Math 321 Fall 2011
Homework 7
Due: October 28, 2011

Insert the usual blurb here about working together and writing distinct solutions.

1. Suppose that $G_1 \cong G_2$ and $H_1 \cong H_2$. Prove that $G_1 \times H_1 \cong G_2 \times H_2$.

2. Let $\mathbb{R}^+$ be the group of positive real numbers under multiplication and let $\mathbb{R}$ be the group of real numbers under addition.

   (a) Show that the mapping $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ sending x to $\sqrt{x}$ is an isomorphism (or automorphism in this case).

   (b) Show that the mapping $\chi : \mathbb{R}^+ \to \mathbb{R}$ sending x to $\ln x$ is an isomorphism.

3. Define $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by sending (x, y) to $x + y$.

   (a) Prove that ϕ is a surjective homomorphism.

   (b) What is the kernel of this mapping?

4. Prove or disprove that S_4 is isomorphic to D_{12}.

5. Let $G = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{Q}. \right\}$ and let $H = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$

   (a) Prove that G and H are isomorphic under addition.

   (b) Notice that G and H are closed under multiplication (you can just check this for yourself). Does your isomorphism preserve multiplication as well as addition?

6. Suppose that ϕ is an isomorphism from a group G onto a group H.

   (a) pg. 123 #3. Show that if G is abelian then so is H.

   (b) pg. 123 #4. Show that if G is cyclic then so is H.

7. pg. 123 # 9 and #10. Let ϕ be an isomorphism from a group G onto a group H and $K \leq G$.

   (a) Show that $\phi[K] \leq H$.

   (b) If $|G : K|$ is finite, show that $|H : \phi[K]| = |G : K|$.

   (c) Show that if $K \trianglelefteq G$, then $\phi[K] \trianglelefteq H$.

8. Let G be a non-abelian group of order pq where both p and q are primes. Prove that G must have a trivial center.

9. If g and a are elements of a group, prove that $C_G(a)$ is isomorphic to $C_G(gag^{-1})$ (where again $C_G(x) = \{g \in G \mid gxg^{-1} = x\}$ is the *centralizer* of the element a).

10. Let $n \geq 2$ be a positive integer and let k be a positive integer such that $k \mid n$.

    (a) Prove that $(r^k) \triangleleft D_n$.

    (b) Prove that $D_n/(r^k) \cong D_k$.