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Multivariate Time Series Analysis

Solution Exercise Sheet 9

1 Exercise 1: Granger Causality – Theory

Let $z_t = (x_t, y_t)'$ be a stationary time series with two dimensions. Define the forecast errar as the univariare series $e_T(h) = y_{T+h} - y_T(h)$ with $y_T(h) = \mathbb{E}\left(y_{T+h}|\Omega_T\right)$. The information set Ω_T contains all relevant variables available whereas $\Omega_T^{\setminus x} = \Omega_T \setminus \{x_t\}_{t=0}^T$ omits the variable x entirely. (This setting is the univariate equivalent to definition 6.1 on Slide 6-4.)

a) Prove that $\mathbb{E}\left(e_T(h)|\Omega_T^{\setminus x}\right) = 0$.

Solution:

$$z_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

$$\begin{aligned}
\mathbb{E}\left(e_T(h)|\Omega_T^{\setminus x}\right) &= \mathbb{E}\left(y_{T+h} - \mathbb{E}\left(y_{T+h}|\Omega_T\right) \mid \Omega_T^{\setminus x}\right) \\
&\overset{\text{LIE}}{=} \mathbb{E}\left(\mathbb{E}\left(y_{T+h} - y_{T+h}|\Omega_T\right)|\Omega_T^{\setminus x}\right) \\
&\text{since } \Omega_T^{\setminus x} \subseteq \Omega_T \\
&\text{LIE} = \text{Law of Iterated Expectations} \\
&= 0
\end{aligned}$$

b) Prove that $\text{Var}\left(e_t(h)|\Omega_T\right) \leq \text{Var}\left(e_t(h)|\Omega_T^{\setminus x}\right)$

Solution:

2 Theorems necessary for the proof:

1 Conditional Jensen's Inequality

$g(\cdot) : \mathbb{R}^m \to \mathbb{R}$ is convex (like χ^2), then for any random vectors (y,x) for which $\mathbb{E}(||y||) < \infty$ and $\mathbb{E}(||g(y)||) < \infty$, $g\left(\mathbb{E}(y|x)\right) \leq \mathbb{E}\left((g(y)|x)\right)$. It is the other way around for concave functions.

2 Conditioning Theorem

If $\mathbb{E}(|y|) < \infty$, then $\mathbb{E}(g(x)y|x) = g(x) \cdot \mathbb{E}(y|x)$. If in addition $\mathbb{E}(|g(x)y|) < \infty$, then $\mathbb{E}(g(x)y) = \mathbb{E}(g(x)\mathbb{E}(y|x))$.

Back to Granger:

$e_T(h) = y_{T+h} - y_T(h)$ is a scalar. We know that $\mathbb{E}\left(e_T(h)|\Omega_T^{\setminus x}\right) = 0$, $\mathbb{E}\left(e_T(h)|\Omega_T\right) = 0$ and $\text{Var}(e_T(h)) < \infty$ since y_t is a weakly stationary (w.s.) process. Furthermore, w.s. implies that $\mathbb{E}(y_t) < \infty, \mathbb{E}(y_t^2) < \infty$.

From Jensen's Inequality it follows:

$$
\begin{aligned}
\left[\mathbb{E}\left(y_{T+h}|\Omega_T^{\setminus x}\right)\right]^2 &\overset{\text{LIE}}{=} \left[\mathbb{E}\left[\mathbb{E}\left(y_{T+h}|\Omega_T\right)|\Omega_T^{\setminus x}\right]\right]^2 \\
&\leq \mathbb{E}\left[\left[\mathbb{E}(y_{t+h}|\Omega_T)\right]^2|\Omega_t^{\setminus x}\right]
\end{aligned}
$$

Taking conditional expactations:

$$
\mathbb{E}\left[\left(\mathbb{E}\left[y_{T+h|\Omega_T^{\setminus x}}\right]\right)^2\right] \leq \mathbb{E}\left(\left[\mathbb{E}\left(y_{T+h}|\Omega_T\right)\right]^2\right)
$$

This extends to:

$$
\begin{aligned}
\left[\mathbb{E}(y_{T+h})\right]^2 &\leq \mathbb{E}\left(\left[\mathbb{E}\left(y_{T+h}|\Omega_T^{\setminus x}\right)\right]^2\right) \\
\text{Since } \mathbb{E}(y_{T+h}) &= \mathbb{E}\left[\mathbb{E}\left(y_{T+h}|\Omega_T^{\setminus x}\right)\right]
\end{aligned}
$$