Exercise Sheet 13

1

a) We know how a VECM for x_t looks like, so let's start from there:

$$\Delta x_t = \alpha\beta' x_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta x_{t-j} + a_t$$

Now use:

$$\Delta z_t = \cancel{\mu_0 - \mu_0} + \mu_1 t - \mu_1 (t-1) + x_t - x_{t-1}$$
$$= \mu_1 + \Delta x_t$$

$$\Rightarrow \Delta z_t - \mu_1 = \alpha\beta' (z_{t-1} - \mu_0 - \mu_1 (t-1))$$
$$+ \sum_{j=1}^{p-1} \Gamma_j (\Delta z_{t-j} - \mu_1) + a_t$$

$$\Leftrightarrow \Delta z_t = \underbrace{\underbrace{-\alpha\beta' \mu_0}_{\nu_0} + \left(I - \sum_{j=1}^{p-1} \Gamma_j\right) \mu_1}_{=: \nu}$$

$$+ \alpha \underbrace{\left(-\beta' \mu_1\right)}_{\eta'} (t-1)$$

$$\underbrace{+ \alpha\beta' z_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta z_{t-j} + a_t}_{\text{case 1}}$$

$$= \nu + \alpha \eta' (t-1)$$

$$+ \alpha\beta' z_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta z_{t-j} + a_t$$

b) $\mu_1 = 0$ to set $\eta' = 0_{1\times 2}$

in order to rule out a trend in the differences. Either $\mu_0 = 0$

as well (case 1) or

$\nu^* = \nu_0 = -\alpha\beta' \mu_0$ (case 2).

$\Rightarrow$ Check mean of $\beta' z_t$ for a decision!

c) Check two indicators:

1. Does the trend appear quadratic?
2. Does $\beta' z_t$ entail a trend?

<u>If (No, No) $\Rightarrow$ case 3:</u>

$\mu_1 = 0$, $\mu_0 \neq 0$ <u>but</u> the interrupt $\nu^* \neq \nu_0$!

(ν^* cannot be absorbed in the

cointegrating relationship.)

If $(N_0, Y_{cs}) \Rightarrow$ case 4:

$\mu_1 \neq 0$, $\mu_0 = ?$ and

ν, η' as defined in a).

(Restrictions for trend and intercept have to hold.)

If $(Y_{es}, Y_{es}) \Rightarrow$ Case 5:

Drop μ_1, $\nu = \nu_0$, add $\nu_1 \cdot t$ as new trend in the differences. (A linear trend is then integrated to a quadratic one.)

d) $$\begin{pmatrix} \Delta z_{1,t} \\ \Delta z_{2,t} \end{pmatrix} = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \boxed{\Pi_{21}} \overset{!}{=} 0 & \Pi_{22} \end{pmatrix} \begin{pmatrix} z_{1,t-1} \\ z_{2,t-1} \end{pmatrix} + \sum_{j=1}^{p-1} \begin{pmatrix} \rho_{11,j} & \rho_{12,j} \\ \boxed{\rho_{21,j}} \overset{!}{=} 0 & \rho_{22,j} \end{pmatrix} \begin{pmatrix} \Delta z_{1,t-j} \\ \Delta z_{2,t-j} \end{pmatrix} + \nu + \alpha\eta'(t-1) + a_t$$

Practically the same conditions as for a stationary VAR.
(You could even derive the non-stationary VAR and inspect ϕ_1.)

$\boxed{2}$

Basic concept for Simulation

Set $z_0 = 0$.

Define $\alpha, \beta, \mu_0, \mu_1$ and so on.
(Compute ν, η if needed.)

Iterate for $t = 1, \ldots, T$:

→ Draw a_t
→ Compute Δz_t
→ obtain $z_t = \Delta z_t + z_{t-1}$

Regarding $\alpha_\perp$

$\alpha_\perp' \alpha = 0 \Rightarrow \alpha_\perp'$ is <u>not</u>

uniquely determined!

(e.g. $(-\alpha_\perp')\alpha = 0$ holds, too)

$\rightarrow R$

$\boxed{3}$ a) $\Pi = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow rk(\Pi) < 2$, at least z_t is not a two-dimensional stationary process.

(Note that $\left| I - \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \right|$

$= \begin{vmatrix} 1 & -2 \\ 0 & 1-2 \end{vmatrix} = (1-2) \overset{!}{=} 0$

which means there is actually a unit root!)

b) $\alpha\beta' = \Pi \Rightarrow \beta' = (1 \quad -1)$

$\quad \alpha = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ ✓

c) Back to the VAR formulation:

$\Phi_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \Theta_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Theta_1 = \phi_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\Theta_2 = \phi_1^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \Theta_j = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\forall j \geq 1$$

$$\Rightarrow \underline{\Sigma_{e_T}(h)} = \sum_{j=0}^{h-1} \Theta_j \Sigma_a \Theta_j'$$

$$= \sum_{j=0}^{h-1} \Theta_j \Theta_j'$$

$$= I_{2\times 2} + (h-1) \overbrace{\phi_1 \phi_1'}^{= \Theta_1 \Theta_1'}$$

$$\underline{= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (h-1) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$$

$$\lim_{h \to \infty} \underbrace{(h-1)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}}_{} + \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\longrightarrow \begin{pmatrix}+\infty & +\infty\\ +\infty & +\infty\end{pmatrix}$$

d) $\beta' z_t = \beta' \left(\phi_1 z_{t-1} + a_t \right)$

$t := T+h$

$$= \underbrace{\boxed{(1\ \ {-1})\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}}}_{=(0\ \ 0)} \begin{pmatrix}z_{1,t-1}\\ z_{2,t-1}\end{pmatrix} + (1\ \ {-1})\begin{pmatrix}a_{1,t}\\ a_{2,t}\end{pmatrix}$$

$$= a_{1,t} - a_{2,t}$$

$$\mathbb{E}(\beta' z_{T+h} \mid \beta' z_T) = \mathbb{E}(a_{1,T+h} - a_{2,T+h} \mid a_{1,T}, a_{2,T})$$

$$= 0$$

$$\Rightarrow y_T(h) = \mathbb{E}(y_{T+h} \mid y_T) = \mathbb{E}(\beta' z_{T+h} \mid \beta' z_T)$$

$$= 0$$

<u>Confidence Interval</u>

$$\{ y_T(h) - z_{\frac{\alpha}{2}} \cdot Var(y_T(h) - y_{T+h}) \; ;$$

$$y_T(h) + z_{\frac{\alpha}{2}} \cdot Var(y_T(h) - y_{T+h}) \}$$

$$= \{ 0 - z_{\frac{\alpha}{2}} \cdot 2 \; ; \; 0 + z_{\frac{\alpha}{2}} \cdot 2 \}$$

since $Var(y_T(h) - y_{T+h})$

$$= Var(a_{1,T} - a_{2,T}) = Var(a_{1,T}) + Var(a_{2,T}) - 2\,Cov(a_{1,T}, a_{2,T})$$

$$= 1 + 1 - 0 = 2$$

$\boxed{4}$ a) $\phi_1 = \Pi + I$

$$= \begin{pmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$$

$$\Rightarrow z_t = \Delta z_t + z_{t-1} = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix} z_{t-1} + a_t$$

b) $$|I - \phi_1 z| = \begin{vmatrix} 1-0.9z & -0.1z \\ -0.1z & 1-0.9z \end{vmatrix}$$

$$= (1-0.9z)^2 - (0.1z)^2$$

$$= 1 - 1.8z + 0.8z^2 \overset{!}{=} 0$$

$$\rightarrow z_1 = 1, \quad z_2 = 1.25$$

c) $\Theta_0 = I, \quad \Theta_i = \phi_1^i, \quad \forall i \geq 1$

$$z_T(h) = \mathbb{E}(z_{T+h} \mid z_T) = \phi_1^h z_T$$

$$\Rightarrow e_T(h) = \sum_{i=0}^{h-1} \Theta_i \, a_{T+h-i}$$

$$\Rightarrow \Sigma_e(h) = \sum_{i=0}^{h-1} \Theta_i \Sigma_a \Theta_i'$$

The confidence interval is centered around $\phi_1^h z_T$ and the ellipsoid is determined by

$$\left\{ z \in \mathbb{R}^2 : (z_T(h) - z)' \Sigma_e^{-1}(h) (z_T(h) - z)' \leq \chi^2_{k, 1-\alpha} \right\}$$

d) $\begin{pmatrix} Z_{1,T+h} \\ Z_{2,T+h} \end{pmatrix} = \Theta_h \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Thus it depends on

$\Theta_h = \Phi_1^h$. Since $Z_{1,t} \rightarrow Z_{2,t+h}$

(Granger causation), there is an effect. And because the variables are cointegrated, this effect is permanent.

(It would be different if $Z_{1,t}$ was a random walk and $Z_{2,t}$ some independent stationary process.)

Note that $\lim_{h \to \infty} \phi_1^h = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$.

The same argument for

$z_{2,T} \longrightarrow z_{1,T+h}$.

$\boxed{\text{3 a) con't}}$

$$z_t = \phi_1 z_{t-1} + a_t$$

$$\Leftrightarrow z_t - z_{t-1} = \underbrace{(\phi_1 - I)}_{=: \Pi} z_{t-1} + a_t$$

$$\iff \underbrace{\Delta z_t = \Pi z_{t-1} + a_t}_{= \text{VECM}}$$