1. Evaluate the following integral.

$$\int_{\frac{4}{\sqrt{3}}}^{4} \frac{\sqrt{x^2-4}}{x}\,dx$$

$x = 2\sec\theta$ trig substitution

$dx = 2\sec\theta\tan\theta\,d\theta$

When $\underline{x=4}$, $\sec\theta = 2$, $\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$

When $\underline{x = 4/\sqrt{3}}$, $\sec\theta = 2/\sqrt{3}$, $\cos\theta = \frac{\sqrt{3}}{2}$, $\theta = \pi/6$

So the integral becomes

$$\int_{\pi/6}^{\pi/3} \frac{\overbrace{\sqrt{4\sec^2\theta - 4}}^{\sqrt{x^2-4}}}{\underbrace{\cancel{2}\cancel{\sec\theta}}_{x}} \cdot \underbrace{\cancel{2}\cancel{\sec\theta}\tan\theta\,d\theta}_{dx}$$

Since $\sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$ we get

$$\int_{\pi/6}^{\pi/3} 2\tan\theta \cdot \tan\theta\,d\theta = \int_{\pi/6}^{\pi/3} 2\tan^2\theta\,d\theta = \int_{\pi/6}^{\pi/3} 2(\sec^2\theta - 1)\,d\theta$$

$$= 2(\tan\theta - \theta)\Big]_{\pi/6}^{\pi/3} = 2(\tan\pi/3 - \pi/3) - 2(\tan\pi/6 - \pi/6)$$

$$= \boxed{2(\sqrt{3} - \pi/3) - 2(1/\sqrt{3} - \pi/6)}$$

2. Evaluate the following integral.

$$\int \tan^3 x\ \sec^3 x\ dx$$

derivative of sec x

$$= \int \tan^2 x\ \sec^2 x\ (\tan x \cdot \sec x)\,dx$$

Since $\tan^2 x = \sec^2 x - 1$ We have

$$\int (\sec^2 x - 1)\cdot \sec^2 x \cdot (\tan x \cdot \sec x)\,dx$$

$$= \int (\sec^4 x - \sec^2 x)\cdot(\tan x \cdot \sec x)\,dx$$

$u = \sec x$

$du = \tan x \cdot \sec x\,dx$

$$= \int u^4 - u^2\,du = \frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$$

3. Evaluate the following integral.

$\int \frac{x^3+3x-2}{x^2-x}\, dx$ partial fractions

First, long division

$$\begin{array}{r} x+1 \\ x^2-x \,\overline{)\, x^3+3x-2} \\ -(x^3-x^2) \\ \hline x^2+3x \\ -(x^2-x) \\ \hline 4x-2 \end{array}$$

So $\frac{x^3+3x-2}{x^2-x} = x+1+\frac{4x-2}{x^2-x}$

Hence $\int \frac{x^3+3x-2}{x^2-x}\, dx = \int x+1+\frac{4x-2}{x^2-x}\, dx = \frac{x^2}{2}+x+\underline{\int \frac{4x-2}{x^2-x}\, dx}$

$\frac{4x-2}{x(x-1)} = \frac{A}{x}+\frac{B}{x-1}$

So $4x-2 = A(x-1)+Bx$

when $\underline{x=1}$ we get

$2=B$

when $\underline{x=0}$ we get

$-2=-A$

$A=2$

So $\int \frac{x^3+3x-2}{x^2-x}\, dx = \frac{x^2}{2}+x+\int \frac{2}{x}+\frac{2}{x-1}\, dx$

$= \boxed{\frac{x^2}{2}+x+2\ln|x|+2\ln|x-1|+C}$

4. (This is really problem 5, but its solution fits better here) Evaluate the following integral.

$\int \frac{4x^2-5x-15}{x^3-4x^2-5x}\, dx$ partial fractions

$x^3-4x^2-5x = x(x^2-4x-5) = x(x-5)(x+1)$

so $\frac{4x^2-5x-15}{x(x-5)(x+1)} = \frac{A}{x}+\frac{B}{x-5}+\frac{C}{x+1}$

or $4x^2-5x-15 = A(x-5)(x+1)+Bx(x+1)+Cx(x-5)$

When $\underline{x=0}$ we get $-15 = A(-5)\cdot 1$ or $A=3$

When $\underline{x=5}$ we get $4\cdot 25-25-15 = B\cdot 5\cdot 6$

$60 = 30B$ or $B=2$

When $\underline{x=-1}$ we get $4+5-15 = C(-1)(-6)$

$-6 = 6C$ or $C=-1$

So $\int \frac{4x^2-5x-15}{x^3-4x^2-5x}\, dx = \int \frac{3}{x}+\frac{2}{x-5}-\frac{1}{x+1}\, dx$

$= \boxed{3\ln|x|+2\ln|x-5|-\ln|x+1|+C}$

5. Evaluate the following integral (really problem 4).

$\int \sin^4 2x\ \cos^2 2x\ dx$

Since both sin and cos are to even powers, we need to use the half angle formula.

$$\cos^2 2x = \frac{1+\cos 4x}{2}$$

$$\sin^4 2x = (\sin^2 2x)^2 = \left(\frac{1-\cos 4x}{2}\right)^2 = \frac{1-2\cos 4x+\cos^2 4x}{4}$$

So the integral becomes

$$\int \left(\frac{1-2\cos 4x+\cos^2 4x}{4}\right)\cdot\left(\frac{1+\cos 4x}{2}\right) dx =$$

$$\frac{1}{8}\int 1 - 2\cancel{\cos 4x} + \cancel{\cos^2 4x} + \cancel{\cos 4x} - 2\cancel{\cos^2 4x} + \cos^3 4x\ dx =$$

$$\frac{1}{8}\int 1 - \cos 4x - \cos^2 4x + \cos^3 4x\ dx =$$

$$\frac{1}{8}\left(x - \frac{1}{4}\sin 4x - \int \cos^2 4x\, dx + \int \cos^3 4x\, dx\right) =$$

We compute the last two integrals.

$$\int \cos^2 4x\, dx = \frac{1}{4}\int \cos^2 u\, du = \frac{1}{4}\left(\frac{1}{2}u + \frac{1}{4}\sin 2u\right) + C = \frac{1}{8}\cdot 4x + \frac{1}{16}\sin 8x$$

$u = 4x$

$du = 4dx$

$$\int \cos^3 4x\, dx = \int (\cos 4x)(\cos^2 4x)\, dx = \int (\cos 4x)(1-\sin^2 4x)\, dx$$

$$= \int \cos 4x\, dx - \int \sin^2 4x \cos 4x\, dx$$

$u = 4x \quad du = 4\,dx$

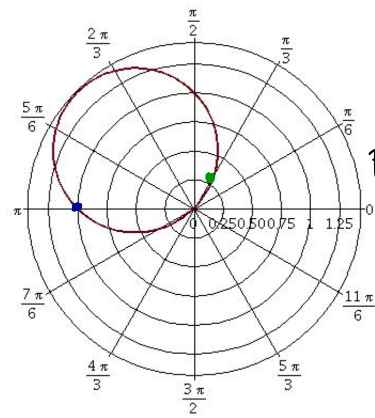
$$= \frac{1}{4}\sin 4x - \frac{1}{4}\int \sin^2 u \cos u\, du$$

$$= \frac{1}{4}\sin 4x - \frac{1}{4}\,\frac{\sin^3 4x}{3} + C$$

Putting it all together

$$\boxed{\frac{1}{8}\left(x - \frac{1}{4}\sin 4x - \left(\frac{1}{2}x + \frac{1}{16}\sin 8x\right) + \frac{1}{4}\sin 4x - \frac{1}{12}\sin^3 4x\right) + C}$$

6. (a) Sketch the curve with the polar equation $r = \sin\theta - \cos\theta$.



For example, when $\theta=0$, $r=\sin 0-\cos 0 = 0-1=-1$.
We note the point $(-1,0)$ in blue.
Or when $\theta = \pi/3$, $r=\sin\frac{\pi}{3}-\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}-\frac{1}{2} \approx 0.366$
We note the point $(\frac{\sqrt{3}-1}{2}, \pi/3)$ in green.

(b) How would you describe the line $y = \sqrt{3}x$ in polar coordinates?

We could let $x = r\cos\theta$ and $y = r\sin\theta$ to get

$$r\sin\theta = \sqrt{3}\, r\cos\theta$$

so $\tan\theta = \sqrt{3}$ or $\boxed{\theta = \pi/3}$

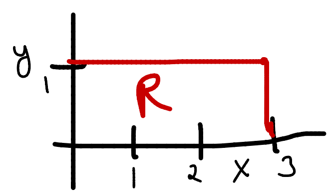
(c) What's another way to describe the line in (b) in polar coordinates?

We could add or subtract π. So

$$\theta = \pi/3 + \pi = \boxed{4\pi/3} \text{ or } \theta = \pi/3 - \pi = \boxed{-2\pi/3}$$

7. Find the volume of the solid in the first octant bounded by the surface $z = 6 + (x-5)^2 + 4y$ and the planes $x = 3$ and $y = 1$.

We compute $\iint_R 6+(x-5)^2+4y \, dA$ where R is



This becomes the iterated integral

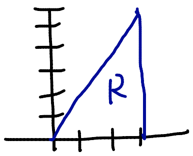
$$\int_0^1\int_0^3 6+(x-5)^2+4y\,dx\,dy = \int_0^1 6x + \frac{(x-5)^3}{3} + 4yx \Big]_0^3$$

$$= \int_0^1 18 - \frac{8}{3} + 12y + \frac{125}{3}\,dy = \int_0^1 \cancel{\frac{171}{3}}_{57} + 12y\,dy = 57y + 6y^2\Big]_0^1 = 57+6-0 = \boxed{63}$$

8. Evaluate the integral

$$\int_0^6 \int_{y/2}^3 \frac{y}{x^3+1}\,dx\,dy$$

We can't integrate this the way it is so we switch the order of integration.



$$\int_0^3 \int_{y=0}^{y=2x} \frac{y}{x^3+1}\,dy\,dx = \int_0^3 \left.\frac{y^2}{2(x^3+1)}\right]_{y=0}^{y=2x} dx = \int_0^3 \frac{4x^2}{2(x^3+1)}\,dx$$

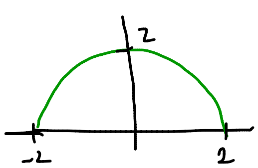
$u = x^3+1$
$du = 3x^2\,dx$

$$= \frac{2}{3}\ln|x^3+1|\Big]_0^3 = \boxed{\frac{2}{3}\ln 28} - \frac{2}{3}\underbrace{\cancel{\ln 1}}_{0}$$

9. Evaluate the integral

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$$

We need to convert to polar.



$$= \int_0^\pi \int_0^2 r \cdot r\,dr\,d\theta = \int_0^\pi \left.\frac{r^3}{3}\right]_0^2 d\theta = \int_0^\pi \frac{8}{3}\,d\theta =$$

$$\frac{8}{3}\theta\Big]_0^\pi = \boxed{\frac{8}{3}\pi}$$

10. Calculate the following integrals.

(a) $\int\limits_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \int\limits_{0}^{\sqrt{2}} \frac{y}{1+x^2}\, \mathrm{d}y\, \mathrm{d}x$ $= \displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{y^2}{2(1+x^2)}\Big]_0^{\sqrt{2}} dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2}{2(1+x^2)} - 0\, dx$

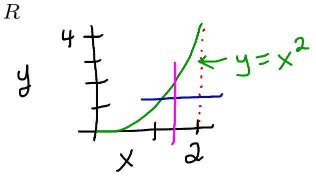
$$= \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1+x^2}\, dx = \tan^{-1} x\Big]_{1/\sqrt{3}}^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1/\sqrt{3}$$

$$= \pi/3 - \pi/6 = \boxed{\pi/6}$$

(b) $\int\limits_{2}^{5} \int\limits_{1}^{4} \frac{x}{y} + \frac{y}{x}\, \mathrm{d}y\, \mathrm{d}x$ $= \displaystyle\int_2^5 x\ln y + \frac{y^2}{2x}\Big]_{y=1}^{y=4} dx = \int_2^5 x\ln 4 + \frac{8}{x} - x\underbrace{\cancel{\ln 1}}_{0} - \frac{1}{2x}\, dx$

$$= \int_2^5 x\ln 4 + \frac{15}{2x}\, dx = \frac{\ln 4}{2}x^2 + \frac{15}{2}\ln x\Big]_2^5 = \boxed{\frac{\ln 4}{2}\cdot 25 + \frac{15}{2}\ln 5 - \frac{\ln 4}{2}\cdot 4 - \frac{15}{2}\ln 2}$$

(c) $\iint\limits_R x\cos y\, \mathrm{d}A$ where R is the region bounded by $y = 0$, $y = x^2$ and $x = 2$.



$\displaystyle\int_0^4 \int_{x=\sqrt{y}}^{x=2} x\cos y\, dx\, dy$ or $\displaystyle\int_0^2 \int_{y=0}^{y=x^2} x\cos y\, dy\, dx$

We'll solve blue.

$$\int_0^4 \frac{x^2}{2}\cos y\Big]_{x=\sqrt{y}}^{x=2} dy = \int_0^4 \frac{4}{2}\cos y - \frac{y}{2}\cos y\, dy$$

parts

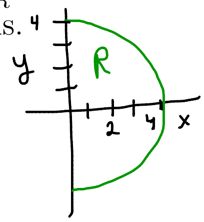
$u = y/2$ $\quad v = \sin y$

$du = \frac{1}{2}dy$ $\quad dv = \cos y\, dy$

$$= 2\sin y\Big]_0^4 - \left(\frac{y}{2}\sin y\Big]_0^4 - \int_0^4 \frac{1}{2}\sin y\, dy\right) = 2\sin y - \frac{y}{2}\sin y - \frac{1}{2}\cos y\Big]_0^4$$

$$= 2\cancel{\sin 4} - 2\cancel{\sin 4} - \frac{1}{2}\cos 4 - (0 - 0 - \frac{1}{2}\underbrace{\cancel{\cos 0}}_{1}) = \boxed{\frac{1}{2} - \frac{1}{2}\cos 4}$$

(d) $\iint\limits_R e^{-x^2-y^2}\ dA$ where R is the region bounded by the semicircle $x = \sqrt{16-y^2}$ and the y-axis.



polar coordinates

$$\int_{-\pi/2}^{\pi/2}\int_0^4 e^{-(r\cos\theta)^2-(r\sin\theta)^2}\cdot r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^4 e^{-r^2(\cancel{\cos^2\theta+\sin^2\theta})^{1}}\cdot r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^4 re^{-r^2}\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} -\frac{1}{2}e^{-r^2}\Big]_0^4 d\theta$$

$u = r^2$
$du = 2r\,dr$

$$= \int_{-\pi/2}^{\pi/2} -\frac{1}{2}e^{-16}+\frac{1}{2}\,d\theta = -\frac{1}{2}e^{-16}\theta + \frac{1}{2}\theta\Big]_{-\pi/2}^{\pi/2} = -\frac{\pi}{4}e^{-16} + \frac{\pi}{4} - \frac{\pi}{4}e^{-16} + \frac{\pi}{4}$$

$$= \boxed{\pi/2 - \pi/2\, e^{-16}}$$

11. Evaluate the following integrals

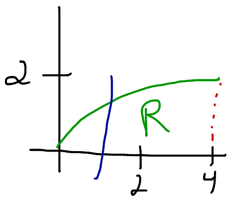
(a) $\int_{-1}^{1}\int_{2}^{4}\int_{0}^{2} \frac{x}{(y+z)^2}\ dx\ dy\ dz.$ $= \int_{-1}^{1}\int_{2}^{4} \frac{x^2}{2(y+z)^2}\Big]_0^2 dy\,dz = \int_{-1}^{1}\int_{2}^{4}\frac{2}{(y+z)^2}\,dy\,dz$

$$= \int_{-1}^{1} \frac{-2}{(y+z)}\Big]_{y=2}^{y=4} dz = \int_{-1}^{1} \frac{-2}{4+z} + \frac{2}{2+z}\,dz =$$

$$-2\ln|4+z| + 2\ln|2+z|\Big]_{-1}^{1} = -2\ln 5 + 2\ln 3 + 2\ln 3 - 2\cancel{\ln 1}^{0}$$

$$= \boxed{4\ln 3 - 2\ln 5}$$

(b) $\iiint\limits_R 3xy \; \mathrm{d}V$ where R lies under the plane $z = 5 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$ and $x = 4$.



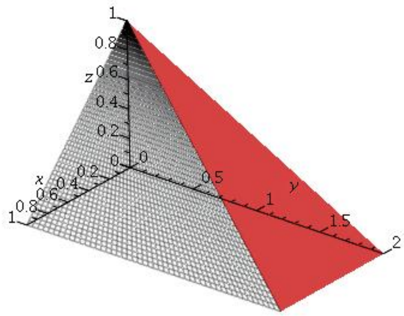
$$\int_{x=0}^{x=4}\int_{y=0}^{y=\sqrt{x}}\int_{z=0}^{z=5+x+y} 3xy\,dz\,dy\,dx = \int_0^4\int_{y=0}^{y=\sqrt{x}} 3xyz\Big]_0^{5+x+y} dy\,dx =$$

$$\int_0^4\int_{y=0}^{y=\sqrt{x}} 3xy(5+x+y)\,dy\,dx = \int_0^4 \frac{15xy^2}{2} + \frac{3x^2y^2}{2} + \frac{\cancel{3}xy^3}{\cancel{3}}\Big]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^4 \frac{15}{2}x^2 + \frac{3}{2}x^3 + x^{5/2}\,dx = \frac{\overset{5}{\cancel{15}}}{2\cdot\cancel{3}}x^3 + \frac{3}{2\cdot 4}x^4 + \frac{x^{7/2}}{7/2}\Big]_0^4$$

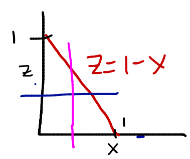
$$= 5/2\,(4)^3 + \frac{3}{8}(4)^4 + \frac{2}{7}(4)^{7/2} = \boxed{160 + 96 + \frac{256}{7}}$$

12. Let $\mathcal{R}$ be the region in the first octant bounded by the planes $z = 1 - x$ and $y = 2 - 2z$.



Express, **but do not evaluate** the triple integrals $\iiint_{\mathcal{R}} f(x,y,z)\ \mathrm{d}V$ as an iterated integral in each of the six possible ways.

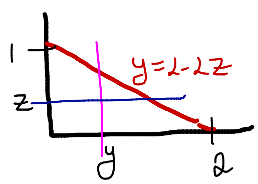
I. Project onto xz



$$\int_{z=0}^{z=1}\int_{x=0}^{x=1-z}\int_{y=0}^{y=2-2z} f(x,y,z)\,dy\,dx\,dz$$

$$\int_{x=0}^{x=1}\int_{z=0}^{z=1-x}\int_{y=0}^{y=2-2z} f(x,y,z)\,dy\,dz\,dx$$

II Project onto yz

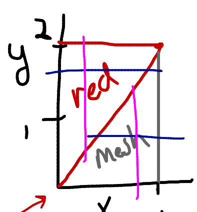


$$\int_{z=0}^{z=1}\int_{y=0}^{y=2-2z}\int_{x=0}^{x=1-z} f(x,y,z)\,dx\,dy\,dz$$

$$\int_{y=0}^{y=2}\int_{z=0}^{z=\frac{2-y}{2}}\int_{x=0}^{x=1-z} f(x,y,z)\,dx\,dz\,dy$$

III Project onto xy

We need to split this one up.



$$\underbrace{\int_{y=0}^{y=2}\int_{x=0}^{x=y/2}\int_{z=0}^{z=\frac{2-y}{2}} f(x,y,z)\,dz\,dx\,dy}_{\text{red}} + \underbrace{\int_{y=0}^{y=2}\int_{x=y/2}^{x=1}\int_{z=0}^{z=1-x} f(x,y,z)\,dz\,dx\,dy}_{\text{mesh}}$$

red

mesh

$$\int_{x=0}^{x=1}\int_{y=2x}^{y=2}\int_{z=0}^{z=\frac{2-y}{2}} f(x,y,z)\,dz\,dy\,dx + \int_{x=0}^{x=1}\int_{y=0}^{y=2x}\int_{z=0}^{z=1-x} f(x,y,z)\,dz\,dy\,dx$$

The divider is where $z=1-x$ and $z=\frac{2-y}{2}$ so

$\cancel{1}-x = \frac{2-y}{2} = \cancel{1}-\frac{y}{2}$

$y=2x$