

08a_tasks

November 21, 2025

```
[2]: import numpy as np
import matplotlib.pyplot as plt
```

1 Tasks

Let $f(x) = 2x \sin(2\pi x) + 3$ be a function defined on the interval $I = [0, 1]$. Consider a uniform mesh of I with $n = 5$ subintervals, which has 6 nodes $\{x_i\}_{i=0}^5$.

Find: 1. The continuous piecewise linear interpolant, $\pi_h f(x) = \sum_{i=0}^5 f(x_i) \phi_i(x)$. 2. The L_2 -projection, $P_h f(x)$, onto the same mesh.

```
[9]: def interpolant_1d(x_nodes, f_vals, x_eval):
    x_eval = np.asarray(x_eval)
    y = np.empty_like(x_eval, dtype=float)

    h = x_nodes[1] - x_nodes[0]
    n = len(x_nodes) - 1

    for k, x in enumerate(x_eval):
        if x <= x_nodes[0]:
            y[k] = f_vals[0]
            continue
        if x >= x_nodes[-1]:
            y[k] = f_vals[-1]
            continue

        # subinterval index
        i = int((x - x_nodes[0]) // h)
        if i == n: # boundary case
            i = n-1

        x_i, x_ip1 = x_nodes[i], x_nodes[i+1]

        y[k] = f_vals[i] * (x_ip1 - x)/h \
            + f_vals[i+1] * (x - x_i)/h

    return y
```

```
[11]: def l2_projector_1d(x, f_vals, f):
    """
    Compute the  $L^2$ -projection of  $f$  onto the  $P_1$  finite element space
    defined by the nodal points  $x$ .

    Parameters
    -----
    x : array_like
        Nodal points (size  $n+1$ ).
    f_vals : array_like
        Values of  $f$  at the nodes (same size as  $x$ ).
    f : callable
        Original function  $f(x)$ . Not required for the projection itself,
        but kept to match the function signature.

    Returns
    -----
    c : ndarray
        Coefficients of the  $L^2$  projection, i.e., nodal values of  $P_h f$ .
    """
    n = len(x) - 1          # number of elements
    M = np.zeros((n + 1, n + 1))  # global mass matrix
    b_vec = np.zeros(n + 1)      # global load vector

    for i in range(n):
        h = x[i+1] - x[i]

        # Local mass matrix:  $(h/6) * \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ 
        M_loc = (h / 6.0) * np.array([[2.0, 1.0],
                                       [1.0, 2.0]])

        # Assemble into global mass matrix
        M[i:i+2, i:i+2] += M_loc

        # Local load vector:  $(h/2) * [f(x_i), f(x_{i+1})]^T$ 
        f0, f1 = f_vals[i], f_vals[i+1]
        b_loc = (h / 2.0) * np.array([f0, f1])

        # Assemble into global load vector
        b_vec[i:i+2] += b_loc

    # Solve the linear system  $M c = b$ 
    c = np.linalg.solve(M, b_vec)
    return c
```

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[12]: # Input data
n = 5  # number of subintervals
```

```

a, b = 0, 1
f = lambda x: 2*x*np.sin(2*np.pi*x) + 3

# nodal points and function values
x = np.linspace(a, b, n + 1)
f_vals = f(x)

#  $L^2$  projection
Pf = l2_projector_1d(x,f_vals,f)

# linear interpolation
xe = np.linspace(a, b, 100)
If = interpolant_1d(x,f_vals,xe)

# plot  $L^2$  projection
plt.figure(figsize=(10, 6))
plt.plot(x, Pf, 'o-', label='$L_2$ Projection ($P_h f$)')
plt.plot(xe, If, '-', label='Interpolant $\pi f(x)$')
plt.plot(x, f(x), 'o', label='Data Points')

# Plot the original function for comparison
plt.plot(xe, f(xe), 'r--', label='Original f(x)')

plt.title('$L_2$ Linear Approx. vs. Linear Interpolation')
plt.xlabel('x')
plt.ylabel('f(x)')
plt.legend()
plt.grid(True)
plt.show()

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/var/folders/jq/xshncj9j3571ygk72m_rkz000000gn/T/ipykernel_9049/122025404.py:21:
SyntaxWarning: invalid escape sequence '\p'
    plt.plot(xe, If, '-', label='Interpolant $\pi f(x)$')

```

