Exercise Sheet 11

$\boxed{1}$ a)

(1) $$x_t = 0.7x_{t-1} + 0.3x_{t-2} + a_t$$
$$= (0.7L + 0.3L^2)x_t + a_t$$

$$\Longleftrightarrow (1 - 0.7L - 0.3L^2)x_t = a_t$$

$\xrightarrow[\text{characteristic polynomial}]{\text{derive the (reverse)}}$ $1 - 0.7z - 0.3z^2 \overset{!}{=} 0$

$\xrightarrow{\text{"polyroot"}}$ $z_1 = 1 + 0i$ ⚡
$z_2 = -\frac{10}{3} + 0i$

There is 1 unit root, hence the process is not stationary.

(2) $Y_t = 2LY_t - L^2 Y_t + b_t$

$\Leftrightarrow (1-2L+L^2)Y_t = b_t$

$\longrightarrow \underbrace{1-2z+z^2}_{=(1-z)\cdot(1-z)} \overset{!}{=} 0$

$\hookrightarrow z_1 = z_2 = 1$

There are 2 unit roots, the process is not stationary!

b) (1) $d=1$

(2) $d=2$

c)

$$\underbrace{X_t - X_{t-1}}_{=\Delta X_t} = \underbrace{0.7X_{t-1} - X_{t-1} + 0.3X_{t-2}}_{} + a_t$$

$$= \underbrace{0.7x_{t-1} - 0.7x_{t-1}}_{=0} \underbrace{- 0.3x_{t-1} + 0.3x_{t-2}}_{= -0.3\Delta X_{t-1}}$$

$$\iff \boxed{\Delta X_t} = -0.3\boxed{\Delta X_{t-1}} + a_t$$

$$\iff (1+0.3L)\Delta X_t = a_t$$

$$\longrightarrow 1 + 0.3z \overset{!}{=} 0 \iff |z| = \frac{10}{3} > 1 \checkmark$$

Since a_t is white noise, ΔX_t follows a weakly stationary process.

d) $\rightarrow$ R

e) $\longrightarrow R$

No, this is regarded as a "spurious regression". Obviously, both time series are mutually independent, so there should not be any significant relationship in levels.

f) $\longrightarrow R$

Since the t-statistics diverges with $\sqrt{T}$ in distribution, its variance diverges with T.

$\boxed{2}$ a) $x_t = 1.5 L x_t - 0.5 L^2 x_t + a_t$

$\Leftrightarrow (1 - 1.5L + 0.5L^2) x_t = a_t$

$\rightarrow$ (Reverse) characteristic polynomial:

$$1 - 1.5z + 0.5z^2 \overset{!}{=} 0$$

$\Leftrightarrow (1 - 0.5z)(1 - z) = 0$

You can also use 'polyroot'

$\Leftrightarrow z_1 = 2, \; z_2 = 1$

$\Rightarrow$ There is one unit root and one root outside the unit circle. Hence this process is $I(1)$ since its differences are stationary.

b) $(1-0.5L)\underbrace{(1-L)X_t}_{=\Delta x_t} = a_t$

$\Rightarrow \Delta x_t = 0.5\Delta x_{t-1} + a_t$

$\rightarrow (1-0.5z) \stackrel{!}{=} 0 \Leftrightarrow |z| = 2 > 1$ ✓

This process is stationary. Thus x_t must be $I(1)$ and Δx_t is $I(0)$.

(Note that not every process with one unit root is $I(1)$, for example if $\Delta x_t = 5\Delta x_{t-1} + a_t$: x_t is integrated, but Δx_t is not stationary!)

$\boxed{3}$

a) $$\left(I - \begin{pmatrix} 1.1\underline{L} & -0.2\underline{L} \\ -0.2\underline{L} & 1.4\underline{L} \end{pmatrix} \right) z_t = a_t$$

$\longrightarrow$ (Reverse) characteristic polynomial:

$$\begin{vmatrix} 1-1.1z & 0.2z \\ 0.2z & 1-1.4z \end{vmatrix} \overset{!}{=} 0$$

$$\Leftrightarrow (1-1.1z)(1-1.4z) - (0.2z)^2 \overset{!}{=} 0$$

$$\Leftrightarrow 1 - 2.5z + 1.54z^2 - 0.04z^2 \overset{!}{=} 0$$

$$\Leftrightarrow 1 - 2.5z + 1.5z^2 \overset{!}{=} 0$$

$$\Leftrightarrow (1-1.5z)\underline{(1-z)} \overset{!}{=} 0$$

$$\Rightarrow z_1 = \frac{2}{3} \qquad z_2 = 1$$

$\longrightarrow$ There is one unit root!

b) $$\phi(L) = \begin{pmatrix} 1-1.1L & 0.2L \\ 0.2L & 1-1.4L \end{pmatrix}$$

$$\phi^{adj}(L) = \begin{pmatrix} 1-1.4L & -0.2L \\ -0.2L & 1-1.1L \end{pmatrix}$$

$$\underbrace{\phi^{adj}(L)\ \phi(L)}_{=\ |\phi(L)|\ I}\ z_t = \phi^{adj}(L)\ a_t$$

$$\Rightarrow \begin{pmatrix} 1-1.4L & -0.2L \\ -0.2L & 1-1.1L \end{pmatrix} \begin{pmatrix} 1-1.1L & 0.2L \\ 0.2L & 1-1.4L \end{pmatrix}$$

$$= \begin{pmatrix} (1-1.4L)(1-1.1L)-(0.2L)^2 & 0 \\ 0 & (1-1.4L)(1-1.1L)-(0.2L)^2 \end{pmatrix}$$

$\rightarrow$ This is the same as the characteristic polynomial in a), thus there is 1 unit root in each ARMA(2,1) process.

c) The impulse responses converge to zero if the coefficients of the causal representation converge to zero.

$|\theta_t| \rightarrow 0$ as $t \rightarrow \infty$

That is what we need to check.

In the VAR(1) case this means:

$$\phi_1^t \overset{?}{\rightarrow} 0 \quad \text{as } t \to \infty$$

$\rightarrow$ R

We find that $\lim_{t \to \infty} |\phi_1^t| \to \infty$,

hence the impulse responses on z_t do not vanish over time.