Math 218: Elementary Number Theory

HOMEWORK 8 : DUE OCTOBER 14

2.2 #2. Prove that $10^k \equiv 1 \bmod 9$ for every integer $k > 0$.

2.2 #9. In ordinary arithmetic, if $a^2 = b^2$, then $a = \pm b$. Is the analogous statement true in the ring of residues mod m, i.e. if $a^2 \equiv b^2 \bmod m$ does that mean $a \equiv \pm b \bmod m$?

2.2 #11. (a) If p is a prime, prove that the binomial coefficient $\binom{p}{r} \equiv 0 \bmod p$ for $r = 1, 2, 3, \ldots, p-1$.

(b) Use (a) to prove that $(a+b)^p \equiv a^p + b^p \bmod p$.

1. Suppose that you are creating a password from 26 letters, 10 numbers, and 15 special characters. How many such 10-character passwords are possible if they must have exactly 6 letters, 2 numbers, and 2 special characters, and they must consist of 10 distinct symbols in the password (so *P1ssw0rd!!* is not a legitimate password because it contains two s's and two !'s)?

2. (a) Prove that
$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$
(b) Generalize part (a) to find the sum
$$\sum_{k=0}^{n} \binom{n}{k} r^k$$
for any real number r.

3. For $n \geq 1$, prove
$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0.$$

4. By integrating the equation in the binomial theorem (and setting $y = 1$), prove that, for a positive integer n,
$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}.$$