Exercise Sheet 3 continued

$\boxed{3}$ e) $Z_t = \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix}$:

$$\begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix} = \begin{pmatrix} \phi_0 \\ 0_{2\times1} \end{pmatrix} + \begin{pmatrix} \phi_1 & \phi_2 \\ I_{2\times2} & 0_{2\times2} \end{pmatrix} \begin{pmatrix} z_{t-1} \\ z_{t-2} \end{pmatrix} + \begin{pmatrix} a_t \\ 0_{2\times1} \end{pmatrix}$$

$$\overset{\mathbb{E}(\cdot)}{\Longrightarrow} \begin{pmatrix} \mu \\ \mu \end{pmatrix} = \begin{pmatrix} \phi_0 \\ 0_{2\times1} \end{pmatrix} + \begin{pmatrix} \phi_1 & \phi_2 \\ I_{2\times2} & 0_{2\times2} \end{pmatrix} \begin{pmatrix} \mu \\ \mu \end{pmatrix} + \begin{pmatrix} 0_{2\times1} \\ 0_{2\times1} \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} \mu \\ \mu \end{pmatrix} = \left[\begin{pmatrix} I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} \end{pmatrix} - \begin{pmatrix} \phi_1 & \phi_2 \\ I_{2\times2} & 0_{2\times2} \end{pmatrix} \right]^{-1} \begin{pmatrix} \phi_0 \\ 0_{2\times1} \end{pmatrix}$$

$$= \begin{pmatrix} 1-0.5 & -0.1 & 0 & 0 \\ -0.4 & 1-0.25 & -0.25 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{220}{37} \\ \frac{360}{37} \\ \frac{220}{37} \\ \frac{360}{37} \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} \frac{1}{1} \\ \frac{1}{1} \end{matrix}}\right\} \mu_2 \\ \left.\vphantom{\begin{matrix} \frac{1}{1} \\ \frac{1}{1} \end{matrix}}\right\} \mu_2 \end{matrix}$$

f) Like in task 1, part c):

$$\Gamma_0 = \Phi \, \Gamma_0 \, \Phi + \Sigma_b$$

with $\Phi = \begin{pmatrix} \phi_1 & \phi_2 \\ I_{2\times 2} & 0_{2\times 2} \end{pmatrix}$

and $$\tilde{z}_t \tilde{z}_t = \begin{pmatrix} \tilde{z}_t \\ \tilde{z}_{t-1} \end{pmatrix} \begin{pmatrix} \tilde{z}_t' & \tilde{z}_{t-1}' \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{z}_t \tilde{z}_t' & \tilde{z}_t \tilde{z}_{t-1}' \\ \tilde{z}_{t-1} \tilde{z}_t' & \tilde{z}_{t-1} \tilde{z}_{t-1}' \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \Gamma_0 & \Gamma_1 \\ \Gamma_1' & \Gamma_0 \end{pmatrix}}_{=: \Gamma_0^*}$$

gives: $$\left(I_{4\times 4} - \underline{\Phi} \otimes \underline{\Phi} \right) vec(\Gamma_0^*)$$

$$= vec(\Sigma_b)$$

$$\Leftrightarrow \operatorname{vec}(\Gamma_0^*) = (I_{4\times 4} - \Phi \otimes \Phi)^{-1} \operatorname{vec}(\Sigma_b)$$

$\Rightarrow$ extract top left or bottom right matrix

$\longrightarrow R$