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Winter Term 2019/2020

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Multivariate Time Series Analysis

Exercise Sheet 2

1 Exercise 1: Moments and Simulation of a VAR(1) Process

Take the model from Example 2.4 on Slide 2-6:

a) Derive a formula to obtain the population cross-covariance matrices for the lags 1 to 10 and compute them using R.

   Hint: A glance at the slides and a loop might save you some time.

b) Based on your results, compute the cross-correlation matrices.

Solution:

$$z_t = \phi_1 z_{t-1} + a_t$$

$$\begin{aligned}
\Gamma_0 &= \phi_1 \; \Gamma_0 \; \phi_1^{'} + \Sigma_a \\
\Leftrightarrow \operatorname{vec}(\Gamma_0) &= (\phi_1 \otimes \phi_1) \cdot \operatorname{vec}(\Gamma_0) + \operatorname{vec}(\Sigma_a) \\
\Leftrightarrow \operatorname{vec}(\Gamma_0) &= (I_{K^2} - \phi_1 \otimes \phi_1) \cdot \operatorname{vec}(\Sigma_a) \\
\Rightarrow \Gamma_1 &= \phi_1 \Gamma_0 \\
\Rightarrow \Gamma_l &= \phi_{l-1} \Gamma_{l-1} = \phi_1^l \Gamma_0
\end{aligned}$$

```r
# First define parameters/coefficients:
Phi <- matrix(data = c(0.2, -0.6, 0.3, 1.1),
              byrow = FALSE, nrow = 2) # VAR coefficients
k <- ncol(Phi)
```

```r
Sigma_a <- matrix(data = c(1, 0.8, 0.8, 2.0),
                  byrow = FALSE, nrow = 2) # innovations' covariances

Phi_kron <- kronecker(X = Phi, Y = Phi) # kronecker product
# Phi %x% Phi # alternative command

Ident <- diag(ncol(Phi_kron))
# identity matrix with the same dimensions as Phi_kron

Gamma0.vec <- solve(Ident - Phi_kron) %*% c(Sigma_a)
# c() works like the "vec" operator

Gamma0.mat <- matrix(data = Gamma0.vec, nrow = 2, byrow = FALSE)
```

The Γ_0 matrix is than:

```
##          [,1]     [,2]
## [1,] 2.288889 3.511111
## [2,] 3.511111 8.622222
```

To derive the Γ_1 matrix, we can apply the following formula:

$$\Gamma_1 = \phi_1 \Gamma_0$$

```r
Gamma1.mat <- Phi %*% Gamma0.mat
```

Γ_1 is than:

```
##          [,1]     [,2]
## [1,] 1.511111 3.288889
## [2,] 2.488889 7.377778
```

To derive all desired l^{th} laged covariance matrices we can use the general equation and program a loop over all desired lags:

$$\Rightarrow \Gamma_l = \phi_{l-1} \Gamma_{l-1} = \phi_1^l \Gamma_0$$

To derive the correlation matrices ρ_l we devide the covariances by the standard deviations:

$$\rho_l = \begin{pmatrix} \frac{\mathrm{Cov}(x_t, x_{t-l})}{\sqrt{\mathrm{Var}(x_t) \cdot \mathrm{Var}(x_{t-l})}} & \frac{\mathrm{Cov}(x_t, y_{t-l})}{\sqrt{\mathrm{Var}(x_t) \cdot \mathrm{Var}(y_{t-l})}} \\ \frac{\mathrm{Cov}(y_t, x_{t-l})}{\sqrt{\mathrm{Var}(y_t) \cdot \mathrm{Var}(x_{t-l})}} & \frac{\mathrm{Cov}(y_t, y_{t-l})}{\sqrt{\mathrm{Var}(y_t) \cdot \mathrm{Var}(y_{t-l})}} \end{pmatrix}$$

```r
# compute further (lagged) cross-covariance-functions
# preparing the variables to store the matrices into
#covariance
ccovf.list <- list()
#correlation
ccorf.list <- list()

# obtaining standard deviations for the single variables (parts of z)
# and putting them in a diagonal matrix
D.inv <- solve(sqrt(Gamma0.mat * diag(ncol(Phi))))
for (i in 1:10){
  ccovf.list[[i]] <- Phi%^%i %*% Gamma0.mat
  ccorf.list[[i]] <- D.inv %*% ccovf.list[[i]] %*% D.inv
}
ccovf.list # cross lagged covariances
```

```
## [[1]]
##          [,1]     [,2]
## [1,] 1.511111 3.288889
## [2,] 2.488889 7.377778
##
## [[2]]
##          [,1]     [,2]
## [1,] 1.048889 2.871111
## [2,] 1.831111 6.142222
##
## [[3]]
##           [,1]     [,2]
## [1,] 0.7591111 2.416889
## [2,] 1.3848889 5.033778
##
```

```
## [[4]]
##           [,1]     [,2]
## [1,] 0.5672889 1.993511
## [2,] 1.0679111 4.087022
##
## [[5]]
##           [,1]     [,2]
## [1,] 0.4338311 1.624809
## [2,] 0.8343289 3.299618
##
## [[6]]
##           [,1]     [,2]
## [1,] 0.3370649 1.314847
## [2,] 0.6574631 2.654694
##
## [[7]]
##           [,1]     [,2]
## [1,] 0.2646519 1.059378
## [2,] 0.5209705 2.131255
##
## [[8]]
##           [,1]      [,2]
## [1,] 0.2092215 0.8512522
## [2,] 0.4142764 1.7087543
##
## [[9]]
##           [,1]      [,2]
## [1,] 0.1661272 0.6828767
## [2,] 0.3301711 1.3688784
##
## [[10]]
##           [,1]      [,2]
## [1,] 0.1322768 0.5472389
## [2,] 0.2635119 1.0960403
```

```
ccorf.list # cross lagged correlations
```

```
## [[1]]
##           [,1]      [,2]
## [1,] 0.6601942 0.7403332
```

```
## [2,] 0.5602522 0.8556701
##
## [[2]]
##           [,1]      [,2]
## [1,] 0.4582524 0.6462909
## [2,] 0.4121855 0.7123711
##
## [[3]]
##           [,1]      [,2]
## [1,] 0.3316505 0.5440449
## [2,] 0.3117403 0.5838144
##
## [[4]]
##           [,1]      [,2]
## [1,] 0.2478447 0.4487420
## [2,] 0.2403882 0.4740103
##
## [[5]]
##           [,1]      [,2]
## [1,] 0.1895379 0.3657466
## [2,] 0.1878085 0.3826876
##
## [[6]]
##           [,1]      [,2]
## [1,] 0.1472614 0.2959738
## [2,] 0.1479958 0.3078898
##
## [[7]]
##           [,1]      [,2]
## [1,] 0.1156246 0.2384673
## [2,] 0.1172711 0.2471817
##
## [[8]]
##            [,1]      [,2]
## [1,] 0.09140746 0.1916180
## [2,] 0.09325416 0.1981803
##
## [[9]]
##            [,1]      [,2]
## [1,] 0.07257985 0.1537165
```

```
## [2,] 0.07432195 0.1587617
##
## [[10]]
##             [,1]      [,2]
## [1,] 0.05779083 0.1231842
## [2,] 0.05931687 0.1271181
```

c) Draw a corresponding innovation sequence at for 300 periods from a (multivariate) Gaussian distribution and simulate the given VAR(1) process without any further built-in functions.

   Hint: 'mvrnorm' and 'for' are still allowed

Solution:

- Steps:
  1. Set T (in code N)
  2. Draw $\{a_1, \ldots, a_T\} \sim [\mu, \Sigma_a]$
  3. Set $z_0 = \mathbb{E}\left(z_t\right)$
  4. $z_1 = \phi_1 z_0 + a_1$
  5. Repeat Sep 4 $(T-1)$ times
  6. (Discard first few observations to minimise effects of z_0 on the results)

     ⇒ *Note* that it is generally advised to discard the first few observations to eliminate the influence of the arbitrary starting point! In our case we skipped this step to keep things short and clear.

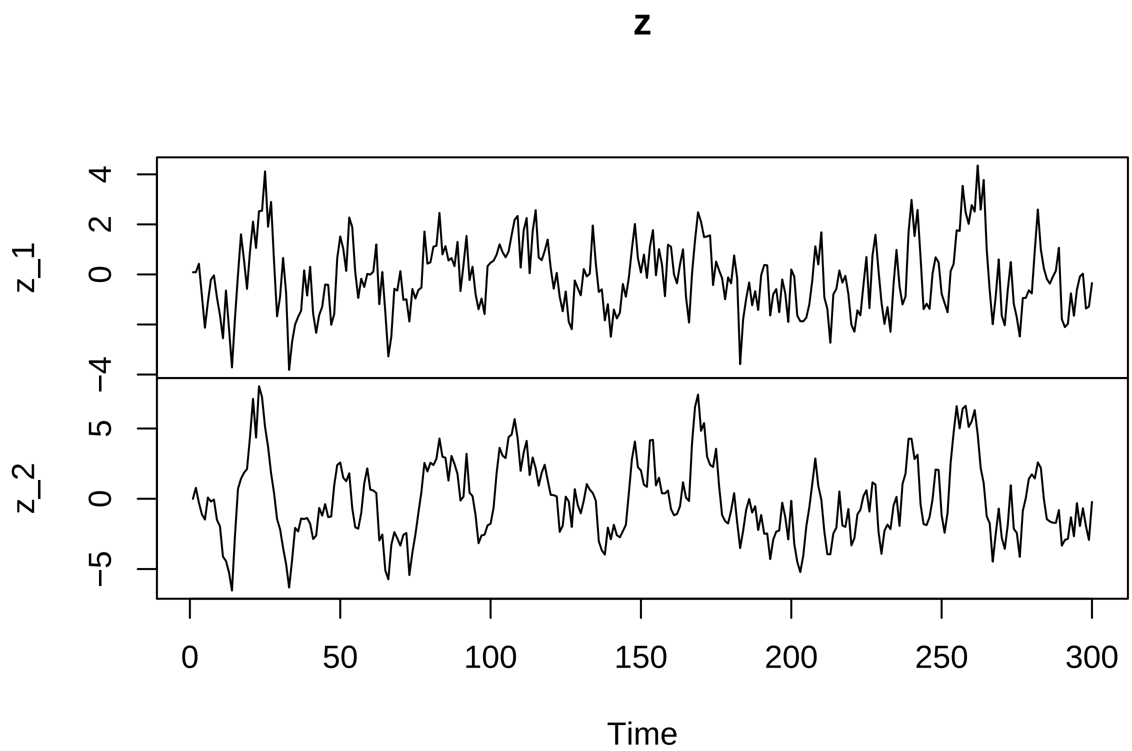
```r
# i) Set sample size and further coefficients
N <- 300
# ii) The Basis: Innovations
set.seed(2^9-1) # for replication: the 'random' numbers drawn
# up from this point will always be the same (given you use
# the same command to draw them....)
a <- mvrnorm(n = N, mu = c(0,0), Sigma = Sigma_a)
# iii) Writing a function to generate 'z' realisations from 'a' using Phi
var1gen <- function(coef.mat, z.lag, innovation){
  z.current <- coef.mat %*% z.lag + innovation
  # z_(t) = phi * z_(t-1) + a_(t)
```

```r
  return(z.current)
}
# iv) Prepare variable for 'z'
z <- ts(data = matrix(data = NA, nrow = (N), ncol = k)
        ,start = 1, names = c("z_1", "z_2"))
# v) Set starting values for z (at mean + innovation)
z[1,] <- c(0,0) + a[1,]
# vi) Generate z repeatedly and store it
for (i in 2:(N)){
  z[i,] <- var1gen(coef.mat = Phi, z.lag = z[(i-1),], innovation = a[i,])
}
```

d) Plot the multivariate time series you have just created. Does it look stationary?

Solution:

```r
# The ordinary 'plot' command works because 'z' is already
# a 'ts'-class object (among other classes) and NOT a data frame!
plot(z)
```



At least it looks stable hence we cannot rule out stationarity.

e) Estimate the sample cross-covariance and cross-correlation matrices. Compare these with the population moment matrices from task a)

Solution:

$$
\begin{aligned}
\widehat{\Gamma_0} &= \tilde{z}_T' \tilde{z}_{T-1} \cdot (T-1)^{-1} \\
\tilde{z}_T' &= z_T - \widehat{\mu}_z \\
&Z \text{ is a } T \times 2 \text{ matrix} \\
&z_t \text{ is a } 2 \times 1 \text{ vector} \\
z_t &:= \begin{pmatrix} x_t \\ y_t \end{pmatrix} \leftarrow \text{variables} \\
Z_t &:= \begin{pmatrix} X_t & Y_t \\ X_{t-1} & Y_{t-1} \\ X_{t-2} & Y_{t-2} \\ \vdots & \vdots \\ X_{t-t} & Y_{t-t} \end{pmatrix} \leftarrow \text{sample data} \\
\widehat{\mathrm{Cov}(x_t, y_{t-1})} &= \frac{1}{(T-1)} \sum_{t=1}^{T} \left(\tilde{X}_t \cdot \tilde{Y}_{t-1} \right) \\
&\Rightarrow \text{is part of:} \frac{1}{T-1} \cdot \tilde{Z}_t' \tilde{Z}_{t-1} = \widehat{\Gamma_1}
\end{aligned}
$$

To estimate moments from simulates data, we first do it by "hand" to get a surrow understanding and see the conection to the Yule-Walker equation.

First we need to demeaning the multivariate time series. Therefore, we calculate the colum means.

```
# compute the individual vector means
mu <- colMeans(z)
```

To demeaning the series we habe now two choicesto proceed, one would be to itterating over the rows and subtracting the mean vector everytime.

```
z.demeaned <- t(apply(X = z, MARGIN = c(1), FUN = function(x) x - mu))
```

The other option would be to subtracting the column means from each column individually.

```
z.demeaned <- cbind(z[,1]-mu[1], z[,2]-mu[2])
```

Now we are able to compute $\hat{\Gamma_0}$.

```
Gamma0.hat <- t(z.demeaned) %*% z.demeaned / (N-1) # sample covariance with
# correction for degrees of freedom (we've demeaned the series!)
```

Please also note that now the first entry is transposed! (`z.demeaned` is a data matrix and not a random vector anymore!)

To obtain $\hat{\rho}_0$ we have to standardise $\hat{\Gamma}_0$.

```
standardisation <- matrix(data = c(Gamma0.hat[1,1],
                                   sqrt( Gamma0.hat[1,1] * Gamma0.hat[2,2]),
                                   sqrt(Gamma0.hat[2,2] * Gamma0.hat[1,1]),
                                   Gamma0.hat[2,2]),
                          nrow = 2, byrow = FALSE)
# just another way to program it,
# maybe it is more visible what's inside D.inv %*% D.inv
Rho0.hat <- Gamma0.hat / standardisation
```

Then ρ_0 is:

```
##          z_1      z_2
## z_1 1.000000 0.809102
## z_2 0.809102 1.000000
```

Since we simulated the trajectory we know the true values and can compare the results. The estimated mean values for the two series are slightly negativ (-0.1107541, -0.084682)$'$ so the differences are also slighlty negative since we simulated the time series without a mean.

For the covariance Γ_0 we computed the analytical solution in part b of this exercise. The differences $(\widehat{\Gamma}_0 - \Gamma_0)$ are:

```
##            z_1       z_2
## z_1 -0.2899134 -0.354941
## z_2 -0.3549410 -1.010082
```

The same applies for the comparison of the correlations ρ_0 and the estimated correlations. The differences $(\widehat{\rho}_0 - \rho_0)$ are:

```
##              z_1          z_2
## z_1 1.110223e-16 1.874622e-02
## z_2 1.874622e-02 2.220446e-16
```

2 Exercise 2: Checking VAR(1) Stationarity

Recall the conditions to check if a VAR(1) process is stationary. Now assume the VAR(1) model $z_t = \phi_1 z_{t-1} + a_t$ with a_t as a sequence of i.i.d innovations:

a) Do you need to make further assumptions on the cross-correlations of a_t to ensure stationarity?

Solution:

$$\begin{aligned} Z_t &= \phi_1 Z_{t-1} + a_t \\ a_t &\overset{i.i.d.}{\sim} [\mu_a, \Sigma_a] \\ i.i.d. : \mathrm{Cov}(a_t, a_{t-1}) &= 0 \end{aligned}$$

Generally not, since i.i.d. errors induce no dynamic strucutre. Still, finite 1^{st} and 2^{nd} moments are required for weak stationarity! (Gaussian innovations fulfill that condition, of course).

b) Which of the following processes are stationary? $\phi_1 = \ldots$

i) $\begin{pmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{pmatrix}$

ii) $\begin{pmatrix} 0.5 & 0.3 \\ 0 & -0.3 \end{pmatrix}$

iii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

iv) $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

v) $\begin{pmatrix} 1 & -0.5 \\ -0.5 & 0 \end{pmatrix}$

Solution:

$$\begin{aligned} Z_t &= \phi_1 z_{t-1} + a_t \\ &= \phi_1 \cdot (\phi_1 z_{t-2} + a_{t-1}) + a_t \\ &= \underbrace{\phi_1^p \; z_{t-p}}_{\text{stable ?}} + \underbrace{\sum_{i=0}^{p} \phi_1^i \; a_{t-1}}_{\text{summable ?}} \end{aligned}$$

$$\lim_{p\to\infty}\phi_1^p \longrightarrow 0_{k\times k}$$

$$\Rightarrow \text{eigenvalues !}$$

$$\Rightarrow \phi_1 x = \lambda x \Rightarrow \phi_1^p x = \lambda^p x$$

$$\Rightarrow \text{solve:} \left(\phi_1 - I_K\lambda\right) x = 0$$

$$\text{for } x \neq 0 : |\phi_1 - I_K\lambda| \overset{!}{=} 0$$

$$\text{stability (for stationarity) } |\lambda_1|, \ldots, |\lambda_k| < 1$$

i) $\begin{pmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{pmatrix}$

$$\begin{aligned}
&= \begin{pmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\
&= \begin{vmatrix} (0.2-\lambda) & 0.3 \\ -0.6 & (1.1-\lambda) \end{vmatrix} \\
&= (0.2-\lambda)(1.1-\lambda) - (-0.6)(0.3) \\
&= \lambda^2 - 1.3\lambda + 0.4 \overset{!}{=} 0 \\
\text{pq-Formel} \Rightarrow \lambda_{1,2} &= -\left(\frac{-1.3}{2}\right) \pm \sqrt{\left(\frac{-1.3}{2}\right)^2 - 0.4} \\
&= 0.65 \pm 0.15 \\
\lambda_1 &= 0.8 \\
\lambda_2 &= 0.5 \\
& \quad |\lambda_1| < 1, |\lambda_2| < 1, \text{ stationary}
\end{aligned}$$

ii) $\begin{pmatrix} 0.5 & 0.3 \\ 0 & -0.3 \end{pmatrix}$

$$\begin{aligned}
&\begin{vmatrix} 0.5-\lambda & 0.3 \\ 0 & -0.3-\lambda \end{vmatrix} \\
=&(0.5-\lambda)(-0.3-\lambda) - 0.3\cdot 0 \overset{!}{=} 0 \\
\Rightarrow&\lambda_1 = 0.5,\ \lambda_2 = 0.3 \Rightarrow \text{stationary}
\end{aligned}$$

iii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}$$

$$=(1-\lambda)(1-\lambda) - 0 \cdot 0 \overset{!}{=} 0$$

$$\Rightarrow \lambda_1 = 1,\ \lambda_2 = 1 \Rightarrow \text{not stationary}$$

iv) $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix}$$

$$=(1-\lambda)(-1-\lambda) - 0 \cdot 0 \overset{!}{=} 0$$

$$=\lambda^2 + \lambda - \lambda - 1 + 1$$

$$=\lambda^2 \overset{!}{=} 0$$

$$\Rightarrow \lambda_1 = 0,\ \lambda_2 = 0 \Rightarrow \ \text{stationary}$$

v) $\begin{pmatrix} 1 & -0.5 \\ -0.5 & 0 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & -0.5 \\ -0.5 & 0-\lambda \end{vmatrix}$$

$$=(1-\lambda)(-\lambda) - 0.5 \cdot 0.5 \overset{!}{=} 0$$

$$=\lambda^2 + \lambda - \lambda - 0.25 \overset{!}{=} 0$$

$$\Rightarrow \lambda_1 = 0.207,\ \lambda_2 = 1.207 \Rightarrow \text{not stationary}$$