Math 321 Fall 2011
Homework 11
Due: December 9, 2011

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books outside of our textbook or other people, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to $\LaTeX$.

1. If the center of G is of index n, prove that every conjugacy class has at most n elements.

2. (a) Assume n is an even positive integer and show that D_n acts on the set consisting of pairs of opposite vertices of a regular n-gon.
   For example, if $n = 6$ label the vertices $\{a, b, c, d, e, f\}$ in order around the hexagon. Then the set A would be: $\{(a, d), (b, e), (c, f)\}$ and r would act on those vertices by $r \cdot (a, d) = (b, e)$ or $r \cdot (c, f) = (a, d)$.

   (b) Find the kernel of this action.

3. Let G be a group and let $G = A$.

   (a) Show that if G is non-abelian then the map defined by $g \cdot a = ag$ for all $g, a \in G$ does not satisfy the axioms of a group action of G on itself.

   (b) Show that the map defined by $g \cdot a = ag^{-1}$ does satisfy the axioms of a group action of G on itself.

4. Define A to be the set of ordered pairs with entries from the set $\{1, 2, 3, \}$,

$$A = \{(i, j) \mid 1 \leq i, j \leq 3\}.$$

   Let S_3 act on A by taking a $\sigma \in S_3$ and defining $\sigma \cdot (i, j) = (\sigma(i), \sigma(j))$. So if $\sigma = (1\ 2)$ then $\sigma \cdot (1, 3) = (2, 3)$ and if $\sigma = (1\ 2\ 3)$ then $\sigma \cdot (1, 3) = (2, 1)$.

   (a) Find the orbits of S_3 on A.

   (b) For each orbit $\mathcal{O}$ from (a), pick some $a \in \mathcal{O}$ and find the stabilizer of $a \in S_3$.

5. Find all conjugacy classes and the Class Equation for the following groups. Justify your work:

   (a) D_5 (Hint: The challenge problem from Homework #3 might help.)

   (b) $S_3 \times \mathbb{Z}/2\mathbb{Z}$

   Note: Theory is your friend here! Try to minimize the number of computations you must do.

6. Find (with proof) all finite groups which have exactly two conjugacy classes.