# Design and Implementation of Efficient Resampling Filters Using Polyphase Recursive All-pass Filters

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# DESIGN AND IMPLEMENTATION OF EFFICIENT RESAMPLING FILTERS USING POLYPHASE RECURSIVE ALL-PASS FILTERS

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## 1. ABSTRACT

Computationally efficient re-sampling filters are formed from sets of recursive all-pass sub-filters operating in a polyphase structure. These sub-filters, exhibiting unity magnitude response, have phase shifts which add constructively in the passband(s) and destructively in the stop-band(s). The computational burden for these filters is one-fifth to one-tenth of conventional resampling filters.

## 2. INTRODUCTION

A finite impulse response (FIR) filter can be modeled, as indicated in Fig. 1, as the weighted summation of the contents of a tapped delay line. This summation is shown in (1) and the transfer function of this filter is shown in (2).

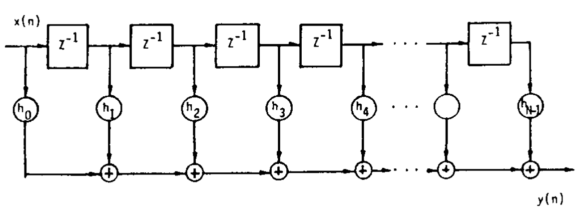


FIGURE 1. TAPPED DELAY LINE FIR FILTER

$$y(n) = \sum_{m=0}^{M-1} h(m-n)\, x(m) \qquad (1)$$

$$Y(Z) = X(Z) \sum_{n=0}^{M-1} h(n)\, Z^{-n} \qquad (2)$$

The time delays between successive samples in the tapped delay line results in a linearly increasing phase shift proportional to input frequency. The summation of these phase shifted samples is responsible for the frequency dependent gain of the filter. This (weighted) phaser summation is shown in Fig. 2 for an input signal at two different frequencies. Note that it is the phase shift, not the weighting terms, responsible for the frequency dependent gain.

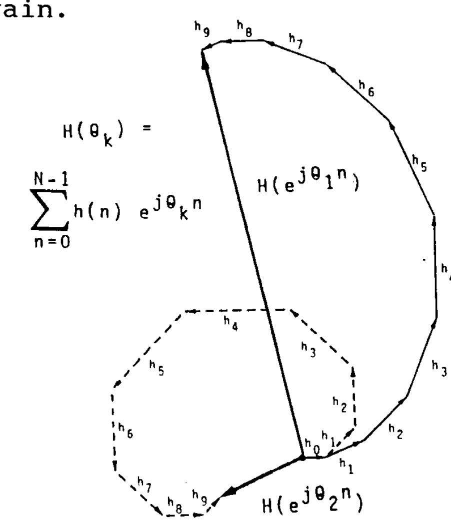


FIGURE 2. SUMMATION OF TIME DELAYED PHASORS AT TWO FREQUENCIES



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Bandwidth of a FIR filter can also be controlled by a non uniform phase shift within the weights (as commonly done in spread spectrum communications [1]). The filters described here use (unity gain) all-pass sub-filters to obtain this frequency dependent phase shift. The all-pass sub-filters are, as shown in (3) and Fig. 3, first order polynomials in $Z^{-M}$ where M is the number of taps in the structure. This phase-shift for weights filter form is indicated in Fig. 4a with a modified form reflecting a polyphase structure in 4b. The transfer function of this structure is shown in (4).

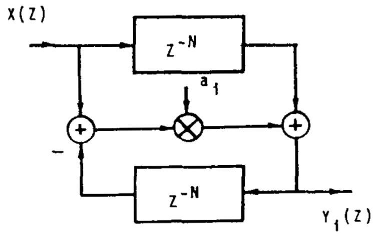


FIGURE 3. ALL-PASS SUBFILTER

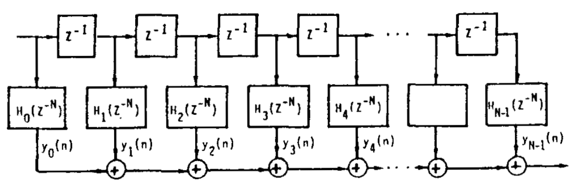

FIGURE 4a. ALL-PASS PHASE SHIFT FOR WEIGHTS FILTER STRUCTURE

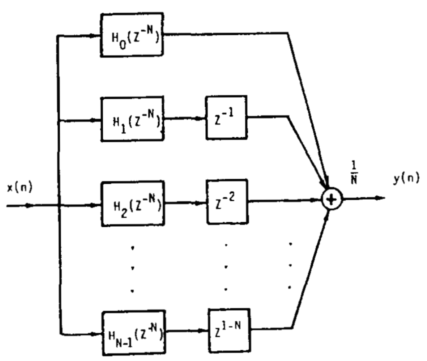


FIGURE 4b. POLYPHASE STRUCTURE

$$H_n(Z^{-M}) = \frac{a_n + Z^{-M}}{1 + a_n Z^{-M}} \qquad (3)$$

$$Y(Z) = X(Z) \sum_{n=0}^{M-1} H_n(Z^{-M}) Z^{-n} \qquad (4)$$

The roots of (3), the M roots of $-a_n$ and (reciprocals), are equally spaced about the origin as shown in Fig. 5 for the indicated degrees M. Note that we realize M poles and M zeros per multiplication in each all-pass stage. Thus we can form two poles **and** two zeros per multiplication in the sub-filters of a two-path filter rather than the one pole **(or** zero) per multiplication of a standard canonic form.

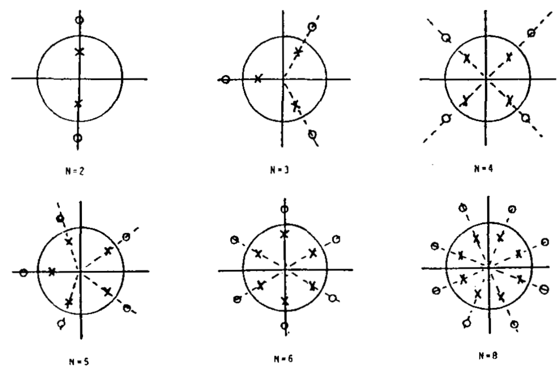

FIGURE 5. POLE-ZERO PLACEMENT FOR ALL-PASS SUBFILTER

As we traverse the unit circle we visualize that the all-pass sub-filters exhibit rapid phase change in the neighborhood of each pole-zero pair. By proper choice of the pole positions, the phase shift of each leg of the M-path filter can be made to match or to differ by multiples of $2\pi/M$ over selected spectral intervals between the pole-zero pairs. This is suggested in Fig. 6 for M=2 and M=4. The coefficients of the all-pass subfilters can be determined by standard algorithms [2] or by a new algorithm developed by harris and d'Oreye and reported in a paper recently submitted for publication (Signal Processing).

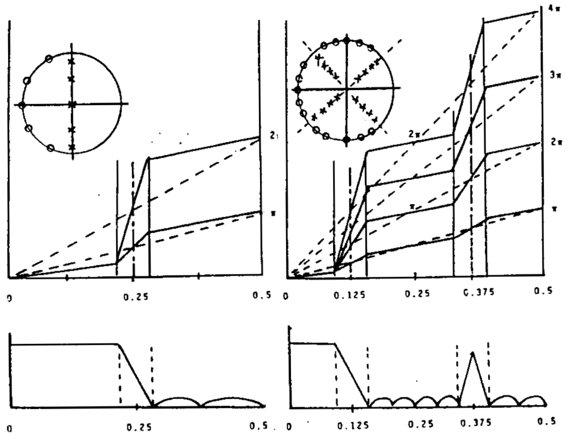

FIGURE 6. TYPICAL PHASE SHIFT VERSUS FREQUENCY FOR ALL-PASS SUB-FILTERS

Figure 7 is an example of an equal ripple design obtained from this new algorithm for a 2-path filter with 2-stages per path and for a 5-path filter with 3-stages per path. Interacting constraints (described in the aforementioned paper) limit the number of stages per path for the 2-path filter to the choices (1,0), (1,1), (2,1), (2,2), (3,2), (3,3), etc. In a similar manner, the 5-path filter choices are (1,1,1,1,0), (2,2,2, 1,1), (3,3,2,2,2), etc., so this filter example uses three less stages than allocated to the design.

### 3. TWO PATH FILTERS

The 2-path version of this filter is simple and is very interesting. This structure is a half bandwidth filter with the 3-dB bandwidth at 0.25 $f_s$. When the zeros are restricted to the half sampling frequency, the filter is the half bandwidth Butterworth

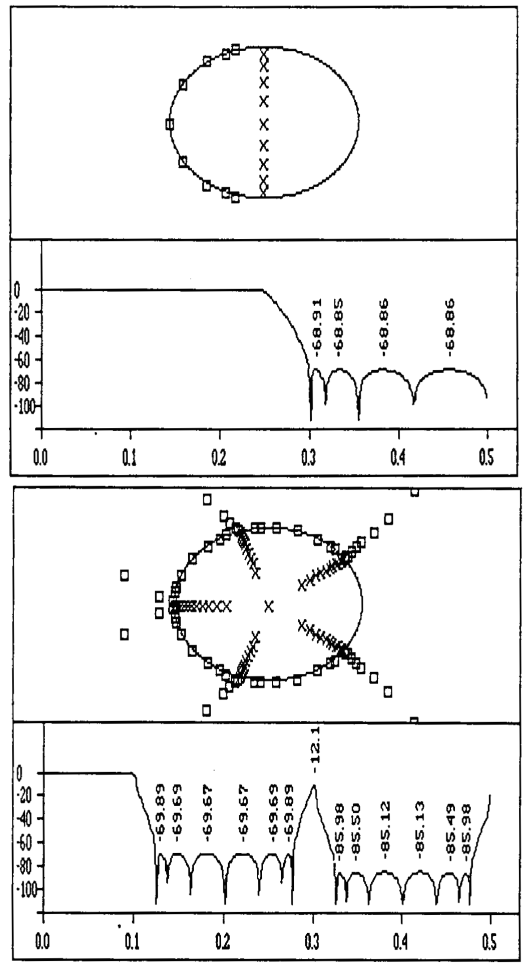

FIGURE 7. FREQUENCY RESPONSE OF 2-PATH AND 5-PATH FILTERS

obtained by the standard bilinear transformation. For this special case, the real parts of the pole locations are zero. If the zeros are distributed around the unit circle (as for equal ripple stopband behavior) the filter is a constrained Elliptic filter.

The constraint is related to a property of complementary all-pass filters. We define the all-pass sections for the 2-path filter as $A_0(\theta)$ and $A_1(\theta)$ and, as indicated in Fig. 8, the scaled (by 1/2) sum and difference of these paths by $H_A(\theta)$ and $H_B(\theta)$, lowpass and highpass filters, respectively. We know that the all-pass sections satisfy (5) from which we derive the power relationship between the lowpass and highpass filters as in (6).

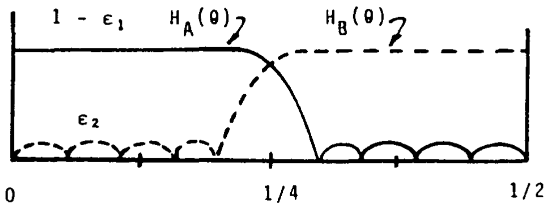


FIGURE 8. COMPLEMENTARY FILTER FREQUENCY RESPONSES

$$|A_0(\theta)|^2 = |A_2(\theta)|^2 = 1 \qquad (5)$$

$$|H_A(\theta)|^2 + |H_B(\theta)|^2 = \qquad (6a)$$

$$|0.5[A_1(\theta) + A_2(\theta)]|^2$$

$$+ |0.5[A_1(\theta) - A_2(\theta)]|^2$$

$$= 0.5\ [|A_1(\theta)|^2 + |A_1(\theta)|^2] \qquad (6b)$$

$$= 1 \qquad (6c)$$

Now to interpret how this relationship impacts the 2-path filter. For each complementary filter we define the minimum passband gain by $1-\epsilon_1$ and peak stopband gain by $\epsilon_2$. Substituting these gains in (6) we obtain (7).

$$(1-\epsilon_1)^2 + (\epsilon_2)^2 = 1 \qquad (7a)$$

$$(1-\epsilon_1)^2 = 1 - (\epsilon_2)^2 \qquad (7b)$$

$$1-2\epsilon_1 + (\epsilon_1)^2 = 1 - (\epsilon_2)^2 \qquad (7c)$$

For small $\epsilon_1$, we can ignore $(\epsilon_1)^2$ which leads to (8).

$$\epsilon_1 = 0.5\ (\epsilon_2)^2 \qquad (8)$$

Thus if the stopband attenuation is selected to be 0.001 (-60 dB), the passband ripple is 0.0000005 (-126 dB) or the passband minimum gain is 0.9999995 (-0.00000022 dB). These filters do indeed have flat passbands and cross at their 3-dB points.

There are two significant differences between the 2-path and the standard implementation of the equivalent elliptic filter. The first is a five-to-one savings in multiplications; a fifth order half bandwidth 2-path filter requires only two coefficients as opposed to the direct implementation which requires ten (scaling included). The second is that all-pass structures exhibit unity gain to all internal states hence do not require extended precision registers to store intermediate results as do cascade 2-nd order implementations.

As with any filter design, for a fixed order, the transition bandwidth can be traded for out-of-band attenuation or transition bandwidth can be reduced for a fixed attenuation by increasing filter order. While nomagraphs exist for the elliptic filters which can be implemented by the 2-path structure, a simple approximate relationship for the equal ripple filter is given in (9), where A(dB) is attenuation in dB, $\Delta f$ is transition bandwidth, and N is the total number of all-pass segments in the filter.

$$A(\mathrm{dB}) = (72\ \Delta f + 10)\ N \qquad (9)$$

**4. INTERPOLATION AND DECIMATION**

Any half bandwidth filter can be used to perform 2-to-1 resampling (up or down, usually designated interpolation and decimation). Finite impulse response (FIR) filters can be operated with a polyphase partition which avoids processing of inserted zero input points (up-sampling) or the computation of discarded output points (down-sampling). While this option is not generally available for the recursive filter, it can be applied to the recursive all-pass M-path filter. The M-path filter can be partitioned into polyphase segments because the segments are polynomials in $Z^{-M}$.

How this relationship interacts with the resampling is particularly easy to see when the 2-path filter is used for an interpolator. An input sample applied to the filter makes a contribution to the output at the same time via the upper path but not from the lower path till the next sample due to its extra $Z^{-1}$ delay. The next input sample is zero so at this time, the upper path offers no contribution to the output as there is no $Z^{-1}$ path associated with its $Z^{-2}$ polynomial while the lower path does offer a response to the previous input via its $Z^{-1}$. Thus the output response of the filter at successive indices are obtained as the response to the same input from alternate paths of the filter.

Up-sampling is performed by delivering non zero packed input data to the 2-path filter and obtaining output samples alternately, via a commutator, from the two sub-filters (in the order 0-1) at an output rate twice the input rate. Down-sampling is accomplished by reversing the flow just described. Alternate input samples are delivered to the two sub-filters via a commutator (in the order 1-0). Each sub-filter, operating at half the input rate, contribute samples to an output adder to form the down-sampled output data. The 2-path resampling structures are shown in figure 9.

The computational efficiency of

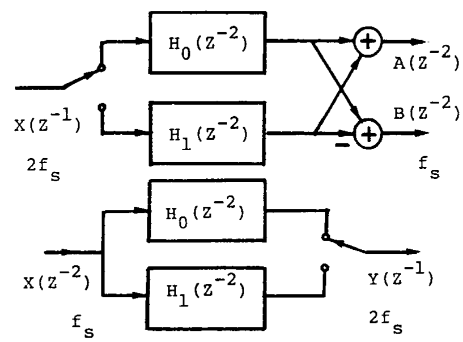


FIGURE 9. UP-SAMPLING AND DOWN-SAMPLING ALL-PASS STRUCTURES

these filters can be seen from the following example. Assume a signal, sampled at 2.56 times the input bandwidth, is to be down-sampled 8-to-1 as part of a zoom transform. (The 2.56 is the standard sampling relationship required for a 400-line spectrum analyzer with a 1024 point FFT.) Specifications for each half-band stage of this filter are seen in Table 1.

| | |
|---|---|
| Passband Ripple: | 0.05 dB |
| Stopband Attenuation: | 80.0 dB |
| Passband Frequency: | 0.1953 |
| Stopband Frequency: | 0.3047 |
| Transition Bandwidth: | 0.1094 |

**TABLE 1. 2-to-1 RESAMPLING FILTER SPECIFICATIONS**

A FIR filter of length 33 (-85 dB) can meet these specifications and when partitioned into a polyphase form, requires 16.5 multiplications per output point. A 2-path all-pass filter which meets these same specifications requires 5-stages, a (3,2) partition, (-89 dB) and thus requires 2.5 multiplications per output point. The frequency response of these filters are presented in figure 10. The 3-stage half band cascade requires seven iterations of the 2-to-1 resampling filter at successively lower sample rates. Translated to the input rate, the computational burden for the FIR filter and all-pass implementations are approximately 173 and 26 multiplications per output point respectively.

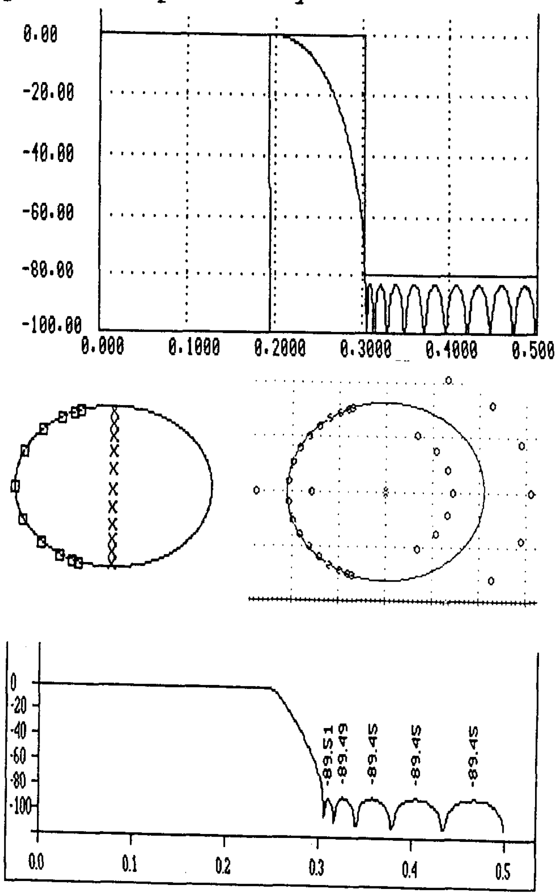


FIGURE 10. RESAMPLING HALF-BAND FILTER RESPONSES (FIR & ALL-PASS)

**5. CONCLUSIONS**

We have presented the form and usefulness of the polyphase all-pass filter as efficient resampling filters. The primary advantage of these structures is the relatively low workload per output point. A secondary advantage is the low sensitivity to finite arithmetic and the lack of bit growth in internal states. Disadvantages include the non uniform phase response of the filter, as well as newness of idea and difficulty in determining coefficients for the filter. The first objection can be addressed with equalization or with a slightly less efficient linear phase version of the all-pass structure (both of which will be addressed in a following paper). The second is being addressed by excellent survey papers [3,4] and by the increased availability of good design routines.

**ACKNOWLEDGEMENT**

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