

Winter Term 2019/2020

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Multivariate Time Series Analysis

Solution Exercise Sheet 3

1 Exercise 1: VAR(1) Moments and Stationarity

Take the VAR(1) model $z_t = \phi_0 + \phi_1 z_{t-1} + a_t$ with the following parameterisation:

$$\phi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 0.75 & 0 \\ -0.25 & 0.5 \end{pmatrix}, \quad \Sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

a) Compute the mean of the process.

Solution:

$$\begin{aligned} E(\cdot) \quad \underbrace{\mathbb{E}(z_t)}_{\mu} &= \mathbb{E} \left(\underbrace{\phi_0}_{\phi_0} + \underbrace{\phi_1 z_{t-1}}_{\phi_1 \cdot \mu} + \underbrace{a_t}_0 \right) \end{aligned}$$

[Key assumption? $\mathbb{E}(z_t) = \mathbb{E}(z_{t-1})$]

$$\Leftrightarrow (I - \phi_1) \cdot \mu = \phi_0$$

$$\Leftrightarrow \mu = (I - \phi_1)^{-1} \cdot \phi_0$$

plugging in ϕ_1 and ϕ_0 :

$$\begin{aligned} \mu &= \begin{pmatrix} 0.25 & 0 \\ 0.25 & 0.5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} \end{aligned}$$

b) Show that the process is stationary.

Solution:

Eigenvalues of ϕ_1 :

$$\begin{aligned}
 |\phi_1 - I\lambda| &\stackrel{!}{=} 0 \\
 &= \begin{vmatrix} 0.75 - \lambda & 0 \\ -0.25 & 0.5 - \lambda \end{vmatrix} \\
 &= (0.75 - \lambda)(0.5 - \lambda) \stackrel{!}{=} 0 \\
 \Rightarrow \lambda_1 &= 0.75 \\
 \lambda_2 &= 0.5
 \end{aligned}$$

Both Eigenvalues are inside the unit circle therefore the process is stationary.

- c) Derive the Yule-Walker equations for the lag $l = \{0, 1, 2\}$ and show that the solution for Γ_0 coincides with equation (2.3) on slide 2-15.

$$\begin{aligned}
 \tilde{z}_t &:= z_t - \mu \\
 \Rightarrow \tilde{z}_t &= \phi_1 \tilde{z}_{t-1} + a_t & | \cdot \tilde{z}'_{t-1} \\
 \Leftrightarrow \tilde{z}_t \tilde{z}'_t &= \phi_1 \tilde{z}_t \tilde{z}'_t + a_t \tilde{z}'_t \\
 \stackrel{\mathbb{E}(\cdot)}{\Rightarrow} \Gamma_l &= \phi_1 \cdot \Gamma_{l-1} + \begin{cases} l = 0 : \Sigma_a \\ l \neq 0 : 0_K \end{cases} \\
 l = 0 : \Gamma_0 &= \phi_1 \cdot \Gamma_{-1} + \Sigma_a \\
 l = 1 : \Gamma_1 &= \phi_1 \cdot \Gamma_0 + 0_K \\
 l = 2 : \Gamma_2 &= \phi_1 \cdot \Gamma_1 + 0_K \\
 \text{using } \Gamma_{-1} &= \Gamma'_1 \\
 \Gamma_0 &= \phi_1 \cdot (\phi_1 \Gamma_0)' + \Sigma_a \\
 \Leftrightarrow \Gamma_0 &= \phi_1 \Gamma'_0 \phi'_1 + \Sigma_a \\
 \text{and } \Gamma'_0 &= \Gamma_0 \text{ since } \Gamma_0 \text{ is symmetric !}
 \end{aligned}$$

- d) Compute Γ_0 and Γ_1 by hand based on your results from c).

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} 0.75 & 0 \\ -0.25 & 0.5 \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix} \begin{pmatrix} 0.75 & -0.25 \\ 0 & 0.5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} (0.75)^2 \gamma_{11} & (-0.75 \cdot 0.25 \gamma_{11}) + 0.5 \cdot 0.75 \gamma_{12} \\ (0.75 \cdot 0.25 \gamma_{11}) + 0.5 \cdot 0.75 \gamma_{12} & (-0.25)^2 \gamma_{11} - 0.25 \cdot 0.5 \gamma_{12} - 0.5 \cdot 0.25 \gamma_{12} + (0.5)^2 \gamma_{22} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \gamma_{11} = (0.75)^2 \gamma_{11} + 1 \\
&\Leftrightarrow \gamma_{11} = \frac{16}{7} \\
&\Rightarrow \gamma_{12} = \frac{1}{4} \cdot \left(-\frac{3}{4}\right) \cdot \frac{16}{7} + \frac{2}{4} \cdot \frac{3}{4} \gamma_{12} \\
&\Leftrightarrow \gamma_{12} = -\frac{24}{35} \\
&\Rightarrow \gamma_{22} = \frac{1}{16} \gamma_{11} - 2 \cdot \frac{1}{2} \cdot \frac{1}{4} \gamma_{12} + \frac{1}{4} \gamma_{22} + 1 \\
&\quad = \frac{1}{16} \cdot \frac{16}{7} - 2 \cdot \frac{1}{2} \cdot \frac{1}{4} \left(-\frac{24}{35}\right) + \frac{1}{4} \gamma_{22} + 1 \\
&\quad \gamma_{22} = \frac{184}{105}
\end{aligned}$$

$$\begin{aligned}
\Gamma_1 &= \phi_1 \cdot \Gamma_0 \\
&= \begin{pmatrix} 0.75 & 0 \\ -0.25 & 0.5 \end{pmatrix} \cdot \begin{pmatrix} \frac{16}{7} & -\frac{24}{35} \\ -\frac{24}{35} & \frac{184}{105} \end{pmatrix}
\end{aligned}$$

2 Exercise 2: Stationarity of VAR(p) Processes

Using the notation of Slide 2-27, prove that $|I_{kp} - \Phi_1 z| = |I_k - \phi_1 z - \dots - \phi_p z^p|$. Recall that $|A|$ denotes the determinant of the matrix A

Hint: Derive Φ_1 and keep it mind that adding multiplies of columns/rows to other columns/rows does not affect the determinate! The plan is to end up with a special matrix.

Solution:

$$z_t = \phi_0 + \phi_1 z_{t-1} + \dots + \phi_p z_{t-p} + a_t$$

$$\begin{aligned} \text{as VAR (1): } Z_t &= \begin{pmatrix} \phi_0 \\ 0 \\ \vdots \end{pmatrix} + \Phi_1 Z_{t-1} + \begin{pmatrix} a_t \\ 0 \\ \vdots \end{pmatrix} \\ \text{with } Z_t &= \begin{pmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-p+1} \end{pmatrix} \text{ and } Z_{t-1} = \begin{pmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-p+1} \end{pmatrix} \\ \Phi_1 &= \begin{pmatrix} \phi_1 & \phi_2 & \dots & \dots & \phi_p \\ I_K & 0_K & \dots & \dots & 0_k \\ 0_K & I_K & 0_k & \dots & 0_k \\ \vdots & \ddots & \ddots & & \vdots \\ 0_k & \dots & 0_k & I_k & 0_k \end{pmatrix} \end{aligned}$$

\Rightarrow Now one can apply the formulas for a VAR(1) to check stationarity.

$$\begin{aligned} |\Phi_1 - \lambda I| &\stackrel{!}{=} 0 \\ \Leftrightarrow (-1)^k |\Phi_1 - \lambda I| &\stackrel{!}{=} 0 \\ \Leftrightarrow \left| \lambda \left(I - \Phi_1 \frac{1}{\lambda} \right) \right| &\stackrel{!}{=} 0 && \text{since } \lambda \text{ is a scalar } I = I_{kp} \\ \Leftrightarrow \lambda^{kp} \left| I - \Phi_1 \frac{1}{\lambda} \right| &\stackrel{!}{=} 0 && \text{and let } \frac{1}{\lambda} =: z \\ \Rightarrow |I - \Phi_1 z| &\stackrel{!}{=} 0 \end{aligned}$$

\Rightarrow Stationarity if all $|z_i| > 1$ ($|\lambda_i| < 1$)

$$I_{kp} - \Phi_1 z = \begin{pmatrix} I_k - \phi_1 z & -\phi_2 z & -\phi_3 z & \dots & -\phi_{p-1} z & -\phi_p z \\ -I_k z & I_k & 0_k & \dots & 0_k & 0_k \\ 0_k & -I_k z & I_k & \dots & 0_k & \vdots \\ \vdots & 0_k & -I_k z & \dots & \vdots & \vdots \\ 0_k & 0_k & 0_k & \dots & -I_k z & I_k \end{pmatrix}$$

$$\Rightarrow -I_k \cdot z + I_k \cdot z = 0_k$$

Since adding multiples of columns to other columns does not affect the determinant:

\Rightarrow column “i” $\cdot z$ + column “i - 1” $\forall i \in \{p, \dots, 2\}$ yields a triangular matrix

$$\Rightarrow \begin{pmatrix} I_k - \phi_1 z & -\phi_2 z^2 & \dots & -\phi_p z & -\phi_1 & -\phi_2 z - \dots - \phi_p z^{p-1} \\ & 0_k & & & I_k & \\ & 0_k & & & & I_k \end{pmatrix}$$

$$\begin{aligned} \Rightarrow |I_{kp} - \Phi_1 z| &= (I_K - \phi_1 z - \dots - \phi_p z^p) \cdot \prod_{i=1}^{p-1} I_k \quad \text{since it is a triangular matrix} \\ &= |I_k - \phi_1 z - \dots - \phi_p z^p| \end{aligned}$$

3 Exercise 3: VAR(2) Moments and Stationarity

Consider the following VAR(2) model with i.i.d. innovations:

$$\phi_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.4 & 0.5 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 & 0 \\ 0.25 & 0 \end{pmatrix}, \quad \Sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

a) Show that the process is stationary.

Solution:

$$\begin{aligned} & |I_2 - \phi_1 z - \phi_2 z^2| \stackrel{!}{=} 0 \\ \Rightarrow & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5z & 0.1z \\ 0.4z & 0.5z \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0.25z & 0 \end{pmatrix} \stackrel{!}{=} 0 \\ \Leftrightarrow & \left| \begin{pmatrix} 1 - 0.5z & -0.1z \\ -0.4z - 0.25z^2 & 1 - 0.5z \end{pmatrix} \right| \stackrel{!}{=} 0 \\ = & (1 - 0.5z)^2 - (0.4z + 0.25z^2) \cdot 0.1z \stackrel{!}{=} 0 \\ = & 1 - z + 0.21z^2 - 0.025z^3 \stackrel{!}{=} 0 \end{aligned}$$

b) Determine the mean vector.

c) Derive the Yule-Walker equations for the lag $l = \{0, 1, 2\}$ for a general VAR(2) process.

d) Suppose we only Γ_0, Γ_1 and Γ_2 - how can we estimate ϕ_1 and ϕ_2 from it?

e) Write the process as a VAR(1) and calculate the mean vector again.

f) Compute Γ_0 based on the VAR(1) formulation.

Hint: You can use R for the calculations.