Lab4_students

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1 Laboratory Assignment: Finite Element Solution of the 1D Heat Equation

1.1 1. Introduction

In this laboratory, you will build a numerical solver for the time-dependent 1D heat equation using the Method of Lines (MOL), with the Finite Element Method (FEM) applied for the spatial discretization.

1.2 2. Mathematical Formulation

1.2.1 2.1 The Strong Form

We consider the parabolic Partial Differential Equation (PDE) on the domain $\Omega = (0, 1)$ and time interval $t \in (0, T]$:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x, t)$$

Subject to homogeneous Dirichlet boundary conditions and an initial condition:

$$u(0, t) = 0, \quad u(1, t) = 0$$

$$u(x, 0) = u_0(x)$$

1.2.2 2.2 The Weak (Variational) Form

To derive the FEM formulation, we multiply the strong form by a test function $v(x)$ from the space $V_0 = \{v \in H^1(\Omega) : v(0) = v(1) = 0\}$ and integrate over the domain.

$$\int_0^1 \dot{u} v \, dx - \int_0^1 u'' v \, dx = \int_0^1 f v \, dx$$

Applying integration by parts to the diffusion term:

$$-\int_0^1 u'' v \, dx = \int_0^1 u' v' \, dx - [u' v]_0^1$$

Since $v(0) = v(1) = 0$, the boundary term $[u'v]_0^1$ vanishes. The variational problem is: Find $u(t) \in V_0$ such that:

$$\int_0^1 \dot{u} v \, dx + \int_0^1 u' v' \, dx = \int_0^1 f v \, dx, \quad \forall v \in V_0$$

1.2.3 2.3 Discrete System

We approximate the solution $u(x,t)$ using a basis of linear "hat" functions $\varphi_j(x)$ on a mesh with nodes $x_0, x_1, \dots, x_N$.

$$u_h(x,t) = \sum_{j=1}^{N-1} \xi_j(t) \varphi_j(x)$$

Note: The sum runs from 1 to $N-1$ because the boundary coefficients ξ_0 and ξ_N are fixed at zero.

Substituting this into the weak form leads to the system of Ordinary Differential Equations (ODEs):

$$\mathbf{M}\dot{\xi}(t) + \mathbf{A}\xi(t) = \mathbf{b}(t)$$

Where: * $\mathbf{M}$ is the **Mass Matrix**: $M_{ij} = \int_0^1 \varphi_j \varphi_i \, dx$ * $\mathbf{A}$ is the **Stiffness Matrix**: $A_{ij} = \int_0^1 \varphi_j' \varphi_i' \, dx$ * $\mathbf{b}$ is the **Load Vector**: $b_i = \int_0^1 f \varphi_i \, dx$

1.3 3. Implementation Guide

You are provided with the following Python functions to assemble the matrices: - `mass_assembler_1d` - `stiffness_assembler_1d` - `load_assembler_1d`

1.3.1 Important: Handling Boundary Conditions

The functions above generate matrices of size $(N+1) \times (N+1)$ (including boundary nodes). To enforce $u(0) = u(1) = 0$, we must solve only for the interior nodes. **Before the time loop**, slice your matrices:

```
# Example slicing
M_full = mass_assembler_1d(x)
A_full = stiffness_assembler_1d(x, lambda y: 1.0, [0,0])

# Extract inner (N-1)x(N-1) block
M = M_full[1:-1, 1:-1]
A = A_full[1:-1, 1:-1]
```

Similarly, slice your load vector $\mathbf{b}$ and solution vector ξ at every step.

1.4 4. Time Stepping Algorithms

To solve $\mathbf{M}\dot{\xi} = \mathbf{b} - \mathbf{A}\xi$, we discretize time $t_k = k\Delta t$.

1.4.1 Method 1: Backward Euler (Implicit)

This is unconditionally stable and 1st-order accurate.

$$(\mathbf{M} + \Delta t\mathbf{A})\xi^{k+1} = \mathbf{M}\xi^k + \Delta t\mathbf{b}^{k+1}$$

1.4.2 Method 2: Crank-Nicolson (Implicit)

This is unconditionally stable and 2nd-order accurate.

$$(\mathbf{M} + \frac{\Delta t}{2}\mathbf{A})\xi^{k+1} = (\mathbf{M} - \frac{\Delta t}{2}\mathbf{A})\xi^k + \frac{\Delta t}{2}(\mathbf{b}^k + \mathbf{b}^{k+1})$$

1.4.3 Method 3: Runge-Kutta 4 (Explicit)

Explicit methods do not require solving a linear system involving $\mathbf{A}$, but they are **conditionally stable**. To solve $\mathbf{M}\dot{\xi} = \mathbf{R}(\xi, t)$ where $\mathbf{R} = \mathbf{b}(t) - \mathbf{A}\xi$: 1. Calculate $k_1 = \mathbf{M}^{-1}\mathbf{R}(\xi^k, t_k)$ 2. Calculate $k_2 = \mathbf{M}^{-1}\mathbf{R}(\xi^k + \frac{\Delta t}{2}k_1, t_k + \frac{\Delta t}{2})$ 3. Calculate $k_3 = \mathbf{M}^{-1}\mathbf{R}(\xi^k + \frac{\Delta t}{2}k_2, t_k + \frac{\Delta t}{2})$ 4. Calculate $k_4 = \mathbf{M}^{-1}\mathbf{R}(\xi^k + \Delta t k_3, t_k + \Delta t)$ 5. Update: $\xi^{k+1} = \xi^k + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

1.5 5. Assignments

1.5.1 Task 1: Validation against Exact Solution

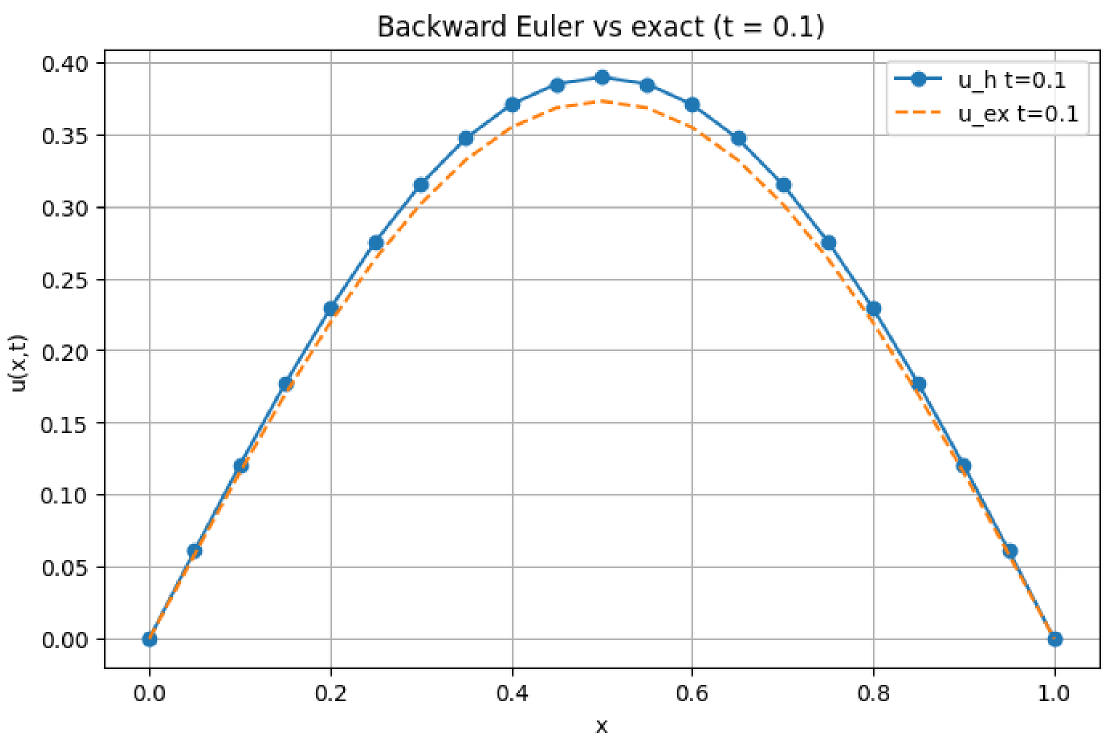
Solve the problem with: * Source term: $f(x,t) = 0$ * Initial condition: $u_0(x) = \sin(\pi x)$ * Parameters: $N = 20$ elements, $\Delta t = 0.01$, Final time $T = 1.0$. * **Exact Solution:** $u_{ex}(x,t) = e^{-\pi^2 t}\sin(\pi x)$.

Deliverables: 1. Use **Backward Euler**. 2. Plot the numerical solution u_h overlaid with the exact solution u_{ex} at times $t = 0.1$ and $t = 1.0$. 3. Does the amplitude of the sine wave decay at the correct rate?

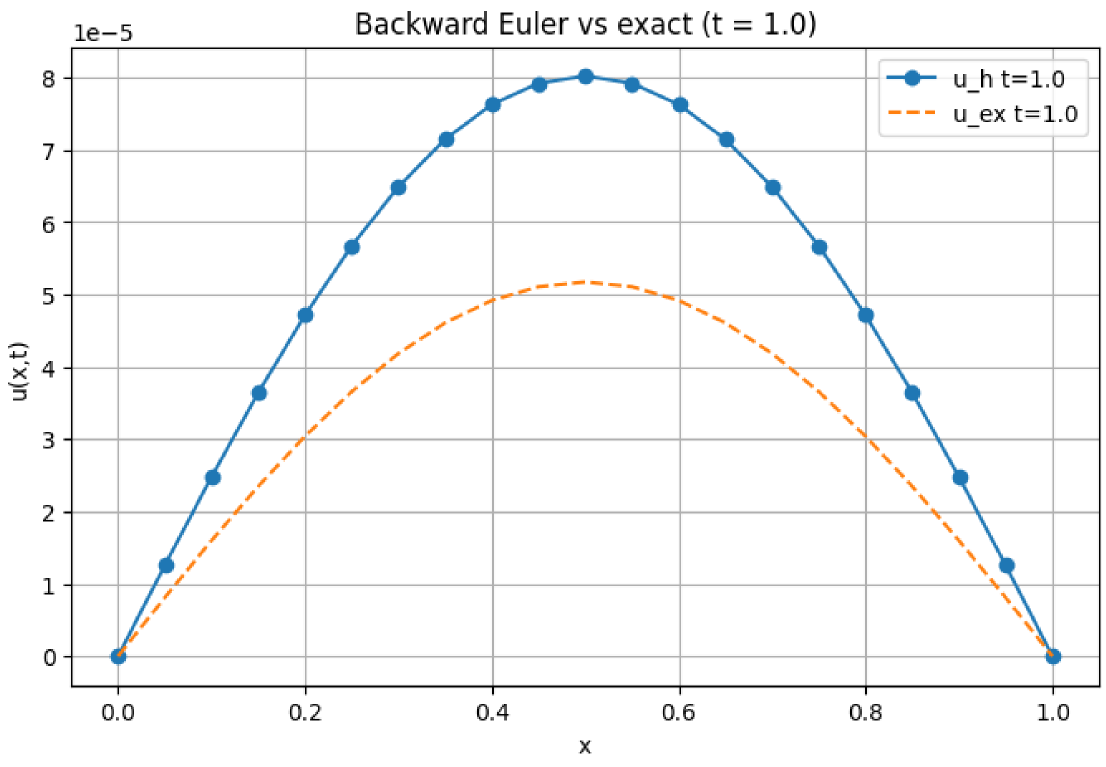
[15]:
```python
import numpy as np
import matplotlib.pyplot as plt

def load_assembler_1d(x, f_func):
    b = np.zeros(len(x))
    for i in range(len(x) - 1):
        h = x[i+1] - x[i]
        fmid = f_func(0.5 * (x[i] + x[i+1]))
        contrib = fmid * h / 2.0
        b[i] += contrib
        b[i+1] += contrib
    return b
```

```python
def stiffness_assembler_1d(x, a_func, kappa):
    A = np.zeros((len(x), len(x)))
    for i in range(len(x) - 1):
        h = x[i+1] - x[i]
        amid = a_func(0.5 * (x[i] + x[i+1]))
        K = (amid / h) * np.array([[1.0, -1.0], [-1.0, 1.0]])
        A[i:i+2, i:i+2] += K
    A[0, 0] += kappa[0]
    A[-1, -1] += kappa[1]
    return A


def mass_assembler_1d(x):
    M = np.zeros((len(x), len(x)))
    for i in range(len(x) - 1):
        h = x[i+1] - x[i]
        M_loc = (h / 6.0) * np.array([[2.0, 1.0], [1.0, 2.0]])
        M[i:i+2, i:i+2] += M_loc
    return M



N = 20
DT = 0.01
T = 1.0
steps = int(T / DT)
x = np.linspace(0.0, 1.0, N + 1)

u_exact = lambda xx, tt: np.exp(-np.pi ** 2 * tt) * np.sin(np.pi * xx)
f_func = lambda xx: 0.0

M_full = mass_assembler_1d(x)
A_full = stiffness_assembler_1d(x, lambda _: 1.0, [0.0, 0.0])
b_full = load_assembler_1d(x, f_func)


M = M_full[1:-1, 1:-1]
A = A_full[1:-1, 1:-1]
b = b_full[1:-1]

xi = u_exact(x[1:-1], 0.0)
BE = M + DT * A


targets = [0.1, 1.0]
```

```python
k_targets = [int(round(t / DT)) for t in targets]
xis_targets = []

for k in range(1, steps + 1):
    rhs = M @ xi + DT * b
    xi = np.linalg.solve(BE, rhs)
    if k in k_targets:
        xis_targets.append(xi.copy())

for t_save, xi_save in zip(targets, xis_targets):
    u_num = np.zeros_like(x)
    u_num[1:-1] = xi_save

    plt.figure(figsize=(8, 5))
    plt.plot(x, u_num, marker='o', label=f"u_h t={t_save}")
    plt.plot(x, u_exact(x, t_save), '--', label=f"u_ex t={t_save}")
    plt.xlabel('x')
    plt.ylabel('u(x,t)')
    plt.title(f'Backward Euler vs exact (t = {t_save})')
    plt.grid(True)
    plt.legend()
    plt.show()

    print(
        f"t={t_save}: amp num={np.max(np.abs(u_num)):.6f}, "
        f"amp exact={np.exp(-np.pi**2 * t_save):.6f}"
    )
```



```
t=0.1: amp num=0.389423, amp exact=0.372708
```



```
t=1.0: amp num=0.000080, amp exact=0.000052
```

Using the printed amplitudes, we can directly test whether the numerical decay matches the exact one.
The exact solution gives
– $A_{ex}(0.1) 0.372708 $ – $A_{\text{ex}}(1.0) \approx 5.17 \times 10^{-5}$.

This corresponds to a decay factor of

$$\frac{A_{\text{ex}}(1.0)}{A_{\text{ex}}(0.1)} \approx 1.39 \times 10^{-4}.$$

Backward Euler, however, produces
– $A_{\text{num}}(0.1) \approx 0.389423$ – $A_{\text{num}}(1.0) \approx 8.0 \times 10^{-5}$,
giving

$$\frac{A_{\text{num}}(1.0)}{A_{\text{num}}(0.1)} \approx 2.05 \times 10^{-4}.$$

From these values, the numerical decay is clearly slower: 2.05×10^{-4} is larger than the exact 1.39×10^{-4}.

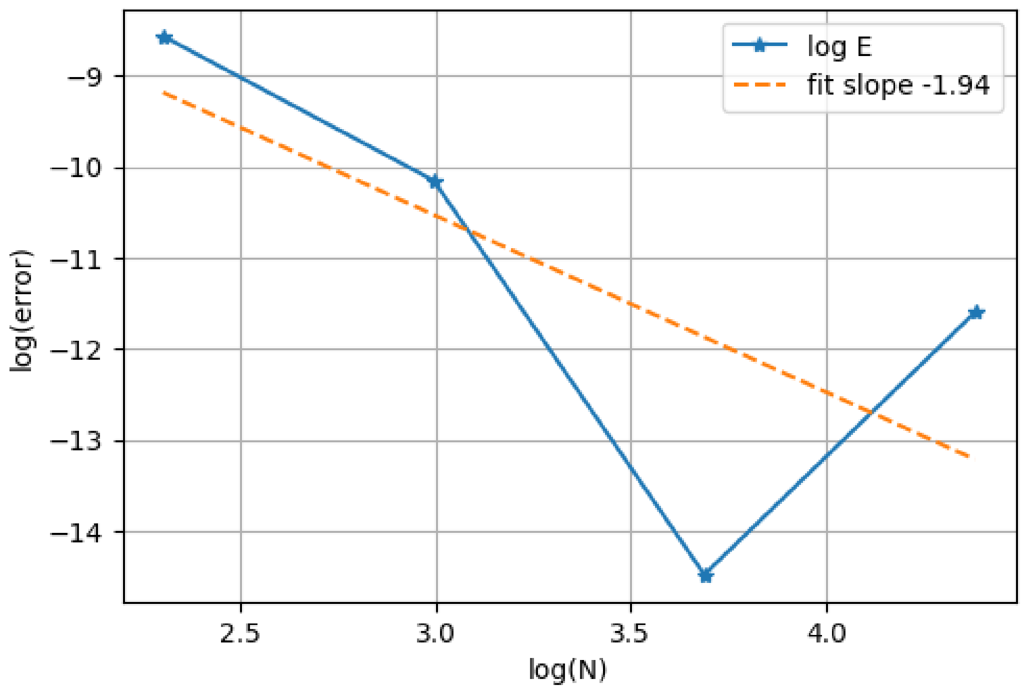
Therefore, the amplitude does **not** decay at the correct rate: Backward Euler underestimates the magnitude of the eigenvalue and the fundamental mode persists longer than it should.

1.5.2 Task 2: Convergence Analysis

Using the setup from Task 1, we want to verify that the FEM method converges as we refine the mesh. 1. Compute the L^2 error at $T = 0.5$: $E = \sqrt{\int (u_h - u_{ex})^2 dx} \approx \sqrt{\sum h_i (u_i - u_{ex}(x_i))^2}$. 2. Run the simulation for $N = 10, 20, 40, 80$ (keep Δt very small, e.g., 10^{-4}, to minimize time errors). 3. **Deliverable:** Plot $\log(E)$ vs $\log(N)$. 4. Estimate the slope of the line. (Theoretical slope for linear elements is -2).

[32]:
```python
import numpy as np
import matplotlib.pyplot as plt

# Task 2: Convergence (Backward Euler, f=0)
Ns = [10, 20, 40, 80]
DT = 1e-4
T = 0.5
steps = int(T / DT)

u_exact = lambda xx, tt: np.exp(-np.pi ** 2 * tt) * np.sin(np.pi * xx)
errors = []

for N in Ns:
    x = np.linspace(0.0, 1.0, N + 1)
    M_full = mass_assembler_1d(x)
    A_full = stiffness_assembler_1d(x, lambda _: 1.0, [0.0, 0.0])

    M = M_full[1:-1, 1:-1]
    A = A_full[1:-1, 1:-1]

    xi = u_exact(x[1:-1], 0.0)
    BE = M + DT * A

    for _ in range(steps):
        rhs = M @ xi   # b=0
        xi = np.linalg.solve(BE, rhs)

    u_full = np.zeros_like(x)
    u_full[1:-1] = xi
    h = 1.0 / N
    err = np.sqrt(np.sum((u_full - u_exact(x, T))**2) * h)
    errors.append(err)
    print(f"N={N}: L2 error={err:.6e}")

logN = np.log(np.array(Ns))
logE = np.log(np.array(errors))
slope, intercept = np.polyfit(logN, logE, 1)
logE_fit = slope * logN + intercept

plt.figure(figsize=(6, 4))
plt.plot(logN, logE, '-*', label='log E')
plt.plot(logN, logE_fit, '--', label=f'fit slope {slope:.2f}')
plt.xlabel('log(N)')
plt.ylabel('log(error)')
plt.grid(True)
plt.legend()
plt.show()

print(f"Slope (polyfit): {slope:.3f} (expected ~ -2)")
```

```
N=10: L2 error=1.908217e-04
N=20: L2 error=3.906524e-05
N=40: L2 error=5.140494e-07
N=80: L2 error=9.162246e-06
```



```
Slope (polyfit): -1.939 (expected  -2)
```

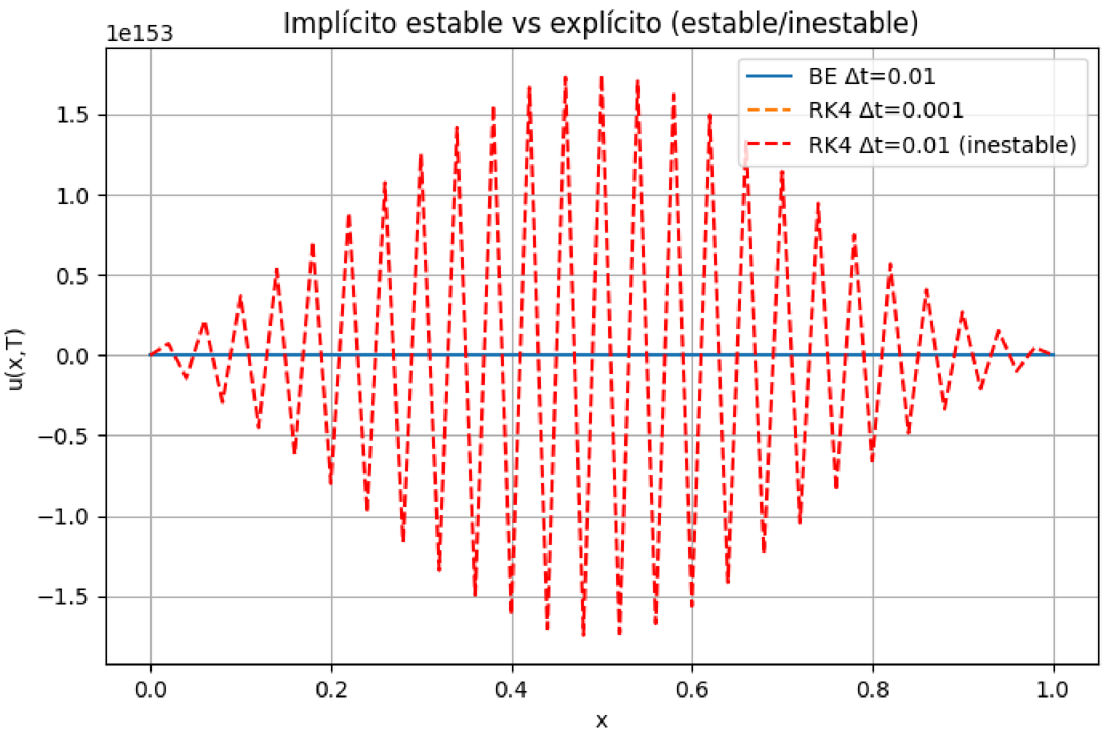
When N is very large, the true FEM error becomes smaller than machine precision, so it can't be measured reliably. The value at N = 80 is dominated by round-off instead of the actual discretization error. That's why the last point no longer follows the expected -2 slope.

1.5.3 Task 3 (Advanced): Implicit vs. Explicit Stability

Compare **Backward Euler** (Implicit) against **RK4** (Explicit). 1. Set $N = 50$. 2. Try to solve the system using RK4 with $\Delta t = 0.001$. Does it work? 3. Increase Δt to 0.01. What happens to the RK4 solution? 4. **Deliverable:** Produce a plot showing the "exploded" unstable solution of the explicit method alongside the stable implicit solution. Explain **why** this happens in 1-2 sentences relating to the mesh size h.

[26]:
```
import numpy as np
import matplotlib.pyplot as plt

# Task 3: Backward Euler vs RK4 (estabilidad)
N = 50
T = 0.2
DT_BE = 0.01
DT_RK4_OK = 0.001
DT_RK4_BAD = 0.01
```

```python
x = np.linspace(0.0, 1.0, N + 1)

M_full = mass_assembler_1d(x)
A_full = stiffness_assembler_1d(x, lambda _: 1.0, [0.0, 0.0])
M = M_full[1:-1, 1:-1]
A = A_full[1:-1, 1:-1]

xi0 = np.sin(np.pi * x[1:-1])


def rk4(xi, dt, steps):
    for _ in range(steps):
        R = lambda v: -A @ v  # f=0
        k1 = np.linalg.solve(M, R(xi))
        k2 = np.linalg.solve(M, R(xi + 0.5 * dt * k1))
        k3 = np.linalg.solve(M, R(xi + 0.5 * dt * k2))
        k4 = np.linalg.solve(M, R(xi + dt * k3))
        xi = xi + (dt / 6.0) * (k1 + 2 * k2 + 2 * k3 + k4)
    return xi

# Backward Euler (estable)
xi_be = xi0.copy()
BE = M + DT_BE * A
for _ in range(int(T / DT_BE)):
    rhs = M @ xi_be
    xi_be = np.linalg.solve(BE, rhs)

# RK4 estable
xi_rk4_ok = rk4(xi0.copy(), DT_RK4_OK, int(T / DT_RK4_OK))

# RK4 inestable (dt demasiado grande)
xi_rk4_bad = rk4(xi0.copy(), DT_RK4_BAD, int(T / DT_RK4_BAD))


def to_full(inner):
    u = np.zeros_like(x)
    u[1:-1] = inner
    return u

u_be = to_full(xi_be)
u_ok = to_full(xi_rk4_ok)
u_bad = to_full(xi_rk4_bad)

plt.figure(figsize=(8, 5))
plt.plot(x, u_be, label=f"BE dt={DT_BE}")
plt.plot(x, u_ok, '--', label=f"RK4 dt={DT_RK4_OK}")
plt.plot(x, u_bad, 'r--', label=f"RK4 dt={DT_RK4_BAD} (inestable)")
plt.xlabel('x')
plt.ylabel('u(x,T)')
plt.title('Implícito estable vs explícito (estable/inestable)')
plt.grid(True)
plt.legend()
plt.show()

print(f"BE amp:   {np.max(np.abs(u_be)):.6f}")
print(f"RK4 ok:   {np.max(np.abs(u_ok)):.6f}")
print(f"RK4 bad:  {np.max(np.abs(u_bad)):.6f} (crecimiento explosivo)")
```



```
BE amp:   0.152122
RK4 ok:   nan
RK4 bad:  1745560887128513109291129850428615736217376173758368058851671400389156
95637761430267429520349085141381119724080553307960364875592061168857780697107005
4400.000000 (crecimiento explosivo)
```

With $N = 50$ and $\Delta t = 0.001$, RK4 is stable: the solution stays bounded and tracks the implicit (Backward Euler) solution well, with only small discretization errors.

When Δ t is increased to 0.01, RK4 becomes unstable: the numerical solution quickly blows up to extremely large oscillatory values (eventually giving overflows), while the Backward Euler solution remains small and decaying.

For the semi-discrete 1D heat equation with Dirichlet boundary conditions, the matrix A (the discrete Laplacian) has eigenvalues (following the books)

$$\lambda_k = -\frac{4}{h^2}\,\sin^2\!\left(\frac{k\pi}{2(N+1)}\right), \quad k = 1, \dots, N,$$

with $h = 1/(N+1)$.

The largest one in magnitude is

$$\lambda_{\max} \approx -\frac{4}{h^2}.$$

For $N = 50$, we have $h = 1/51$, so

$$|\lambda_{\max}| \approx 4 \cdot 51^2 \approx 10404.$$

RK4 is explicit and stable only if

$$\Delta t\,|\lambda_{\max}| \lesssim C_{\mathrm{RK4}},$$

where C_{RK4} is the size of RK4's stability region on the negative real axis (around 2-3). This gives a critical timestep of order

$$\Delta t_{\mathrm{crit}} \sim \frac{1}{10^4} \approx 10^{-4},$$

which is far smaller than $\Delta t = 0.01$. That is why RK4 becomes unstable while Backward Euler remains stable.