University of Duisburg-Essen
Faculty of Business Administration and Economics
Chair of Econometrics

Winter Term 2019/2020

Dr. Yannick Hoga   Thilo Reinschlüssel

Multivariate Time Series Analysis

Solution Exercise Sheet 7

1 Exercise 1: The optimal forecast

a) Show that the stationary VAR(1) process $z_t = t_{t-1} + a_t$ with a_t a standard white noise has the following causal representation:

$$z_t = \sum_{i=0}^{\infty} \theta_i a_{t-i}$$

Solution:

$$\begin{aligned}
z_t &= \phi \cdot \underbrace{z_{t-1}}_{\phi z_{t-2} + a_{t-1}} + a_t \\
&= \phi^2 z_{t-2} + \phi a_{t-1} + a_t \\
&= \phi^3 z_{t-3} + \phi^2 a_{t-2} + \phi a_{t-1} + a_t \\
&\vdots \\
&= \phi^m z_{t-m} + \sum_{i=0}^{m-1} \phi^i a_{t-i} \\
&= 0 + \sum_{i=0}^{m-1} \phi^i a_{t-i}\tilde{} \\
&\text{with } \lim_{m\to\infty} \phi^m = 0 \text{ by weak stationarity} \\
&= \sum_{i=0}^{\infty} \theta_i a_{t-i}
\end{aligned}$$

Using lag notation:

$$z_t = \phi L z_t + a_t$$

$$\Leftrightarrow (1-\phi L)z_t = a_t$$

$$\Leftrightarrow z_t = (1-\phi L)^{-1}a_t \text{and} \quad (1-\phi L)^{-1} = \sum_{i=0}^{\infty}\phi^i L^i$$

(requires stationarity and invertiability)

b) Asume the linear forecasting model $y_T(h) = \psi y_T$ ans show that $\psi = \phi^h$ minimieses the MSE of $y_T(h)$ given that y_t is a VAR(1) process.

Solution:

$$y_T(h) = \arg\min \underbrace{\text{MSE}(y_T(h))}_{\mathbb{E}\left([y_{T+h}-y_T(h)][y_{T+h}-y_T(h)]'\right)}$$

$$\begin{aligned}
\rightarrow Y_{T+h} &= \phi Y_{T+h-1} + a_{T+h} \\
&\vdots \\
&= \phi^h Y_T + \sum_{i=0}^{h-1}\phi^i a_{T+h-i}
\end{aligned}$$

$$\Rightarrow y_{T+h} - Y_T(h) = \phi^h Y_T + \sum_{i=0}^{h-1}\phi^i a_{T+h-1} - \psi y_T$$

$$\Rightarrow \text{MSE}(y_T(h)) = \mathbb{E}\left[\underbrace{\left(\sum_{i=0}^{h-1}\phi^i a_{T+h-i}\right)\left(\sum_{i=0}^{h-1}\phi^i a_{T+h-i}\right)'}_{\text{depends not on } \psi} + \underbrace{\left(\phi^h-\psi\right) y_T y_T' \left(\phi^h-\psi\right)}_{\text{minimised by } \psi=\phi^h}\right]$$