Exercise Sheet 4

$\boxed{1}$

a) Model: $z_t = \phi_0 + \phi_1 z_{t-1} + a_t$

Lag notation: $z_t = \phi_0 + \phi_1 L z_t + a_t$

$\iff z_t - \phi_1 L z_t = \phi_0 + a_t$

b) $\underbrace{(I - \phi_1 L)}_{=:\phi(L)} z_t = \phi_0 + a_t$

$$\Rightarrow \phi(L) = \begin{pmatrix} 1-0.75L & 0 \\ -0.25L & 1-0.5L \end{pmatrix}$$

$$\Rightarrow \phi^{adj}(L) = \begin{pmatrix} 1-0.5L & 0 \\ 0.25L & 1-0.75L \end{pmatrix}$$

adjoint !

c)

$$\begin{pmatrix} 1-0.5L & 0 \\ 0.25L & 1-0.75L \end{pmatrix} \begin{pmatrix} 1-0.75L & 0 \\ -0.25L & 1-0.5L \end{pmatrix} z_t = \begin{pmatrix} 1-0.5L & 0 \\ 0.25L & 1-0.75L \end{pmatrix} \cdot \left(\phi_0 + a_t \right)$$

$$\Leftrightarrow \begin{pmatrix} (1-0.5L)(1-0.75L) & 0 \\ (0.25L)(1-0.75L)+(1-0.75L)\cdot(-0.25L) & (1-0.75L)(1-0.5L) \end{pmatrix} z_t$$

$$= \begin{pmatrix} (1-0.5L)\cdot 1 \\ 0.25L\cdot 1+(1-0.75L)\cdot 0 \end{pmatrix} + \begin{pmatrix} (1-0.5L)\cdot a_{1,t} \\ (0.25L)a_{1,t}+(1-0.75L)a_{2,t} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} z_{1,t} - 1.25 z_{1,t-1} + 0.375 z_{1,t-2} \\ z_{2,t} - 1.25 z_{2,t-1} + 0.375 z_{2,t-2} \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.25 \end{pmatrix} + \begin{pmatrix} a_{1,t} - 0.5 a_{1,t-1} \\ 0 + 0.25 a_{1,t-1} + a_{2,t} - 0.75 a_{2,t-1} \end{pmatrix}$$

d) ARMA(2,1)

e), f), g): $\longrightarrow$ R

h) $z_{1,t}$ is predicted similarly well by both models, but $z_{2,t}$ is predicted much better by the VAR(1).

Reasons: $z_{1,t}$ is a genuine AR(1) process independent of $a_{2,t}$ whereas $z_{2,t}$ depends on $a_{2,t}$ and $a_{1,t}$ through $z_{1,t}$. But a univariate model allows only to estimate the aggregated innovation sequence when the VAR estimates $k=2$ sequences.

$\to$ R

i) Option 1: $a_{1,t} \stackrel{!}{=} a_{2,t} =: \tilde{a}_t$

$$\Rightarrow \Sigma_a = \begin{pmatrix} \sigma_a^2 & \sigma_a^2 \\ \sigma_a^2 & \sigma_a^2 \end{pmatrix} = \boxed{\sigma_a^2 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$$

Option 2: Let $a_{2,t}$ dominate $a_{1,t}$ by a larger variance to marginalise $a_{1,t}$.

$$\stackrel{\text{example}}{\Rightarrow} \Sigma_a = \boxed{\begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}}$$

$\rightarrow R$

$\boxed{2}$

a) $\to$ R

The coefficients coincide pretty well.

b) Trick: $\operatorname{vec}(ABC) = (C' \otimes A)\operatorname{vec}(B)$

$$\Rightarrow X\beta = X\beta I_k \Rightarrow \operatorname{vec}(X\beta) = \operatorname{vec}(X\beta I_k)$$
$$= (I_k \otimes X)\operatorname{vec}(\beta)$$

Here: $Z = X\beta + A$, $\hat{\beta} = (X'X)^{-1}X'Z$

and β is a $(Kp+1) \times K$ matrix.

If I want to predict z_j (column 'j' in Z), I can rewrite the estimator to: $\operatorname{vec}(\hat{\beta}) = \left(I_k \otimes (X'X)^{-1}X'\right)\operatorname{vec}(Z)$

I am interested in column 'j' of Z and $\hat{\beta}_1$ that means I have to look at the 'j^{th}' row of matrices in $\left(I_k \otimes (X'X)^{-1}X'\right)$. This is done by inspecting $I_k(j,j)$. The result is just $(X'X)^{-1}X'$ since $I_k(j,l)=0$ $\forall l \neq j$.

$$\operatorname{vec}(\hat{\beta}')$$

$$= \begin{pmatrix} (X'X)^{-1}X' & & \cdots & 0_{T-p} \\ 0_{T-p} & (X'X)^{-1}X' & & \vdots \\ \vdots & \ddots & \boxed{(X'X)^{-1}X'} & \vdots \\ \vdots & & \ddots & 0_{T-p} \\ & & & (X'X)^{-1}X' \\ 0_{T-p} & \text{—} & \cdots & 0_{T-p} \end{pmatrix} \begin{pmatrix} z_{1,p+1} \\ \vdots \\ z_{1,T} \\ z_{2,p+1} \\ \vdots \\ z_{2,T} \\ \vdots \\ \boxed{\begin{matrix} z_{j,p+1} \\ \vdots \\ z_{j,T} \end{matrix}} \\ z_{j+1,p+1} \\ \vdots \\ z_{K,T} \end{pmatrix}$$

$$= \begin{pmatrix} \phi_{0.1} & \phi_{0.2} & \cdots & \phi_{0.j} & \cdots & \phi_{0.K} \\ \phi_{1.11} & \cdots & \cdots & \phi_{1.1j} & \cdots & \phi_{1.1K} \\ \vdots & & & & & \\ \phi_{1.K1} & \cdots & \cdots & \phi_{1.Kj} & \cdots & \phi_{1.KK} \\ \vdots & & & \vdots & & \\ \phi_{p.K1} & \cdots & \cdots & \phi_{p.Kj} & \cdots & \phi_{p.KK} \end{pmatrix}$$

That's why $\hat{\beta}_j = (X'X)^{-1}X'Z_j$

$\boxed{3}$

a) Random Sampling:

$$\mathcal{L} = \prod_{t=1}^{T} f(\varepsilon_t ; \mu, \sigma^2)$$

standard normal

$$= \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\left(\frac{\varepsilon_t-\mu}{\sigma}\right)^2}$$

$$\overset{\log(\cdot)}{\Longrightarrow} ll = \sum_{t=1}^{T}\left[-\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2}\left(\frac{\varepsilon_t-\mu}{\sigma}\right)^2\right]$$

$$= -\frac{T}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\cdot\sum_{t=1}^{T}(\varepsilon_t-\mu)^2$$

$\underline{\text{FOCs:}}$ $$\frac{\partial ll}{\partial \mu} = -\frac{1}{2\sigma^2}\cdot(-2)\cdot\sum_{t=1}^{T}(\varepsilon_t-\mu) \overset{!}{=} 0$$

$$\Leftrightarrow \sum_{t=1}^{T}\varepsilon_t - \sum_{t=1}^{T}\mu \overset{!}{=} 0 \Leftrightarrow \sum_{t=1}^{T}\varepsilon_t \overset{!}{=} T\cdot\mu$$

$$\Leftrightarrow \underline{\underline{\mu = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t}}$$

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{T}{2}\cdot \cancel{2\pi}\cdot\frac{1}{\cancel{2\pi}\sigma^2} - \frac{1}{2}\cdot(-1)\cdot\sum_{t=1}^{T}\frac{(\varepsilon_t-\mu)^2}{\sigma^4} \overset{!}{=} 0$$

$$\Leftrightarrow -\frac{T}{2\sigma^2} + \frac{1}{2}\frac{1}{\sigma^4}\sum_{t=1}^{T}(\varepsilon_t-\mu)^2 \overset{!}{=} 0$$

$$\Leftrightarrow \sum_{t=1}^{T}(\varepsilon_t-\mu)^2 = \frac{\cancel{2}T\sigma^4}{\cancel{2}\sigma^2}$$

$$\Leftrightarrow \sigma^2 = \frac{1}{T}\sum_{t=1}^{T}(\varepsilon_t-\mu)^2$$

b) In general, let $f(x,y,z)$ be a joint density.

$$f(x|y,z) = \underbrace{\frac{f(x,y,z)}{f(y,z)}}_{=: f_{x|y=y,z=z}(x)}\cdot f(y,z)$$

$\left(\text{in short: } f_{x|y,z}(x)\right)$

$$= f_{X|Y,Z}(x) \cdot f_{Y|Z}(y) \cdot f_{Z}(z)$$

Here:

$$f_{Z_{p+1:T}|Z_{1:p}}(z_{p+1}, \ldots, z_T) = f_{Z_T|Z_{1:T-1}}(z_T) \cdot f_{Z_{p+1:T-1}|Z_{1:p}}(z_{p+1}, \ldots, z_{T-1})$$

$$= f_{Z_T|Z_{1:T-1}}(z_T) \cdot f_{Z_{T-1}|Z_{1:T-2}}(z_{T-1}) \cdot \ldots \cdot f_{Z_{p+1}|Z_{1:p}}(z_{p+1})$$

$$= \prod_{t=p+1}^{T} f_{Z_t|Z_{1:t-1}}(z_t)$$

$$Z_{p+1} - \mathbb{E}(Z_{p+1} | Z_p, \ldots, Z_1)$$