
$$4\left(\tfrac{1}{4}\right) + 2 = 3 = 4\left(\tfrac{7}{4}\right) - 4 \checkmark$$

$z = -t_2$

So these particles collide at $\boxed{x = -\frac{1}{4} \quad y = 3 \quad z = -\frac{7}{4}}$

7. Find the derivative of the vector valued function $\mathbf{r}(t) = \langle \ln t, \tan 3t, e^{2t}\rangle$.

$\vec{r}\,'(t)$ is just the vector valued function with the derivative in each component. So

$$\boxed{\vec{r}\,'(t) = \left\langle \frac{1}{t}, 3\sec^2 3t, 2e^{2t} \right\rangle}$$

8. Given the parametric equations $x = t^2 - 9$ and $y = t^2 - 8t$

(a) Find where the tangent is horizontal or vertical.

We first rewrite these parametric equations as a vector valued function, so $\vec{r}(t) = \langle t^2-9, t^2-8t \rangle$. Then any questions about tangent lines will require the derivative. So

$$\vec{r}\,'(t) = \langle 2t, 2t-8 \rangle$$

Vertical tangent lines occur when there is no change in the x direction, so when $2t=0$ or $\boxed{t=0}$

horizontal tangent lines occur when there is no change in the y direction, so when $2t-8=0$ or $\boxed{t=4}$

(b) Find the equation of the tangent line at $t = 4$.

When $t=4$, $\vec{r}(4) = \langle 16-9, 16-32 \rangle = \langle 7, -16 \rangle$

$\vec{r}\,'(4) = \langle 8, 0 \rangle$

So this line has corresponding vector $\langle 8,0 \rangle$ and point $(7,-16)$

using the vector equation for a line we get

$$\langle 7,-16 \rangle + \langle 8,0 \rangle t$$
$$= \boxed{\langle 7+8t, -16 \rangle}$$

Alternatively, a horizontal line at $t=4$ must go through the point $(7,-16)$ so the line is $\boxed{y=-16.}$

9. Find the scalar and vector projections of $\mathbf{b} = \langle 0, 1, 1 \rangle$ onto $\mathbf{a} = \langle 4, -1, 0 \rangle$

$$|\text{proj}_{\vec{a}}\vec{b}| = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{4 \cdot 0 + (-1) \cdot 1 + 0 \cdot 1}{\sqrt{4^2 + (-1)^2 + 0^2}} = \boxed{\frac{-1}{\sqrt{17}}}$$

then

$$\text{proj}_{\vec{a}}\vec{b} = |\text{proj}_{\vec{a}}\vec{b}| \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{-1}{\sqrt{17}} \cdot \frac{\langle 4,-1,0 \rangle}{\sqrt{17}} = \boxed{\left\langle \frac{-4}{17}, \frac{1}{17}, 0 \right\rangle}$$