08a_piecewise_polynomial_approximation_students

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0.1 Piecewise Polynomial Approximation in 1D

This notes introduces the mathematical and computational framework for approximating continuous functions $f(x)$ using continuous piecewise linear functions. It is based on: - **Chapter 1**, *The Finite Element Method: Theory, Implementation, and Applications*, by Mats G. Larson and Fredrik Bengzon.

0.1.1 1. Function Spaces

1.1 The Space $P_1(I)$: Linear Polynomials Let $I = [x_0, x_1]$ be a single interval. * **Definition:** The space $P_1(I)$ is the vector space of all linear polynomials on I.

$$P_1(I) = \{v(x) \mid v(x) = c_0 + c_1 x,\ x \in I,\ c_0, c_1 \in \mathbb{R}\}$$

This space has a dimension of 2.

- **Monomial Basis:** The set $\{1, x\}$ is a basis. Any $v \in P_1(I)$ can be determined by its values at x_0 and x_1, $v_0 = v(x_0)$ and $v_1 = v(x_1)$. This leads to a linear system for the coefficients c_0, c_1:

$$\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$$

  The system matrix is a **Vandermonde matrix** with determinant $x_1 - x_0 = h$ (the interval length). Since $h > 0$, a unique solution for c_0, c_1 always exists.

- **Nodal Basis (Lagrange Basis):** A more convenient basis is the **nodal basis** $\{\lambda_0, \lambda_1\} \subset P_1(I)$, defined by the property:

$$\lambda_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

  These basis functions are explicitly:

$$\lambda_0(x) = \frac{x_1 - x}{x_1 - x_0} \quad \text{and} \quad \lambda_1(x) = \frac{x - x_0}{x_1 - x_0}$$

  Using this basis, any $v \in P_1(I)$ has a direct representation:

$$v(x) = v(x_0)\lambda_0(x) + v(x_1)\lambda_1(x)$$

1.2 The Space V_h: Continuous Piecewise Linear Functions Let $I = [0, L]$ be partitioned by a **mesh** $\mathcal{T}_h = \{I_i\}_{i=1}^n$ of n subintervals, defined by $n+1$ nodes $0 = x_0 < x_1 < ... < x_n = L$. Let $h_i = x_i - x_{i-1}$ and $h = \max_i h_i$.

- **Definition:** The space V_h is the set of all continuous functions on I that are linear on each subinterval I_i.
$$V_h = \{v \in C^0(I) \mid v|_{I_i} \in P_1(I_i) \text{ for } i = 1, ..., n\}$$
This is a vector space. A function $v \in V_h$ is uniquely determined by its $n+1$ nodal values $\{v(x_i)\}_{i=0}^n$. Thus, $\dim(V_h) = n + 1$.

- **Basis ("Hat Functions"):** The standard basis for V_h is the set of **hat functions** $\{\phi_i\}_{i=0}^n$. Each $\phi_i \in V_h$ is defined by the nodal property $\phi_i(x_j) = \delta_{ij}$. The explicit formula for an interior hat function $\phi_i(x)$ (for $i = 1, ..., n-1$) is:
$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i} & \text{if } x \in I_i = [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{h_{i+1}} & \text{if } x \in I_{i+1} = [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$
(The endpoint functions ϕ_0 and ϕ_n are "half-hats").

- Any $v \in V_h$ can be written as a linear combination of these basis functions, with the coefficients being the nodal values:
$$v(x) = \sum_{i=0}^n v(x_i)\phi_i(x)$$

0.1.2 2. Approximation Methods

We seek an approximation $f_h \in V_h$ for a given continuous function $f \in C^0(I)$.

2.1 Interpolation

- **Definition:** The **linear interpolant** $\pi_h f \in V_h$ is defined as the function that matches f at all nodes.
$$\pi_h f(x_i) = f(x_i) \quad \text{for } i = 0, ..., n$$
Using the hat function basis, its formula is:
$$\pi_h f(x) = \sum_{i=0}^n f(x_i)\phi_i(x)$$

- **Error Estimates:** We measure the approximation error using norms. The L_2**-norm** is defined as:
$$||v||_{L_2(I)} = \left(\int_I v(x)^2 \, dx \right)^{1/2}$$
Proposition 1.2: For a function f with a continuous second derivative ($f \in C^2(I)$), the interpolation error is bounded by:
$$||f - \pi_h f||_{L_2(I)} \leq Ch^2 ||f''||_{L_2(I)}$$
where C is a constant independent of the mesh size h. This shows the error converges **quadratically** as the mesh is refined.

2.2 L_2-Projection

- **Definition:** We define the L_2**-inner product** for two functions v, w as:

$$(v, w) = \int_I v(x)w(x)\ dx$$

  The L_2**-projection** $P_h f \in V_h$ is the function that satisfies the **orthogonality condition**:

$$(f - P_h f, v) = 0 \quad \forall v \in V_h$$

  This means the error $f - P_h f$ is orthogonal to the entire approximation space V_h.

- **Optimality (Theorem 1.1):** The L_2-projection is the **best possible approximation** in V_h when measured by the L_2-norm.

$$||f - P_h f||_{L_2(I)} = \min_{v \in V_h} ||f - v||_{L_2(I)}$$

- **Error Estimate (Theorem 1.2):** The L_2-projection has the same optimal convergence rate as interpolation:

$$||f - P_h f||_{L_2(I)} \leq Ch^2 ||f''||_{L_2(I)}$$

0.1.3 3. Computation of the L_2-Projection

We must solve for $P_h f$. 1. **Ansatz:** We seek $P_h f$ as an unknown linear combination of the basis functions. Let $\xi = [\xi_0, ..., \xi_n]^T$ be the vector of unknown coefficients:

$$P_h f(x) = \sum_{j=0}^{n} \xi_j \phi_j(x)$$

2. **Derivation of the Linear System:** We substitute this into the orthogonality condition $(f - P_h f, v) = 0$. Since this must hold for *all* $v \in V_h$, it must hold for each basis function $v = \phi_i$ $(i = 0, ..., n)$:

$$(f - \sum_{j=0}^{n} \xi_j \phi_j, \phi_i) = 0 \quad \text{for } i = 0, ..., n$$

   By the linearity of the inner product:

$$(f, \phi_i) - \sum_{j=0}^{n} \xi_j (\phi_j, \phi_i) = 0$$

   Rearranging, we get a linear system for the ξ_j:

$$\sum_{j=0}^{n} (\phi_i, \phi_j) \xi_j = (f, \phi_i) \quad \text{for } i = 0, ..., n$$

3. **Matrix Form $M\xi = b$:** This is a linear system of $n + 1$ equations for the $n + 1$ unknowns in ξ.

- M **(Mass Matrix):**

$$M_{ij} = (\phi_i, \phi_j) = \int_I \phi_i(x)\phi_j(x)\ dx$$

- b **(Load Vector):**

$$b_i = (f, \phi_i) = \int_I f(x)\phi_i(x)\ dx$$

- ξ **(Solution Vector):** ξ_j are the coefficients of $P_h f$. Note: $\xi_j \neq f(x_j)$ in general.

0.1.4 4. Practical Implementation: Assembly

We build M and b using an **assembly** process, looping over each element $I_i = [x_{i-1}, x_i]$ (length h_i).

4.1 Assembly of Mass Matrix M On element I_i, the only non-zero basis functions are ϕ_{i-1} and ϕ_i. Their restrictions to I_i are just the local nodal basis functions: * $\phi_{i-1}(x) = \frac{x_i - x}{h_i}$ (local form of ϕ_{i-1} on I_i) * $\phi_i(x) = \frac{x - x_{i-1}}{h_i}$ (local form of ϕ_i on I_i)

We compute the 2x2 **local element mass matrix** M^i by integrating these local functions over I_i:

$$M_{11}^i = \int_{I_i} \lambda_0^2\ dx = \int_{x_{i-1}}^{x_i} \left(\frac{x_i - x}{h_i}\right)^2\ dx = \frac{h_i}{3}$$

$$M_{12}^i = \int_{I_i} \lambda_0\lambda_1\ dx = \int_{x_{i-1}}^{x_i} \frac{(x_i - x)(x - x_{i-1})}{h_i^2}\ dx = \frac{h_i}{6}$$

$$M_{21}^i = M_{12}^i = \frac{h_i}{6}$$

$$M_{22}^i = \int_{I_i} \lambda_1^2\ dx = \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{h_i}\right)^2\ dx = \frac{h_i}{3}$$

So,

$$M^i = \frac{h_i}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Assembly Algorithm: 1. Initialize a global $M = 0$. 2. Loop $i = 1$ to n (over all elements): * Compute M^i. * Add its entries to the global matrix: - $M_{i-1,i-1}$ += M_{11}^i - $M_{i-1,i}$ += M_{12}^i - $M_{i,i-1}$ += M_{21}^i - $M_{i,i}$ += M_{22}^i

The resulting global M is symmetric, positive definite, and tridiagonal.

4.2 Assembly of Load Vector b We must compute $b_i = (f, \phi_i)$. These integrals are typically approximated using **numerical quadrature**.

Let's use the **Trapezoidal Rule** on each element I_i for the **local element load vector** b^i.

$$b_1^i = \int_{I_i} f(x)\lambda_0(x)\ dx \approx \frac{h_i}{2}[f(x_{i-1})\lambda_0(x_{i-1}) + f(x_i)\lambda_0(x_i)]$$

$$b_1^i \approx \frac{h_i}{2}[f(x_{i-1}) \cdot 1 + f(x_i) \cdot 0] = \frac{h_i}{2} f(x_{i-1})$$

$$b_2^i = \int_{I_i} f(x)\lambda_1(x)\ dx \approx \frac{h_i}{2}[f(x_{i-1})\lambda_1(x_{i-1}) + f(x_i)\lambda_1(x_i)]$$

$$b_2^i \approx \frac{h_i}{2}[f(x_{i-1}) \cdot 0 + f(x_i) \cdot 1] = \frac{h_i}{2} f(x_i)$$

So,

$$b^i \approx \frac{h_i}{2} \begin{bmatrix} f(x_{i-1}) \\ f(x_i) \end{bmatrix}$$

Assembly Algorithm: 1. Initialize a global $b = 0$. 2. Loop $i = 1$ to n (over all elements): * Compute b^i (using quadrature). * Add its entries to the global vector: - b_{i-1} += b_1^i - b_i += b_2^i

Finally, we solve the tridiagonal system $M\xi = b$ to find the coefficient vector ξ, which defines our approximation $P_h f$.

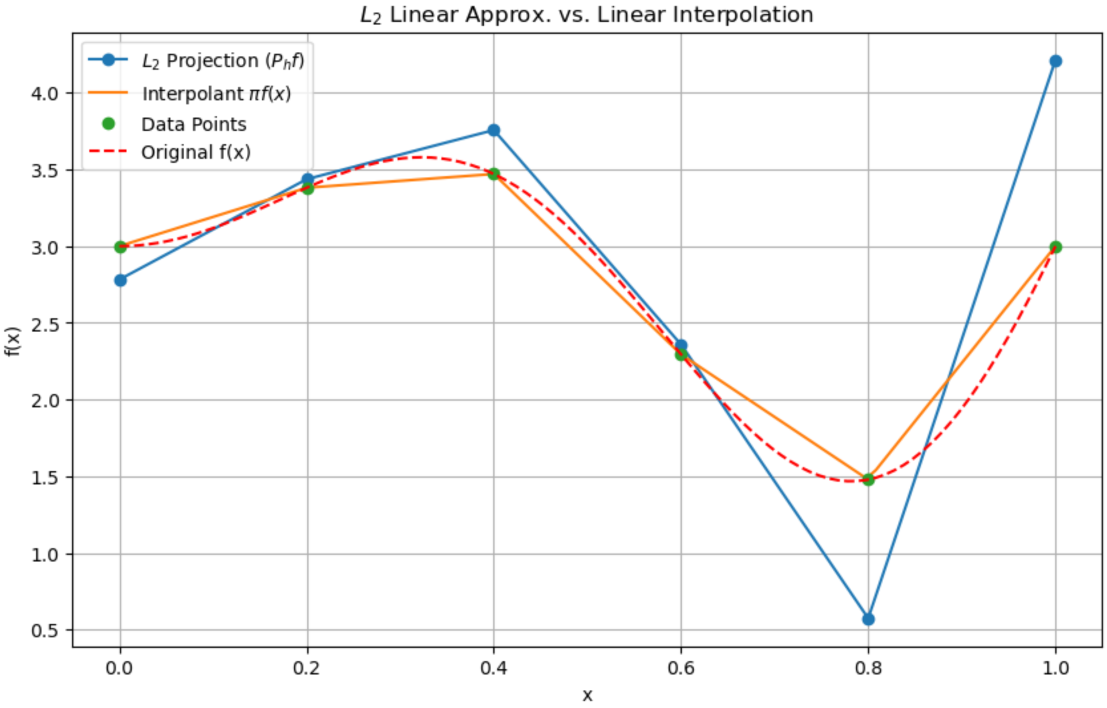
Practice Assignment Let $f(x) = 2x\sin(2\pi x) + 3$ be a function defined on the interval $I = [0, 1]$. Consider a uniform mesh of I with $n = 5$ subintervals, which has 6 nodes $\{x_i\}_{i=0}^5$.

Find: 1. The continuous piecewise linear interpolant, $\pi_h f(x) = \sum_{i=0}^5 f(x_i)\phi_i(x)$. 2. The $L_2$-projection, $P_h f(x)$, onto the same mesh.

[ ]:

[2]:

```
# Input data
n = 5   # number of subintervals
a, b = 0, 1
f = lambda x: 2*x*np.sin(2*np.pi*x) + 3

# nodal points and function values
x = np.linspace(a, b, n + 1)
f_vals = f(x)

# L^2 projection
Pf = l2_projector_1d(x,f_vals,f)

# linear interpolation
xe = np.linspace(a, b, 100)
If = interpolant_1d(x,f_vals,xe)


# plot L^2 projection
plt.figure(figsize=(10, 6))
plt.plot(x, Pf, 'o-', label='$L_2$ Projection ($P_h f$)')
plt.plot(xe, If, '-', label='Interpolant $\pi f(x)$')
plt.plot(x, f(x), 'o', label='Data Points')

# Plot the original function for comparison
plt.plot(xe, f(xe), 'r--', label='Original f(x)')

plt.title('$L_2$ Linear Approx. vs. Linear Interpolation')
plt.xlabel('x')
plt.ylabel('f(x)')
plt.legend()
plt.grid(True)
plt.show()
```


$L_2$ Linear Approx. vs. Linear Interpolation
$L_2$ Projection ($P_h f$)
Interpolant $\pi f(x)$
Data Points
Original f(x)
f(x)
x