Exercise Sheet 3

$\boxed{1}$ a) $z_t = \phi_0 + \phi_1 z_{t-1} + a_t$

$\overset{\mathbb{E}(\cdot)}{\Longrightarrow}$ $\underbrace{\mathbb{E}(z_t)}_{\mu} = \mathbb{E}\Big(\underbrace{\phi_0}_{\phi_0} + \underbrace{\phi_1 z_{t-1}}_{\phi_1 \cdot \mu} + \underbrace{a_t}_{0}\Big)$

[Key assumption? $\mathbb{E}(z_t) = \mathbb{E}(z_{t-1})$]

$\Longleftrightarrow (I - \phi_1) \cdot \mu = \phi_0$

$\Longleftrightarrow \mu = (I - \phi_1)^{-1} \cdot \phi_0$

Plugging in:

$$\mu = \begin{pmatrix} 0.25 & 0 \\ 0.25 & 0.5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

b) Eigenvalues of ϕ_1:

$$|\phi_1 - I\lambda| \stackrel{!}{=} 0$$

$$= \begin{vmatrix} 0.75-\lambda & 0 \\ -0.25 & 0.5-\lambda \end{vmatrix}$$

$$= (0.75 - \lambda)(0.5 - \lambda) \overset{!}{=} 0$$

$$\left.\begin{aligned} \Rightarrow \lambda_1 &= 0.75 \\ \lambda_2 &= 0.5 \end{aligned}\right\} \text{stationary}$$

c) Yule-Walker:

$$\tilde{z}_t := z_t - \mu$$

$$\Rightarrow \tilde{z}_t = \phi_1 \tilde{z}_{t-1} + a_t \quad \Big| \cdot \tilde{z}'_{t-l}$$

$$\Leftrightarrow \tilde{z}_t \tilde{z}'_{t-l} = \phi_1 \tilde{z}_{t-1} \tilde{z}'_{t-l} + a_t \tilde{z}'_{t-l}$$

$$\underline{\mathbb{E}(\cdot)} \Rightarrow \Gamma_l = \phi_1 \cdot \Gamma_{l-1} + \begin{cases} l=0: \Sigma_a \\ l\neq 0: 0_k \end{cases}$$

$$l=0: \Gamma_0 = \phi_1 \cdot \Gamma_{-1} + \Sigma_a$$

$$l=1: \Gamma_1 = \phi_1 \cdot \Gamma_0 + 0_k$$

$$l=2: \Gamma_2 = \phi_1 \cdot \Gamma_1 + 0_k$$

using $\Gamma_{-1} = \Gamma_1'$:

$$\Gamma_0 = \phi_1 \cdot (\phi_1 \cdot \Gamma_0)' + \Sigma_a$$

$$\Leftrightarrow \Gamma_0 = \phi_1 \Gamma_0' \phi_1' + \Sigma_a$$

and $\Gamma_0' = \Gamma_0$ since Γ_0 is symmetric!

d) $$\begin{pmatrix} \gamma_{11} & \underline{\gamma_{12}} \\ \underline{\gamma_{21}} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0.75 & 0 \\ -0.25 & 0.5 \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix} \begin{pmatrix} 0.75 & -0.25 \\ 0 & 0.5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (0.75)^2 \gamma_{11} & (-0.75 \cdot 0.25 \gamma_{11}) + 0.5 \cdot 0.75 \gamma_{12} \\ (-0.75 \cdot 0.25 \gamma_{11}) + 0.5 \cdot 0.75 \gamma_{12} & (-0.25)^2 \gamma_{11} - 0.25 \cdot 0.5 \gamma_{12} - 0.5 \cdot 0.25 \gamma_{12} + (0.5)^2 \gamma_{22} \end{pmatrix}$$

$$+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \gamma_{11} = (0.75)^2 \gamma_{11} + 1$$

$$\Leftrightarrow \gamma_{11} = \frac{16}{7}$$

$$\Rightarrow \gamma_{12} = \frac{1}{4} \cdot \left(-\frac{3}{4}\right) \cdot \frac{16}{7} + \frac{2}{4} \cdot \frac{3}{4} \cdot \gamma_{12}$$

$$\Leftrightarrow \gamma_{12} = -\frac{24}{35}$$

$$\Rightarrow \gamma_{22} = \frac{1}{16}\gamma_{11} \cancel{- 2 \cdot \frac{1}{2}} \cdot \frac{1}{4}\gamma_{12}$$

$$+ \frac{1}{4}\gamma_{22} + 1$$

$$= \frac{1}{\cancel{16}} \cdot \frac{\cancel{16}}{7} - \frac{1}{4} \cdot \left(-\frac{24}{35}\right) + \frac{1}{4} \gamma_{22} + 1$$

$$\Leftrightarrow \gamma_{22} = \frac{184}{105}$$

$$\Gamma_1 = \phi_1 \cdot \Gamma_0$$

$$= \begin{pmatrix} 0.75 & 0 \\ -0.25 & 0.5 \end{pmatrix} \cdot \begin{pmatrix} \frac{16}{7} & \frac{-24}{35} \\ \frac{-24}{35} & \frac{184}{105} \end{pmatrix}$$

$\boxed{2}$ $z_t = \phi_0 + \phi_1 \cdot z_{t-1} + \dots + \phi_p \cdot z_{t-p} + a_t$

as VAR(1): $Z_t = \begin{pmatrix} \phi_0 \\ 0 \\ \vdots \end{pmatrix} + \Phi_1 Z_{t-1} + \begin{pmatrix} a_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

with $Z_t = \begin{pmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-p+1} \end{pmatrix}$ and $Z_{t-1} = \begin{pmatrix} z_{t-1} \\ z_{t-2} \\ \vdots \\ z_{t-p} \end{pmatrix}$

$$\Phi_1 = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & & \phi_p \\ I_k & 0_k & \cdots & & 0_k \\ 0_k & I_k & 0_k & \cdots & 0_k \\ \vdots & & \ddots & \ddots & \vdots \\ 0_k & \cdots & 0_k & I_k & 0_k \end{pmatrix}$$

$\Rightarrow$) can apply the formulas for a VAR(1) to check stationarity!

$$|\Phi_1 - \lambda I| \overset{!}{=} 0$$

$$\Leftrightarrow (-1)^k \, |\lambda I - \Phi_1| \overset{!}{=} 0$$

$$\Rightarrow \left| \lambda \left(I - \Phi_1 \frac{1}{\lambda} \right) \right| \overset{!}{=} 0$$

since λ is scalar, $I = I_{kp}$

$$\Leftrightarrow \lambda^{kp} \left| I - \Phi_1 \frac{1}{\lambda} \right| \overset{!}{=} 0$$

and let $\frac{1}{\lambda} =: z$

$$\Rightarrow \left| I - \Phi_1 z \right| \overset{!}{=} 0$$

stationarity iff:

all($|z_i| > 1$

($|\lambda_i| < 1$)

$$I_{kp} - \Phi_1 Z$$

$$= \begin{pmatrix} I_k - \phi_1 z & -\phi_2 z & -\phi_3 z & \cdots & -\phi_{p-1} z & -\phi_p z \\ -I_k z & I_k & 0_k & & 0_k & 0_k \\ 0_k & -I_k z & I_k & & 0_k & \vdots \\ \vdots & 0_k & -I_k z & & \vdots & \vdots \\ \vdots & \vdots & 0_k & & 0_k & \vdots \\ \vdots & \vdots & \vdots & & I_k & 0_k \\ 0_k & 0_k & 0_k & \cdots & -I_k z & I_k \end{pmatrix}$$

$$\Rightarrow -I_k \cdot z + I_k \cdot z = 0_k$$

Since adding multiples of columns to other columns does not affect the determinant:

$\Rightarrow$ column 'i' $\cdot z$ + column 'i-1' $\forall i \in \{k, \dots, 2\}$

yields a triangular matrix

$$\Rightarrow \left(\begin{array}{c|c} I_k - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p & -\phi_1 - \phi_2 z - \cdots - \phi_p z^{p-1} \quad \cdots \\ 0_k & I_k \\ \vdots & \ddots \\ 0_k & I_k \end{array}\right)$$

$$\Rightarrow \left| I_{kp} - \Phi_1 z \right| = \left(I_k - \phi_1 z - \dots - \phi_p z^p \right) \cdot \prod_{i=1}^{p-1} I_k$$

since it is a triangular matrix

$$= \left| I_k - \phi_1 z - \dots - \phi_p z^p \right|$$

$\boxed{3}$ a) check if

$$\left| I_2 - \phi_1 z - \phi_2 z^2 \right| \overset{!}{=} 0$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5z & 0.1z \\ 0.4z & 0.5z \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0.25z & 0 \end{pmatrix} \overset{!}{=} 0$$

$$\Leftrightarrow \left| \begin{pmatrix} 1-0.5z & -0.1z \\ -0.4z-0.25z^2 & 1-0.5z \end{pmatrix} \right|$$

$$= (1-0.5z)^2 - (0.4z+0.25z^2) \cdot 0.1z$$

$$= 1 - z + 0.21z^2 - 0.025z^3 \overset{!}{=} 0$$

$\to R \Rightarrow$ stationary

b) $z_t = \phi_0 + \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t$

$\overset{\mathbb{E}(\cdot)}{\Longrightarrow} \dots \Rightarrow \mu = (I - \phi_1 - \phi_2)^{-1} \phi_0$

$$= \begin{pmatrix} 1-0.5 & -0.1 \\ -0.4-0.25 & 1-0.5 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{228}{37} \\ \frac{360}{37} \end{pmatrix}$$

$$\mathbb{E}(\tilde{z}_t \tilde{z}_{t-l}') = \phi_1 \mathbb{E}(\tilde{z}_{t-1} \tilde{z}_{t-l}')$$

$$+ \phi_2 \mathbb{E}(\tilde{z}_{t-2} \tilde{z}_{t-l}') + \mathbb{E}(a_t \tilde{z}_{t-l}')$$

$$\Rightarrow l=0: \Gamma_0 = \phi_1 \Gamma_{-1} + \phi_2 \Gamma_{-2} + \Sigma_a$$

$$l=1: \Gamma_1 = \phi_1 \Gamma_0 + \phi_2 \Gamma_{-1} + 0_{2\times2}$$

$$l=2: \Gamma_2 = \phi_1 \Gamma_1 + \phi_2 \Gamma_0 + 0_{2\times2}$$

d) $\overbrace{(\Gamma_1, \Gamma_2)}^{\text{row vector}} = (\phi_1, \phi_2) \begin{pmatrix} \Gamma_0 & \Gamma_1 \\ \Gamma_1' & \Gamma_0 \end{pmatrix}$

$\Leftrightarrow (\phi_1, \phi_2) = \begin{pmatrix} \Gamma_0 & \Gamma_1 \\ \Gamma_1' & \Gamma_0 \end{pmatrix}^{-1} (\Gamma_1, \Gamma_2)$