Math 321 Fall 2011
Homework 4
Due: September 23, 2011

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books outside of our textbook or other people, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

1. For each D_n, prove that $r^i s = sr^{-i}$ for all $i \in \mathbb{Z}^+$.

2. pg. 92. # 2. Let G be a cyclic group of finite order. Prove that if $H < G$ then $|H| \mid |G|$. (Hint: Ignore the hint, it's for #1.)

3. pg. 92. # 3. Let G be a cyclic group of order n and let $k \mid n$. Prove that there is one and only one subgroup H of G such that $|H| = k$.

4. pg. 94. # 6. Let $H < G$ where $H \neq G$. Prove that the set $S = G - H$ (the complement of H relative to G) is a set of generators of G.

5. (a) Let G be an abelian group. Prove that the set of all elements that satisfy the equation $x^n = e$ is a subgroup of G.

   (b) Find an example of a group G where the elements of G that satisfy the equation $x^2 = e$ do not form a subgroup of G.

6. Prove that if H and K are subgroups of G then so is their intersection $H \cap K$.

7. Determine whether each of the following relations is reflexive, symmetric, and transitive (you should check each individual property, not all three at once). If a certain property fails, you should give a specific counterexample.

   a. $S = \mathbb{Z}$ where $a \sim b$ means $a - b \neq 1$.

   b. $S = \mathbb{Z}$ where $a \sim b$ means that both a and b are even.

   c. $S = \mathbb{Z}$ where $a \sim b$ means $a \mid b$.

   d. (pg 26 #4.) $S = \mathbb{Z}$ where for any positive integer $n \geq 2$, $a \sim_n b$ means $n \mid a - b$.

8. As in class, let $S = \mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ and let $\sim$ be the equivalence relation given by $(a, b) \sim (c, d)$ if $ad = bc$. Let Q be the set of equivalence classes of A under $\sim$.

   a. Show that if $(a_1, b_1) \sim (c_1, d_1)$ and $(a_2, b_2) \sim (c_2, d_2)$, then $(a_1a_2, b_1b_2) \sim (c_1c_2, d_1d_2)$.

   b. Show that if $(a_1, b_1) \sim (c_1, d_1)$ and $(a_2, b_2) \sim (c_2, d_2)$, then $(a_1b_2 + a_2b_1, b_1b_2) \sim (c_1d_2 + c_2d_1, d_1d_2)$.

   continued . . .

The above parts show that the operations of addition and multiplication of fractions you learned in grade school are indeed well-defined on Q. In other words, the following definitions make sense:

$$\overline{(a,b)} \cdot \overline{(c,d)} = \overline{(ac,bd)} \qquad \overline{(a,b)} + \overline{(c,d)} = \overline{(ad+bc,bd)}.$$

c. Let $a, b \in \mathbb{Z}$ with $b \neq 0$. Show that $\overline{(a,b)} + \overline{(-a,b)} = \overline{(0,1)}$

d. Let $a, b \in \mathbb{Z}$ with both $a, b \neq 0$. Show that $\overline{(a,b)} \cdot \overline{(b,a)} = \overline{(1,1)}$.

Thus, every element has an additive inverse and every nonzero element (i.e. every element other than $\overline{(0,1)}$) has a multiplicative inverse.

Challenge

1. We showed in class that D_n is not abelian for $n \geq 3$. What elements in D_n commute with every other element in D_n? Prove your assertion for all $n \geq 3$.