Math 321 Fall 2011
Homework 9
Due: November 18, 2011

Insert the usual blurb here about working together and writing distinct solutions.

1. pg 184 # 3. Let D be an integral domain and let U be the set of units in D. Prove that U is a group with respect to (the restriction of) multiplication as group operation.

2. pg 185 # 1. Prove: If a, b are non-zero elements in a PID, then there are elements s and t in the domain such that $sa + tb = \gcd(a, b)$.

3. Let R be an integral domain. Prove that $(a) = (b)$ for some elements a, $b \in R$ if and only if $a = ub$ for some unit u of R.

4. Let R be a ring and I_n a countable collection of ideals of R. Prove that the set $I = \bigcup_{n=1}^{\infty} I_n$ is an ideal.

5. Supose that a and b belong to an integral domain, $b \neq 0$ and a is not a unit. Show that (ab) is a proper subset of (b).

6. (a) Give an example of a ring that has exactly two maximal ideals.

   (b) Suppose that R is a commutative ring and $|R| = 30$. If I is an ideal of R and $|I| = 10$, prove that I is maximal ideal.

   (Hint: Rings are abelian groups.)

7. (a) In $\mathbb{Z} \times \mathbb{Z}$ let $I = \{(a, 0) \mid a \in \mathbb{Z}\}$. Show that I is a prime ideal. Is I maximal?

   (b) Show that $I = \{(3x, y) \mid x, y \in \mathbb{Z}\}$ is a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$.

8. Let R be a ring and let I be an ideal of R. Prove that the factor ring R/I is commutative if and only if $rs - sr \in I$ for all r and s in R.

9. If R is a PID and I is an ideal of R, prove that every ideal of R/I is principal.

10. An ideal A of a commutative ring R with unity is said to be finitely generated if there exist elements $a_1, a_2, \dots a_n$ of A such that $A = (a_1, a_2, \dots, a_n)$. An integral domain R is said to satisfy the ascending chain condition if every strictly increasing chain of ideals $I_1 \subset I_2 \cdots$ must be finite in length. Show that an integral domain R satisfies the ascending chain condition if and only if every ideal of R is finitely generated.

Challenge

1. Let R be an integral domain with fraction field F and let P be a prime ideal of R. Let R_P be the subest of F defined by $R_P = \{a/d \mid a, d \in R, d \notin P\}$. (This subset is called the *localization* of R at P.)

   (a) Prove that R_P is a subring of F.

   (b) Determine all maximal ideals of R_P.