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Multivariate Time Series Analysis

Exercise Sheet 1

1 Exercise 1: Matrix Operations

Prove properties 3,4 and 5 from Proposition 1.2 (Slide 1-11). Are there any requirements regarding the matrix dimensions?

Solution:

i) Property 3: $(A \otimes B)(F \otimes G) = (AF) \otimes (BG)$

$$\text{Let } A = \begin{pmatrix} a_{11} & \dots & a_{1q} \\ \vdots & \ddots & \vdots \\ a_{p1} & \dots & a_{pq} \end{pmatrix} \text{ and } F = \begin{pmatrix} f_{11} & \dots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \dots & f_{mn} \end{pmatrix}$$

$$\text{hence}(A \otimes B) = \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{pmatrix} \text{ and } (F \otimes G) \text{ analsgously}$$

$$\begin{aligned}
(A \otimes B)(F \otimes G) &= \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{pmatrix} \begin{pmatrix} f_{11}G & \dots & f_{1n}G \\ \vdots & \ddots & \vdots \\ f_{m1}G & \dots & f_{mn}G \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}Bf_{11}G + \ldots + a_{1q}Bf_{m1}G) & \dots & (a_{11}Bf_{1n}G + \ldots + a_{1q}Bf_{mn}G) \\ \vdots & \ddots & \vdots \\ (a_{p1}Bf_{11}G + \ldots + a_{pq}Bf_{m1}G) & \dots & (a_{p1}Bf_{1n}G + \ldots + a_{pq}Bf_{mn}G) \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}f_{11} + \ldots + a_{1q}f_{m1}) & \dots & (a_{11}f_{1n} + \ldots + a_{1q}f_{mn}) \\ \vdots & \ddots & \vdots \\ (a_{p1}f_{11} + \ldots + a_{pq}f_{m1}) & \dots & (a_{p1}f_{1n} + \ldots + a_{pq}f_{mn}) \end{pmatrix} \otimes (BG)
\end{aligned}$$

$$= \begin{pmatrix} \sum_{i=1}^{q=m} a_{1i} f_{i1} & \dots & \sum_{i=1}^{q=m} a_{1i} f_{1i} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{q=m} a_{pi} f_{i1} & \dots & \sum_{i=1}^{q=m} a_{pi} f_{in} \end{pmatrix} \otimes (BG)$$

$$= (AF) \otimes (BG)$$

Dimensions:

$A : p \times q$	$F : m \times n$
$B : c \times d$	$G : h \times k$

$\Rightarrow \; dim(A \otimes B) = pc \times qd, dim(F \otimes G) = mh \times kn$

ii) Property 4: $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
$\Rightarrow$ Claim and verify
The inverse is defined as following:
$(A \otimes B)(A \otimes B)^{-1} = I$ where I is the identitiy matrix
Then $(A \otimes B)(A^{-1} \otimes B^{-1}) = I$ must hold if the claim was true
We know from Property 3 that $(A \otimes B)(A^{-1} \otimes B^{-1}) = (AA^{-1} \otimes BB^{-1}) = I \otimes I = I$
Dimensions: A and B must be non-singular square matrices

iii) Property 3: $tr(A \otimes C) = tr(A) \cdot tr(C)$ for square matrices A and C

$$tr(A \otimes C) = tr \begin{pmatrix} a_{11}C & \dots & a_{1n}C \\ \vdots & \ddots & \vdots \\ a_{n1}C & \dots & a_{nn}C \end{pmatrix} = \sum_{i=1}^{n} (a_{ii} tr(C)) = tr(C) \sum_{i=1}^{n} a_{ii} = tr(C) tr(A)$$

2 Exercise 2: Bivariate Functions

Find the extrema of the following functions (using pen and paper). Determine whether these points constitute minima, maxima or saddle points:

a) $f(x, y) = (x - 2)^2 + (y - 5)^2 + xy$

b) $g(x, y) = (x - 1)^3 - (4y + 1)^2$

Solution:

Solution concept:

1. FOC: first derivatives $\overset{!}{=} 0$

2. SOC: check the determinant of the Hessian matrix

a) $f(x,y) = (x-2)^2 + (y-5)^2 + xy$

$$f(x,y) = (x-2)^2 + (y-5)^2 + xy$$

$$\frac{\partial f(x,y)}{\partial x} = 2(x-2) + y \overset{!}{=} 0 \frac{\partial f(x,y)}{\partial y} \qquad = 2(y-5) + x \overset{!}{=} 0$$

– Solving the equation system yields:

$$x = 2 - \frac{y}{2} \Rightarrow 2y - 10 + 2 - \frac{y}{2} = 0 \Rightarrow y^* = \frac{16}{3}$$
$$\Rightarrow x^* = 2 - \frac{16}{3 \cdot 2} = -\frac{2}{3}$$

– Evaluating the Hessian matrix:

$$\frac{\partial f(x,y)}{\partial x^2} = 2 \qquad \frac{\partial f(x,y)}{\partial xy} = 1$$
$$\frac{\partial f(x,y)}{\partial yx} = 1 \qquad \frac{\partial f(x,y)}{\partial y^2} = 2$$
$$\Rightarrow H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $det(H) = 2 \cdot 2 - 1 \cdot 1 = 3 > 0$ which indicates a minimum

b) $g(x,y) = (x-1)^3 - (4y+1)^2$

$$g(x,y) = (x-1)^3 + (4y-1)^2$$

$$\frac{\partial g(x,y)}{\partial x} = 3(x-1)^2 \overset{!}{=} 0 \Leftrightarrow x^* = 1$$
$$\frac{\partial g(x,y)}{\partial y} = 2 \cdot 4(4y+1) + x \overset{!}{=} 0 \Leftrightarrow y^* = -\frac{1}{4}$$

– Evaluating the Hessian matrix:

$$\frac{\partial g(x,y)}{\partial x^2} = 6x - 6 \qquad \frac{\partial g(x,y)}{\partial xy} = 0$$
$$\frac{\partial g(x,y)}{\partial yx} = 0 \qquad \frac{\partial g(x,y)}{\partial y^2} = 32$$
$$\Rightarrow H = \begin{pmatrix} 6x-6 & 0 \\ 0 & 32 \end{pmatrix}$$

and $det(H)|_{x=x^*,y=y^*} = (6-6) \cdot 32 - 0 \cdot 0 = 0$ which indicates a saddle point.

Thus we did not find an extremal point.

3 Exercise 3: Stationarity

a) Are weakly stationary processes always strictly stationary? Construct an example to support your argument

b) Is weak stationarity a necessary condition for strict stationarity? Bring an example.

Hint: How many moments does a distribution require?

Solution:

a) No. A time series of length T drawing from $N(0,1)$ for $t \in \left[0, \frac{T}{2}\right]$ and drawing from Student's t-distribution for $t \in \left(\frac{T}{2}, T\right]$ has a constant mean $\mu = 0$ and variance $\sigma^2 = 1$, but the kurtosis (4^{th} moment) changes throughout time. In consequence the joint distribution of a subsequence $x_{t-p}, \ldots, x_{t+p}$ is not independent of t. Therefore it is not strictly stationary

b) No. Take the Cauchy distribution as an example: $f(x) = \frac{1}{\pi} \cdot \frac{s}{s^2 + (x-t)^2}$. Any *i.i.d.* sample from this distribution would be obviously strictly stationary. Yet this distribution has no existing moments at all (the integral diverges), hence it cannot exhibit a constant expected value or variance over time. Therefore it is only strictly stationary, but not weakly stationary! (Other example: t_1 distribution, where only the mean but not the variance exists).

4 Exercise 4: Covariance Matrices under Stationarity

Referring to Remark 1.13: Show that $\Gamma_l = \Gamma_{-l}^T$ holds for all weakly stationary processes.

(*Two dimensions suffice*)

Solution:

Without loss of generality assume $\mu = 0$ everywhere and assume z to be a bivariate vector $(x, y)^T$. Let $\Gamma_{l,t}$ be the covariance matrix of the l^{th} lag at time t:

$$\Gamma_{l,t} = \begin{bmatrix} \mathbb{E}(x_t \cdot x_{t-l}) & \mathbb{E}(x_t \cdot y_{t-l}) \\ \mathbb{E}(y_t \cdot x_{t-l}) & \mathbb{E}(y_t \cdot y_{t-l}) \end{bmatrix} \quad \text{and} \quad \Gamma_{l,t}^T = \begin{bmatrix} \mathbb{E}(x_{t-l} \cdot x_t) & \mathbb{E}(x_{t-l} \cdot y_t) \\ \mathbb{E}(y_{t-l} \cdot x_t) & \mathbb{E}(y_{t-l} \cdot y_t) \end{bmatrix} = \Gamma_{-l,t-l}$$

Since weak stationarity has been assumed, the covariance matrix is constant across time and $\Gamma_{-l,t-l} = \Gamma_{-l} = \Gamma_l^T$ and vice versa.

5 Exercise 5: Ljung-Box Test in R

Load the package *MTS* and open the associated data pool 'mts-examples' (Slide 1-8). We are interested in the time series 'GS', 'MS' and 'JPM' from the dataset 'tenstocks':

a) First apply the Ljung-Box test on each time series individually. What do the results imply?

b) Now apply the multivariate Ljung-Box test on all three time series together. Compare the results with those from the univariate test and comment on it.

Solution:

Firstly, we need to importe the example datasets from the *MTS* package, which includes the tenstocks data set.

```
data("mts-examples")
```

If we take a look at the description of the *tenstock* dataset[1], we can see that it contains 11 variables and that we have 132 monthly simple returns from January 2001 to December 2011 for each of the 10 companies (one variable is the time vector).

a)

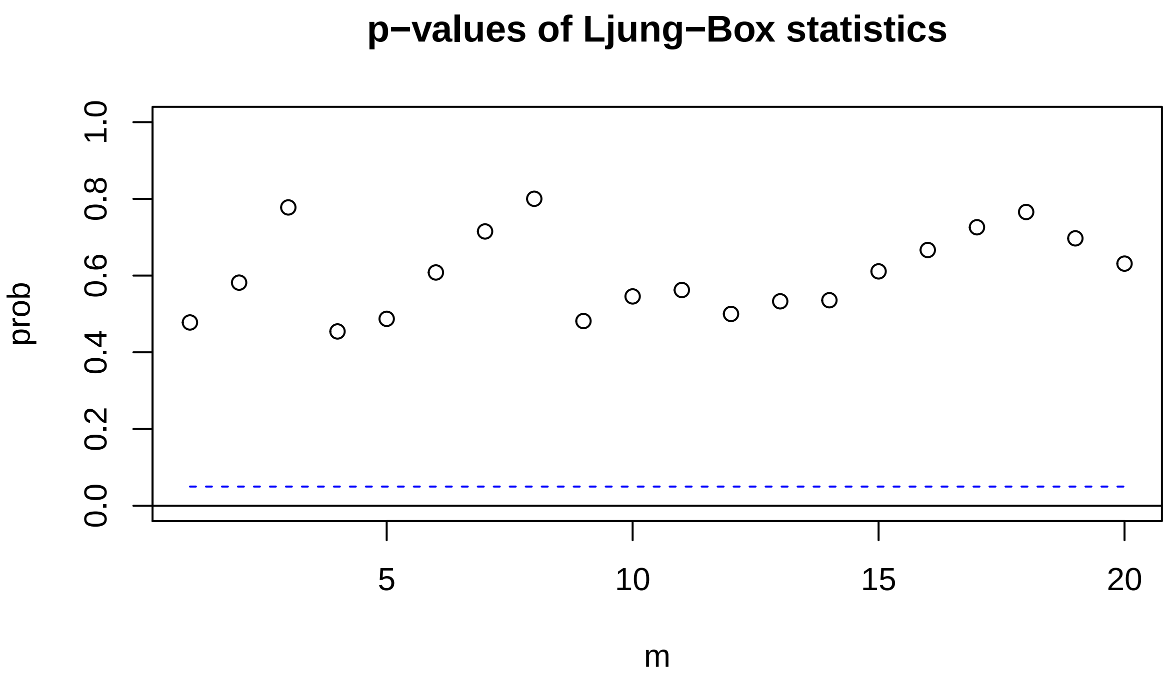
First, we will have a look at the simple returns for *JP-Morgan Chase & Co. (JPM)*.

```
mq(x = tenstocks$JPM, lag = 20)
```

```
## Ljung-Box Statistics:
##          m     Q(m)     df    p-value
##  [1,]  1.000   0.504   1.000    0.48
##  [2,]  2.000   1.084   2.000    0.58
##  [3,]  3.000   1.097   3.000    0.78
##  [4,]  4.000   3.657   4.000    0.45
##  [5,]  5.000   4.445   5.000    0.49
##  [6,]  6.000   4.509   6.000    0.61
##  [7,]  7.000   4.547   7.000    0.72
##  [8,]  8.000   4.592   8.000    0.80
##  [9,]  9.000   8.533   9.000    0.48
## [10,] 10.000   8.857  10.000    0.55
```

[1]To access help-file: `?tenstock()`

```
## [11,] 11.000     9.647  11.000      0.56
## [12,] 12.000    11.340  12.000      0.50
## [13,] 13.000    11.935  13.000      0.53
## [14,] 14.000    12.882  14.000      0.54
## [15,] 15.000    12.887  15.000      0.61
## [16,] 16.000    13.083  16.000      0.67
## [17,] 17.000    13.152  17.000      0.73
## [18,] 18.000    13.425  18.000      0.77
## [19,] 19.000    15.397  19.000      0.70
## [20,] 20.000    17.334  20.000      0.63
```



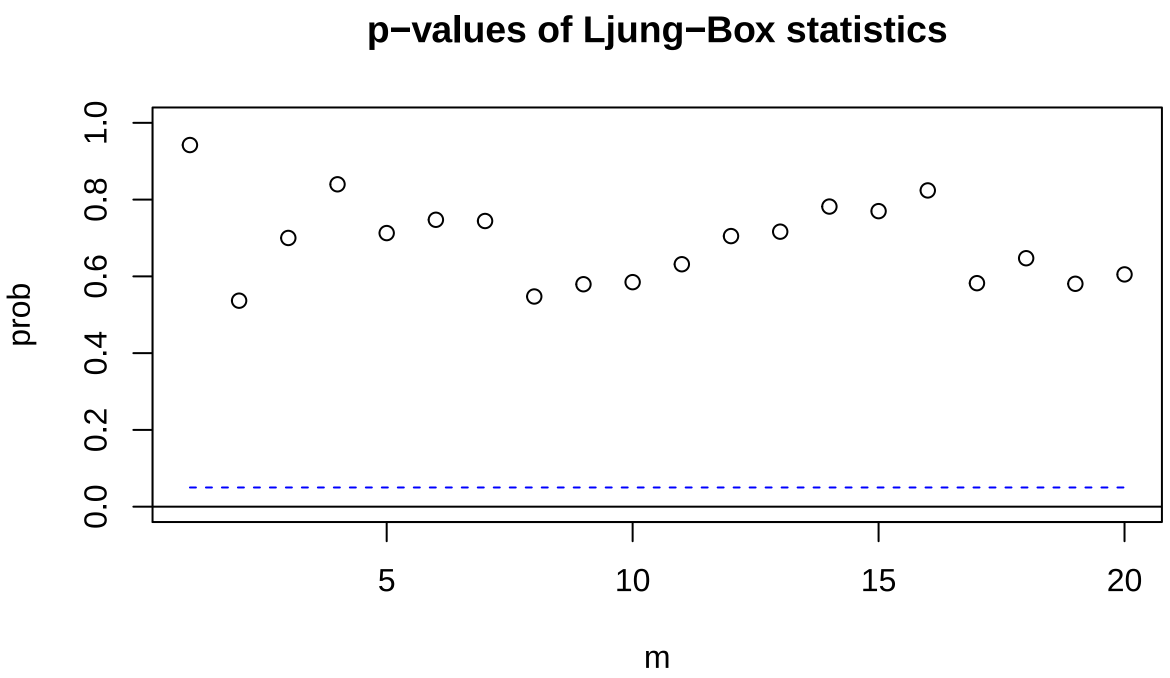
For this time series there are no autocorrelation for the first 20 lags.

Next, we will have a look at time series for *Morgan Stanley (MS)*.

```
mq(x = tenstocks$MS, lag = 20)
```

```
## Ljung-Box Statistics:
##              m        Q(m)       df     p-value
##   [1,]  1.00000   0.00526  1.00000      0.94
##   [2,]  2.00000   1.24473  2.00000      0.54
##   [3,]  3.00000   1.42333  3.00000      0.70
##   [4,]  4.00000   1.42489  4.00000      0.84
##   [5,]  5.00000   2.91701  5.00000      0.71
```

```
## [6,] 6.00000 3.47398 6.00000 0.75
## [7,] 7.00000 4.30334 7.00000 0.74
## [8,] 8.00000 6.89871 8.00000 0.55
## [9,] 9.00000 7.55530 9.00000 0.58
## [10,] 10.00000 8.45013 10.00000 0.58
## [11,] 11.00000 8.89468 11.00000 0.63
## [12,] 12.00000 8.97639 12.00000 0.70
## [13,] 13.00000 9.72283 13.00000 0.72
## [14,] 14.00000 9.72714 14.00000 0.78
## [15,] 15.00000 10.75241 15.00000 0.77
## [16,] 16.00000 10.76237 16.00000 0.82
## [17,] 17.00000 15.18350 17.00000 0.58
## [18,] 18.00000 15.21323 18.00000 0.65
## [19,] 19.00000 17.13504 19.00000 0.58
## [20,] 20.00000 17.73024 20.00000 0.61
```



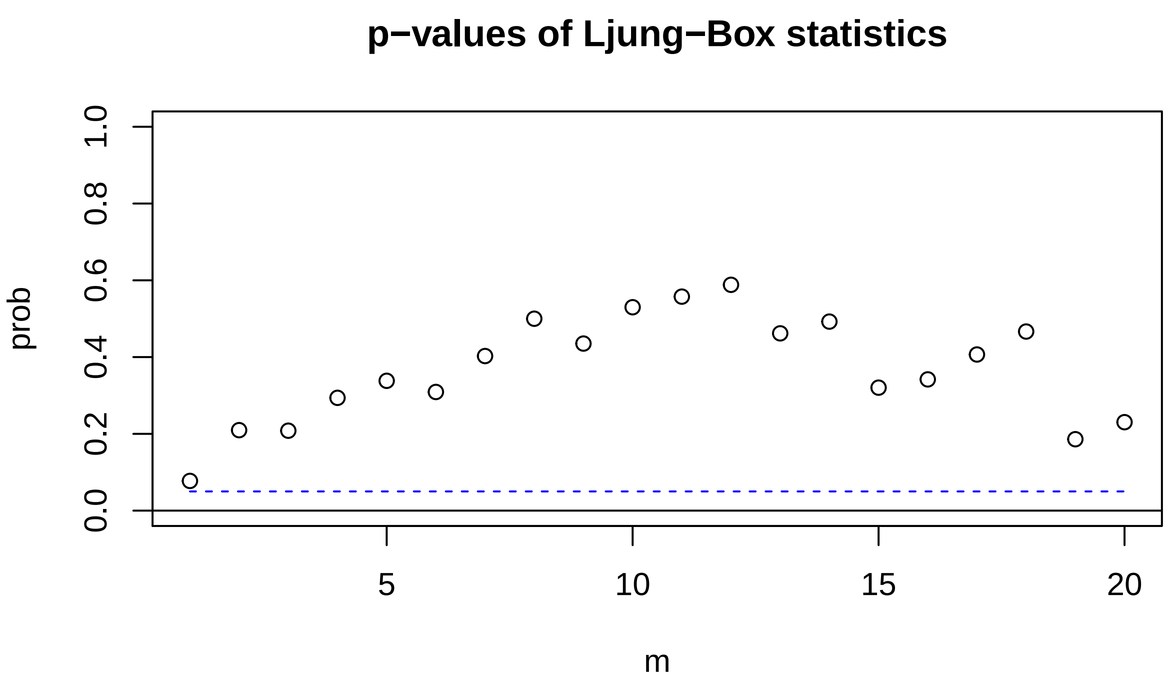
The results for the *MS* time series are similar to those of the *JPM*.

Lastly, there is only the time series from *Goldman Sachs Group Inc (GS)* left to be analysed.

```
mq(x = tenstocks$GS, lag = 20)
```

```
## Ljung-Box Statistics:
##          m       Q(m)      df     p-value
```

```
## [1,]  1.00   3.12  1.00   0.08
## [2,]  2.00   3.12  2.00   0.21
## [3,]  3.00   4.55  3.00   0.21
## [4,]  4.00   4.94  4.00   0.29
## [5,]  5.00   5.68  5.00   0.34
## [6,]  6.00   7.13  6.00   0.31
## [7,]  7.00   7.26  7.00   0.40
## [8,]  8.00   7.34  8.00   0.50
## [9,]  9.00   9.02  9.00   0.44
## [10,] 10.00   9.02 10.00   0.53
## [11,] 11.00   9.70 11.00   0.56
## [12,] 12.00  10.32 12.00   0.59
## [13,] 13.00  12.82 13.00   0.46
## [14,] 14.00  13.44 14.00   0.49
## [15,] 15.00  16.97 15.00   0.32
## [16,] 16.00  17.70 16.00   0.34
## [17,] 17.00  17.72 17.00   0.41
## [18,] 18.00  17.84 18.00   0.47
## [19,] 19.00  24.27 19.00   0.19
## [20,] 20.00  24.28 20.00   0.23
```



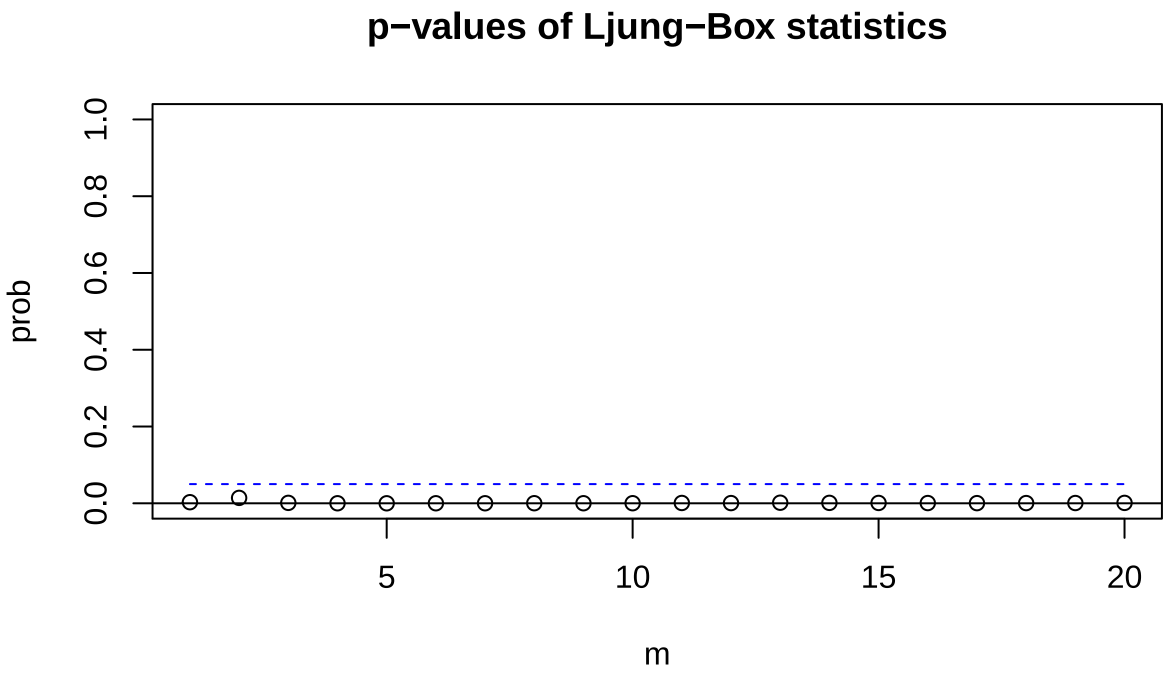
Only the first lag is *relatively* close to significant to a 5 percent significance level

b)

Now we take a look at the combined test.

```r
mq(x = cbind(tenstocks$JPM, tenstocks$MS, tenstocks$GS), lag = 20)
```

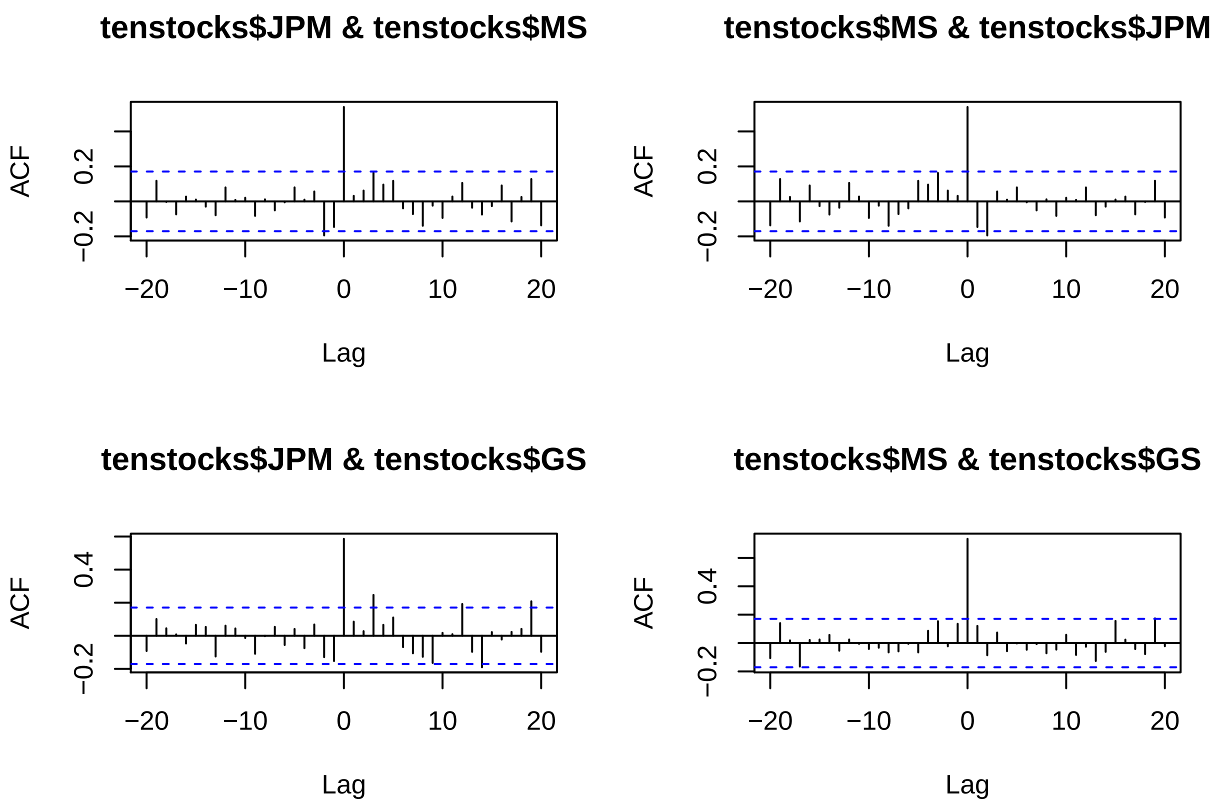
```
## Ljung-Box Statistics:
##          m       Q(m)     df    p-value
##  [1,]    1.0     25.1     9.0     0.00
##  [2,]    2.0     33.6    18.0     0.01
##  [3,]    3.0     55.2    27.0     0.00
##  [4,]    4.0     78.1    36.0     0.00
##  [5,]    5.0     95.3    45.0     0.00
##  [6,]    6.0    103.4    54.0     0.00
##  [7,]    7.0    113.7    63.0     0.00
##  [8,]    8.0    122.9    72.0     0.00
##  [9,]    9.0    135.2    81.0     0.00
## [10,]   10.0    145.1    90.0     0.00
## [11,]   11.0    149.4    99.0     0.00
## [12,]   12.0    162.5   108.0     0.00
## [13,]   13.0    167.4   117.0     0.00
## [14,]   14.0    180.3   126.0     0.00
## [15,]   15.0    192.0   135.0     0.00
## [16,]   16.0    205.4   144.0     0.00
## [17,]   17.0    218.8   153.0     0.00
## [18,]   18.0    227.4   162.0     0.00
## [19,]   19.0    236.5   171.0     0.00
## [20,]   20.0    244.1   180.0     0.00
```



All p-values are below 5%. Since the time series are not much autocorrelated (univariate!), there must be cross-correlations which cause the Null hypothesis to be rejected. So there is a dynamic pattern which might be explained using multivariate time series models.

Lets have a look at the correlation to may see some patterns.

```r
ccf(x = tenstocks$JPM, y = tenstocks$MS, lag.max = 20)
ccf(y = tenstocks$JPM, x = tenstocks$MS, lag.max = 20)
ccf(x = tenstocks$JPM, y = tenstocks$GS, lag.max = 20)
ccf(x = tenstocks$MS, y = tenstocks$GS, lag.max = 20)
```



But keep in mind that the *Ljung-Box* test does not take ρ_0 into consideration. Lastly, we will plot the times series with the command `plot.ts()`.

```
plot.ts(cbind(tenstocks$JPM, tenstocks$MS, tenstocks$GS))
```