University of Duisburg-Essen
Faculty of Business Administration and Economics
Chair of Econometrics

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Dr. Yannick Hoga Thilo Reinschlüssel

Multivariate Time Series Analysis

Solution Exercise Sheet 9

1 Exercise 1: Granger Causality – Theory

Let $z_t = (x_t, y_t)'$ be a stationary time series with two dimensions. Define the forecast errar as the univariare series $e_T(h) = y_{T+h} - y_T(h)$ with $y_T(h) = \mathbb{E}\left(y_{T+h}|\Omega_T\right)$. The information set Ω_T contains all relevant variables available whereas $\Omega_T^{\setminus x} = \Omega_T \setminus \{x_t\}_{t=0}^T$ omits the variable x entirely. (This setting is the univariate equivalent to definition 6.1 on Slide 6-4.)

a) Prove that $\mathbb{E}\left(e_T(h)|\Omega_T^{\setminus x}\right) = 0$.

Solution:

$$z_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

$$\begin{aligned}
\mathbb{E}\left(e_T(h)|\Omega_T^{\setminus x}\right) &= \mathbb{E}\left(y_{T+h} - \mathbb{E}\left(y_{T+h}|\Omega_T\right) \mid \Omega_T^{\setminus x}\right) \\
&\overset{\text{LIE}}{=} \mathbb{E}\left(\mathbb{E}\left(y_{T+h} - y_{T+h}|\Omega_T\right) |\Omega_T^{\setminus x}\right) \\
&\text{since } \Omega_T^{\setminus x} \subseteq \Omega_T \\
&\text{LIE} = \text{Law of Iterated Expectations} \\
&= 0
\end{aligned}$$

b) Prove that $\mathrm{Var}\left(e_t(h)|\Omega_T\right) \leq \mathrm{Var}\left(e_t(h)|\Omega_T^{\setminus x}\right)$

Solution:

2 Theorems necessary for the proof:

1. Conditional Jensen's Inequality

$g(\cdot):\mathbb{R}^m \to \mathbb{R}$ is convex (like χ^2), then for any random vectors (y,x) for which $\mathbb{E}(||y||) < \infty$ and $\mathbb{E}(||g(y)||) < \infty$, $g\left(\mathbb{E}(y|x)\right) \leq \mathbb{E}\left((g(y)|x)\right)$. It is the other way around for concave functions.

2. Conditioning Theorem

If $\mathbb{E}(|y|) < \infty$, then $\mathbb{E}(g(x)y|x) = g(x)\cdot\mathbb{E}(y|x)$. If in addition $\mathbb{E}(|g(x)y|) < \infty$, then $\mathbb{E}(g(x)y) = \mathbb{E}(g(x)\mathbb{E}(y|x))$.

Back to Granger:

$e_T(h) = y_{T+h} - y_T(h)$ is a scalar. We know that $\mathbb{E}\left(e_T(h)|\Omega_T^{\setminus x}\right) = 0$, $\mathbb{E}\left(e_T(h)|\Omega_T\right) = 0$ and $\mathrm{Var}(e_T(h)) < \infty$ since y_t is a weakly stationary (w.s.) process. Furthermore, w.s. implies that $\mathbb{E}(y_t) < \infty, \mathbb{E}(y_t^2) < \infty$.

From Jensen's Inequality it follows:

$$\begin{aligned}\left[\mathbb{E}\left(y_{T+h}|\Omega_T^{\setminus x}\right)\right]^2 &\overset{\text{LIE}}{=} \left[\mathbb{E}\left[\mathbb{E}\left(y_{T+h}|\Omega_T\right)|\Omega_T^{\setminus x}\right]\right]^2 \\ &\leq \mathbb{E}\left[\left[\mathbb{E}(y_{t+h}|\Omega_T)\right]^2|\Omega_t^{\setminus x}\right]\end{aligned}$$

Taking conditional expectations:

$$\mathbb{E}\left[\left(\mathbb{E}\left[y_{T+h|\Omega_T^{\setminus x}}\right]\right)^2\right] \leq \mathbb{E}\left(\left[\mathbb{E}\left(y_{T+h}|\Omega_T\right)\right]^2\right) \tag{1.1}$$

This extends to:

$$\begin{aligned}\left[\mathbb{E}(y_{T+h})\right]^2 &\leq \mathbb{E}\left(\left[\mathbb{E}\left(y_{T+h}|\Omega_T^{\setminus x}\right)\right]^2\right) \qquad (1.2)\\ \text{Since }\ \mathbb{E}(y_{T+h}) &= \mathbb{E}\left[\mathbb{E}\left(y_{T+h}|\Omega_T^{\setminus x}\right)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left(y_{T+h}|\Omega_T\right)\right]\end{aligned}$$

the inequations (1.1) and (1.2) imply similar ranking for the variances:

$$\begin{aligned}0 \leq \mathrm{Var}\left(\mathbb{E}\left[y_{T+h}|\Omega_T^{\setminus x}\right]\right) &\leq \mathrm{Var}\left(\mathbb{E}\left[y_{T+h}|\Omega_T\right]\right) \\ \text{since}\,\mathrm{Var}(z) = \mathbb{E}(z^2) - \left[\mathbb{E}(z)\right]^2\end{aligned}$$

Consider the decomposition below:

$$y_{T+h} - \mu = \underbrace{y_{T+h} - \mathbb{E}\left(y_{T+h}|\Omega\right)}_{e_T(h)|\Omega} + \underbrace{\mathbb{E}\left(y_{T+h}|\Omega\right) - \mu}_{u_T(h)|\Omega}$$

Remember that:

$$\mathbb{E}[e_T(h)|\Omega] = 0 \quad \text{for} \quad \Omega = \left\{\Omega_T, \Omega_T^{\setminus x}\right\}$$
$$\text{and } \mathbb{E}[e_T(h) * u_T(h)] = 0 \Rightarrow \text{Cov}(e_T(h), u_T(h)) = 0$$

Thus:

$$\begin{aligned} \text{Var}\left(y_{T+h} - \mu|\Omega\right) &= \text{Var}\left(e_T(h) + u_T(h)|\Omega\right) \\ &= \text{Var}\left(e_T(h)|\Omega\right) + \text{Var}\left(u_T(h)|\Omega\right) \end{aligned}$$

Since μ is a constant and y_{T+h} does not depend on Ω:

$$\begin{aligned} \text{Var}\left(y_{T+h} - \mu|\Omega\right) &= \text{Var}\left(y_{T+h}\right) \\ \text{Var}\left(u_T(h)|\Omega\right) &= \text{Var}\left(\mathbb{E}[y_{T+h}|\Omega]\right) \\ \text{Var}\left(y_{T+h}\right) &= \text{Var}\left(e_T h|\Omega\right) + \text{Var}\left(\mathbb{E}[y_{T+h}|\Omega]\right) \end{aligned}$$

We have already shown that

$$\text{Var}\left(\mathbb{E}(y_{T+h}|\Omega_T)\right) \geq \text{Var}\left(\mathbb{E}\left(y_{T+h}|\Omega_T^{\setminus x}\right)\right)$$

and we know that $\text{Var}(y_{T+h}) = \sigma^2$ is constant. This implies:

$$\text{Var}\left(e_T(h)|\Omega_T\right) \leq \text{Var}\left(e_T(h)|\Omega_T^{\setminus x}\right) \qquad \square$$

2 Exercise 2: Granger Causality and IRFs in Data

We return to the dataset `fx_series.Rda` and examine Granger (Non)-Causality and the Impulse Response Functions (IRFs). Remember that this dataset contains two time series of exchange rates.

a) Do you find any Granger Causality in a VAR(1) model? Which zero restrictions are implied for the coefficient matrix ϕ_1?

Solution:

```r
# VAR(1) with intercept, no trends
var1.fit <- VAR(fx_series, p = 1, type = "const")
# there is conditional heteroscedasticity in the data,
#that's why employing a HC VCOV matric helps
causality(x = var1.fit, cause = "lr.Eu", vcov. = vcovHC(var1.fit))
```

```
## $Granger
##
##  Granger causality H0: lr.Eu do not Granger-cause lr.Ja
##
## data:  VAR object var1.fit
## F-Test = 0.00011986, df1 = 1, df2 = 10034, p-value = 0.9913
##
##
## $Instant
##
##  H0: No instantaneous causality between: lr.Eu and lr.Ja
##
## data:  VAR object var1.fit
## Chi-squared = 370.25, df = 1, p-value < 2.2e-16
```

```r
causality(x = var1.fit, cause = "lr.Ja", vcov. = vcovHC(var1.fit))
```

```
## $Granger
##
##  Granger causality H0: lr.Ja do not Granger-cause lr.Eu
##
## data:  VAR object var1.fit
## F-Test = 2.9687, df1 = 1, df2 = 10034, p-value = 0.08492
##
##
## $Instant
##
```

```
##  H0: No instantaneous causality between: lr.Ja and lr.Eu
##
## data:  VAR object var1.fit
## Chi-squared = 370.25, df = 1, p-value < 2.2e-16
```

The H_0 of Granger *Non*-Causality is never rejected.

$$z_t = \phi_0 + \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} z_{t-1} + a_t$$

b) Is there evidence for instantaneous causality? Whar are the implications regarding Σ_a?

Solution:

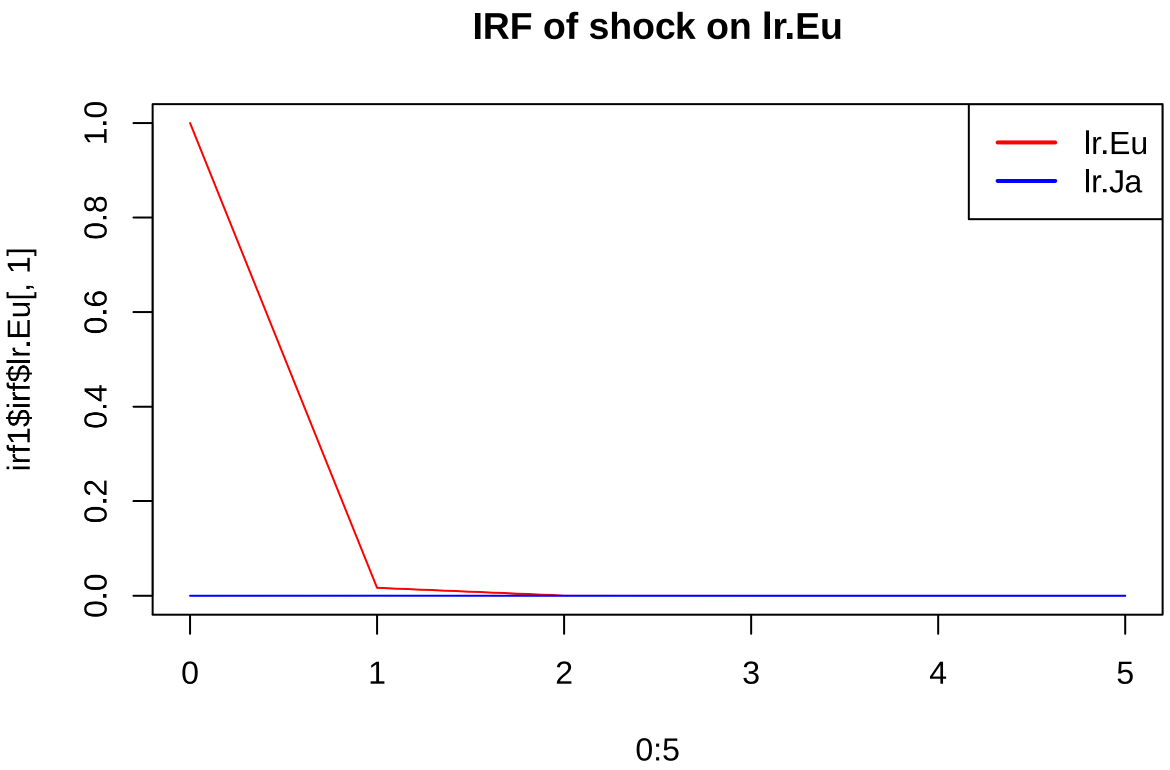
Yes, the H_0 is rejected (in the test from a)).There is evidence for instantaneous Causality meaning that Σ_a has non-zero entries of the main diagonal.

$$\Sigma_a = \begin{pmatrix} * & \sigma_{12} \\ \sigma_{12} & * \end{pmatrix}, \quad \sigma_{12} \neq 0$$

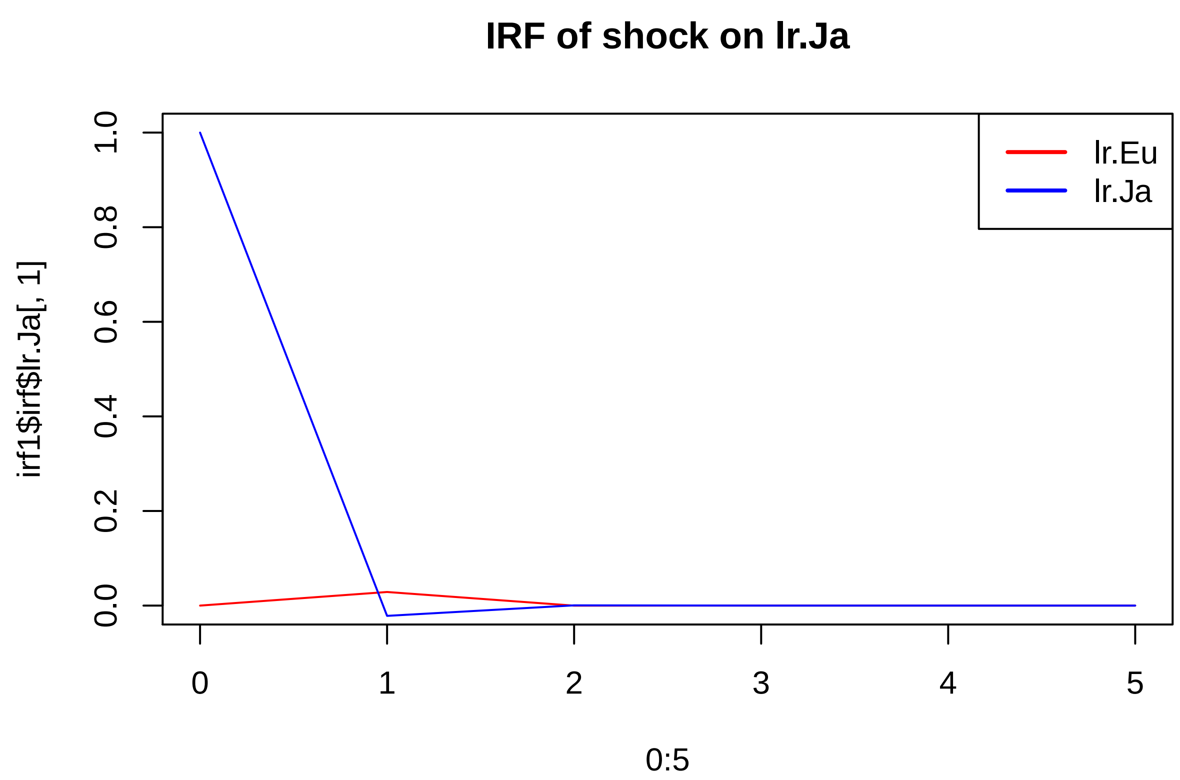
c) Before you plot the IRFs, make a guess about their appearance based on Granger Causality. Then compute the IRFs (do *not* use orthogonal innovations!) for five periods and comment.

Solution:

```
irf1 <- irf(x = var1.fit, ortho = FALSE, n.ahead = 5)
# plot IRFs of shock on first exchange rate
plot(x = 0:5, irf1$irf$lr.Eu[,1], ylim = c(0,1), type = "l", col = "red"
     , main = "IRF of shock on lr.Eu")
points(x = 0:5, y = irf1$irf$lr.Eu[,2], type = "l", col = "blue")
legend("topright", legend = c("lr.Eu","lr.Ja"), lwd = c(2,2), col = c("red", "blue"))
```




```r
# plot IRFs of shock on second exchange rate
plot(x = 0:5, irf1$irf$lr.Ja[,1], ylim = c(0,1), type = "l", col = "red"
     , main = "IRF of shock on lr.Ja")
points(x = 0:5, y = irf1$irf$lr.Ja[,2], type = "l", col = "blue")
legend("topright", legend = c("lr.Eu","lr.Ja"), lwd = c(2,2), col = c("red", "blue"))
```




```
irf1$irf$lr.Eu
```

```
##              lr.Eu         lr.Ja
## [1,] 1.000000e+00  0.000000e+00
## [2,] 1.659978e-02  2.009948e-04
## [3,] 2.813309e-04 -1.032811e-06
## [4,] 4.640339e-06  7.899758e-08
## [5,] 7.929965e-08 -7.845867e-10
## [6,] 1.293801e-09  3.299438e-11
```

```
irf1$irf$lr.Ja
```

```
##               lr.Eu         lr.Ja
## [1,]  0.000000e+00  1.000000e+00
## [2,]  2.874839e-02 -2.173827e-02
## [3,] -1.477234e-04  4.783307e-04
## [4,]  1.129906e-05 -1.042778e-05
## [5,] -1.122198e-07  2.289529e-07
## [6,]  4.719202e-09 -4.999595e-09
```

No evidence for Granger Causality between $z_{1,t}$ and $z_{2,t}$

$\Rightarrow a_{1,t}$ barely influences $z_{2,t}$ if we control for $a_{2,t}$ and vice versa.

As expected, unit impulses on either $a_{1,t}/a_{2,t}$ did not affect $z_{2,t}/z_{1,t}$ by much. The impulse vanishes quickly.

3 Exercise 3: Granger Non-Causality and IRFs

Consider a general three-dimensional VAR(1) in which the first variable $z_{1,t}$ does not Granger cause the other variables. Show that a shock $a_{1,T}$ does not affect $\{z_{2,t}\}_{t=T}^{\infty}$ and $\{z_{3,t}\}_{t=T}^{\infty}$.

Solution:

$$z_t = \phi_1 z_{t-1} + a_t$$

$$\{z_{1,t}\} \not\rightarrow \{z_{2,\cdot}, z_{3,\cdot}\}$$

$$\Rightarrow \phi_1 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix}$$

Take θ_S from the causal representation:

$$\frac{\partial z_{t,s}}{\partial a_t} = \theta_s$$
$$z_{t+s} = \sum_{i=0}^{t+s-1} \theta_i a_{t+s-i}$$

Special case for VAR(1):

$\theta_s = \phi_1^2$ and $\theta_0 = I_{3\times 3}$ which fulfills the restrictions trivially as θ_1 does.

$$\theta_2 = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix}$$
$$= \begin{pmatrix} * & * & * \\ 0 \cdot a + d \cdot 0 + e \cdot 0 & * & * \\ 0 \cdot a + f \cdot 0 + g \cdot 0 & * & * \end{pmatrix}$$

$$= \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

$\Rightarrow \theta_i = \phi_1 \cdot \theta_{i-1}$ does always fulfill the restirctions for $i \geq 0$. Therefore $a_{1,t}$ does never influences $z_{2,\cdot}$ or $z_{3,\cdot}$. $\square$