Exercise Sheet 7

$\boxed{1}$ a)

$$
\begin{aligned}
z_t &= \phi \underbrace{z_{t-1}}_{= \phi z_{t-2} + a_{t-1}} + a_t \\
&= \phi^2 z_{t-2} + \phi a_{t-1} + a_t \\
&= \phi^3 z_{t-3} + \phi^2 a_{t-2} + \phi a_{t-1} + a_t \\
&\vdots \\
&= \phi^m z_{t-m} + \sum_{i=0}^{m-1} \phi^i a_{t-i} \\
&= 0 + \sum_{i=0}^{\infty} \phi^i a_{t-i}
\end{aligned}
$$

[with $\lim_{m \to \infty} \phi^m = 0$ by weak stationarity!]

$$= \sum_{i=0}^{\infty} \theta_i a_{t-i}$$

Using Lag notation:

$$z_t = \phi L z_t + a_t$$

$$\Leftrightarrow (1-\phi L) z_t = a_t \Leftrightarrow z_t = (1-\phi L)^{-1} a_t$$

and $(1-\phi L)^{-1} = \sum_{i=0}^{\infty} \phi^i L^i$ (requires stationarity and invertibility)

b) $Y_T(h) = \arg\min \underbrace{MSE(Y_T(h))}_{E([Y_{T+h} - Y_T(h)][Y_{T+h} - Y_T(h)]')}$

$$\rightarrow Y_T(h) = \underline{\psi} \; Y_T$$

$$\Leftrightarrow Y_{T+h} = \phi Y_{T+h-1} + a_{T+h}$$

$$\vdots$$

$$= \phi^h Y_T + \sum_{i=0}^{h-1} \phi^i a_{T+h-i}$$

$$\Rightarrow Y_{T+h} - Y_T(h) = \phi^h Y_T + \sum_{i=0}^{h-1} \phi^i a_{T+h-i} - \psi Y_T$$

$$\Rightarrow MSE(Y_T(h)) = \mathbb{E}\left(\left(\sum_{i=0}^{h-1} \phi^i a_{T+h-i}\right)\left(\sum_{i=0}^{h-1} \phi^i a_{T+h-i}\right)' + (\phi^h - \psi) Y_T Y_T' (\phi^h - \psi)' \right)$$

since $Y_T \perp\!\!\!\perp a_{T+i} \quad \forall\, i > 0$

: not depending on ψ

: minimised by $\psi = \phi^h$

$\boxed{2}$ a) $Z_T = \phi_0 + \phi_1 z_{T-1} + \dots + \phi_p z_{T-p} + a_T$

and forecasts: $z_{T-1}(1) = \phi_0 + \phi_1 z_{T-1} + \dots + \phi_p z_{T-p}$

$\Rightarrow Z_{T+1} - Z_T(1) = a_{T+1} \qquad = e_T(1)$

$\Rightarrow Z_{T+2} - Z_T(2) = \phi_1 \cdot (z_{T+1} - z_T(1)) + a_{T+2}$

$= \phi_1 \; a_{T+1} + a_{T+2}$

$\Rightarrow Z_{T+3} - Z_T(3) = \phi_1 (z_{T+2} - z_T(2)) + \phi_2 (z_{T+1} - z_T(1)) + a_{T+3}$

$= \phi_1 (\phi_1 a_{T+1} + a_{T+2}) + \phi_2 a_{T+2} + a_{T+3}$

$= \underbrace{(\phi_1^2 + \phi_2)}\, a_{T+1} + \underbrace{\phi_1}\, a_{T+2} + \underbrace{1}\, a_{T+3}$

$= \theta_2 \; a_{T+1} + \theta_1 \, a_{T+2} + \theta_0 \, a_{T+3}$

$$Z_{T+h} - Z_T(h) = \theta_{h-1} a_{T+1} + \theta_{h-2} a_{T+2} + \dots$$

$$+ \theta_1 a_{T+h-1} + \underbrace{\theta_0}_{=I} a_{T+h}$$

b) $\mathbb{E}(Z_{T+h} - Z_T(h))$

$$= \sum_{i=0}^{h-1} \theta_i \underbrace{\mathbb{E}(a_{T+h-i})}_{=0} = 0$$

$\mathrm{Cov}(Z_{T+h} - Z_T(h))$

$$= \mathbb{E}\left(\left(\sum_{i=0}^{h-1} \theta_i a_{T+h-i}\right)\left(\sum_{j=0}^{h-1} \theta_j a_{T+h-j}\right)'\right)$$

$$= \mathbb{E}\left(\sum_{i=0}^{h-1} \theta_i a_{T+h-i} a_{T+h-i}' \theta_i'\right)$$

$$= \sum_{i=0}^{h-1} \theta_i \mathbb{E}(a_{T+h-i} a_{T+h-i}') \theta_i'$$

$$= \sum_{i=0}^{h-1} \Theta_i \Sigma_a \Theta_i' = \Sigma_e(h)$$

since $\mathbb{E}(a_{T+h-i} \, a_{T+h-j}) = 0$
if $j \neq i$!

By using the fact that a sum of iid normally distributed variables follows a normal distribution:

$$e_T(h) \sim N(0, \Sigma_e(h))$$

with $\Sigma_e(h) = \sum_{i=0}^{h-1} \Theta_i \Sigma_a \Theta_i'$

c) $\lim_{h \to \infty} Cov(e_T(h))$

$$= \mathbb{E}\left(\left(\sum_{i=0}^{\infty} \theta_i a_{T+h-i}\right)\left(\sum_{i=0}^{\infty} \theta_i a_{T+h-i}\right)' \right)$$

$$= \lim_{h \to \infty} \mathbb{E}(z_{T+h} z_{T+h}')$$

$$= \lim_{h \to \infty} \Gamma_0^{T+h} = \Gamma_0$$

by weak stationarity.

$\boxed{3}$ $e_T(h) \sim N(0, Cov(e_T(h)))$

For each element it holds that:

$$\frac{e_T^{(i)}(h)}{\sqrt{Var(e_T^{(i)}(h))}} \sim N(0,1)$$

$\Rightarrow$ We need $\sqrt{Cov(e_T(h))}$!

$\hookrightarrow$ Cholesky decomposition of a positive definite matrix A:

$$A = U D U', \quad \text{with } D \text{ a diagonal}$$

matrix and U a lower triangular matrix.

D can be split further into $D^{\frac{1}{2}} \cdot D^{\frac{1}{2}}$ (note that $D' = D$ and $D^{\frac{1}{2}\,'} = D^{\frac{1}{2}}$)

$$\Rightarrow A = \underbrace{U \cdot D^{\frac{1}{2}}} \underbrace{D^{\frac{1}{2}} U'} = U \cdot D^{\frac{1}{2}} (U D^{\frac{1}{2}})'$$
$$= L L'$$

Using $Cov(e_T(h)) = Cov(e_T(h))'$ by symmetry

$$\Rightarrow e_T(h)' \, Cov(e_T(h))^{-1} \cdot e_T(h)$$

$$= e_T(h)' \underbrace{Cov(e_T(h))^{-\frac{1}{2}\,'}}_{=L} \cdot \underbrace{Cov(e_T(h))^{-\frac{1}{2}}}_{=L'} \cdot e_T(h)$$

$$= \underbrace{\left(e_T(h) \, Cov(e_T(h))^{-\frac{1}{2}}\right)'}_{\sim N(0, I)} \underbrace{\left(e_T(h) \, Cov(e_T(h))^{-\frac{1}{2}}\right)}_{\sim N(0, I)}$$

Both distributions are multivariate with K variables. Due to the inner product we have a sum of K squared standard normal variables.

$\Rightarrow$ this follows a χ^2_K distribution!

The ellipsoid can then be set up:

$$\left\{ z \in \mathbb{R}^K : e_T(h)' \, Cov(e_T(h))^{-1} e_T(h) \leq \chi^2_{K, 1-\alpha} \right\}$$

$\boxed{4}$ a) $Y_t = \phi_1 X_t$

constant!

$$\mathbb{E}(Y_t) = \mathbb{E}(\phi_1 X_t) = \phi_1 \mathbb{E}(X_t) = \phi_1 \mu_X$$

$$Y_t - \mathbb{E}(Y_t) = \tilde{Y}_t = \phi_1 (X_t - \mu_X) = \phi_1 \tilde{X}_t$$

$$Cov(Y_t) = Cov(\tilde{Y}_t) = Cov(\phi_1 \tilde{X}_t)$$

$$= \phi_1 \mathbb{E}(\tilde{X}_t \tilde{X}_t') \phi_1' = \phi_1 \Sigma_X \phi_1'$$

b) X_t is iid distributed, $\mathbb{E}(x_t) < \infty$

$Cov(x_t) < \infty$

$\Rightarrow$ a CLT applies!

$$\sqrt{T}\left(\bar{Y}_T - \mathbb{E}(Y)\right) \xrightarrow{d} N(0, \phi_1 \Sigma_x \phi_1')$$

c) $f(x) = f(x_1, \dots, x_k)$

$$= \begin{pmatrix} f_1(x_1, \dots, x_k) \\ \vdots \\ f_k(x_1, \dots, x_k) \end{pmatrix}$$

1st order Taylor expansion:

$$f(x) \approx$$

$$f(\mu_x) + \underbrace{\underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_K} \\ \vdots & & \\ \frac{\partial f_K}{\partial x_1} & \cdots & \frac{\partial f_K}{\partial x_K} \end{pmatrix}}_{\text{Jacobian Matrix}} \Bigg|_{x=\mu_x}}_{=: J} \bullet \begin{pmatrix} x_1 - \mu_{x_1}^{(1)} \\ \vdots \\ x_K - \mu_{x_K}^{(K)} \end{pmatrix}$$

a constant because it was evaluated!

d) Use $x = \overline{x_T}$

$$\Rightarrow f(\overline{x_T}) \approx f(\mu_x) + J\,(\overline{x_T} - \mu_x)$$

$$\Leftrightarrow f(\overline{x_T}) - f(\mu_x) \approx J \cdot (\overline{x_T} - \mu_x)$$

$$\Rightarrow \mathbb{E}\left(f(\bar{x}_T)-f(\mu_x)\right)=\mathbb{E}\left(J\cdot \bar{x}_T-\mu_x\right)$$

$$=J\cdot 0 \;=0$$

$$\Rightarrow \operatorname{Cov}\left(f(\bar{x}_T)-f(\mu_x)\right)$$

$$=\mathbb{E}\Big(\underbrace{\left(f(\bar{x}_T)-f(\mu_x)\right)}\left(f(\bar{x}_T)-f(\mu_x)\right)'\Big)$$

$$=\mathbb{E}\Big(\overbrace{J\cdot(\bar{x}_T-\mu_x)}(\bar{x}_T-\mu_x)'J'\Big)$$

$$=J\cdot \mathbb{E}\left(\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_t\right)\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_t\right)'\right)J$$

$$=J\cdot\mathbb{E}\Big(\frac{1}{T}\big(\underbrace{\sum_{t=1}^{T}\tilde{x}_t\tilde{x}_t'}_{=\Sigma_x}+\underbrace{\sum_{t\neq s}\tilde{x}_t\tilde{x}_s}_{\mathbb{E}(\cdot)=0}\big)\Big)\cdot J\cdot\frac{1}{T}$$

since $x_t \perp\!\!\!\perp x_s \;\forall s\neq t$

$$=\frac{1}{T}\, J\,\Sigma_x\, J'$$

$$\Rightarrow \sqrt{T}\left(f(\bar{x}_T)-f(\mu_x)\right)\xrightarrow{d} N(0, J\,\Sigma_x\, J')$$

$$e)\quad \hat{Y}_t - Y_t = \overbrace{(\hat{\phi}_1 - \phi_1)}^{\text{stochastic}} \overset{\text{deterministic}}{X_t}$$

$$\mathbb{E}(\hat{Y}_t - Y_t) = \underbrace{\mathbb{E}(\hat{\phi}_1 - \phi_1)}_{=0} \cdot X_t = 0$$

$$\begin{aligned} Cov(\hat{Y}_t - Y_t) &= Cov((\hat{\phi}_1 - \phi_1)\, X_t) \\ &= X_t' \, Cov(\hat{\phi}_1 - \phi_1)\, X_t \\ &= X_t' \, \Sigma_\phi \, X_t \end{aligned}$$

and since $\hat{\phi}_1$ follows an normal distribution:

$$\hat{Y}_t - Y_t \sim N(0,\ x' \Sigma_{\phi_1} x)$$