Math 133
Exam 2 Review Solutions

1. Compute all the first and second partial derivatives of the following functions.

(a) $f(x, y) = x\ln(x^2y) - 3y$

$$f_x(x,y) = x \cdot \frac{1}{x^2y} \cdot 2xy + \ln(x^2y) = \boxed{2 + \ln(x^2y)}$$

$$f_y(x,y) = x \cdot \frac{1}{x^2y} \cdot x^2 - 3 = \boxed{\frac{x}{y} - 3}$$

$$f_{xx} = \frac{1}{x^2y} \cdot 2xy = \boxed{\frac{2}{x}}$$

$$f_{xy} = f_{yx} = \boxed{\frac{1}{y}}$$

$$f_{yy} = \boxed{\frac{-x}{y^2}}$$

(b) $f(x, y) = e^{\sqrt{x^2+y^2}}$

$$f_x(x,y) = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}x\, e^{\sqrt{x^2+y^2}} = \boxed{\frac{x e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}}$$

$$f_y(x,y) = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}y\, e^{\sqrt{x^2+y^2}} = \boxed{\frac{y e^{\sqrt{x^2+y^2}}}{\sqrt{y^2+y^2}}}$$

$$f_{xx} = \frac{\sqrt{x^2+y^2}\left(x \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}x e^{\sqrt{x^2+y^2}} + e^{\sqrt{x^2+y^2}}\right) - \frac{1}{\cancel{2}}\sqrt{x^2+y^2} \cdot \cancel{2}x \cdot x \cdot e^{\sqrt{x^2+y^2}}}{\left(\sqrt{x^2+y^2}\right)^2} = \boxed{\frac{e^{\sqrt{x^2+y^2}}\left(x^2 + \sqrt{x^2+y^2} - x^2\sqrt{x^2+y^2}\right)}{x^2+y^2}}$$

Since x any y are symmetric we know $f_{yy} = \boxed{\frac{e^{\sqrt{x^2+y^2}}\left(y^2 + \sqrt{x^2+y^2} - y^2\sqrt{x^2+y^2}\right)}{x^2+y^2}}$

$$f_{xy} = f_{yx} = x\left(\frac{\cancel{\sqrt{x^2+y^2}} \cdot \frac{1}{\cancel{2}}\cancel{(x^2+y^2)^{-1/2}}\cancel{2}y e^{\sqrt{x^2+y^2}} - e^{\sqrt{x^2+y^2}} \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}y}{\left(\sqrt{x^2+y^2}\right)^2}\right) = \boxed{\frac{xy e^{\sqrt{x^2+y^2}}\left(1 - \frac{1}{\sqrt{x^2+y^2}}\right)}{x^2+y^2}}$$

2. Compute the gradient for the function $f(x, y) = \cos(x^2 + y)$.

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$
$$= \boxed{\langle -\sin(x^2+y) \cdot 2x, -\sin(x^2+y) \rangle}$$

3. Given the function $f(x, y) = \frac{x}{\sqrt{y}}$, find an equation of the tangent plane at the point $(4, 4)$.

$f_x(x,y) = \frac{1}{\sqrt{y}}$ $\qquad f_x(4,4) = \frac{1}{2}$

$f_y(x,y) = -\frac{1}{2} x y^{-3/2} = \frac{-x}{2\sqrt{y^3}}$ $\qquad f_y(4,4) = \frac{-4}{2 \cdot 8} = -\frac{1}{4}$

$f(4,4) = 2$

Putting this together, we get

$$\boxed{z = 2 + \tfrac{1}{2}(x-4) - \tfrac{1}{4}(y-4)}$$

4. Let $f(x, y) = 4x - 3x^3 - 2xy^2$.

(a) Find the critical points of $f(x, y)$.

$f_x(x,y) = 4 - 9x^2 - 2y^2$

$f_y(x,y) = -4xy$

when $x = 0$, $4 - 2y^2 = 0$
$4 = 2y^2$
$y = \pm\sqrt{2}$

when $y = 0$ $\quad 4 - 9x^2 = 0$
$4 = 9x^2$
$x = \pm 2/3$

So $\boxed{\begin{matrix}(0, \pm\sqrt{2}) \\ (\pm 2/3, 0)\end{matrix}}$

(b) Are they local minima, local maxima, or saddle points? Why?

$f_{xx} = -18x \quad f_{yy} = -4x \quad f_{xy} = -4y$

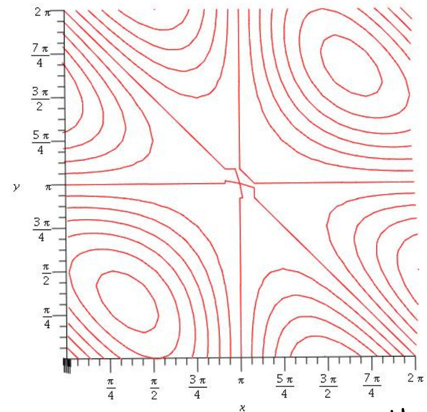
at $(0, \sqrt{2})$ $\quad D = \boxed{- \text{ so saddle point}}$

at $(0, -\sqrt{2})$ $\quad D = \boxed{- \text{ so also saddle}}$

at $(2/3, 0)$ $\quad D = +$ $\quad f_{xx} = \boxed{- \quad \text{so max}}$

at $(-2/3, 0)$ $\quad D = +$ $\quad f_{xx} = \boxed{+ \quad \text{so min}}$

5. Below is a picture of the level curves of a function $f(x, y)$. Based on the picture, where does this function have a local max or a local min? Where does it have a saddle point?



It appears to have a saddle point at $\boxed{(\pi, \pi)}$

It appears to have a max or a min at about $\boxed{\left(\frac{3\pi}{8}, \frac{3\pi}{8}\right) \text{ and about } \left(\frac{13\pi}{8}, \frac{13\pi}{8}\right)}$

6. (a) Use the chain rule to find $\frac{df}{dt}$ when $f(x, y) = \ln x + \ln y$, $x = \cos t$, and $y = t^2$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{x} \cdot (-\sin t) + \frac{1}{y} \cdot 2t$$

$$= \frac{1}{\cos t} (-\sin t) + \frac{1}{t^2} \cdot 2t = \boxed{-\tan t + \frac{2}{t}}$$

(b) Use the chain rule to find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ where $f(x, y) = x^2 + \sin(xy)$, $x = e^{s+t}$, and $y = s + t$.

We need 4 pieces of information:

$$\frac{\partial f}{\partial x} = 2x + y\cos(xy) \qquad \frac{\partial x}{\partial s} = e^{s+t} \qquad \frac{\partial y}{\partial s} = 1$$

$$\frac{\partial f}{\partial y} = x\cos(xy) \qquad \frac{\partial x}{\partial t} = e^{s+t} \qquad \frac{\partial y}{\partial t} = 1$$

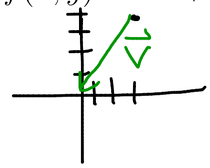
So $\frac{\partial f}{\partial s} = (2x + y\cos(xy))e^{s+t} + x\cos(xy) \cdot 1$

$$\boxed{= 2e^{s+t} + (s+t)\cos(e^{s+t}(s+t))e^{s+t} + e^{s+t}\cos(e^{s+t}(s+t))}$$

In fact, since $\frac{\partial x}{\partial s} = \frac{\partial x}{\partial t}$ and $\frac{\partial y}{\partial s} = \frac{\partial y}{\partial t}$ what we wrote above is also $\boxed{\frac{\partial f}{\partial t}}$

7. (a) Find the directional derivative of $f(x, y) = x^2 + 4y^2$ at the point $(3, 4)$ in the direction pointing toward the origin.

First, what is $\vec{u}$?



Points in direction of green

$\vec{v} = \langle -3, -4 \rangle$ But we need the unit vector. $|\vec{v}| = 5$

So $\vec{u} = \langle \frac{-3}{5}, \frac{-4}{5} \rangle$.

Second, what is $\nabla_f(3,4)$?

$\nabla_f = \langle 2x, 8y \rangle$ hence $\nabla_f(3,4) = \langle 6, 32 \rangle$

Then $D_{\vec{u}} f = \nabla_f \cdot \vec{u}$

$= \langle 6, 32 \rangle \cdot \langle \frac{-3}{5}, \frac{-4}{5} \rangle$

$= \frac{-18}{5} - \frac{128}{5} = \boxed{\frac{-146}{5}}$

(b) Is this function increasing or decreasing at the point $(3, 4)$ in the direction pointing toward the origin?

Since the value we found in ⓐ is negative, the rate of change is negative and so the function is $\boxed{\text{decreasing.}}$

8. Find the linearization $L(x, y)$ of $f(x, y) = x^2y^3$ at the point $(2, 1)$.

$f_x(x,y) = 2xy^3$ $\quad f_x(2,1) = 2 \cdot 2 \cdot 1 = 4$

$f_y(x,y) = 3x^2y^2$ $\quad f_y(2,1) = 3 \cdot 4 \cdot 1 = 12$

$f(2,1) = 4$

Putting everything together gives

$L(x,y) = f_x(2,1)(x-2) + f_y(2,1)(x-1) + f(2,1)$

$= \boxed{4(x-2) + 12(x-1) + 4}$

Not on exam.

9. Find an equation of the tangent plane to the surface $x^2 + z^2 e^{y-x} = 13$ at the point $\left(2, 3, \frac{3}{\sqrt{e}}\right)$.

This surface is a level surface to the function

$g(x,y,z) = x^2 + z^2 e^{y-x}$.

So the gradient of $g(x,y,z)$ will be normal to the tangent plane.

$\nabla g = \langle 2x - z^2 e^{y-x}, z^2 e^{y-x}, 2z e^{y-x} \rangle$

$\nabla g(2,3,\frac{3}{\sqrt{e}}) = \langle 4 - \frac{9}{e} \cdot e, \frac{9}{e} \cdot e, 2 \cdot \frac{3}{\sqrt{e}} e \rangle = \langle -5, 9, 6\sqrt{e} \rangle$

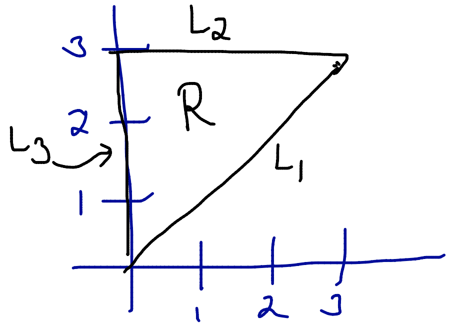
We combine this normal vector with the point $(2,3,\frac{3}{\sqrt{e}})$ to get the following equation of a plane:

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$\boxed{-5(x-2) + 9(y-3) + 6\sqrt{e}(z - \frac{3}{\sqrt{e}}) = 0}$

10. Find the absolute maximum and absolute minimum values of the function $f(x,y) = x^2 + y^2 - 2x - 4y$ on the region defined by $x \geq 0$, $0 \leq y \leq 3$, and $y \geq x$.

We first find local max/min inside R.



$\frac{\partial f}{\partial x} = 2x - 2 = 0$ when $x = 1$

$\frac{\partial f}{\partial y} = 2y - 4 = 0$ when $y = 2$

Hence a critical point at $(1,2)$ and $f(1,2) = 1 + 4 - 2 - 8 = -5$

Now check L_1. Along this line $y = x$ for $0 \leq x \leq 3$.

So $f(x,x) = 2x^2 - 6x$. Call $g(x) = 2x^2 - 6x$ then

$g'(x) = 4x - 6$ $\quad 4x - 6 = 0$ if $x = 3/2$

$- - - \mid + + + +$ at $3/2$, so $x = 3/2$ is min

max occurs at endpoint $x = 3$

So we record max and min values

$f(3,3) = 9 + 9 - 6 - 12 = 0$

$f(3/2, 3/2) = 9/4 + 9/4 - 3 - 6 = -9/2$

Next check L_2. Here $y = 3$ for $0 \leq x \leq 3$. So $f(x,3) = x^2 + 9 - 2x - 12$
$= x^2 - 2x - 3$

Call $g(x) = x^2 + 2x - 3$

$g'(x) = 2x - 2$ $\quad$ This is 0 when $x = 1$

$- - - \mid + + + +$ at 1 $\rightarrow$ min here so max at endpoint $x = 3$

We record max/min values.

$f(1,3) = 1 + 9 - 2 - 12 = -4$

$f(3,3)$ done before

Finally check L_3. Here $x = 0$ where $0 \leq y \leq 3$. So $f(0,y) = y^2 - 4y$

Call $g(y) = y^2 - 4y$ and $g'(y) = 2y - 4$ This is 0 when $y = 2$ $\quad - - - \mid + + +$ at 2 $\rightarrow$ min and max at $y = 0$.

We record max/min values.

$f(0,0) = 0$

$f(0,2) = 4 - 8 = -4$

Putting this all together we get $\boxed{\text{a max value of } 0 \quad \text{a min value of } -5}$

11. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 + 6x + 6y^2$ subject to the constraint $2x^2 + 3y^2 = 18$.

$g(x,y) = 2x^2 + 3y^2$

$$\left.\begin{array}{l} \nabla f = \langle 2x+6,\ 12y \rangle \\ \lambda \nabla g = \langle 4\lambda x,\ 6\lambda y \rangle \end{array}\right] \to \text{So} \quad \begin{array}{ll} 2x+6 = 4\lambda x & (1) \\ 12y = 6\lambda y & (2) \\ \text{and } 2x^2+3y^2 = 18 & (3) \end{array}$$

From (2), $\lambda = 2$ or $y = 0$.

If $\lambda = 2$, then by (1)

$2x + 6 = 8x$

$6x = 6$

$x = 1$

So by (3)

$2 + 3y^2 = 18$

$y^2 = 16/3$

$y = \pm 4/\sqrt{3}$

If $y = 0$ then by (3)

$2x^2 = 18$

$x^2 = 9$

$x = \pm 3$

This gives us several points to test:

$f(3,0) = 9 + 18 + 0 = 27$

$f(-3,0) = 9 - 18 + 0 = -9$

$f(1, \pm 4/\sqrt{3}) = 1 + 6 + 6(16/3) = 39$

$\boxed{\text{So the maximum value is } 39 \text{ and the minimum value is } -9}$

12. Evaluate the following integrals.

(a) $\int \sin 2x \cos^3 2x \, dx$

U-substitution

$u = \cos 2x$

$du = -2\sin 2x \, dx$

$-\frac{1}{2} du = \sin 2x \, dx$

So the integral becomes $\int u^3 \left(-\frac{1}{2}\right) du = -\frac{1}{2} \cdot \frac{1}{4} u^4 + C$

$= \boxed{-\frac{1}{8}\cos^4 2x + C}$

(b) $\int \frac{x}{\sqrt{9-x^4}} \, dx$

u-substitution and inverse trig

$u = x^2$

$du = 2x \, dx$

$\frac{1}{2} du = x \, dx$

So the integral becomes $\int \frac{1}{\sqrt{9-u^2}} \cdot \frac{1}{2} du$

3 from $\sqrt{9}$

$= \frac{1}{2}\int \frac{1}{\sqrt{9(1-u^2/9)}} du = \frac{1}{6}\int \frac{1}{\sqrt{1-\left(\frac{u}{3}\right)^2}} du =$

$\frac{1}{6}\sin^{-1}\left(\frac{u}{3}\right) \cdot 3 = \boxed{\frac{1}{2}\sin^{-1}\frac{x^2}{3} + C}$

chain rule

(c) $\int x \sec^2 x \, dx$

integration by parts

$u = x$ $\quad v = \tan x$

$du = 1\,dx$ $\quad dv = \sec^2 x \, dx$

So the integral becomes

$uv - \int v\,du = x\tan x - \int \tan x \, dx = \boxed{x \tan x - \ln|\sec x| + C}$