Exercise Sheet 12

1

$$Z_{1,t} = \underbrace{\sum_{i=1}^{t} a_{3,i}}_{=: X_t \text{ (random walk)}} + a_{1,t}$$

$$Z_{2,t} = \frac{1}{2} \sum_{i=1}^{t} a_{3,i} + a_{2,t}$$

$=: X_t$ (random walk)

For cointegration, we need $Z_t \sim I(1)$.

$\Longrightarrow$ $\Delta Z_t \sim I(0)$ must be verified!

$$\Delta Z_{1,t} = \sum_{i=1}^{t} a_{3,i} - \sum_{j=1}^{t-1} a_{3,j} + a_{1,t} - a_{1,t-1}$$

$$= a_{3,t} + a_{1,t} - a_{1,t-1}$$

$$\Delta z_{2,t} = \frac{1}{2} a_{3,t} + a_{2,t} - a_{2,t-1}$$

$$\Rightarrow \Delta z_t = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a_{1,t} \\ a_{2,t} \\ a_{3,t} \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_{1,t} \\ a_{2,t} \\ a_{3,t} \end{pmatrix} + a_t$$

Which is trivially stationary.

Thus $\Delta z_t \sim I(0)$ and $z_t \sim I(1)$.

$\longrightarrow$ Cointegration rank r ?

Since $K=2$ and because z_t is $I(1)$, $r < K$ has to hold in case of cointegration, which is obviously given. $\Rightarrow r = 1$

$\rightarrow$ Cointegration vector?

$$\beta_{(1)} z_{1,t} + \beta_{(2)} z_{2,t}$$

$$= \beta_{(1)} (\boxed{x_t} + a_{1,t}) + \beta_{(2)} (\tfrac{1}{2} \boxed{x_t} + a_{2,t})$$

$$\overset{!}{=} \beta_{(1)} a_{1,t} + \beta_{(2)} a_{2,t} \sim I(0)$$

$$\Rightarrow \beta_{(1)} = -\tfrac{1}{2} \beta_{(2)}$$

Norming $\beta_{(1)} = 1$ yields $\beta_{(2)} = -2$.

$$\Rightarrow \beta = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\boxed{2}$ From Sheet 11, Ex. 3 we know there is one unit root.

But is there cointegration?

Cointegration would mean:

$$r > 0 \Rightarrow |\Pi| = |\Phi_1 - I| = 0$$

since $rk(\Pi) < K = 2$

$$\Pi = \Phi_1 - I = \begin{pmatrix} 0.1 & -0.2 \\ -0.2 & 0.4 \end{pmatrix}$$

$$|\Pi| = 0.1 \cdot 0.4 - 0.2 \cdot 0.2 = 0 \checkmark$$

It turns out that column #2

$= -2 \cdot$ column #1 and row #2

$= -2$ row #1.

$\Rightarrow \Pi = \alpha\beta' = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix} \begin{pmatrix} 1 & -2 \end{pmatrix}$

Hence $r = 1$ since α, β are $K \times r$ matrices.

$\Rightarrow$ The VAR(1) process can be written as VECM. ✓

Note that $\Gamma_i \Delta Z_{t-i}$ does not arise here, since

$\Gamma_i = -(\phi_{i+1} + \ldots + \phi_p)$ but we only have ϕ_1.

$\boxed{3}$ $\bar{r}$ denotes the maximum rank of cointegration.

a) $\bar{r} = 0$. There is no $I(1)$ process at all.

b) $\bar{r} = 0$. We need at least two $I(1)$ processes which can be made stationary by subtraction, but we only have one $I(1)$ variable.

c) $\bar{r} = 1$. Either $z_{1,t}$ and $z_{3,t}$ <u>or</u> $z_{2,t}$ and $z_{3,t}$ are cointegrated.
It cannot be both at the same time since the $z_{1,t}$ and $z_{2,t}$ would be

cointegrated as well.

d) $\bar{r} = 1$. $(z_{1,t}, z_{3,t})$ can still be cointegrated, with $z_{2,t}$ as some independent $I(1)$ process.

Note that generally $\bar{r} = K-1$.
(see Slide 7-32)

$\boxed{4}$ a) Linear dependence:
(a, b, c are scalars)

$$\beta_1 = a \cdot \beta_2 (+ b\beta_3 + c\beta_4 + \dots)$$

but $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \neq a \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

because of the zeros.

b) $a \cdot \beta_1' z_t = a(z_{1,t} - z_{2,t})$

$= a\, v_{1,t} \sim I(0)$

and $a \beta_2' z_t = a(z_{2,t} - z_{3,t})$

$= a\, v_{2,t} \sim I(0)$.

We use the result that rescaled stationary processes remain stationary.

c) $a\beta_1 + (1-a)\beta_2 \quad , a \in \mathbb{R}$

$$\left(a\beta_1 + (1-a)\beta_2\right)' z_t = \begin{pmatrix} a \\ 1-2a \\ a-1 \end{pmatrix}' z_t$$

$$= a z_{1,t} + z_{2,t} - 2a z_{2,t} + a z_{3,t} - z_{3,t}$$

$$= a \underbrace{\left(z_{1,t} - z_{2,t}\right)}_{= v_{1,t}} + \underbrace{\left(z_{1,t} - z_{3,t}\right)}_{v_{2,t}}$$

$$-\alpha \underbrace{\left(z_{2,t} - z_{3,t}\right)}_{= v_{2,t}}$$

$$= a\, v_{1,t} + (1-a) v_{2,t} \sim I(0)$$

Using the fact that (i) multiples of stationary processes are stationary and (ii) sums of stationary processes are stationary.