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Multivariate Time Series Analysis

Solution Exercise Sheet 3

1 Exercise 1: VAR(1) Moments and Stationarity

Take the VAR(1) model $z_t = \phi_0 + \phi_1 z_{t-1} + a_t$ with the following parameterisation:

$$\phi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 0.75 & 0 \\ -0.25 & 0.5 \end{pmatrix}, \quad \Sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

a) Compute the mean of the process.

Solution:

$$\underset{\Rightarrow}{E(\cdot)} \quad \underbrace{\mathbb{E}(z_t)}_{\mu} = \mathbb{E}\left(\underbrace{\phi_0}_{\phi_0} + \underbrace{\phi_1 z_{t-1}}_{\phi_1 \cdot \mu} + \underbrace{a_t}_{0} \right)$$

[Key assumption? $\mathbb{E}(z_t) = \mathbb{E}(z_{t-1})$]

$$\Leftrightarrow (I - \phi_1) \cdot \mu = \phi_0$$

$$\Leftrightarrow \mu = (I - \phi_1)^{-1} \cdot \phi_0$$

plugging in ϕ_0 and ϕ_1 :

$$\mu = \begin{pmatrix} 0.25 & 0 \\ 0.25 & 0.5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

b) Show that the process is stationary.

Solution:

Eigenvalues of ϕ_1 :

$$
\begin{aligned}
|\phi_1 - I\lambda| &\overset{!}{=} 0 \\
&= \begin{vmatrix} 0.75-\lambda & 0 \\ -0.25 & 0.5-\lambda \end{vmatrix} \\
&= (0.75-\lambda)(0.5-\lambda) \overset{!}{=} 0 \\
\Rightarrow \lambda_1 &= 0.75 \\
\lambda_2 &= 0.5
\end{aligned}
$$

Both eigenvalues lie within the unit circle, therefore the process is stationary.

c) Derive the Yule-Walker equations for the lags $l = \{0, 1, 2\}$ and show that the solution for Γ_0 coincides with equation (2.3) on slide 2-15.

$$
\begin{aligned}
\tilde{z}_t &:= z_t - \mu \\
\Rightarrow \tilde{z}_t &= \phi_1 \tilde{z}_{t-1} + a_t && | \cdot \tilde{z}_{t-1}^{'} \\
\Leftrightarrow \tilde{z}_t \tilde{z}_t^{'} &= \phi_1 \tilde{z}_t \tilde{z}_t^{'} + a_t \tilde{z}_t^{'} \\
\overset{\mathbb{E}(\cdot)}{\Rightarrow} \Gamma_l &= \phi_1 \cdot \Gamma_{l-1} + \begin{cases} l & = 0 : \Sigma_a \\ l & \neq 0 : 0_k \end{cases} \\
l = 0 : \Gamma_0 &= \phi_1 \cdot \Gamma_{-1} + \Sigma_a \\
l = 1 : \Gamma_1 &= \phi_1 \cdot \Gamma_0 + 0_K \\
l = 2 : \Gamma_2 &= \phi_1 \cdot \Gamma_1 + 0_K \\
\text{using } \Gamma_{-1} &= \Gamma_1^{'} \\
\Gamma_0 &= \phi_1 \cdot (\phi_1 \Gamma_0)^{'} + \Sigma_a \\
\Leftrightarrow \Gamma_0 &= \phi_1 \Gamma_0^{'} \phi_1^{'} + \Sigma_a \\
\text{and } \Gamma_0^{'} &= \Gamma_0 \text{ since } \Gamma_0 \text{ is symmetric !}
\end{aligned}
$$

d) Compute Γ_0 and Γ_1 by hand based on your results from c).

$$\Gamma_0 = \Gamma_0'$$
$$\Rightarrow \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0.75 & 0 \\ -0.25 & 0.5 \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix} \begin{pmatrix} 0.75 & -0.25 \\ 0 & 0.5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} (0.75)^2\gamma_{11} & (-0.75 \cdot 0.25\gamma_{11}) + 0.5 \cdot 0.75\gamma_{12} \\ (0.75 \cdot 0.25\gamma_{11}) + 0.5 \cdot 0.75\gamma_{12} & (-0.25)^2\gamma_{11} - 0.25 \cdot 0.5\gamma_{12} - 0.5 \cdot 0.25\gamma_{12} + (0.5)^2\gamma_{22} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
\Rightarrow \gamma_{11} &= (0.75)^2\gamma_{11} + 1 \\
\Leftrightarrow \gamma_{11} &= \frac{16}{7} \\
\Rightarrow \gamma_{12} &= \frac{1}{4} \cdot \left(-\frac{3}{4}\right) \cdot \frac{16}{7} + \frac{2}{4} \cdot \frac{3}{4}\gamma_{12} \\
\Leftrightarrow \gamma_{12} &= -\frac{24}{35} \\
\Rightarrow \gamma_{22} &= \frac{1}{16}\gamma_{11} - 2 \cdot \frac{1}{2} \cdot \frac{1}{4}\gamma_{12} + \frac{1}{4}\gamma_{22} + 1 \\
&= \frac{1}{16} \cdot \frac{16}{7} - 2 \cdot \frac{1}{2} \cdot \frac{1}{4}\left(-\frac{24}{35}\right) + \frac{1}{4}\gamma_{22} + 1 \\
\gamma_{22} &= \frac{184}{105}
\end{aligned}$$

$$\begin{aligned}
\Gamma_1 &= \phi_1 \cdot \Gamma_0 \\
&= \begin{pmatrix} 0.75 & 0 \\ -0.25 & 0.5 \end{pmatrix} \cdot \begin{pmatrix} \frac{16}{7} & \frac{-24}{35} \\ \frac{-24}{35} & \frac{184}{105} \end{pmatrix} \\
&= \begin{pmatrix} \frac{12}{7} & \frac{-18}{35} \\ \frac{-32}{35} & \frac{22}{21} \end{pmatrix}
\end{aligned}$$

2 Exercise 2: Stationarity of VAR(p) Processes

Using the notation of Slide 2-27, prove that $|I_{kp} - \mathbf{\Phi_1} z| = |I_k - \phi_1 z - \ldots - \phi_p z^p|$. Recall that $|A|$ denotes the determinant of the matrix A

Hint: Derive $\mathbf{\Phi}_1$ and keep it mind that adding multiplies of columns/rows to other columns/rows does not affect the determinate! The plan is to end up with a special matrix.

Solution:

$$
\begin{aligned}
z_t =& \phi_0 + \phi_1 z_{t-1} + \ldots + \phi_p z_{t-p} + a_t \\
\text{as VAR (1):} Z_t =& \begin{pmatrix} \phi_0 \\ 0 \\ \vdots \end{pmatrix} + \Phi_1 Z_{t-1} + \begin{pmatrix} a_t \\ 0 \\ \vdots \end{pmatrix} \\
\text{with} Z_t =& \begin{pmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-p+1} \end{pmatrix} \text{ and } Z_{t-1} = \begin{pmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-p} \end{pmatrix} \\
\Phi_1 =& \begin{pmatrix} \phi_1 & \phi_2 & \ldots & \ldots & \phi_p \\ I_K & 0_K & \ldots & \ldots & 0_k \\ 0_K & I_K & 0_k & \ldots & 0_k \\ \vdots & \ddots & \ddots & & \vdots \\ 0_k & \ldots & 0_k & I_k & 0_k \end{pmatrix}
\end{aligned}
$$

$\Rightarrow$ Now one can simply apply the formula for a VAR(1) to check if the process is stationary.

$$
\begin{aligned}
|\Phi_1 - \lambda I| &\stackrel{!}{=} 0 \\
\Leftrightarrow (-1)^k |\Phi_1 - \lambda I| &\stackrel{!}{=} 0 \\
\Leftrightarrow \left|\lambda \left(I - \Phi_1 \frac{1}{\lambda}\right)\right| &\stackrel{!}{=} 0 && \text{since } \lambda \text{ is a scalar } I = I_{kp} \\
\Leftrightarrow \lambda^{kp} \left|I - \Phi_1 \frac{1}{\lambda}\right| &\stackrel{!}{=} 0 && \text{and let} \frac{1}{\lambda} =: z \\
\Rightarrow |I - \Phi_1 z| &\stackrel{!}{=} 0
\end{aligned}
$$

$\Rightarrow$ Stationarity if all $|z_i| > 1$ $(|\lambda_i| < 1)$

$$
I_{kp} - \Phi_1 z = \begin{pmatrix} I_k - \phi_1 z & -\phi_2 z & -\phi_3 z & \ldots & -\phi_{p-1} z & -\phi_p z \\ -I_k z & I_k & 0_k & \ldots & 0_k & 0_k \\ 0_k & -I_k z & I_k & \ldots & 0_k & \vdots \\ \vdots & 0_k & -I_k z & \ldots & \vdots & \vdots \\ 0_k & 0_k & 0_k & \ldots & -I_k z & I_k \end{pmatrix}
$$

$\Rightarrow -I_k \cdot z + I_k \cdot z = 0_k$

Since adding multiples of columns to other columns does not affect the determinant:

$\Rightarrow$ column "i" $\cdot z$ + column "i - 1" $\forall i \in \{p, \dots, 2\}$ yields a triangular matrix

$$\Rightarrow \begin{pmatrix} I_k - \phi_1 z & -\phi_2 z^2 & \dots & -\phi_p z & -\phi_1 & -\phi_2 z - \dots - \phi_p z^{p-1} \\ & 0_k & & & I_k & \\ & 0_k & & & & I_k \end{pmatrix}$$

$$\Rightarrow |I_{kp} - \Phi_1 z| = (I_K - \phi_1 z - \dots - \phi_p z^p) \cdot \prod_{i=1}^{p-1} I_k \qquad \text{since it is a triangular matrix}$$
$$= |I_k - \phi_1 z - \dots - \phi_p z^p|$$

3 Exercise 3: VAR(2) Moments and Stationarity

Consider the following VAR(2) model with i.i.d. innovations:

$$\phi_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.4 & 0.5 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 & 0 \\ 0.25 & 0 \end{pmatrix}, \quad \Sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

a) Show that the process is stationary.

Solution:

$$\begin{aligned} \left| I_2 - \phi_1 z - \phi_2 z^2 \right| &\overset{!}{=} 0 \\ \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5z & 0.1z \\ 0.4z & 0.5z \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0.25z & 0 \end{pmatrix} &\overset{!}{=} 0 \\ \Leftrightarrow \left| \begin{pmatrix} 1 - 0.5z & -0.1z \\ -0.4z - 0.25z^2 & 1 - 0.5z \end{pmatrix} \right| &\overset{!}{=} 0 \\ = (1 - 0.5z)^2 - (0.4z + 0.25z^2) \cdot 0.1z &\overset{!}{=} 0 \\ = 1 - z + 0.21z^2 - 0.025z^3 &\overset{!}{=} 0 \end{aligned}$$

```r
roots <- polyroot(c(1, -1, 0.21, 0.025))
roots # there are some imaginary parts attached to it
```

```
## [1]   1.804197+0.27546i   1.804197-0.27546i -12.008393+0.00000i
```

```r
sum(abs(roots) < 1) # count how many roots lie inside the unit circle
```

```
## [1] 0
```

Since all roots are outside the unit circle the process is stationary. Alternatively, we could also use the VAR(1) approach and compute the eigenvalues.

```r
phi_1 <- matrix(data = c(0.5, 0.4, 0.1, 0.5), nrow = 2)
phi_2 <- matrix(data = c(0, 0.25, 0, 0), nrow = 2)
I2x2 <- diag(2)
O2x2 <- matrix(data = rep(0, 4), nrow = 2)
Phi <- rbind( cbind(phi_1, phi_2), cbind(I2x2, O2x2) )
Phi
```

```
##      [,1] [,2] [,3] [,4]
## [1,]  0.5  0.1 0.00    0
## [2,]  0.4  0.5 0.25    0
## [3,]  1.0  0.0 0.00    0
## [4,]  0.0  1.0 0.00    0
```

```r
var1.eigen <- eigen(Phi)
sum(abs(var1.eigen$values) < 1) # How many eigenvalues lie inside the unit circle?
```

```
## [1] 4
```

All 4 eigenvalues are within the unit circle, so the process is stationary and we get the same result as before.

b) Determine the mean vector.

Solution:

$$
\begin{aligned}
z_t &= \phi_0 + \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t \\
\overset{E(\cdot)}{\Rightarrow} \ldots \Rightarrow \mu &= (I - \phi_1 - \phi_2)^{-1} \phi_0 \\
&= \begin{pmatrix} 1-0.5 & -0.1 \\ -0.4-0.25 & 1-0.5 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\end{aligned}
$$

```r
phi_0 <- c(2,1)
mu <- solve((I2x2 - phi_1 - phi_2)) %*% phi_0
fractions(mu)
```

```
##        [,1]
## [1,] 220/37
## [2,] 360/37
```

c) Derive the Yule-Walker equations for the lags $l = \{0, 1, 2\}$ for a general VAR(2) process.

Solution:

$$\mathbb{E}\left(\tilde{z}_t \tilde{z}'_{t-l}\right) = \phi_1 \mathbb{E}\left(\tilde{z}_t \tilde{z}'_{t-l}\right) + \phi_2 \mathbb{E}\left(\tilde{z}_t \tilde{z}'_{t-l}\right) + \mathbb{E}\left(a_t \tilde{z}'_{t-l}\right)$$

$$\begin{aligned}
\Rightarrow l = 0 &: \Gamma_0 = \phi_1 \Gamma_{-1} + \phi_2 \Gamma_{-2} + \Sigma_a \\
l = 1 &: \Gamma_1 = \phi_1 \Gamma_0 + \phi_2 \Gamma_{-1} + 0_{2\times 2} \\
l = 2 &: \Gamma_1 = \phi_1 \Gamma_1 + \phi_2 \Gamma_0 + 0_{2\times 2}
\end{aligned}$$

d) Suppose we only knew Γ_0, Γ_1 and Γ_2 - how can we estimate ϕ_1 and ϕ_2 from it?

Solution:

$$\begin{aligned}
\overbrace{\begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix}}^{\text{row vector}} &= \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \Gamma_0 & \Gamma_1 \\ \Gamma'_1 & \Gamma_0 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} &= \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix}^{-1} \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix}
\end{aligned}$$

e) Write the process as a VAR(1) and calculate the mean vector again.

Solution:

$$Z_t = \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix}$$

$$\begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix} = \begin{pmatrix} \phi_0 \\ 0_{2\times 1} \end{pmatrix} + \begin{pmatrix} \phi_1 & \phi_2 \\ I_{2\times 2} & 0_{2\times 2} \end{pmatrix} \begin{pmatrix} z_{t-1} \\ z_{t-2} \end{pmatrix} + \begin{pmatrix} a_t \\ 0_{2\times 1} \end{pmatrix}$$

$$
\begin{aligned}
\overset{\mathbb{E}(\cdot)}{\Rightarrow} \begin{pmatrix} \mu \\ \mu \end{pmatrix} &= \begin{pmatrix} \phi_0 \\ 0_{2\times 1} \end{pmatrix} + \begin{pmatrix} \phi_1 & \phi_2 \\ I_{2\times 2} & 0_{2\times 2} \end{pmatrix} \begin{pmatrix} \mu \\ \mu \end{pmatrix} + \begin{pmatrix} 0_{2\times 2} \\ 0_{2\times 2} \end{pmatrix} \\
\begin{pmatrix} \mu \\ \mu \end{pmatrix} &= \left[\begin{pmatrix} I_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & I_{2\times 2} \end{pmatrix} - \begin{pmatrix} \phi_1 & \phi_2 \\ I_{2\times 2} & 0_{2\times 2} \end{pmatrix} \right]^{-1} \begin{pmatrix} \phi_0 \\ 0_{2\times 2} \end{pmatrix} \\
&= \begin{pmatrix} 1-0.5 & -0.1 & 0 & 0 \\ -0.4 & 1-0.5 & -0.25 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

```r
# Alternative solution to b) using the VAR(1) representation:
mu2 <- solve(diag(4) - Phi) %*% c(phi_0, rep(0,2))
fractions(mu2)
```

```
##        [,1]
## [1,] 220/37
## [2,] 360/37
## [3,] 220/37
## [4,] 360/37
```

f) Compute Γ_0 based on the VAR(1) formulation.

Hint: You can use R for the calculations.

Solution:

$$\Gamma_0 = \Phi \Gamma_0 \Phi + \Sigma_b$$

$$\text{with} \Phi = \begin{pmatrix} \phi_1 & \phi_2 \\ I_{2\times 2} & 0_{2\times 2} \end{pmatrix}$$

$$
\begin{aligned}
\text{and } \tilde{Z}_t \tilde{Z}_t &= \begin{pmatrix} \tilde{z}_t \\ \tilde{z}_{t-1} \end{pmatrix} \begin{pmatrix} \tilde{z}_t & \tilde{z}_{t-1} \end{pmatrix} \\
&= \begin{pmatrix} \tilde{z}_t \tilde{z}_t' & \tilde{z}_t \tilde{z}_{t-1}' \\ \tilde{z}_{t-1} \tilde{z}_t' & \tilde{z}_{t-1} \tilde{z}_{t-1}' \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} \Gamma_0 & \Gamma_1 \\ \Gamma_1' & \Gamma_0 \end{pmatrix}}_{=:\Gamma_0^*}
\end{aligned}
$$

$$\text{gives: } (I_{4\times 4} - \Phi \otimes \Phi) \ \text{vec}(\Gamma_0^*) = \ \text{vec}(\Sigma_b)$$
$$\Leftrightarrow \text{vec}(\Gamma_0^*) = (I_{4\times 4} - \Phi \otimes \Phi)^{-1} \ \text{vec}(\Sigma_b)$$

$\Rightarrow$ extract top left or bottom right matrix

```
Sigma_a <- diag(2)
Sigma_b <- rbind( cbind(Sigma_a, O2x2), cbind(O2x2, O2x2) )
Gamma0ast.mat <- matrix(solve(diag(16) - Phi %x% Phi) %*% as.vector(Sigma_b)
                         , nrow = 4)

Gamma0ast.mat
```

```
##           [,1]     [,2]      [,3]      [,4]
## [1,] 1.4776581 0.800842 0.8189132 0.6820146
## [2,] 0.8008420 2.815935 1.1962126 2.0273576
## [3,] 0.8189132 1.196213 1.4776581 0.8008420
## [4,] 0.6820146 2.027358 0.8008420 2.8159354
```