04_tasks

October 28, 2025

[64]:
```python
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

1 Numerical Methods for Ordinary Differential Equations

1.1 Assignment: Solving an Initial Value Problem

Objective: Write a program to numerically solve a given Initial Value Problem (IVP) using various one-step methods, and perform a convergence analysis by comparing the numerical results to the exact solution.

1.1.1 Exercise

Consider the initial value problem (IVP)

$$u' = \frac{u^2 + u}{t}, \quad 1 \le t \le 5, \quad u(1) = -2,$$

whose exact solution is $u(t) = \frac{2t}{1-2t}$.

Part A: Implementation

Write a program (using a language like Python, MATLAB, or C++) to solve this IVP using the numerical methods listed below. Note that for the **Backward Euler method**, you will need to solve a quadratic equation for U^{n+1} at each time step.

1. **Forward Euler method (FE):**

$$U^{n+1} = U^n + kf(U^n, t_n).$$

2. **Backward Euler method (BE):**

$$U^{n+1} = U^n + kf(U^{n+1}, t_{n+1}).$$

   (Hint: This is an implicit method; rearrange it into a quadratic equation $AU^2 + BU + C = 0$ for U^{n+1} and use the quadratic formula.)

3. **Explicit midpoint RK2:**

$$
\begin{aligned}
k_1 &= f(U^n, t_n) \\
k_2 &= f\left(U^n + \frac{k}{2}k_1, t_n + \frac{k}{2}\right) \\
U^{n+1} &= U^n + kk_2
\end{aligned}
$$

4. **Explicit trapezoidal RK2 (Improved Euler):**

$$
\begin{aligned}
k_1 &= f(U^n, t_n) \\
k_2 &= f(U^n + kk_1, t_n + k) \\
U^{n+1} &= U^n + \frac{k}{2}(k_1 + k_2)
\end{aligned}
$$

5. **Heun's method (RK3):**

$$
\begin{aligned}
k_1 &= f(U^n, t_n) \\
k_2 &= f\left(U^n + \frac{k}{3}k_1, t_n + \frac{k}{3}\right) \\
k_3 &= f\left(U^n + \frac{2k}{3}k_2, t_n + \frac{2k}{3}\right) \\
U^{n+1} &= U^n + \frac{k}{4}(k_1 + 3k_3)
\end{aligned}
$$

6. **Standard RK4:**

$$
\begin{aligned}
k_1 &= f(U^n, t_n) \\
k_2 &= f\left(U^n + \frac{k}{2}k_1, t_n + \frac{k}{2}\right) \\
k_3 &= f\left(U^n + \frac{k}{2}k_2, t_n + \frac{k}{2}\right) \\
k_4 &= f(U^n + kk_3, t_n + k) \\
U^{n+1} &= U^n + \frac{k}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

[ ]:
```
def one_step_method_solver(f, k, u0, t0, tf, method):
    """
    Function to implement differents one-step methods and solve the ODE u' =␣
 ↪f(u,t).

    Parameters
    f: callable
        Function that defines the ODE.
    k: float
        Time step.
    u0: float
        Initial condition.
    t0: float
        Initial time.
```

```
    tf: float
        Final time.
    method: str

    """
    N = int((tf-t0)/k)
    U = np.zeros(N+1)
    U[0]=u0
    T = np.linspace(t0, tf, N+1)

    def FE_method(f, k):
        for n in range(0, N):
            U[n+1] = U[n] + k*f(U[n], T[n])
        return U, T

    def BE_method(k):
        for n in range(N):
            a = k / T[n+1]
            U[n+1] = (1 - a - np.sqrt((1 - a)**2 - 4*a*U[n])) / (2*a)   # negative root for continuity
        return U, T

    def RK2_midpoint(f, k):
        for n in range(0, N):
            k1 = f(U[n], T[n])
            k2 = f(U[n] + k*k1/2, T[n] + k/2)
            U[n+1] = U[n] + k*k2
        return U, T

    def RK2_trapezoidal(f, k):
        for n in range(0, N):
            k1 = f(U[n], T[n])
            k2 = f(U[n]+k*k1, T[n] + k)
            U[n+1] = U[n] + k/2 * (k1+k2)
        return U, T

    def RK3(f, k):
        for n in range(0, N):
            k1 = f(U[n], T[n])
            k2 = f(U[n] + k*k1/3, T[n] + k/3)
            k3 = f(U[n] + 2*k*k2/3, T[n] + 2*k/3)
            U[n+1] = U[n] + k/4 * (k1 + 3*k3)
        return U, T

    def RK4(f, k):
        for n in range(0, N):
            k1 = f(U[n], T[n])
```

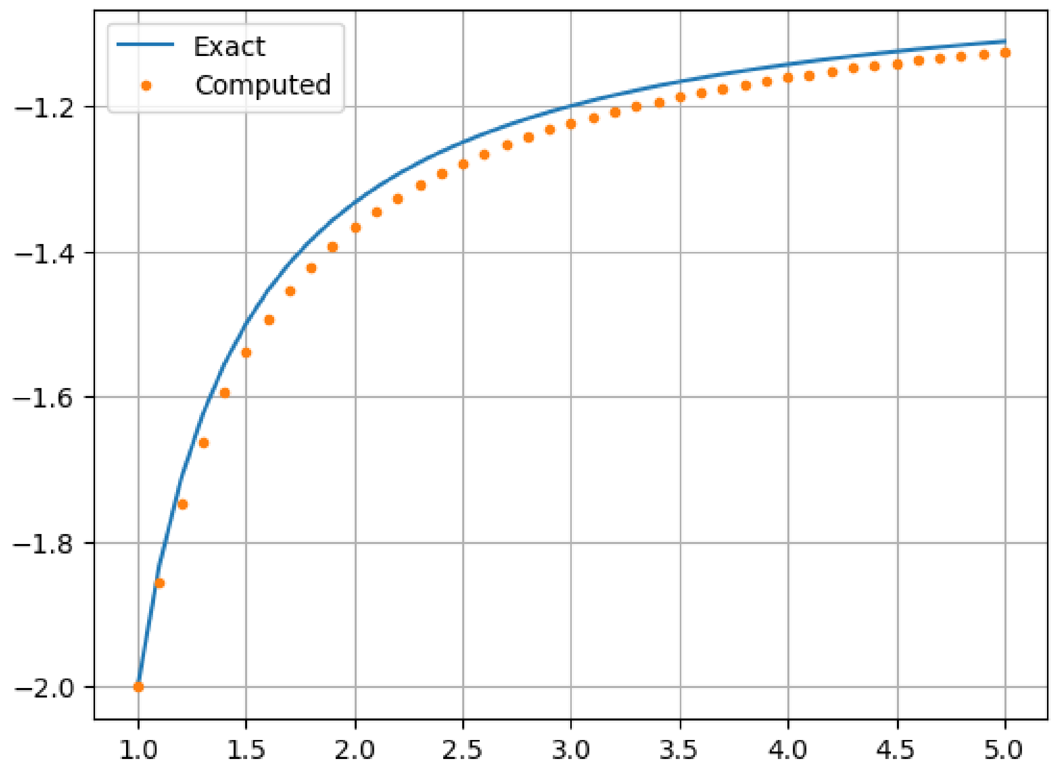
```
            k2 = f(U[n] + k*k1/2, T[n] + k/2)
            k3 = f(U[n] + k*k2/2, T[n] + k/2)
            k4 = f(U[n] + k*k3, T[n] + k)
            U[n+1] = U[n] + k/6 * (k1 + 2*k2 + 2*k3 + k4)
        return U, T


    if method=="FE":
        U_sol, T_sol = FE_method(f, k)

    if method=="BE":
        U_sol, T_sol = BE_method(k)

    if method=='Midpoint':
        U_sol, T_sol = RK2_midpoint(f, k)

    if method=="Trapezoidal":
        U_sol, T_sol = RK2_trapezoidal(f, k)

    if method=="RK3":
        U_sol, T_sol = RK3(f, k)

    if method=="RK4":
        U_sol, T_sol = RK4(f, k)

    return U_sol, T_sol
```

[90]:

```
F = lambda x, t: (x**2 + x)/t
u,t = one_step_method_solver(F, 0.1, -2, 1, 5, method="BE")
```

[91]:

```
t_points= np.linspace(1, 5, len(y))
f_sol = lambda t: 2*t/(1-2*t)

plt.figure()
plt.plot(t_points, f_sol(t_points), label='Exact')
plt.plot(t, u, '.' ,label='Computed')
plt.plot()
plt.legend()
plt.grid()
plt.show()
```



1.1.2 Part B: Convergence Analysis

Analyze the **convergence** of each method. Compare the numerical solution against the exact solution $u(t)$ at the final time $t = 5$ for a sequence of step sizes:

$$k_m = \frac{0.2}{2^m}, \quad \text{for } m = 0, 1, \ldots, 6.$$

The exact value at $t = 5$ is $u(5) = \frac{2(5)}{1-2(5)} = -\frac{10}{9}$.

Required Deliverables:

1. **Tabular Results:** For each method, present the results in a clear table format, showing k_m, the absolute error $\mathcal{E}(k_m) = |U^N(k_m) - u(5)|$, and the **estimated order of convergence** p for consecutive steps.

m	k_m	N	FE Error $\mathcal{E}$	BE Error $\mathcal{E}$	Midpoint RK2 Error $\mathcal{E}$	Trapezoidal RK2 Error $\mathcal{E}$	Heun RK3 Error $\mathcal{E}$	RK4 Error $\mathcal{E}$
0	0.2000	20						
1	0.1000	40						
2	0.0500	80						

m	k_m	N	FE Error $\mathcal{E}$	BE Error $\mathcal{E}$	Midpoint RK2 Error $\mathcal{E}$	Trapezoidal RK2 Error $\mathcal{E}$	Heun RK3 Error $\mathcal{E}$	RK4 Error $\mathcal{E}$
3	0.0250	160						
4	0.0125	320						
5	0.0063	640						
6	0.0031	1280						
Observed p								

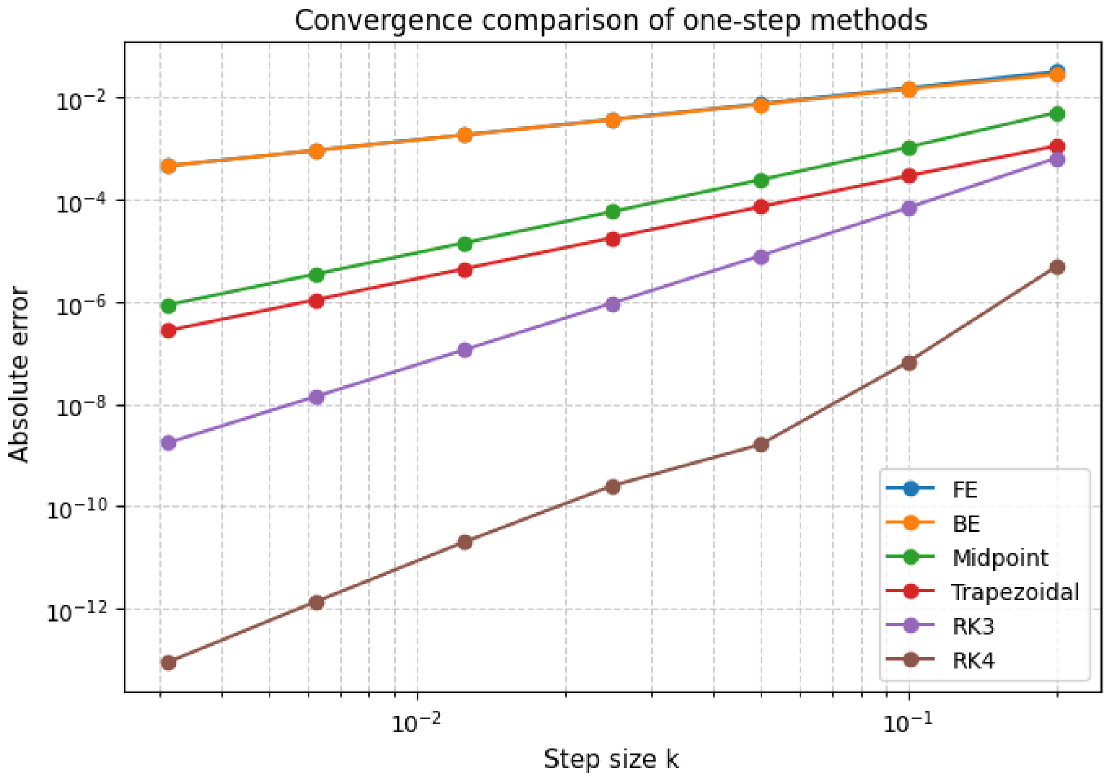
[100]:
```python
f = lambda u,t: (u*u + u)/t
t0, tf, u0 = 1.0, 5.0, -2.0
u_exact_5 = -10.0/9.0

methods = ["FE", "BE", "Midpoint", "Trapezoidal", "RK3", "RK4"]
kms = [0.2/(2**m) for m in range(7)]

rows = []
for m, k in enumerate(kms):
    N = int((tf - t0)/k)
    row = {"m": m, "k_m": k, "N": N}
    for meth in methods:
        U, T = one_step_method_solver(f, k, u0, t0, tf, meth)
        err = abs(U[-1] - u_exact_5)
        row[f"{meth} Error  "] = err
    rows.append(row)

table = pd.DataFrame(rows)

def observed_p(errs):
    p = [np.nan]
    for i in range(1, len(errs)):
        e_old, e_new = errs[i-1], errs[i]
        p.append(np.log(e_old/e_new)/np.log(2) if e_old>0 and e_new>0 else np.
↪nan)
    return p

for meth in methods:
    table[f"{meth} p"] = observed_p(table[f"{meth} Error  "].to_numpy())

table
```

[100]:
```
   m       k_m    N  FE Error    BE Error    Midpoint Error    \
0  0  0.200000   20   0.031373   0.027677      4.957013e-03
1  1  0.100000   40   0.014992   0.014144      1.048329e-03
2  2  0.050000   80   0.007367   0.007159      2.404533e-04
```

```
3  3  0.025000   160    0.003655    0.003603      5.759537e-05
4  4  0.012500   320    0.001821    0.001808      1.409647e-05
5  5  0.006250   640    0.000909    0.000905      3.487104e-06
6  6  0.003125  1280    0.000454    0.000453      8.671988e-07

   Trapezoidal Error    RK3 Error     RK4 Error        FE p      BE p  \
0         1.105883e-03  6.404530e-04  4.867114e-06       NaN       NaN
1         2.891698e-04  6.879651e-05  6.692678e-08  1.065306  0.968436
2         7.164031e-05  7.872264e-06  1.616277e-09  1.025018  0.982313
3         1.772532e-05  9.388160e-07  2.538914e-10  1.011243  0.990525
4         4.403035e-06  1.145520e-07  2.019673e-11  1.005360  0.995078
5         1.096935e-06  1.414510e-08  1.389999e-12  1.002620  0.997489
6         2.737392e-07  1.757309e-09  9.126033e-14  1.001296  0.998731

   Midpoint p  Trapezoidal p     RK3 p     RK4 p
0         NaN            NaN       NaN       NaN
1    2.241379       1.935209  3.218686  6.184339
2    2.124263       2.013074  3.127485  5.371837
3    2.061732       2.014960  3.067864  2.670391
4    2.030619       2.009241  3.034840  3.652017
5    2.015232       2.005020  3.017628  3.860966
6    2.007595       2.002605  3.008863  3.928952
```

2. **Log-Log Plot:** Generate a single log-log plot showing the absolute error $\mathcal{E}$ versus the step size k for **all six methods**.

[98]:
```python
plt.figure(figsize=(7,5))

# Cada método en la misma gráfica
for meth in methods:
    plt.loglog(
        table["k_m"],
        table[f"{meth} Error  "],
        marker="o",
        label=meth
    )

plt.xlabel("Step size k", fontsize=11)
plt.ylabel("Absolute error ", fontsize=11)
plt.title("Convergence comparison of one-step methods", fontsize=12)
plt.grid(True, which="both", ls="--", alpha=0.6)
plt.legend()
plt.tight_layout()
plt.show()
```


Convergence comparison of one-step methods
Absolute error
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-2}
10^{-1}
Step size k
FE
BE
Midpoint
Trapezoidal
RK3
RK4