Exercise Sheet 9

$\boxed{1}$

$$Z_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}$$

a) $$\mathbb{E}\left(e_T(h) \,\middle|\, \Omega_T^{\backslash x}\right)$$

$$= \mathbb{E}\left(Y_{T+h} - \mathbb{E}(Y_{T+h} \mid \Omega_T) \,\middle|\, \Omega_T^{\backslash x}\right)$$

$$\overset{LIE}{=} \mathbb{E}\left(\mathbb{E}\left(Y_{T+h} - Y_{T+h} \mid \Omega_T\right) \,\middle|\, \Omega_T^{\backslash x}\right)$$

(since $\Omega_T^{\backslash x} \subseteq \Omega_T$

LIE = Law of iterated expectations)

$$= 0$$

b) 2 Theorems necessary for the proof:

1. Conditional Jensen's Inequality

$g(\cdot): \mathbb{R}^m \longrightarrow \mathbb{R}$ is convex (like x^2),

then for any random vectors (Y, X)

for which $\mathbb{E}(\|Y\|) < \infty$ and

$\mathbb{E}(\|g(Y)\|) < \infty$,

$$g(\mathbb{E}(Y|X)) \leq \mathbb{E}(g(Y)|X).$$

It is the other way around for

concave functions.

2. Conditioning Theorem

If $\mathbb{E}(|Y|) < \infty$, then scalar!

$$\mathbb{E}(g(x)\, y \mid x) = g(x) \cdot \mathbb{E}(y \mid x).$$

If in addition $\mathbb{E}(|g(x)\, y|) < \infty$, then

$$\mathbb{E}(g(x)\, y) = \mathbb{E}(g(x)\, \mathbb{E}(y \mid x)).$$

Back to Granger:

$e_T(h) = Y_{T+h} - Y_T(h)$ is scalar

We know that $\mathbb{E}(e_T(h) \mid \Omega_T^{u}) = 0$,

$\mathbb{E}(e_T(h) \mid \Omega_T) = 0$ and

$Var(e_T(h)) < \infty$ since Y_t is a weakly stationary (w.s.) process. Furthermore, w.s. implies that $\mathbb{E}(Y_t) < \infty$, $\mathbb{E}(Y_t^2) < \infty$.

From Jensen's Inequality it follows:

$$\left[\mathbb{E}(Y_{T+h} \mid \Omega_T^{1x})\right]^2 \overset{LIE}{=} \left[\mathbb{E}(\mathbb{E}(Y_{T+h} \mid \Omega_T) \mid \Omega_T^{1x})\right]^2$$

$$\leq \mathbb{E}\left(\left[\mathbb{E}(Y_{T+h} \mid \Omega_T)\right]^2 \mid \Omega_T^{1x}\right)$$

Taking unconditional expectations:

$$\mathbb{E}\left(\left[\mathbb{E}(Y_{T+h} \mid \Omega_T^{1x})\right]^2\right)$$

$$\leq \mathbb{E}\left(\left[\mathbb{E}(Y_{T+h} \mid \Omega_T)\right]^2\right) \qquad (\mathrm{I})$$

This extends to: $\quad (\mathrm{II})$

$$\left[\mathbb{E}(Y_{T+h})\right]^2 \leq \mathbb{E}\left(\left[\mathbb{E}(Y_{T+h} \mid \Omega_T^{1*})\right]^2\right)$$

Since $\mathbb{E}(Y_{T+h}) = \mathbb{E}(\mathbb{E}(Y_{T+h} \mid \Omega_T^{1*}))$
$$= \mathbb{E}(\mathbb{E}(Y_{T+h} \mid \Omega_T))$$

the inequalities (I) and (II) imply a similar ranking for the variances:

$$0 \leq \mathrm{Var}(\mathbb{E}(Y_{T+h} \mid \Omega_T^{1*})) \leq \mathrm{Var}(\mathbb{E}(Y_{T+h} \mid \Omega_T))$$

(Since $\mathrm{Var}(z) = \mathbb{E}(z^2) - [\mathbb{E}(z)]^2$

Consider the decomposition below:

$$Y_{T+h} - \mu = \underbrace{Y_{T+h} - \mathbb{E}(Y_{T+h} \mid \Omega)}_{= e_T(h) \mid \Omega} + \underbrace{\mathbb{E}(Y_{T+h} \mid \Omega) - \mu}_{= u_T(h) \mid \Omega}$$

some information set

Remember that $\mathbb{E}(e_T(h) \mid \Omega) = 0$ for $\Omega = \{\Omega_T, \Omega_{T-1}, \ldots\}$

and $\mathbb{E}(e_T(h) \cdot u_T(h)) = 0 \implies Cov(e_T(h), u_T(h)) = 0$.

Thus: $$Var(Y_{T+h} - \mu_Y \mid \Omega) = Var(e_T(h) + u_T(h) \mid \Omega)$$

$$= \underline{Var(e_T(h) \mid \Omega)} + Var(u_T(h) \mid \Omega)$$

Since μ is a constant and Y_{T+h} does not depend on Ω:

$$Var(Y_{T+h} - \mu | \Omega) = Var(Y_{T+h}) \, , \quad Var(u_T(h)|\Omega) = Var(\mathbb{E}(Y_{T+h}|\Omega))$$

$$\underline{Var(Y_{T+h}) = Var(e_T(h)|\Omega) + Var(\mathbb{E}(Y_{T+h}|\Omega))}$$

We have already shown that

$$Var(\mathbb{E}(Y_{T+h}|\Omega_T)) \geq Var(\mathbb{E}(Y_{T+h}|\Omega_T^{1x}))$$

and we know that $Var(Y_{T+h}) = \sigma^2$ is constant. This implies: $\underline{Var(e_T(h)|\Omega_T) \leq Var(e_T(h)|\Omega_T^{1x})}$ $\blacksquare$

2 a) Using $\alpha = 5\%$.

$\rightarrow R$

The H_0 of Granger Non-Causality is never rejected.

$$z_t = \phi_0 + \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} z_{t-1} + a_t$$

b) Yes, the H_0 is rejected. Then is evidence for Instantaneous Causality meaning that Σ_a has non-zero entries off the main diagonal.

$$\Sigma_a = \begin{pmatrix} * & \sigma_{12} \\ \sigma_{12} & * \end{pmatrix}, \quad \sigma_{12} \neq 0$$

c) No evidence for Granger Causality between $z_{1,t}$ and $z_{2,t}$

$\Rightarrow$ $a_{1,t}$ barely influences $z_{2,t}$ if we control for $a_{2,t}$ and vice versa

$\rightarrow$ R

As expected, unit impulses on either $a_{1,t}$ / $a_{2,t}$ did not affect $z_{2,t}$ / $z_{1,t}$ by much. The impulse vanishes quickly.

$\boxed{3}$ $z_t = \phi_1 z_{t-1} + a_t$

$\{z_{1,t}\} \not\rightarrow \{z_{2,\cdot}, z_{3,\cdot}\}$

$$\Rightarrow \phi_1 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix}$$

Take Θ_s from the causal representation:

$$\frac{\partial z_{t+s}}{\partial a_t} = \Theta_s, \quad z_{t+s} = \sum_{i=0}^{t+s-1} \Theta_i \, a_{t+s-i}$$

Special case for VAR(1):

$\Theta_s = \phi_1^s$ and $\Theta_0 = I_{3\times 3}$ which fulfills the restrictions trivially as Θ_1 does.

$$\Theta_2 = \phi_1^2 = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix}$$

$$= \begin{pmatrix} * & * & * \\ \boxed{0 \cdot a + d \cdot 0 + e \cdot 0} & * & * \\ \boxed{0 \cdot a + f \cdot 0 + g \cdot 0} & * & * \end{pmatrix}$$

$$= \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

$\Rightarrow$ $\Theta_i = \phi_1 \cdot \Theta_{i-1}$ does always fulfill the restrictions for $i \geq 0$. Therefore a_{1t} does never influence $z_{2,\cdot}$ or $z_{3,\cdot}$.

$\blacksquare$