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Multivariate Time Series Analysis

Solution Exercise Sheet 7

1 Exercise 1: The optimal forecast

a) Show that the stationary VAR(1) process $z_t = t_{t-1} + a_t$ with a_t a standard white noise has the following causal representation:

$$z_t = \sum_{i=0}^{\infty} \theta_i a_{t-i}$$

Solution:

$$\begin{aligned}
z_t &= \phi \cdot \underbrace{z_{t-1}}_{\phi z_{t-2} + a_{t-1}} + a_t \\
&= \phi^2 z_{t-2} + \phi a_{t-1} + a_t \\
&= \phi^3 z_{t-3} + \phi^2 a_{t-2} + \phi a_{t-1} + a_t \\
&\vdots \\
&= \phi^m z_{t-m} + \sum_{i=0}^{m-1} \phi^i a_{t-i} \\
&= 0 + \sum_{i=0}^{m-1} \phi^i a_{t-i}\tilde{} \\
&\text{with } \lim_{m\to\infty} \phi^m = 0 \text{ by weak stationarity} \\
&= \sum_{i=0}^{\infty} \theta_i a_{t-i}
\end{aligned}$$

Using lag notation:

$$z_t = \phi L z_t + a_t$$

$$\Leftrightarrow (1-\phi L)z_t = a_t$$

$$\Leftrightarrow z_t = (1-\phi L)^{-1}a_t \text{and} \quad (1-\phi L)^{-1} = \sum_{i=0}^{\infty}\phi^i L^i$$

(requires stationarity and invertiability)

b) Assume the linear forecasting model $y_T(h) = \psi y_T$ ans show that $\psi = \phi^h$ minimieses the MSE of $y_T(h)$ given that y_t is a VAR(1) process.

Solution:

$$y_T(h) = \arg\min \underbrace{\text{MSE}(y_T(h))}_{\mathbb{E}\left([y_{T+h}-y_T(h)][y_{T+h}-y_T(h)]'\right)}$$

$$\begin{aligned}\rightarrow Y_{T+h} &= \phi Y_{T+h-1} + a_{T+h}\\ &\vdots\\ &= \phi^h Y_T + \sum_{i=0}^{h-1}\phi^i a_{T+h-i}\end{aligned}$$

$$\Rightarrow y_{T+h} - Y_T(h) = \phi^h Y_T + \sum_{i=0}^{h-1}\phi^i a_{T+h-1} - \psi y_T$$

$$\Rightarrow \text{MSE}(y_T(h)) = \mathbb{E}\left[\underbrace{\left(\sum_{i=0}^{h-1}\phi^i a_{T+h-i}\right)\left(\sum_{i=0}^{h-1}\phi^i a_{T+h-i}\right)'}_{\text{depends not on } \psi} + \underbrace{\left(\phi^h-\psi\right)y_T y_T'\left(\phi^h-\psi\right)}_{\text{minimised by } \psi=\phi^h}\right]$$

2 Exercise 2: Properties of forecast errors

a) Show that for a general VAR(p) process

$$z_{T+h} - z_T(h) = e_T(h) = \sum_{i=0}^{h-1}\theta_i a_{T+h-i}$$

where $z_T(h)$ is assumed to be the optimal forecast.

Solution:

$$z_T = \phi_0 + \phi_1 z_{T-1} + \ldots + \phi_p z_{T-p} + a_t$$
$$\text{and forecast: } z_{T-1}(1) = \phi_0 + \phi_1 z_{T-1} + \ldots + \phi_p z_{T-p}$$

$$\begin{aligned}
\Rightarrow Z_{T+1} - z_T(1) &= a_{T+1} = e_T(1) \\
\Rightarrow z_{T+2} - z_T(2) &= \phi_1 \underbrace{\left(z_{T+1} - z_T(1)\right)} + a_{T+2} \\
&= \phi_1 a_{T+1} + a_{T+2} \\
\Rightarrow z_{T+3} - z_T(3) &= \phi_1 \underbrace{\left(Z_{T+2} - z_T(2)\right)}_{\phi_1 a_{T+1} + a_{T+2}} + \phi_2 \underbrace{\left(z_{T+1} - Z_T(1)\right)}_{a_{T+2}} + a_{T+3} \\
&= \phi_1 \left(\phi_1 a_{T+1} + a_{T+2}\right) + \phi_2 a_{T+2} + a_{T+3} \\
&= \underbrace{\left(\phi_1^2 + \phi_2\right)}_{\theta_2} a_{T+1} + \underbrace{\phi_1}_{\theta_1} a_{T+2} + \underbrace{}_{I} a_{T+3} \\
&= \theta_2 a_{T+1} + \theta_1 a_{T+2} + \underbrace{\theta_0}_{I} a_{T+3} \\
\Rightarrow z_{T+h} - z_T(h) &= \theta_{h-1} a_{T+1} + \theta_{h-2} a_{T+2} + \ldots + \theta_1 a_{T+h-1} + \underbrace{\theta_0}_{I} a_{T+h}
\end{aligned}$$

b) Assume that $a_t \sim N\left(0, \Sigma_a\right)$. Derive the distribution of $e_T(h)$.

Solution:

$$\mathbb{E}\left[z_{T+h} - z_T(h)\right] = \sum_{i=0}^{h-1} \theta_i \underbrace{\mathbb{E}\left[a_{T+h-1}\right]}_{=0} = 0$$

$$\begin{aligned}
\operatorname{Cov}\left[z_{T+} - z_T(h)\right] &= \mathbb{E}\left[\left(\sum_{i=0}^{h-1} \theta_i a_{T+h-i}\right)\left(\sum_{j=0}^{h-1} \theta_j a_{T+h-j}\right)'\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{h-1} \theta_i \; a_{T+h-i} \; a_{T+h-j}^{'} \; \theta_i^{'}\right] \\
&= \sum_{i=0}^{h-1} \theta_i \; \mathbb{E}\left[\; a_{T+h-i} \; a_{T+h-j}^{'}\right] \; \theta_i^{'} \\
&= \sum_{i=0}^{h-1} \theta_i \; \Sigma_a \theta_i^{'} = \Sigma_e(h) \\
&\text{since} \mathbb{E}\left(a_{T+h-i} a_{T+h-j}\right) = 0 \text{if} j \neq i
\end{aligned}$$

By using the fact that a sum of i.i.d. normally distributed variables follows are normal distributed:

$$e_T(h) \sim \left(N, \Sigma_e(h)\right)$$

$$\text{with } \Sigma_e(h) = \sum_{i=0}^{h-1} \theta_i \Sigma_a \theta_i$$

c) Prove that $\mathrm{Cov}\left(e_T(h)\right) \to \Gamma_0$ as $h \to \infty$.

Solution:

$$\begin{aligned}
\lim_{h\to\infty} \mathrm{Cov}\left(e_T(h)\right) &= \mathbb{E}\left[\left(\sum_{i=0}^{\infty} \theta_i a_{T+h-i}\right)\left(\sum_{i=0}^{\infty} \theta_i a_{T+h-i}\right)^{'}\right] \\
&= \lim_{h\to\infty} \mathbb{E}\left(z_{T+h} z_{T+h}^{'}\right) \\
&= \lim_{h\to\infty} \Gamma_0^{T+h} = \Gamma_0
\end{aligned}$$

by weak stationary.

3 Exercise 3: Forecast intervals

Solution:

$$e_T(h) \sim N\left(0, \mathrm{Cov}(e_T(h))\right)$$

For each element it holds that:

$$\frac{e_T^{(i)}(h)}{\sqrt{\mathrm{Var}(e_T^{(i)}(h))}} \sim N(0,1)$$

$\Rightarrow$we need $\sqrt{\mathrm{Cov}(e_T(h))}$

↳ Cholesky decomposition of a positive deinifite matrix A

$A = UDU^{'}$, with D a diagonal matrix and U a lower traingular matrix

D can be split further into $D^{\frac{1}{2}} \cdot D^{\frac{1}{2}}$ (note that $D^{'} = D$ and $D^{\frac{1}{2}'} = D^{\frac{1}{2}}$)

$$\Rightarrow A = UD^{\frac{1}{2}}D^{\frac{1}{2}'} = UD^{\frac{1}{2}}\left(UD^{\frac{1}{2}}\right)^{'}$$

$$= LL^{'}$$

using $\mathrm{Cov}(e_T(h)) = \mathrm{Cov}(e_T(h))^{'}$ by symmetry

$$\Rightarrow e_T(h)' \cdot \mathrm{Cov}(e_T(h))^{-1} \cdot e_T(h) = e_T(h)' \underbrace{\mathrm{Cov}(e_T(h))^{-\frac{1}{2}'}}_{=L} \underbrace{\mathrm{Cov}(e_T(h))^{-\frac{1}{2}}}_{=L'} e_T(h)$$

$$= \underbrace{\left(e_T(h)\ \ \mathrm{Cov}(e_T(h))^{-\frac{1}{2}}\right)'}_{\sim N(0,I)} \underbrace{\left(e_T(h)\ \ \mathrm{Cov}(e_T(h))^{-\frac{1}{2}}\right)}_{\sim N(0,I)}$$

Both distributions are multivariate with K variables. Due to the inner product we have a sum of K squared standard normal variables.

$\Rightarrow$ this follows are χ^2_K distribution!

The ellipsid an then be set up:

$$\left\{z \in \mathbb{R}^K : e_T(h)' \, \mathrm{Cov}(e_T(h))^{-1} e_T(h) \leq \chi^2_{K,1-\alpha}\right\}$$

4 Exercise 4: Delta Method

For this task, assume both y_T and x_t to be $K \times 1$ vectors and $x_t \overset{i.i.d.}{\sim} [\mu_x, \Sigma_x]$.

a) Let $y_t = f(x_t) = \phi_1 x_t$. Compute the mean and variance of y_T.

Solution:

$$y_t = \phi_1 x_t$$

$$\begin{aligned} \mathbb{E} &= \mathbb{E}(\phi_1 x_t) = \phi_1 \mathbb{E}(x_t) = \phi_1 \mu_x \\ y_t - \mathbb{E}(y_t) &= \tilde{y}_t = \phi_1 (x_t - \mu_x) = \phi_1 \tilde{x}_t \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}(y_t) &= \mathrm{Cov}(\tilde{y}_t) = \mathrm{Cov}(\phi_1 \tilde{x}_t) \\ &= \phi_1 \mathbb{E}(\tilde{x}_t \tilde{x}_t') \phi_1' = \phi_1 \Sigma_x \phi_1' \end{aligned}$$

b) Derive the distribution of $\sqrt{T}\,(\bar{y_T} - E(y))$ from your results in a).

Solution:

$$x_t \quad \text{is i.i.d. distributed}, \mathbb{E}(x_t) < \infty, \mathrm{Cov}(x_t) < \infty$$

$$\Rightarrow \text{ a CLT applies!}$$
$$\sqrt{T}\left(\bar{Y}_T - \mathbb{E}(y)\right) \xrightarrow{d} N(0, \phi_1 \Sigma_x \phi_1^{'})$$

c) Now let $f(\cdot)$ be some function $f(x) : \mathbb{R}^K \mapsto \mathbb{R}^K$. Derive the first order Taylor expansion for $f(x)$ at μ_x and write it down in detail.

Solution:

$$\begin{aligned} f(x) &= f(x_1, \ldots, f_k) \\ &= \begin{pmatrix} f_1(x_1, \ldots, f_k) \\ \vdots \\ f_k(x_1, \ldots, f_k) \end{pmatrix} \end{aligned}$$

1st order Tylor expansion:

$$f(x) \approx f(\mu_x) + \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \vdots & & \\ \frac{\partial f_x}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_k} \end{pmatrix} \cdot \begin{pmatrix} x_1 - \mu_{x_1}^{(1)} \\ \vdots \\ x_k - \mu_{x_k}^{(k)} \end{pmatrix}$$

d) Based on the expression in c), show that a CLT applies for $\sqrt{T}\left(f\left(\bar{X_T}\right) - f\left(\mu_x\right)\right)$, and derive the distribution.

Solution:

$$\begin{aligned} \text{Use } x &= \bar{x_T} \\ \Rightarrow f(\bar{x_T}) &\approx f(\mu_x) + J(\bar{x_T} - \mu_x) \\ \Leftrightarrow f(\bar{x_T} - f(\mu_x)) &\approx J(\bar{x_T} - \mu_x) \\ \mathbb{E}\left(f\left(\bar{x_T}\right) - f\left(\mu_x\right)\right) &= \mathbb{E}\left(J \cdot \bar{x}_t - \mu_x\right) \\ &= J \cdot 0 = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \operatorname{Cov}\left(f\left(\bar{x_T}\right) - f\left(\mu_x\right)\right) &= \mathbb{E}\left(\left(f\left(\bar{x_T}\right) - f\left(\mu_x\right)\right)\left(f\left(\bar{x_T}\right) - f\left(\mu_x\right)\right)^{'}\right) \\ &= \mathbb{E}\left(J\left(\tilde{x_T} - \mu_x\right)\left(\tilde{x_T} - \mu_x\right)^{'} J^{'}\right) \\ &= J\ \mathbb{E}\left(\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_t\right)\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_t\right)^{'}\right) J \end{aligned}$$

$$
\begin{aligned}
&= J\ \mathbb{E}\left(\frac{1}{T}\left(\underbrace{\sum_{t=1}^{T}\tilde{x}_t\tilde{x}_t'}_{\Sigma_x}+\underbrace{\sum_{t\neq s}^{T}\tilde{x}_t\tilde{x}_s'}_{E(\cdot)=0}\right)\right)J\cdot\frac{1}{T}\\
&= \frac{1}{T}J\cdot\Sigma_x\cdot J'\\
&\Rightarrow \sqrt{T}\left(f(\bar{x}_T - f(\mu_x))\right)\xrightarrow{d} N(0, J\Sigma_x J')
\end{aligned}
$$

e) Lastly, assume the variable x_t to be known (meaning it is not stochastic). We want to predict y_t using $y_t = \phi_1 x_t$. Unfortunately, we only have $\widehat{\phi_1}$ which is stochastic with $\sqrt{T}\left(\widehat{\phi_1}-\phi_1\right)\xrightarrow{d} N(0,\Sigma_\phi)$. Can we say something about the distribution of the prediction error $\hat{y}_t - y_t$?

Solution:

$$
\begin{aligned}
\hat{y}_t - y_t &= \underbrace{\left(\widehat{\phi_1}-\phi_1\right)}_{\text{stochastic}}\underbrace{x_t}_{\text{deterministic}}\\
\mathbb{E}\left(\widehat{y_t}-y_t\right) &= \underbrace{\mathbb{E}\left(\widehat{\phi_1}-\phi_1\right)}_{=0}x_t = 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}(f(\bar{x_T})-f(\mu_x)) &= \mathrm{Cov}\left(\left(\widehat{\phi_1}-\phi_1\right)x_t\right)\\
&= x_t'\,\mathrm{Cov}\left(\widehat{\phi_1}-\phi_1\right)x_t\\
&= x_t'\Sigma_{\phi_1}x_t
\end{aligned}
$$

and since $\widehat{\phi_1}$ follows a normal distribution:

$$
\widehat{y_t} - y_t \sim N(0, x'\Sigma_{\phi_1}x)
$$