Exercise Sheet 2

$\boxed{1}$ a) $Z_t = \phi_1 Z_{t-1} + a_t$

$$\Gamma_0 = \phi_1 \Gamma_0 \phi_1' + \Sigma_a$$

$$\Leftrightarrow vec(\Gamma_0) = (\phi_1 \otimes \phi_1) \cdot vec(\Gamma_0) + vec(\Sigma_a)$$

$$\Leftrightarrow vec(\Gamma_0) = (I_{k^2} - \phi_1 \otimes \phi_1) vec(\Sigma_a)$$

$$\Gamma_1 = \phi_1 \Gamma_0$$

$$\Rightarrow \Gamma_l = \phi_1 \Gamma_{l-1} = \phi_1^l \Gamma_0$$

$\rightarrow R$

b)

$$\rho_l = \begin{pmatrix} \frac{Cov(x_t, x_{t-l})}{\sqrt{Var(x_t) \cdot Var(x_{t-l})}} & \frac{Cov(x_t, y_{t-l})}{\sqrt{Var(x_t) \cdot Var(y_{t-l})}} \\ \frac{Cov(y_t, x_{t-l})}{\sqrt{Var(y_t) \cdot Var(x_{t-l})}} & \frac{Cov(y_t, y_{t-l})}{\sqrt{Var(y_t) \cdot Var(y_{t-l})}} \end{pmatrix}$$

$\rightarrow R$

c) $z_t = \phi_1 z_{t-1} + a_t$

1. Set T
2. Draw $\{a_1, \dots, a_T\} \sim [\mu, \Sigma_a]$
3. Set $z_0 = \mathbb{E}(z_t)$
4. $z_1 = \phi_1 z_0 + a_1$
5. Repeat Step 4 $(T-1)$ times

(6. Discard first few observations to minimise effects of z_0 on the results.)

$\rightarrow R$

d) At least it looks stable hence we cannot rule out stationarity.

$\rightarrow R$

e) $\hat{\Gamma}_l = \tilde{Z}_T' \tilde{Z}_{T-l} \cdot (T-1)^{-1}$

$\tilde{Z}_T = Z_T - \hat{\mu}_2$

Z is a $T \times 2$ matrix!

z_t is 2×1 vector!

$$z_t := \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$ ← variable

$$Z_t = \begin{pmatrix} x_t & y_t \\ x_{t-1} & y_{t-1} \\ x_{t-2} & y_{t-2} \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix}$$

sample (data)

$$\overbrace{\operatorname{Cov}(x_t, y_{t-1})} = \frac{1}{(T-1)} \sum_{t=1}^{T} \left(\tilde{x}_t \cdot \tilde{y}_{t-1}\right)$$

is part of: $\frac{1}{T-1} \cdot \tilde{z}_t \tilde{z}_{t-1} = \hat{\rho}_1$

$\boxed{2}$ a) $z_t = \phi_1 z_{t-1} + a_t$

$$a_t \overset{iid}{\sim} [\mu_a, \Sigma_a]$$

iid: $\operatorname{Cov}(a_t, a_{t-1}) = 0$

$$\Sigma_a = \begin{pmatrix} Var(a_{1t}) & Cov(a_{1t}, a_{2t}) \\ Cov(a_{2t}, a_{1t}) & Var(a_{2t}) \end{pmatrix}$$

Generally not, since iid errors induce no dynamic structure. Still, finite 1^{st} and 2^{nd} moments are required for weak stationarity!

(Gaussian innovations fulfill that condition, of course)

b) $Z_t = \phi_1 Z_{t-1} + a_t$

$= \phi_1 \cdot (\phi_1 Z_{t-2} + a_{t-1}) + a_t$

$$= \underbrace{\phi_1^p \cdot Z_{t-p}}_{\text{stable?}} + \underbrace{\sum_{i=0}^{p} \phi_1^i a_{t-i}}_{\text{summable?}}$$

$$\lim_{p \to \infty} \phi_1^p \overset{?}{\longrightarrow} 0_{k \times k}$$

$\Longrightarrow$ eigenvalues!

$$\Longrightarrow \underbrace{\phi_1 x}_{\text{vector}} = \underbrace{\lambda x}_{\text{vector}} \Longrightarrow \phi_1^p \cdot x = \lambda^p \cdot x$$

$$\Rightarrow \text{solve: } \underbrace{(\phi_1 - I_k \lambda)}\, x = 0$$

$$\text{for } x \neq 0: \quad |\phi_1 - I_k \lambda| \overset{!}{=} 0$$

stability (for stationarity): $|\lambda_1|, \dots, |\lambda_k| < 1$

i)
$$\left| \begin{pmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right|$$

$$= \begin{vmatrix} (0.2 - \lambda) & 0.3 \\ -0.6 & (1.1 - \lambda) \end{vmatrix}$$

$$= (0.2-\lambda)(1.1-\lambda)$$
$$- (-0.6)\cdot 0.3$$
$$= \lambda^2 - 1.3\lambda + 0.4 \overset{!}{=} 0$$

$$\overset{p-q}{\Longrightarrow} \quad \lambda_{1,2} = -\left(\frac{-1.3}{2}\right) \pm \sqrt{\left(\frac{-1.3}{2}\right)^2 - 0.4}$$
formula

$$= 0.65 \pm 0.15$$
$$= \{0.8, 0.5\}$$

$$|\lambda_1| < 1, |\lambda_2| < 1$$
$$\Rightarrow \text{stationary} \checkmark$$

ii) $\begin{vmatrix} 0.5-\lambda & 0.3 \\ 0 & -0.3-\lambda \end{vmatrix}$

$= (0.5-\lambda)(-0.3-\lambda) \overset{!}{=} 0$

$\Rightarrow \lambda_1 = 0.5, \ \lambda_2 = -0.3$

$\Rightarrow$ stationary ✓

iii) $\lambda_{1,2} = 1$ ↯

not stationary

iv) $\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix}$

$= (1-\lambda)(-1-\lambda) - (-1)\cdot 1$

$= \lambda^2 + \lambda - \lambda - 1 + 1$

$= \lambda^2 \overset{!}{=} 0$

$\lambda_{1,2} = 0 \Rightarrow$ stationary

v) $\begin{vmatrix} 1-\lambda & -0.5 \\ -0.5 & 0-\lambda \end{vmatrix}$

$= (1-\lambda)\cdot(-\lambda) - 0.5\cdot 0.5$

$$= \lambda^2 - \lambda - 0.25 \stackrel{!}{=} 0$$

$$\Rightarrow \lambda_1 = -0.207$$

$$\lambda_2 = 1.207 \quad \lightning$$

not stationary