17-10-2022

Exercise Sheet 1

$\boxed{1}$ a) <u>Idempotency</u> (see Definition 9 from the Matrix Algebra Reader):

$$\underset{(m\times m)}{A^2} = \underset{(m\times m)}{A}\,\underset{(m\times m)}{A} \overset{!}{=} \underset{(m\times m)}{A}$$

for some $(m\times m)$ matrix A.

Eigenvalue λ: Suppose $\lambda \in \mathbb{R}$ (scalar!) is an eigenvalue of A. Then there exists an eigenvector x such that:

$$\underset{(m\times m)}{A}\,\underset{(m\times 1)}{x} = \underset{(1\times 1)}{\lambda}\,\underset{(m\times 1)}{x}$$

λ is scalar!

$$\Rightarrow \underbrace{A}_{=A \text{ because } A \text{ is idempotent!}} Ax = A \lambda x = \lambda A x = \lambda \lambda x = \lambda^2 x$$

$$\Rightarrow \underbrace{Ax}_{= \lambda x} = \ldots = \lambda^2 x$$

$$\Rightarrow \lambda x = \lambda^2 x$$

If x is non-zero (at least one entry

is different from zero), only $\lambda = 0$ and $\lambda = 1$ solve the equation.

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

with $\lambda_1 = 1$ and $\lambda_2 = 0$ since A is a triangular (even diagonal) matrix presenting its eigenvalues on the main diagonal.

b) Symmetry: $A' = A$

$\uparrow$ transposed

Again, x is a non-zero $(m \times 1)$ vector:

$$\underbrace{\underset{(1\times m)}{x'}\,\underset{(m\times m)}{A}\,\underset{(m\times 1)}{x}}_{(1\times 1)} \overset{\text{idempotent}}{=} x' \underbrace{AA}_{=A} x$$

$$\overset{\text{symmetry}}{=} x'A'Ax$$

$$= \underset{(1\times m)}{(Ax)'}\,\underset{(m\times 1)}{Ax} \geq 0$$

Since this is a quadratic form (leading to a sum of squared increments).

[illegible]) $(I_m - A)$ with identity matrix I_m:

$$I_m = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \Rightarrow I_m' = I_m$$

$$I_m I_m = I_m$$

$$\Rightarrow (I_m - A)' = I_m' - A' = I_m - A$$

symmetry ✓

$$\Rightarrow (I_m - A)(I_m - A) = I_m I_m - 2 I_m A + AA$$

$$= I_m - 2A + A$$

$$= I_m - A$$

idempotency ✓

$\Rightarrow$ $(I_m - A)$ is symmetric and idempotent just like matrix A!

$\hookrightarrow$ $x'(I_m - A)x = x'(I_m - A)'(I_m - A)x$

$$= ((I_m - A)x)'(I_m - A)x \geq 0$$

hence it is positive semidefinite as well!

c) orthogonality: $A'A = I_m$

eigenvector

$$Av = \lambda \underset{(m\times 1)}{v}$$

$$\Leftrightarrow \underbrace{(Av)'}_{= v'A'} Av = (Av)' \lambda v = \underbrace{(\lambda v)' \lambda v}_{= \lambda^2 v'v}$$

$$\Leftrightarrow v' \underbrace{A'A}_{= I_m \text{ by orthogonality}} v = \lambda^2 v'v$$

$$\Leftrightarrow v'Iv = \lambda^2 \underset{(1\times m)}{v'} \underset{(m\times 1)}{v}$$

$$\Leftrightarrow \pm\sqrt{1} = \sqrt{\lambda^2}$$

$$\Rightarrow \lambda = \pm 1$$

$\boxed{2}$ Generalised Inverse (unique)

See page 8 in the Matrix Algebra Reader.

Let $X_{n\times k}$ be any matrix.

(a) $\underbrace{X^+}_{k\times n} := \underbrace{(X'X)^{-1}}_{k\times k} \underbrace{X'}_{k\times n}$

(I) $X X^+ X = X (X'X)^{-1} X'X = X$ ✓

(II) $X^+ X X^+ = (X'X)^{-1} X'X (X'X)^{-1} X'$

$= (X'X)^{-1} X' = X^+$ ✓

(III) $(X X^+)' = \left(X (X'X)^{-1} X'\right)' = \underbrace{(X')'}_{=X} \underbrace{((X'X)^{-1})'}_{=(X'X)^{-1}} X'$

symmetry!

$= X (X'X)^{-1} X' = X X^+$ ✓

by Results 20.4: $(AB)' = B'A'$

and Results 21.3: $(A')^{-1} = (A^{-1})'$.

(IV) $(X^+ X)' = \left((X'X)^{-1} X'X\right)' = I_k' = I_k$

$\|$

$X^+ X = (X'X)^{-1} X'X = I_k$ ✓

Extra: The OLS estimator b is expressed as:

$$b = (X'X)^{-1} X'Y = X^+ Y$$

b always minimises the loss function $SSR(\tilde{\beta})$

if $rk(X) = k < n$:

$$\left.\frac{\partial SSR(\tilde{\beta})}{\partial \tilde{\beta}}\right|_b = -2X'y + 2X'Xb \overset{!}{=} 0$$

$$= -2X'y + 2\underbrace{X'X \overbrace{X^+}^{} y}_{= X' \\ = (X'X)(X'X)^{-1}X'y}$$

$$= -2X'y + 2X'y = 0$$

BUT: X^+ is only unique, if $rk(X) = k$!

For any X^+ with $rk(X) < k$,

X^+ is not unique anymore. Consequently b is thus also not uniquely determined anymore!

b) Definition 9: regular matrix $A \Rightarrow \det(A) \neq 0$

A needs to be a square matrix for that.

Suppose X^- is a generalised inverse of X. By regularity of X, the inverse X^{-1} exists and is unique.

$$X X^- X = X$$

has to hold true for any inverse X^-!

$$\Rightarrow \underbrace{X^{-1} X}_{=I} X^- X = \underbrace{X^{-1} X}_{=I}$$

$$X^- X = I$$

$$\Leftrightarrow X^- \underbrace{X X^{-1}}_{=I} = I X^{-1}$$

$$X^- = X^{-1}$$

Thus X^- must be identical to the unique ~~...~~ inverse X^{-1}!

$\boxed{3}$ Expectation, Variance

(a) $a \in \mathbb{R}$ and deterministic:

$$E\left[(Y-a)^2\right] - \left[E(Y-a)\right]^2$$

$$= E\left[Y^2 - 2Ya + a^2\right] - E(Y-a) \cdot E(Y-a)$$

linearity of $E(\cdot)$

$$= E(Y^2) - 2a\,E(Y) + a^2 - \left[E(Y)-a\right]\left[E(Y)-a\right]$$

$$= E(Y^2) - \left[E(Y)\right]^2 + \cancel{a^2} - \cancel{a^2} - \cancel{2a\,E(Y)}$$

$$- \cancel{(-2a\,E(Y))}$$

$$= Var(Y)$$

(b) By definition:

$$Var(aY+b) = E\left[\left(aY+b - \underbrace{E(aY+b)}_{= a \cdot E(Y) + b}\right)^2\right]$$

$$= E\left[a^2 \left(Y - E(Y)\right)^2\right] = a^2 \underbrace{E\left[(Y-E(Y))^2\right]}_{= Var(Y)}$$

$$= a^2 \; Var(Y)$$

(c) Recall that, with $f(x,y)$ denoting the bivariate density of (X,Y):

$$E(Y|X) = \int_{-\infty}^{+\infty} y \, f(y|X) \, dy$$

where $f(y|X) = \dfrac{f(X,y)}{f_X(X)}$.

Note that $E(Y|X)$ is actually a random variable (rv) because it is a function of the rv X!

$$E_X\left[E(Y|X)\right] = \int_{-\infty}^{+\infty} E(Y|X) \, f_X(x) \, dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y \, f(y|x) \, dy \, f_X(x) \, dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y \, f(y|x) \, f_X(x) \, dy \, dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y \, f(x,y) \, dy \, dx$$

$$= \int_{-\infty}^{+\infty} Y \boxed{\int_{-\infty}^{+\infty} f(x,y)\, dx} \, dy$$

$$= \int_{-\infty}^{+\infty} y \; f_Y(y)\, dy$$

$$= E(Y)$$

(d) $$E(Y) = \underset{a \in \mathbb{R}}{\arg\min} \; E\left((Y-a)^2\right)$$

FOC: $$\frac{\partial}{\partial a} E\left((Y-a)^2\right) = E\left(\frac{\partial}{\partial a}(Y-a)^2\right)$$

$$= E\left(-2(Y-a)\right) \overset{!}{=} 0$$

$$\Leftrightarrow \quad E(Y) = a \quad \Rightarrow$$ ~~max~~ extremum!

SOC: $$\frac{\partial^2}{\partial a^2} E\left((Y-a)^2\right) = \frac{\partial}{\partial a} E(-2(Y-a))$$

$$= E(-2(-1)) = 2 > 0$$

$\Rightarrow$ minimum! (since $E((Y-a)^2)$ is convex around $E(Y)=a$)

(e) $X \perp\!\!\!\perp Y$ (independence)

$$\overset{\text{by}}{\underset{\text{independence}}{\Longrightarrow}} f(x,y) = f_X(x) \cdot f_Y(y)$$

Thus: $$E(Y|X) = \int_{-\infty}^{+\infty} y \, f(y|X) \, dy$$

$$= \int_{-\infty}^{+\infty} y \, \frac{f(x,y)}{f_X(x)} \, dy$$

$$= \int_{-\infty}^{+\infty} y \, \frac{f_Y(y) \, f_X(x)}{f_X(x)} \, dy$$

$$= \int_{-\infty}^{+\infty} y \, f_Y(y) \, dy$$

$$= E(Y) .$$

(f) We know that $E(\varepsilon | X) = 0$.

By the ~~[illegible]~~ law of iterated expectations (LIE):

$$E(\varepsilon) = E_X\big(E(\varepsilon | X) \big)$$

$$= E_X(0)$$

$$= 0 .$$

$$Cov(X, \varepsilon) = E(X\varepsilon) - E(X) \cdot E(\varepsilon)$$

$$= E(X\varepsilon) - E(X) \cdot 0$$

$$= E_X\big(E(X\varepsilon | X) \big) - 0$$

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$$= E_X\big(X \, E(\varepsilon | X)\big)$$

$$= E_X(X \cdot 0)$$

$$= 0 .$$

4 Dependence Concepts

(a) See task 3(e). $E(\varepsilon)=0$ is not required!

$$\left(E(\varepsilon|X) = \int_{-\infty}^{+\infty} \varepsilon \, f(\varepsilon|x) \, d\varepsilon = \int_{-\infty}^{+\infty} \varepsilon \, f_\varepsilon(\varepsilon) d\varepsilon = E(\varepsilon) \right)$$

(b) $$E(\varepsilon X) \overset{LIE}{=} E_X\left(E(\varepsilon X|X)\right)$$

$$= E_X\left(X \underbrace{E(\varepsilon|X)}_{= E(\varepsilon) \text{ by assumption.}}\right)$$

$$= E_X\left(X \; E(\varepsilon)\right)$$

$E(\varepsilon)$ that's a constant!

$$= E(\varepsilon) \cdot E_X(X)$$

$$\Rightarrow Cov(\varepsilon, X) = E(\varepsilon X) - E(\varepsilon) E(X)$$

$$= E(\varepsilon) E(\varepsilon) - E(\varepsilon) E(X)$$

$$= 0.$$

c) $X \sim N(0,1) \Rightarrow E(X) = 0$

$E(X^2) = 1$

$E(X^3) = 0$

$$\Rightarrow Cov(X, \varepsilon) = E(X\varepsilon) - E(X)\,E(\varepsilon)$$

Use: $\varepsilon := X^2 - 1$

$$\Rightarrow Cov(X, X^2-1) = E(X(X^2-1)) - E(X)\,E(X^2-1)$$

$$= E(X^3 - X) - \underbrace{E(X)}_{=0}\underbrace{(E(X^2)-1)}_{=0}$$

$$= \underbrace{E(X^3)}_{=0} - \underbrace{E(X)}_{=0}$$

$$= 0.$$

BUT: $E(\varepsilon|X) = E(X^2-1|X)$

$$= (X^2-1)\underbrace{E(1|X)}_{=1} \neq 0$$

because X^2 is still a random variable!

whereas $E(\varepsilon) = E(X^2-1) = 0$ ↯

(d) To show: $E(\varepsilon|X) = E(\varepsilon) \not\Rightarrow \varepsilon \perp\!\!\!\perp X$

$\Rightarrow \varepsilon := XY$ and $XY \not\perp\!\!\!\perp X$ by definition

further: $X \perp\!\!\!\perp Y$ and $E(Y) = 0$.

$$E(\varepsilon|X) = E(XY|X) = X\,E(Y|X)$$

$$= X \cdot E(Y) \quad (\text{as } X \perp\!\!\!\perp Y).$$

$$= 0$$

$$E(\varepsilon) = E(XY) \overset{X \perp\!\!\!\perp Y}{=} E(X) \cdot \underbrace{E(Y)}_{0} = 0$$

Concluding: $E(\varepsilon|X) = E(\varepsilon)$ but $\varepsilon \not\!\perp\!\!\!\perp X$ ↯