Universität Duisburg-Essen
Fachbereich VWL
Lehrstuhl Ökonometrie

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Dr. Yannick Hoga Thilo Reinschlüssel

Multivariate Time Series Analysis
Solution Sheet 1

Exercise 1: Matrix Operations

Prove properties 3,4 and 5 from Proposition 1.2 (Slide 1-11). Are there any requirements regarding the matrix dimensions?

Solution:

i) Property 3: $(A \otimes B)(F \otimes G) = (AF) \otimes (BG)$

Let $A = \begin{pmatrix} a_{11} & \dots & a_{1q} \\ \vdots & \ddots & \vdots \\ a_{p1} & \dots & a_{pq} \end{pmatrix}$ and $F = \begin{pmatrix} f_{11} & \dots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \dots & f_{mn} \end{pmatrix}$,

hence $(A \otimes B) = \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{pmatrix}$ and $(F \otimes G)$ analogously.

$$\Rightarrow (A \otimes B)\,(F \otimes G) = \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{pmatrix} \begin{pmatrix} f_{11}G & \dots & f_{1n}G \\ \vdots & \ddots & \vdots \\ f_{m1}G & \dots & f_{mn}G \end{pmatrix}$$

$$= \begin{pmatrix} (a_{11}Bf_{11}G + \dots + a_{1q}Bf_{m1}G) & \dots & (a_{11}Bf_{1n}G + \dots + a_{1q}Bf_{mn}G) \\ \vdots & \ddots & \vdots \\ (a_{p1}Bf_{11}G + \dots + a_{pq}Bf_{m1}G) & \dots & (a_{p1}Bf_{1n}G + \dots + a_{pq}Bf_{mn}G) \end{pmatrix}$$

$$= \begin{pmatrix} (a_{11}f_{11} + \dots + a_{1q}f_{m1}) & \dots & (a_{11}f_{1n} + \dots + a_{1q}f_{mn}) \\ \vdots & \ddots & \vdots \\ (a_{p1}f_{11} + \dots + a_{pq}f_{m1}) & \dots & (a_{p1}f_{1n} + \dots + a_{pq}f_{mn}) \end{pmatrix} \otimes (BG)$$

$$= \begin{pmatrix} \sum_{i=1}^{q=m} a_{1i}f_{i1} & \dots & \sum_{i=1}^{q=m} a_{1i}f_{in} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{q=m} a_{pi}f_{i1} & \dots & \sum_{i=1}^{q=m} a_{pi}f_{in} \end{pmatrix} \otimes (BG)$$

$$= (AF) \otimes (BG)$$

Dimensions:

$A : p \times q$	$F : m \times n$
$B : c \times d$	$G : h \times k$

$\Rightarrow dim(A \otimes B) = pc \times qd\ ,\ dim(F \otimes G) = mh \times kn$

ii) Property 4: $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
$\Rightarrow$ Claim and verify
The inverse is defined as following:
$(A \otimes B)\,(A \otimes B)^{-1} = I$ where I is the identity matrix
Then $(A \otimes B)\left(A^{-1} \otimes B^{-1}\right) = I$ must hold if the claim was true.
We know from Property 3 that $(A \otimes B)\left(A^{-1} \otimes B^{-1}\right) = \left(AA^{-1} \otimes BB^{-1}\right) = I \otimes I = I$.
Dimension: A and B must be non-singular square matrices.

iii) Property 3: $tr(A \otimes C) = tr(A) \cdot tr(C)$ for square matrices A and C

$$tr(A \otimes C) = tr \begin{pmatrix} a_{11}C & \dots & a_{1n}C \\ \vdots & \ddots & \vdots \\ a_{n1}C & \dots & a_{nn}C \end{pmatrix} = \sum_{i=1}^{n} (a_{ii}\,tr(C)) = tr(C)\,\sum_{i=1}^{n} a_{ii} = tr(C)\,tr(A)$$

Exercise 2: Bivariate Functions

Find the extrema of the following functions (using pen and paper). Determine whether these points constitute minima, maxima or saddle points:

a) $f(x,y) = (x-2)^2 + (y-5)^2 + xy$

b) $g(x,y) = (x-1)^3 - (4y+1)^2$

Solution: Solution concept:

1. FOC: first derivatives $\stackrel{!}{=} 0$

2. SOC: check Hessian (its determinant)

a)

$$\begin{aligned} f(x,y) =& (x-2)^2 + (y-5)^2 + xy \\ f_x(x,y) =& 2(x-2) + y \stackrel{!}{=} 0 \\ f_y(x,y) =& 2(y-5) + x \stackrel{!}{=} 0 \Leftrightarrow 2y - 10 + 2 - \frac{y}{2} = 0 \Leftrightarrow y* = \frac{16}{3} \end{aligned}$$

Solving the equation system yields:

$$\begin{aligned} x = 2 - \frac{y}{2} \Rightarrow 2y - 10 + 2 - \frac{y}{2} = 0 \Rightarrow y* = \frac{16}{3} \\ \Rightarrow x* = 2 - \frac{16}{3 \cdot 2} = -\frac{2}{3} \end{aligned}$$

Evaluating the Hessian to determine whether it is a minimum, maximum or saddle point:

$$f_{xx}(x,y) = 2 \qquad f_{xy}(x,y) = 1$$

$$f_{yx}(x,y) = 1 \qquad f_{yy}(x,y) = 2$$

$\Rightarrow H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $det(H) = 2 \cdot 2 - 1 \cdot 1 = 3 > 0$ which indicates a minimum.

b)

$$\begin{aligned} g(x,y) =& (x-1)^3 - (4y+1)^2 \\ g_x(x,y) =& 3(x-1)^2 \stackrel{!}{=} 0 \Leftrightarrow x* = 1 \\ g_y(x,y) =& 2 \cdot 4(4y+1) \stackrel{!}{=} 0 \Leftrightarrow y* = -\frac{1}{4} \end{aligned}$$

$\begin{matrix} g_{xx}(x,y) = 6x - 6 & g_{xy}(x,y) = 0 \\ g_{yx}(x,y) = 0 & g_{yy}(x,y) = 32 \end{matrix} \quad \Rightarrow det(H)\Big|_{x=x*,y=y*} = (6-6) \cdot 32 - 0 \cdot 0 = 0$ indication a saddle point. Thus we did not find an extremal point!

Exercise 3: Stationarity

a) Are weakly stationary processes always strictly stationary? Construct an example to support your argument

b) Is weak stationarity a necessary condition for strict stationarity? Bring an example.
Hint: How many moments does a distribution require?

Solution:

a) No. A time series of length T drawing from $N(0,1)$ for $t \in \left[0, \frac{T}{2}\right]$ and drawing from Student's t-distribution for $t \in \left(\frac{T}{2}, T\right]$ has a constant mean $\mu = 0$ and variance $\sigma^2 = 1$, but the kurtosis (4^{th} moment) changes throughout time. In consequence the joint distribution of a subsequence $x_{t-p}, \ldots x_{t+q}$ is not independent of t. Therefore it is not strictly stationary.

b) No. Take the Cauchy distribution as an example: $f(x) = \frac{1}{\pi} \cdot \frac{s}{s^2+(x-t)^2}$. Any i.i.d. sample from this distribution would be obviously strictly stationary. Yet this distribution has no existing moments at all (the integral diverges), hence it cannot exhibit a constant expected value or variance over time. Therefore it is only strictly stationary, but not weakly stationary! (Other example: t_1 distribution, where only the mean but not the variance exists).

Exercise 4: Covariance Matrices under Stationarity

Referring to Remark 1.13: Show that $\Gamma_l = \Gamma_{-l}^\top$ holds for all weakly stationary processes.
(Two dimensions suffice)

Solution: Without loss of generality assume $\mu = 0$ everywhere and assume z to be a bivariate vector $(x,y)^\top$. Let $\Gamma_{l,t}$ be the covariance matrix of the l^{th} lag at time t:

$$\Gamma_{l,t} = \begin{bmatrix} \mathbb{E}\left(x_t \cdot x_{t-l}\right) & \mathbb{E}\left(x_t \cdot y_{t-l}\right) \\ \mathbb{E}\left(y_t \cdot x_{t-l}\right) & \mathbb{E}\left(y_t \cdot y_{t-l}\right) \end{bmatrix} \quad \text{and} \quad \Gamma_{l,t}^\top = \begin{bmatrix} \mathbb{E}\left(x_{t-l} \cdot x_t\right) & \mathbb{E}\left(x_{t-l} \cdot y_t\right) \\ \mathbb{E}\left(y_{t-l} \cdot x_t\right) & \mathbb{E}\left(y_{t-l} \cdot y_t\right) \end{bmatrix} = \Gamma_{-l,t-l}$$

Since weak stationarity has been assumed, the covariance matrix is constant across time and $\Gamma_{-l,t-l} = \Gamma_{-l} = \Gamma_l^T$ and vice versa.

Exercise 5: Ljung-Box Test in R

Load the package *MTS* and open the associated data pool 'mts-examples' (Slide 1-8). We are interested in the time series 'GS', 'MS' and 'JPM' from the dataset 'tenstocks':

a) First apply the Ljung-Box test on each time series individually. What do the results imply?

b) Now apply the multivariate Ljung-Box test on all three time series together. Compare the results with those from the univariate test and comment on it.

Solution: Look at the R code in this folder. (If there isn't any, though luck!)

This exercise sheet will be discussed in the exercise course on Wednesday, 23 October 2019