Exercise Sheet 10

1 a) $z_1 \longrightarrow z_2$

$z_2 \not\longrightarrow z_1$

"$\longrightarrow$" = "causes"

and there is instantaneous causality since $cov(a_{1,t}, a_{2,t}) \neq 0$.

b) $z_{1,t} = 0.5 z_{1,t-1} + 0 + a_{1,t}$

$z_{2,t} = 0.25 z_{1,t-1} + 0.5 z_{2,t-1} + a_{2,t}$

and impulse at $t=0$.

$\Rightarrow a_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; z_{-1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow z_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

t	$z_{1,t}$	$z_{2,t}$
0	1	0
1	$\frac{1}{2}$ $\frac{1}{2}\cdot 1+0\cdot 0+0$	$\frac{1}{4}$ $\frac{1}{4}\cdot 1+0.5\cdot 0+0$
2	$\frac{1}{4}$ $\frac{1}{2}\cdot\frac{1}{2}+0\cdot\frac{1}{4}+0$	$\frac{1}{4}$ $\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{4}+0$
3	$\frac{1}{8}$ $\frac{1}{2}\cdot\frac{1}{4}+0\cdot\frac{1}{4}+0$	$\frac{3}{16}$ $\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{16}$
4	$\frac{1}{16}$	$\frac{1}{8}$
5	$\frac{1}{32}$	$\frac{5}{64}$

$\rightarrow$ R

c) We need the Cholesky decomposition:

$LL' = \Sigma_a$ (R: "chol")

$\Rightarrow L = \begin{pmatrix} 0.5 & 0 \\ 0.25 & 0.5 \end{pmatrix}$

Now derive $\tilde{\Theta}_i = \Theta_i L$ from the causal representation using that

$\Theta_i = \phi_1^i$ (VAR(1)!!!)

$\tilde{\Theta}_0 = I_{2\times2} L = L = \begin{pmatrix} 0.5 & 0 \\ 0.25 & 0.5 \end{pmatrix}$

$\tilde{\Theta}_1 = \phi_1 L = \begin{pmatrix} 0.5 & 0 \\ 0.25 & 0.5 \end{pmatrix}\begin{pmatrix} 0.5 & 0 \\ 0.25 & 0.5 \end{pmatrix}$

$= \begin{pmatrix} 0.25 & 0 \\ 0.25 & 0.25 \end{pmatrix}$

$$\tilde{\Theta}_2 = \phi_1^2 L = \begin{pmatrix} 0.125 & 0 \\ 0.1875 & 0.125 \end{pmatrix}$$

$\vdots$

$\rightarrow R$ for $\tilde{\Theta}_3, \dots, \tilde{\Theta}_5$

Note: $a_t = L \eta_t \Leftrightarrow \eta_t = L^{-1} a_t$

We assume $\eta_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ because we work with orthogonal innovations!

$\Rightarrow \eta_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \eta_t = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; \forall t > 0$

$$\hookrightarrow z_t = \tilde{\Theta}_t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underbrace{\tilde{\Theta}_{t-1} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \dots + \tilde{\Theta}_0 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{=0}$$

$$z_0 = \begin{pmatrix} 0.5 & 0 \\ 0.25 & 0.5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.25 \end{pmatrix}$$

(compare with b)!

$$z_1 = \begin{pmatrix} 0.25 & 0 \\ 0.25 & 0.25 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$$

$\vdots$

$\longrightarrow R$

d) 1. The shocked variable turns 1 in $t=0$ for 'ordinary' shocks but only 0.5 using orthogonal shocks.

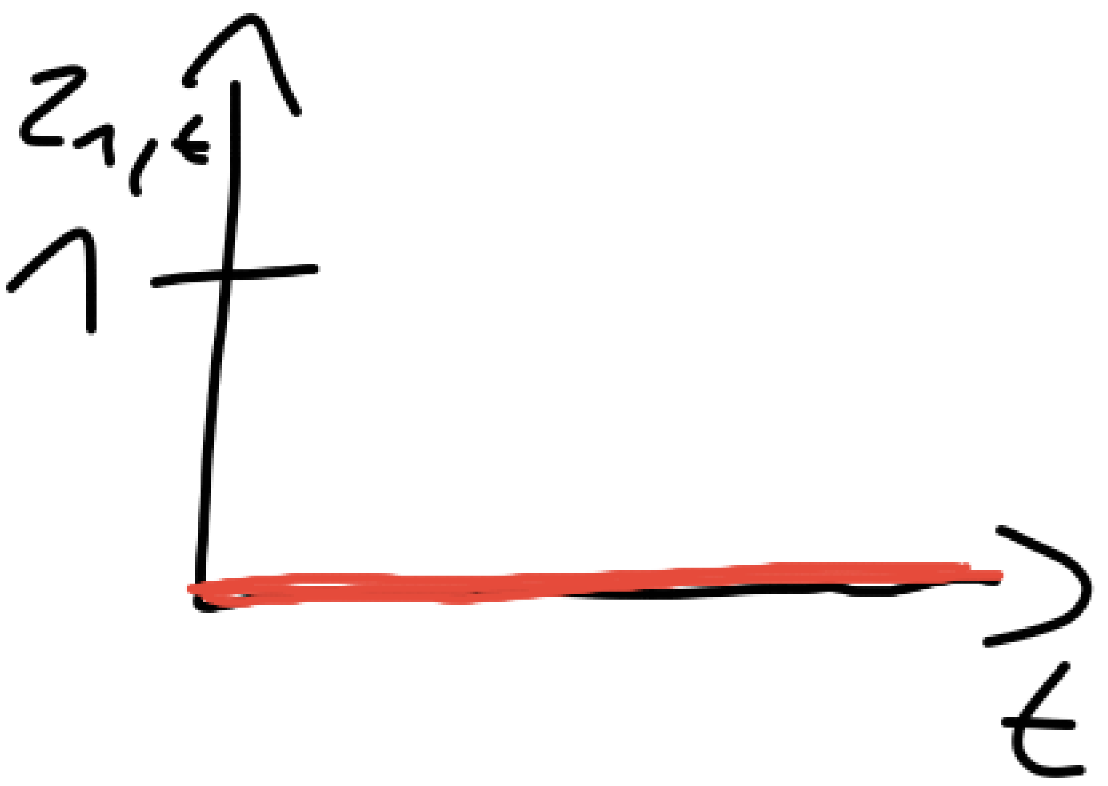
2. In c), both variables $z_{1,t}$ and $z_{2,t}$ were affected in $t=0$.

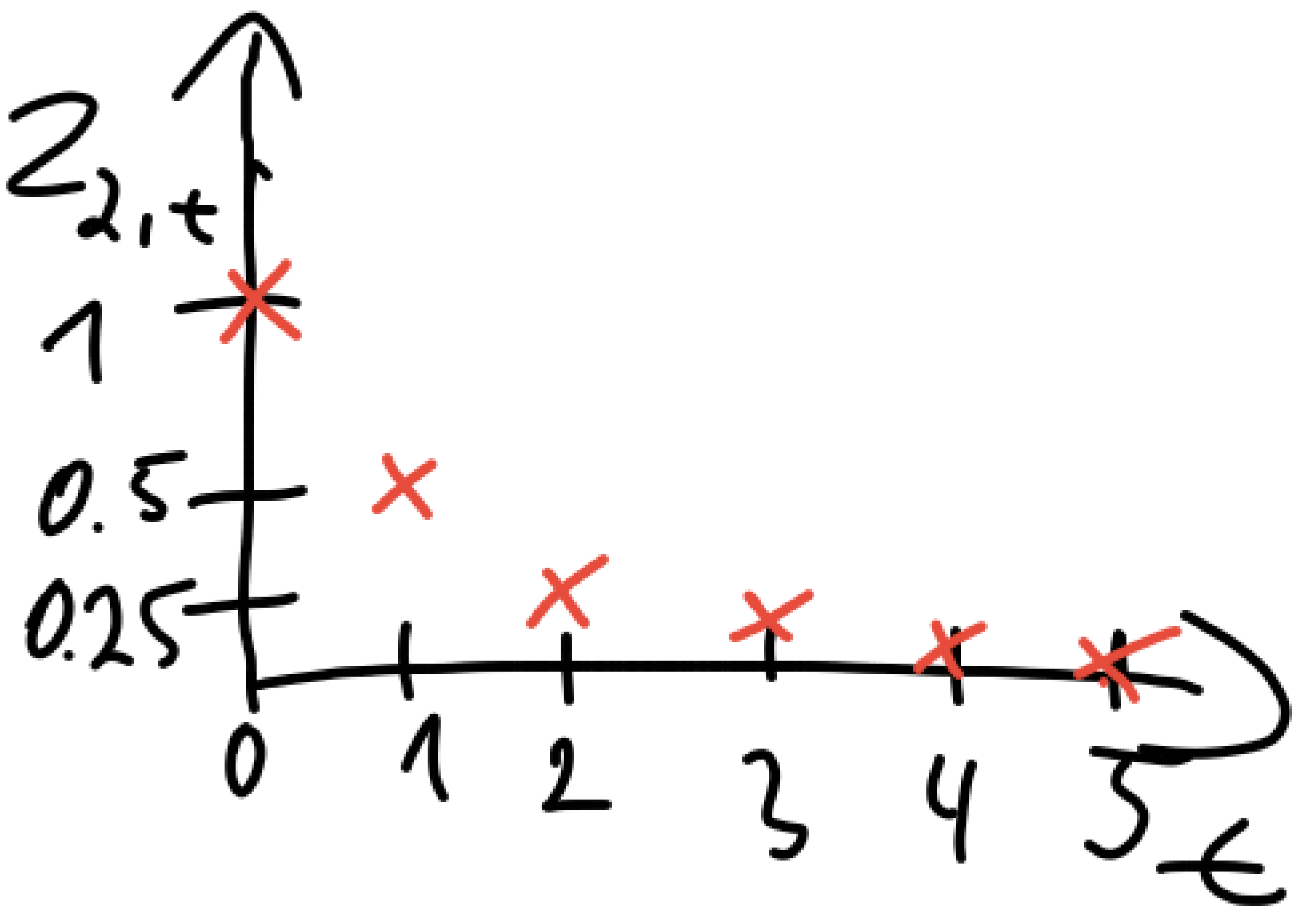
Reasons: Orthogonal IRFs account for instantaneous causality and the innovations' variances by incorporating Σ_u.

e) Only if $a_t \overset{!}{=} \eta_t = L^{-1} a_t$

$$\Rightarrow L^{-1} \overset{!}{=} I \Rightarrow \Sigma_a = LL' = I$$

f) remember: $z_2 \not\to z_1$





$\boxed{2}$ a) $w_{12}(h) = 0 \quad \forall h \in \mathbb{Z}$

because $z_2 \not\to z_1$.

If there is Granger causality,

The corresponding $w_{ij}(h)$ will be non-zero for at least some h.
(Inspect the coefficients of the causal representation!)

b) $\longrightarrow R$

Use: $$Var(e_T^{(i)}(h)) = \sum_{j=1}^{K} \sum_{\ell=0}^{h-1} \tilde{\Theta}_{\ell,ij}^2$$
$$= \sum_{j=1}^{K} w_{ij}(h)$$

and compute $$\frac{w_{ij}(h)}{Var(e_T^{(i)}(h))} = \frac{w_{ij}(h)}{\sum_{j=1}^{K} w_{ij}(h)}$$ row sum!

(Note that $\Theta_0 = I_{k \times k} \Rightarrow \tilde{\Theta}_0 = I\,L$
$\Rightarrow \tilde{\Theta}_0 \tilde{\Theta}_0' = I L L' I = \Sigma_a$!)

$\boxed{3}$ a) No. The variables are uncorrelated, but they still might be autocorrelated.

$$\Phi_1 = \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$$

(See slide 1-24)

b) $$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad W_{ij}(h) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

c) p-value $\to 1$ (Ljung-Box)

$\Leftrightarrow$ no dynamic pattern,

$$\Gamma_i = 0_{k \times k} \quad \forall i \neq 0$$

But if there is no correlation over time, there cannot be Granger causality!

Ljung-Box: $H_0: \phi_i = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & & 0 & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$

Granger

d) As said in a), $H_0^{\text{Ljung-Box}}$ rejected does not imply Granger causality! We only know that <u>some</u> variable responds to the shock in the following period. (<u>not</u> necessarily the shocked one!)

e) Not at all. See slide 1-24 and note that "i" starts at 1 in the sum!

f) i) shock on investment

t	investment	consumption
0	1	0
1	0.5	0
2	0.25	$0.1 \cdot 0.2 = 0.02$

ii) shock on consumption

t	investment	consumption
0	0	1
1	0	0.3
2	0	0.15

$\rightarrow$ R (from the causal representation)

g) $\rightarrow R$

$h=1$: w_{12}, w_{13}, w_{31}

$h=2$: w_{12}, w_{13}

h) $\rightarrow R$

Eventually, it is all about

$$\theta_{t,13} = 0 \quad \forall t$$

and there are multiple matrices ϕ_1 fulfilling this criterion.

$$\begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$$