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Multivariate Time Series Analysis

Exercise Sheet 1

1 Exercise 1: Matrix Operations

Prove properties 3,4 and 5 from Proposition 1.2 (Slide 1-11). Are there any requirements regarding the matrix dimensions?

Solution:

i) Property 3: $(A \otimes B)(F \otimes G) = (AF) \otimes (BG)$

$$\text{Let } A = \begin{pmatrix} a_{11} & \dots & a_{1q} \\ \vdots & \ddots & \vdots \\ a_{p1} & \dots & a_{pq} \end{pmatrix} \text{ and } F = \begin{pmatrix} f_{11} & \dots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \dots & f_{mn} \end{pmatrix}$$

$$\text{hence}(A \otimes B) = \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{pmatrix} \text{ and } (F \otimes G) \text{ analsgously}$$

$$\begin{aligned}
(A \otimes B)(F \otimes G) &= \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{pmatrix} \begin{pmatrix} f_{11}G & \dots & f_{1n}G \\ \vdots & \ddots & \vdots \\ f_{m1}G & \dots & f_{mn}G \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}Bf_{11}G + \ldots + a_{1q}Bf_{m1}G) & \dots & (a_{11}Bf_{1n}G + \ldots + a_{1q}Bf_{mn}G) \\ \vdots & \ddots & \vdots \\ (a_{p1}Bf_{11}G + \ldots + a_{pq}Bf_{m1}G) & \dots & (a_{p1}Bf_{1n}G + \ldots + a_{pq}Bf_{mn}G) \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}f_{11} + \ldots + a_{1q}f_{m1}) & \dots & (a_{11}f_{1n} + \ldots + a_{1q}f_{mn}) \\ \vdots & \ddots & \vdots \\ (a_{p1}f_{11} + \ldots + a_{pq}f_{m1}) & \dots & (a_{p1}f_{1n} + \ldots + a_{pq}f_{mn}) \end{pmatrix} \otimes (BG)
\end{aligned}$$

$$= \begin{pmatrix} \sum_{i=1}^{q=m} a_{1i} f_{i1} & \dots & \sum_{i=1}^{q=m} a_{1i} f_{1i} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{q=m} a_{pi} f_{i1} & \dots & \sum_{i=1}^{q=m} a_{pi} f_{in} \end{pmatrix} \otimes (BG)$$

$$= (AF) \otimes (BG)$$

Dimensions:

$A : p \times q$	$F : m \times n$
$B : c \times d$	$G : h \times k$

$$\Rightarrow \; dim(A \otimes B) = pc \times qd, dim(F \otimes G) = mh \times kn$$

ii) Property 4: $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
$\Rightarrow$ Claim and verify
The inverse is defined as following:
$(A \otimes B)(A \otimes B)^{-1} = I$ where I is the identitiy matrix
Then $(A \otimes B)(A^{-1} \otimes B^{-1}) = I$ must hold if the claim was true
We know from Property 3 that $(A \otimes B)(A^{-1} \otimes B^{-1}) = (AA^{-1} \otimes BB^{-1}) = I \otimes I = I$
Dimensions: A and B must be non-singular square matrices

iii) Property 3: $tr(A \otimes C) = tr(A) \cdot tr(C)$ for square matrices A and C

$$tr(A \otimes C) = tr \begin{pmatrix} a_{11}C & \dots & a_{1n}C \\ \vdots & \ddots & \vdots \\ a_{n1}C & \dots & a_{nn}C \end{pmatrix} = \sum_{i=1}^{n} (a_{ii} tr(C)) = tr(C) \sum_{i=1}^{n} a_{ii} = tr(C) tr(A)$$

2 Excerise 2: Bivariate Functions

Find the extrema of the following functions (using pen and paper). Determine whether these points constitute minima, maxima or saddle points:

a) $f(x, y) = (x - 2)^2 + (y - 5)^2 + xy$

b) $g(x, y) = (x - 1)^3 - (4y + 1)^2$

Solution:
Soluton concept:

1. FOC: first derivatives $\overset{!}{=} 0$
2. SOC: check the determinant of the Hessian matrix

a) $f(x, y) = (x-2)^2 + (y-5)^2 + xy$

$$f(x, y) = (x-2)^2 + (y-5)^2 + xy$$

$$\frac{\partial f(x,y)}{\partial x} = 2(x-2) + y \stackrel{!}{=} 0 \frac{\partial f(x,y)}{\partial y} \qquad = 2(y-5) + x \stackrel{!}{=} 0$$

– Solving the equation system yields:

$$x = 2 - \frac{y}{2} \Rightarrow 2y - 10 + 2 - \frac{y}{2} = 0 \Rightarrow y^* = \frac{16}{3}$$
$$\Rightarrow x^* = 2 - \frac{16}{3 \cdot 2} = -\frac{2}{3}$$

– Evaluting the Hessian matrix:

$$\frac{\partial f(x,y)}{\partial x^2} = 2 \qquad \frac{\partial f(x,y)}{\partial xy} = 1$$
$$\frac{\partial f(x,y)}{\partial yx} = 1 \qquad \frac{\partial f(x,y)}{\partial y^2} = 2$$
$$\Rightarrow H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $det(H) = 2 \cdot 2 - 1 \cdot 1 = 3 > 0$ which indicates a minimum

b) $g(x, y) = (x-1)^3 - (4y+1)^2$

$$g(x, y) = (x-1)^3 + (4y-1)^2$$

$$\frac{\partial g(x,y)}{\partial x} = 3(x-1)^2 \stackrel{!}{=} 0 \Leftrightarrow x^* = 1$$
$$\frac{\partial g(x,y)}{\partial y} = 2 \cdot 4(4y-5) + x \stackrel{!}{=} 0 \Leftrightarrow y^* = -\frac{1}{4}$$

– Evaluting the Hessian matrix:

$$\frac{\partial g(x,y)}{\partial x^2} = 6x - 6 \qquad \frac{\partial g(x,y)}{\partial xy} = 0$$
$$\frac{\partial g(x,y)}{\partial yx} = 0 \qquad \frac{\partial g(x,y)}{\partial y^2} = 32$$
$$\Rightarrow H = \begin{pmatrix} 6x-6 & 0 \\ 0 & 32 \end{pmatrix}$$

and $det(H)|_{x=x^*, y=y^*} = (6-6) \cdot 32 - 0 \cdot 0 = 0$ which indicates a saddle point.

Thus we did not find an extremal point.

3 Excerise 3: Stationarity

a) Are weakly stationary processes always strictly stationary? Construct an example to support your argument

b) Is weak stationarity a necessary condition for strict stationarity? Bring an example.

Hint: How many moments does a distribution require?

Solution:

a) No. A time series of length T drawing from $N(0,1)$ for $t \in \left[0, \frac{T}{2}\right]$ and drawing from Student's t-distribution for $t \in \left(\frac{T}{2}, T\right]$ has a constant mean $\mu = 0$ and variance $\sigma^2 = 1$, but the kurtosis (4^{th} moment) changes throughout time. In consequence the joint distribution of a subsequence $x_{t-p}, \ldots, x_{t+p}$ is not independent of t. Therefore it is not strictly stationary

b) No. Take the Cauchy distribution as an example: $f(x) = \frac{1}{\pi} \cdot \frac{s}{s^2 + (x-t)^2}$. Any *i.i.d.* sample from this distribution would be obviously strictly stationary. Yet this distribution has no existing moments at all (the integral diverges), hence it cannot exhibit a constant expected value or variance over time. Therefore it is only strictly stationary, but not weakly stationary! (Other example: t_1 distribution, where only the mean but not the variance exists).

4 Excerise 3: Covariance Matrices under Stationarity

Referring to Remark 1.13: Show that $\Gamma_l = \Gamma_{-l}^T$ holds for all weakly stationary processes.

(*Two dimensions suffice*)

Solution:

Without loss of generality assume $\mu = 0$ everywhere and assume z to be a bivariate vector $(x, y)^T$. Let $\Gamma_{l,t}$ be the covariance matrix of the l^{th} lag at time t:

$$\Gamma_{l,t} = \begin{bmatrix} \mathbb{E}(x_t \cdot x_{t-l}) & \mathbb{E}(x_t \cdot y_{t-l}) \\ \mathbb{E}(y_t \cdot x_{t-l}) & \mathbb{E}(y_t \cdot y_{t-l}) \end{bmatrix} \quad \text{and} \quad \Gamma_{l,t}^T = \begin{bmatrix} \mathbb{E}(x_{t-l} \cdot x_t) & \mathbb{E}(x_{t-l} \cdot y_t) \\ \mathbb{E}(y_{t-l} \cdot x_t) & \mathbb{E}(y_{t-l} \cdot y_t) \end{bmatrix} = \Gamma_{-l,t-l}$$

Since weak stationarity has been assumed, the covariance matrix is constant across time and $\Gamma_{-l,t-l} = \Gamma_{-l} = \Gamma_l^T$ and vice versa.