Math 321 Fall 2011
Homework 10
Due: December 2, 2011

Insert the usual blurb here about working together and writing distinct solutions.

1. pg. 216 #10. Let F be a field. Prove:

   (a) If $p_1, p_2, \ldots, p_n$ are nonzero polynomials in $F[x]$ then these polynomials have a gcd which is unique to within unit factors.

   (b) If d is the gcd (up to units) of $p_1, p_2, \ldots, p_n$ then there exist $q_1, q_2, \ldots, q_n \in F[x]$ such that
   $$d = p_1q_1 + p_2q_2 + \cdots p_nq_n.$$

2. pg. 217 #1. If D is an integral domain, prove the only units in $D[x]$ are the units D.

3. (a) How many roots does $x^2 + 3x + 2$ have in $\mathbb{Z}/6\mathbb{Z}$?

   (b) Find, with proof, all the irreducible polynomials of degree 2 or 3 in $\mathbb{Z}/2\mathbb{Z}[x]$.

   (c) Show that the polynomial $2x + 1$ in $\mathbb{Z}/4\mathbb{Z}[x]$ has a multiplicative inverse in $\mathbb{Z}/4\mathbb{Z}[x]$.

4. Determine the greatest common divisor of $f(x) = x^5 + 2x^3 + x^2 + x + 1$ and $g(x) = x^5 + x^4 + 2x^3 + 2x^2 + 2x + 1$ in $\mathbb{Q}[x]$. Then find $s(x)$ and $t(x)$ in $\mathbb{Q}[x]$ so that the gcd$= g(x)s(x) + f(x)t(x)$.

5. Let F be a field and let
   $$I = \{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 \mid a_i \in F \text{ and } a_n + \cdots + a_0 = 0\}.$$
   Prove that I is an ideal of $F[x]$. Find a generator of I.

6. (a) If I is an ideal of a ring R, prove that $I[x]$ is an ideal of $R[x]$.

   (b) Let R be a commutative ring with unity. If I is a prime ideal of R, prove that $I[x]$ is a prime ideal of $R[x]$.

7. Let F be a field. Show that there are infinitely many primes in $F[x]$.

8. Let F be a field. Prove that the set R of polynomials in $F[x]$ whose coefficient of x is equal to 0 is a subring of $F[x]$ and that R is not a UFD. (Hint: Can you factor x^6 two different ways?)

9. Determine (with explanation)whether the following polynomials are irreducible in the rings indicated.

   (a) $x^4 + x + 1 \in \mathbb{Z}/2\mathbb{Z}[x]$

   (b) $x^2 + x + 4 \in \mathbb{Z}/11\mathbb{Z}[x]$

   (c) $x^6 + 30x^5 - 15x^3 + 6x - 120 \in \mathbb{Z}[x]$

   (d) $x^2 + x + 4 \in \mathbb{Z}[x]$

   (e) $\frac{3}{7}x^4 - \frac{2}{7}x^2 + \frac{9}{35}x + \frac{3}{5} \in \mathbb{Q}[x]$ (Hint: Part (a) might come in handy.)