Ruhr Graduate School in Economics
Econometrics I (Winter Term 2022/23)
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Exercise Sheet No. 1

1 Matrix Algebra

(a) Let $\boldsymbol{A}\,(m \times m)$ be an idempotent matrix and $\lambda \in \mathbb{R}$ an eigenvalue of $\boldsymbol{A}$.
Prove that $\lambda = 0$ or $\lambda = 1$.

(b) Let $\boldsymbol{A}\,(m \times m)$ be a symmetric, idempotent matrix.
Prove that the matrices $\boldsymbol{A}$ and $\boldsymbol{I}_m - \boldsymbol{A}$ are positive semidefinite.

(c) Let $\boldsymbol{A}\,(m \times m)$ be an orthogonal matrix and $\lambda \in \mathbb{R}$ an eigenvalue of $\boldsymbol{A}$.
Prove that $\lambda = -1$ or $\lambda = 1$.
Hint: Consider $(\boldsymbol{Av})^T(\boldsymbol{Av})$ for an eigenvector $\boldsymbol{v} \in \mathbb{R}^m$ of the eigenvalue λ.

2 Generalized Inverse

See the Matrix Algebra Reader, p.8. Let $\boldsymbol{X}(n \times k)$ be any matrix.

(a) Show that if $\text{rk}(\boldsymbol{X}) = k < n$, $(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ is the Moore-Penrose inverse to $\boldsymbol{X}$.

(b) Show that if $\boldsymbol{X}$ is regular, there exists no other generalized inverse than $\boldsymbol{X}^{-1}$.

3 Expectation, Variance

Verify the following statements for the random variables (r.v.s) X, Y:

(a) $\text{Var}(Y) = \text{E}\{(Y-a)^2\} - \{\text{E}(Y-a)\}^2, \quad a \in \mathbb{R}$

(b) $\text{Var}(aY+b) = a^2\,\text{Var}(Y) \quad a, b \in \mathbb{R}$

(c) $\text{E}(Y) = \text{E}_X[\text{E}(Y|X)]$
Hint: It suffices to show the result for absolutely continuous r.v.s.

(d) $\mathsf{E}(Y) = \arg\min_{a \epsilon \mathbb{R}} \mathsf{E}\{(Y-a)^2\}$

(e) $\mathsf{E}(Y|X) = \mathsf{E}(Y)$ if X, Y are independent.

(f) $\mathsf{E}(\varepsilon|X) = 0 \Rightarrow \mathsf{E}(\varepsilon) = 0$ & $\operatorname{cov}(X, \varepsilon) = 0$ with $\operatorname{cov}(\cdot, \cdot)$ the covariance.

4 Dependence concepts

Consider the two random variables (r.v.s) ε and X. This exercise extends Exercise 1.1(e) and clarifies the relation between the following dependence concepts, namely

$$\varepsilon, X \text{ are independent} \implies \mathsf{E}(\varepsilon|X) = \mathsf{E}(\varepsilon) \implies \operatorname{cov}(\varepsilon, X) = 0,$$

where the converses do not hold. Show:

(a) If ε and X are independent, then $\mathsf{E}(\varepsilon|X) = \mathsf{E}(\varepsilon)$.
Hint: It suffices to show this for absolutely continuous r.v.s.

(b) If $\mathsf{E}(\varepsilon|X) = \mathsf{E}(\varepsilon)$, then ε and X are uncorrelated, i.e. $\operatorname{cov}(\varepsilon, X) = 0$.

(c) The converse of (b) does not hold, i.e. from $\operatorname{cov}(\varepsilon, X) = 0$ it does not necessarily follow that $\mathsf{E}(\varepsilon|X) = \mathsf{E}(\varepsilon)$.
Hint: Consider $X^2 - 1$ and X, where $X \sim \mathcal{N}(0, 1)$.

(d) The converse of (a) does not hold.
Hint: Consider $\varepsilon = XY$, where Y is centered and independent of X.