University of Duisburg-Essen

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Multivariate Time Series Analysis

Exercise Sheet 1

1 Exercise 1: Matrix Operations

Prove properties 3,4 and 5 from Proposition 1.2 (Slide 1-11). Are there any requirements regarding the matrix dimensions?

Solution:

i) Property 3: $(A \otimes B)(F \otimes G) = (AF) \otimes (BG)$

$$\text{Let } A = \begin{pmatrix} a_{11} & \dots & a_{1q} \\ \vdots & \ddots & \vdots \\ a_{p1} & \dots & a_{pq} \end{pmatrix} \text{ and } F = \begin{pmatrix} f_{11} & \dots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \dots & f_{mn} \end{pmatrix}$$

$$\text{hence} (A \otimes B) = \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{pmatrix} \text{ and } (F \otimes G) \text{ analsgously}$$

$$
\begin{aligned}
(A \otimes B)(F \otimes G) &= \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{pmatrix} \begin{pmatrix} f_{11}G & \dots & f_{1n}G \\ \vdots & \ddots & \vdots \\ f_{m1}G & \dots & f_{mn}G \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}Bf_{11}G + \dots + a_{1q}Bf_{m1}G) & \dots & (a_{11}Bf_{1n}G + \dots + a_{1q}Bf_{mn}G) \\ \vdots & \ddots & \vdots \\ (a_{p1}Bf_{11}G + \dots + a_{pq}Bf_{m1}G) & \dots & (a_{p1}Bf_{1n}G + \dots + a_{pq}Bf_{mn}G) \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}f_{11} + \dots + a_{1q}f_{m1}) & \dots & (a_{11}f_{1n} + \dots + a_{1q}f_{mn}) \\ \vdots & \ddots & \vdots \\ (a_{p1}f_{11} + \dots + a_{pq}f_{m1}) & \dots & (a_{p1}f_{1n} + \dots + a_{pq}f_{mn}) \end{pmatrix} \otimes (BG)
\end{aligned}
$$

$$= \begin{pmatrix} \sum_{i=1}^{q=m} a_{1i}f_{i1} & \cdots & \sum_{i=1}^{q=m} a_{1i}f_{1i} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{q=m} a_{pi}f_{i1} & \cdots & \sum_{i=1}^{q=m} a_{pi}f_{in} \end{pmatrix} \otimes (BG)$$

$$= (AF) \otimes (BG)$$

Dimensions:

$A : p \times q$	$F : m \times n$
$B : c \times d$	$G : h \times k$

$$\Rightarrow dim(A \otimes B) = pc \times qd, dim(F \otimes G) = mh \times kn$$

ii) Property 4: $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

$\Rightarrow$ Claim and verify

The inverse is defined as following:

$(A \otimes B)(A \otimes B)^{-1} = I$ where I is the identitiy matrix

Then $(A \otimes B)(A^{-1} \otimes B^{-1}) = I$ must hold if the claim was true

We know from Property 3 that $(A \otimes B)(A^{-1} \otimes B^{-1}) = (AA^{-1} \otimes BB^{-1}) = I \otimes I = I$

Dimensions: A and B must be non-singular square matrices

iii) Property 3: $tr(A \otimes C) = tr(A) \cdot tr(C)$ for square matrices A and C

$$tr(A \otimes C) = tr \begin{pmatrix} a_{11}C & \cdots & a_{1n}C \\ \vdots & \ddots & \vdots \\ a_{n1}C & \cdots & a_{nn}C \end{pmatrix} = \sum_{i=1}^{n} (a_{ii}tr(C)) = tr(C) \sum_{i=1}^{n} a_{ii} = tr(C)tr(A)$$

2 Exercise 2: Bivariate Functions

Find the extrema of the following functions (using pen and paper). Determine whether these points constitute minima, maxima or saddle points:

a) $f(x, y) = (x - 2)^2 + (y - 5)^2 + xy$

b) $g(x, y) = (x - 1)^2 - (4y + 1)^2$

Solution:

Solution concept:

1. FOC: first derivatives $\overset{!}{=} 0$

2. SOC: Inspectthe Hessian matrix

$\Rightarrow f_{xx}, f_{yy} < (>) 0$ for maxima (minima), but also f_{xy}, f_{yx} matter. Ultimatley, eigenvalues are need to judge correctly in complex situations.

a) $f(x, y) = (x - 2)^2 + (y - 5)^2 + xy$

$$f(x, y) = (x - 2)^2 + (y - 5)^2 + xy$$

$$\frac{\partial f(x, y)}{\partial x} = 2(x - 2) + y \overset{!}{=} 0$$

$$\frac{\partial f(x, y)}{\partial y} = 2(y - 5) + x \overset{!}{=} 0$$

– Solving the equation system yields:

$$x = 2 - \frac{y}{2} \Rightarrow 2y - 10 + 2 - \frac{y}{2} = 0 \Rightarrow y^* = \frac{16}{3}$$

$$\Rightarrow x^* = 2 - \frac{16}{3 \cdot 2} = -\frac{2}{3}$$

– Evaluating the Hessian matrix:

$$\frac{\partial f(x, y)}{\partial x^2} = 2 \qquad \frac{\partial f(x, y)}{\partial xy} = 1$$

$$\frac{\partial f(x, y)}{\partial yx} = 1 \qquad \frac{\partial f(x, y)}{\partial y^2} = 2$$

$$\Rightarrow H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

1. $f_{xx}, f_{yy} > 0 \Rightarrow$ minimum is possible

2. $|H(x, y)|_{x=x^*, y=y^*} = 2 \cdot 2 - 1 \cdot 1 = 3 \Rightarrow$ That constitutes a minimum

3. $\lambda_1 = 3, \lambda_2 = 1 \Rightarrow$ both eigenvalues are positive which is nor surpising since the determinant is also positive

$\Rightarrow$ _Note_ that 1. and 2. are sufficient arguments in this case where $|H(x, y)|$ is positive definite

b) $g(x, y) = (x - 1)^2 - (4y + 1)^2$

$$g(x, y) = (x - 1)^3 + (4y - 1)^2$$

$$\frac{\partial g(x, y)}{\partial x} = 2(x - 1) \overset{!}{=} 0 \Leftrightarrow x^* = 1$$

$$\frac{\partial g(x, y)}{\partial y} = -2 \cdot 4(4y + 1) + x \overset{!}{=} 0 \Leftrightarrow y^* = -\frac{1}{4}$$

– Evaluating the Hessian matrix:

$$\frac{\partial g(x,y)}{\partial x^2} = 2 \qquad \frac{\partial g(x,y)}{\partial xy} = 0$$
$$\frac{\partial g(x,y)}{\partial yx} = 0 \qquad \frac{\partial g(x,y)}{\partial y^2} = -32$$

$$\Rightarrow |H|_{x=1,y=-0,25} = \begin{pmatrix} 2 & 0 \\ 0 & -32 \end{pmatrix}$$

1. $f_{xx}, f_{yy} < 0 \Rightarrow$ no clear minimum / maximum

2. $|H| = 2 \cdot (-32) - 0 \cdot 0 = -64 < 0 \Rightarrow$ indicates a saddle point in the case of two variables!

3. $\lambda_1 = 2, \lambda_2 = -32 \Rightarrow$ One postive eigenvalue, one negative eigenvalue is sufficient for a saddle point

3 Exercise 3: Stationarity

a) Are weakly stationary processes always strictly stationary? Construct an example to support your argument

b) Is weak stationarity a necessary condition for strict stationarity? Bring an example.

   Hint: How many moments does a distribution require?

Solution:

a) No. A time series of length T drawing from $N(0,1)$ for $t \in \left[0, \frac{T}{2}\right]$ and drawing from Student's t-distribution for $t \in \left(\frac{T}{2}, T\right]$ has a constant mean $\mu = 0$ and variance $\sigma^2 = 1$, but the kurtosis (4th moment) changes throughout time. In consequence the joint distribution of a subsequence $x_{t-p}, \ldots, x_{t+p}$ is not independent of t. Therefore it is not strictly stationary

b) No. Take the Cauchy distribution as an example: $f(x) = \frac{1}{\pi} \cdot \frac{s}{s^2 + (x - t)^2}$. Any *i.i.d.* sample from this distribution would be obviously strictly stationary. Yet this distribution has no existing moments at all (the integral diverges), hence it cannot exhibit a constant expected value or variance over time. Therefore it is only strictly stationary, but not weakly stationary! (Other example: t_1 distribution, where only the mean but not the variance exists).

4 Exercise 4: Covariance Matrices under Stationarity

Referring to Remark 1.13: Show that $\Gamma_l = \Gamma_{-l}^T$ holds for all weakly stationary processes.

(*Two dimensions suffice*)

Solution:

Without loss of generality assume $\mu = 0$ everywhere and assume z to be a bivariate vector $(x, y)^T$. Let $\Gamma_{l,t}$ be the covariance matrix of the lth lag at time t:

$$\Gamma_{l,t} = \begin{bmatrix} \mathbb{E}(x_t \cdot x_{t-l}) & \mathbb{E}(x_t \cdot y_{t-l}) \\ \mathbb{E}(y_t \cdot x_{t-l}) & \mathbb{E}(y_t \cdot y_{t-l}) \end{bmatrix} \quad \text{and} \quad \Gamma_{l,t}^T = \begin{bmatrix} \mathbb{E}(x_{t-l} \cdot x_t) & \mathbb{E}(x_{t-l} \cdot y_t) \\ \mathbb{E}(y_{t-l} \cdot x_t) & \mathbb{E}(y_{t-l} \cdot y_t) \end{bmatrix} = \Gamma_{-l,t-l}$$

Since weak stationarity has been assumed, the covariance matrix is constant across time and $\Gamma_{-l,t-l} = \Gamma_{-l} = \Gamma_l^T$ and vice versa.

5 Exercise 5: Ljung-Box Test in R

Load the package *MTS* and open the associated data pool 'mts-examples' (Slide 1-8). We are interested in the time series 'GS', 'MS' and 'JPM' from the dataset 'tenstocks':

a) First apply the Ljung-Box test on each time series individually. What do the results imply?

b) Now apply the multivariate Ljung-Box test on all three time series together. Compare the results with those from the univariate test and comment on it.

Solution:

Firstly, we need to importe the example datasets from the *MTS* package, which includes the tenstocks data set.

```
data("mts-examples")
```

If we take a look at the description of the *tenstock* dataset[1], we can see that it contains 11 variables and 132 observations on monthly simple returns from January 2001 to December 2011 for each of the 10 companies (first variable is the time vector).

a)

---

[1]To access help-file: `?tenstock()`

First, we will perform the Ljung-Box-test for the simple returns of *JP-Morgan Chase &*
Co. (JPM) company.

```
mq(x = tenstocks$JPM, lag = 20)
```

```
## Ljung-Box Statistics:
##              m      Q(m)      df    p-value
##  [1,]   1.000    0.504   1.000     0.48
##  [2,]   2.000    1.084   2.000     0.58
##  [3,]   3.000    1.097   3.000     0.78
##  [4,]   4.000    3.657   4.000     0.45
##  [5,]   5.000    4.445   5.000     0.49
##  [6,]   6.000    4.509   6.000     0.61
##  [7,]   7.000    4.547   7.000     0.72
##  [8,]   8.000    4.592   8.000     0.80
##  [9,]   9.000    8.533   9.000     0.48
## [10,]  10.000    8.857  10.000     0.55
## [11,]  11.000    9.647  11.000     0.56
## [12,]  12.000   11.340  12.000     0.50
## [13,]  13.000   11.935  13.000     0.53
## [14,]  14.000   12.882  14.000     0.54
## [15,]  15.000   12.887  15.000     0.61
## [16,]  16.000   13.083  16.000     0.67
## [17,]  17.000   13.152  17.000     0.73
## [18,]  18.000   13.425  18.000     0.77
## [19,]  19.000   15.397  19.000     0.70
## [20,]  20.000   17.334  20.000     0.63
```

p–values of Ljung–Box statistics
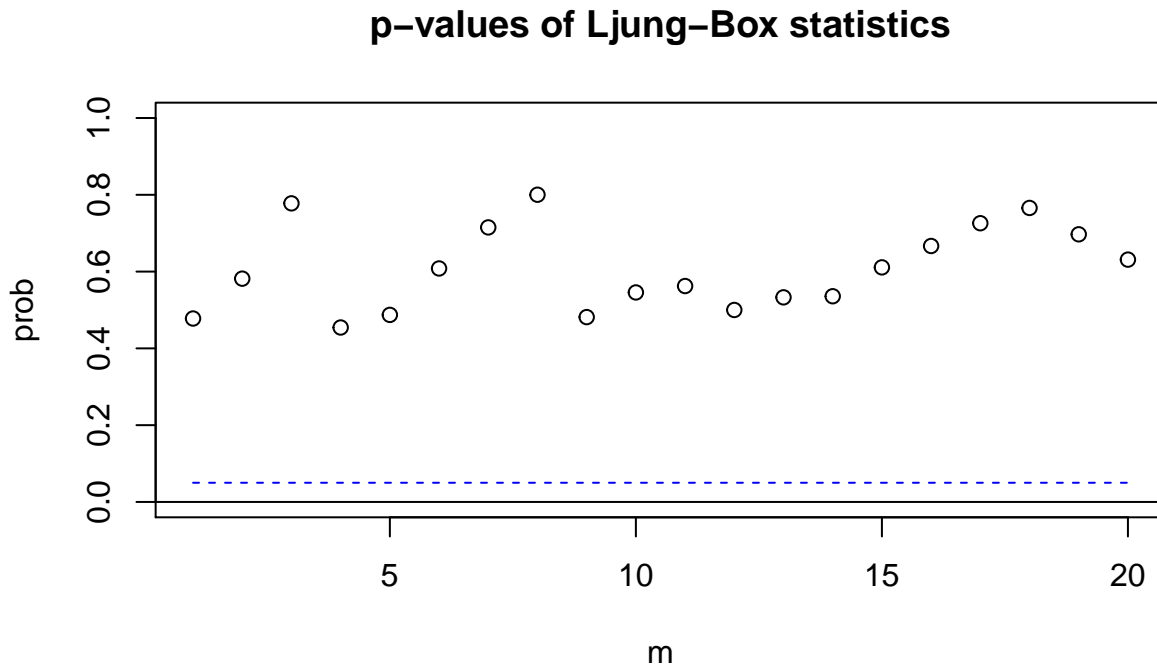
The simple returns for *JP-Morgan Chase & Co. (JPM)* show no autocorrelation for the first 20 lags to a significance level of 5 percent.

Next, we will have a look at the test for the time series for *Morgan Stanley (MS)*.

```
mq(x = tenstocks$MS, lag = 20)
```

```
## Ljung-Box Statistics:
##             m        Q(m)        df    p-value
##  [1,]  1.00000   0.00526   1.00000      0.94
##  [2,]  2.00000   1.24473   2.00000      0.54
##  [3,]  3.00000   1.42333   3.00000      0.70
##  [4,]  4.00000   1.42489   4.00000      0.84
##  [5,]  5.00000   2.91701   5.00000      0.71
##  [6,]  6.00000   3.47398   6.00000      0.75
##  [7,]  7.00000   4.30334   7.00000      0.74
##  [8,]  8.00000   6.89871   8.00000      0.55
##  [9,]  9.00000   7.55530   9.00000      0.58
## [10,] 10.00000   8.45013  10.00000      0.58
## [11,] 11.00000   8.89468  11.00000      0.63
## [12,] 12.00000   8.97639  12.00000      0.70
## [13,] 13.00000   9.72283  13.00000      0.72
## [14,] 14.00000   9.72714  14.00000      0.78
## [15,] 15.00000  10.75241  15.00000      0.77
```

```
## [16,] 16.00000  10.76237 16.00000     0.82
## [17,] 17.00000  15.18350 17.00000     0.58
## [18,] 18.00000  15.21323 18.00000     0.65
## [19,] 19.00000  17.13504 19.00000     0.58
## [20,] 20.00000  17.73024 20.00000     0.61
```

p–values of Ljung–Box statistics
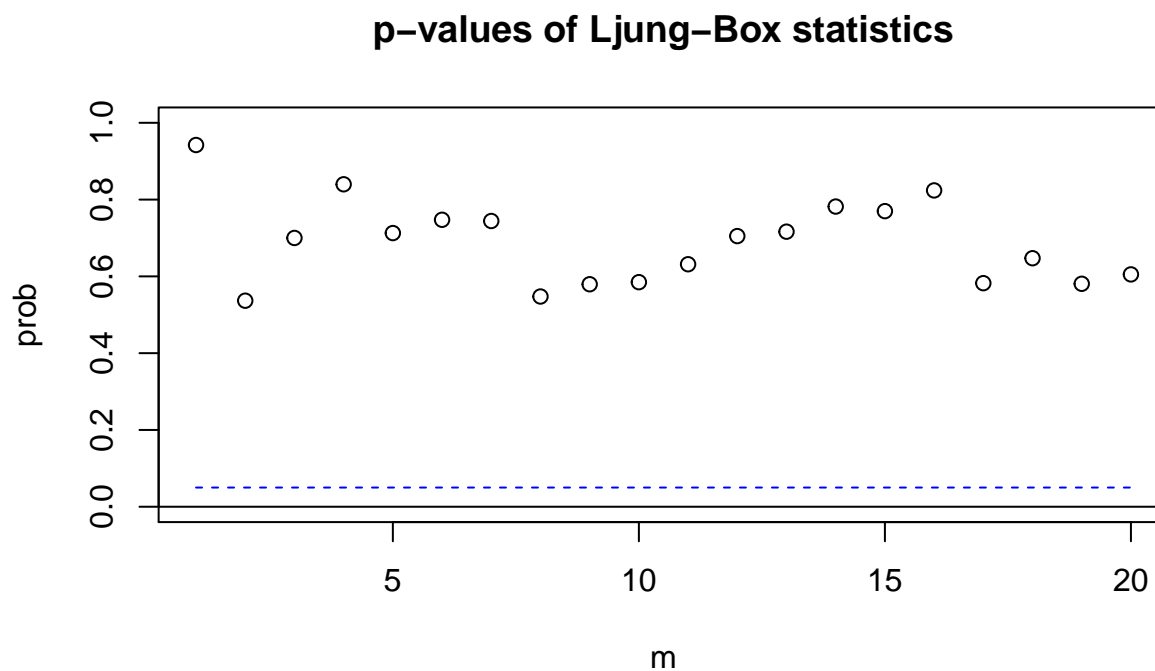
The results for the *MS* time series are similar to those of *JPM*.

Lastly, there is only the time series from *Goldman Sachs Group Inc (GS)* left to be analysed.

```
mq(x = tenstocks$GS, lag = 20)
```

```
## Ljung-Box Statistics:
##           m      Q(m)      df    p-value
##   [1,]  1.00      3.12    1.00     0.08
##   [2,]  2.00      3.12    2.00     0.21
##   [3,]  3.00      4.55    3.00     0.21
##   [4,]  4.00      4.94    4.00     0.29
##   [5,]  5.00      5.68    5.00     0.34
##   [6,]  6.00      7.13    6.00     0.31
##   [7,]  7.00      7.26    7.00     0.40
##   [8,]  8.00      7.34    8.00     0.50
##   [9,]  9.00      9.02    9.00     0.44
## [10,] 10.00      9.02   10.00     0.53
```

```
## [11,] 11.00      9.70   11.00      0.56
## [12,] 12.00     10.32   12.00      0.59
## [13,] 13.00     12.82   13.00      0.46
## [14,] 14.00     13.44   14.00      0.49
## [15,] 15.00     16.97   15.00      0.32
## [16,] 16.00     17.70   16.00      0.34
## [17,] 17.00     17.72   17.00      0.41
## [18,] 18.00     17.84   18.00      0.47
## [19,] 19.00     24.27   19.00      0.19
## [20,] 20.00     24.28   20.00      0.23
```

p–values of Ljung–Box statistics
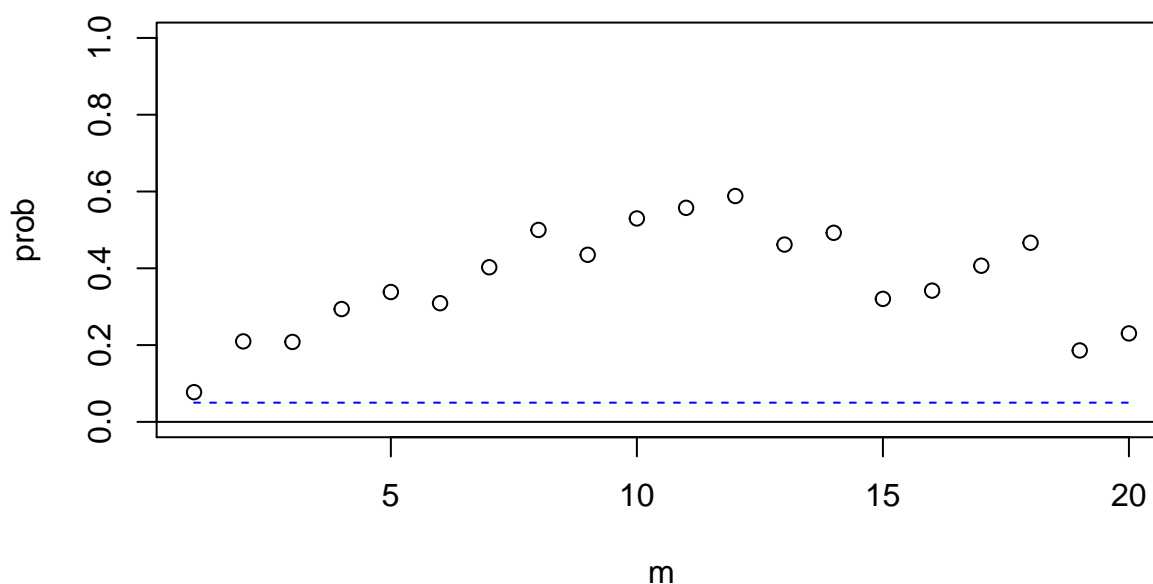
Only the first lag is *relatively* close to be significant at a 5 percent significance level.

b)

Now we take a look at the combined Ljung-Box test.

```
mq(x = cbind(tenstocks$JPM, tenstocks$MS, tenstocks$GS), lag = 20)
```

```
## Ljung-Box Statistics:
##          m       Q(m)      df     p-value
## [1,]    1.0      25.1     9.0      0.00
## [2,]    2.0      33.6    18.0      0.01
```

```
##  [3,]    3.0      55.2    27.0      0.00
##  [4,]    4.0      78.1    36.0      0.00
##  [5,]    5.0      95.3    45.0      0.00
##  [6,]    6.0     103.4    54.0      0.00
##  [7,]    7.0     113.7    63.0      0.00
##  [8,]    8.0     122.9    72.0      0.00
##  [9,]    9.0     135.2    81.0      0.00
## [10,]   10.0     145.1    90.0      0.00
## [11,]   11.0     149.4    99.0      0.00
## [12,]   12.0     162.5   108.0      0.00
## [13,]   13.0     167.4   117.0      0.00
## [14,]   14.0     180.3   126.0      0.00
## [15,]   15.0     192.0   135.0      0.00
## [16,]   16.0     205.4   144.0      0.00
## [17,]   17.0     218.8   153.0      0.00
## [18,]   18.0     227.4   162.0      0.00
## [19,]   19.0     236.5   171.0      0.00
## [20,]   20.0     244.1   180.0      0.00
```

### p–values of Ljung–Box statistics
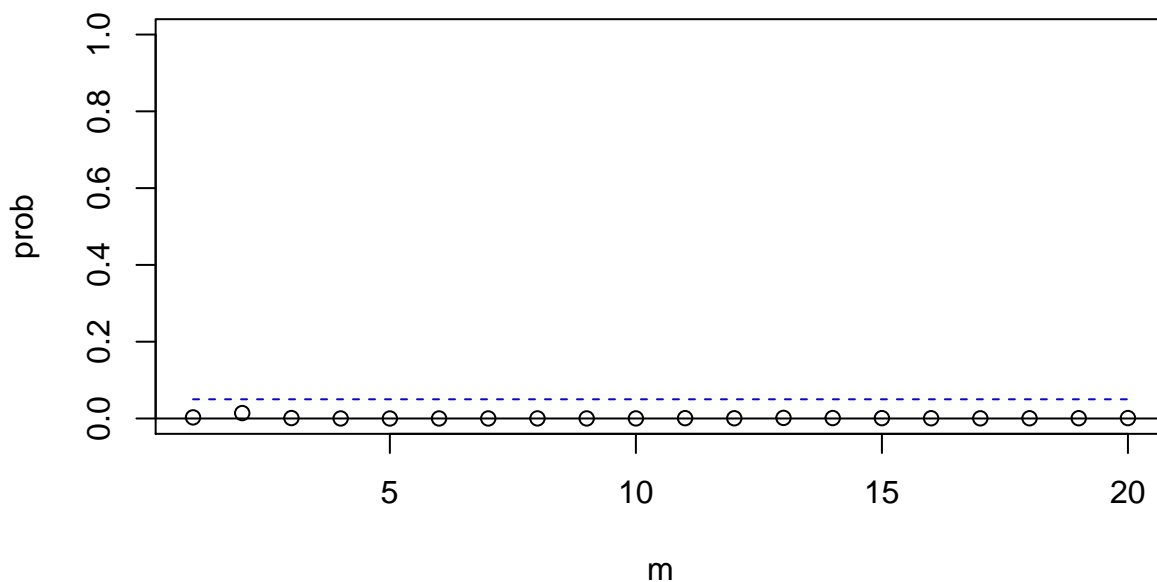
All p-values are below 5%. Since the time series are not much autocorrelated (univariate!), there must be cross-correlations which cause the Null hypothesis to be rejected. So there is a dynamic pattern which might be explained using multivariate time series models.
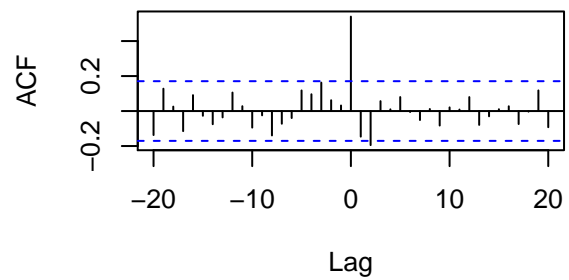
Lets have a look at the correlation as a second look.

```r
ccf(x = tenstocks$JPM, y = tenstocks$MS, lag.max = 20)
ccf(y = tenstocks$JPM, x = tenstocks$MS, lag.max = 20)
ccf(x = tenstocks$JPM, y = tenstocks$GS, lag.max = 20)
ccf(x = tenstocks$MS, y = tenstocks$GS, lag.max = 20)
```
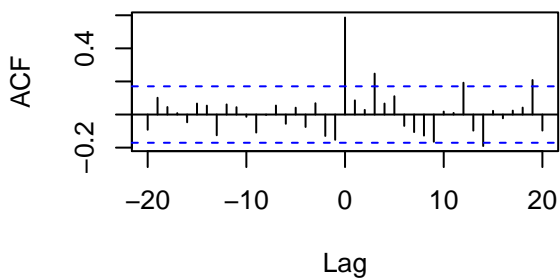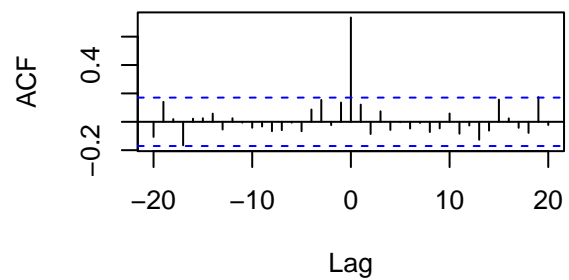
tenstocks$JPM & tenstocks$MS
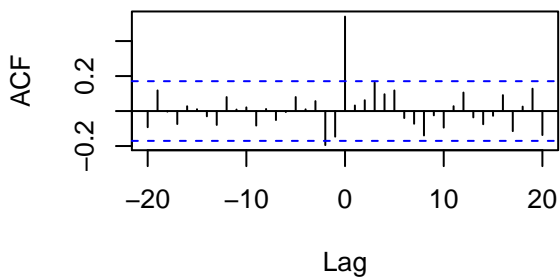
tenstocks$MS & tenstocks$JPM

tenstocks$JPM & tenstocks$GS

tenstocks$MS & tenstocks$GS

But keep in mind that the *Ljung-Box* test does not take ρ_0 into consideration.

Lastly, we will plot the times series with the command `plot.ts()`.

```r
plot.ts(cbind(tenstocks$JPM, tenstocks$MS, tenstocks$GS))
```

cbind(tenstocks$JPM, tenstocks$MS, tenstocks$GS)