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1 Introduction

Meta tests have been shown to be a powerful tool when testing for the null of non-cointegration. The distribution of their test statistic, however, is mostly not available in closed form. This might pose difficulties when implementing the meta tests in econometric software packages, as one has to include the full null distribution for each combination of the underlying tests. Software package size limitations are therefore quickly exceeded.

In this paper we propose supervised Machine Learning Algorithms to approximate the p-values of the meta test by Bayer and Hanck (2012) which tests for the null of non-cointegration. This approach might reduce the size of associated software packages considerably. The algorithms are trained on simulated data for various specifications of the aforementioned test.

Ergebnis der Models (1-2 Sätze)

Inhalt Paper

2 Bayer Hanck Test

The choice as to which of the available cointegration tests to use is a recurrent issue in econometric time series analysis. Bayer and Hanck (2012) propose powerful meta tests which provide unambiguous test decisions. They combine several residual- and system-based tests in the manner of Fisher's (1932) Chi-squared test.

Bayer and Hanck build their paper on previous work from Pesavento (2004), who defines the underlying model as $z_t' = [x_t', y_t]$, with x_t being an $n_1 \times 1$ vector and y_t a scalar, which displays the cointegration relation. They can be written as \begin{subequations}

$$\Delta x_t = \tau_1 + v_{1t} \tag{2.1}$$

$$y_t = (\mu_2 - \gamma'\mu_1) + (\tau_2 - \gamma'\tau_1)t + \gamma' x_t + u_t, \tag{2.2}$$

$$u_t = \rho u_{t-1} + v_{2t}. \tag{2.3}$$

\end{subequation} Δx_t presents the regressor dynamics. μ_1, μ_2, τ_1 and τ_2 are the deterministic parts of the model. They are subject to the following restrictions: (i) $\mu_2 - \gamma'\mu_1$ and $\tau = 0$ which translates to no deterministics, (ii) $\tau = 0$ which corresponds to a constant in the cointegrating vector, (iii) $\tau_2 - \gamma'\tau_1 = 0$, a constant plus trend.

$v_t = [v'_{1t} v_{2t}]'$ with Ω the long-run covariance matrix of v_t. For derivation of v_t see Pesavento (2004). Pesavento shows that $\{v_t\}$ satisfies an FCLT, i.e. $T^{-1/2}\sum_{t=1}^{[T\cdot]} v_t \Rightarrow \Omega^{1/2}W(\cdot)$. It is further assumed that the x_t are not cointegrated.

It clearly follows from (2.3) that z_t is cointegrated if $\rho < 1$. Hence the null hypothesis of no cointegration is $H_0 : p = 1$. Furthermore, Pesavento introduces two other parameters. First, R^2 measures the squared correlation of v_{1t} and v_{2t}. It can be interpreted as the influence of the right-hand side variables in (2.2). It ranks between zero and one. When there is no long-run correlation between those variables and the errors from the cointegration regression, R^2 equals zero. Secondly, the number of lags is approximated by a finite number k.

Assumptions (BH S. 84)?

Bayer and Hanck's (2012) meta test considers the test statistics of up to four stand-alone tests. Namely, these are the tests of Engle and Granger (1987), Johansen (1988), Boswijk (1994) and Banerjee, Dolado, and Mestre (1998). For the sake of brevity the detailed derivation of the underlying tests has been deliberately omitted here.

Engle and Granger (1987) propose a two-step procedure to test the null hypothesis of no cointegration against the alternative of at least one cointegrating vector. First, the long-run relationship between y_t and $\mathbf{x}_t$ is estimated by least squares regression. The obtained residuals $\hat{u}_t$ are then tested for a unit root. For this, Engle and Granger suggest the use of the t-statistic t_γ^{ADF} in the Augmented Dickey-Fuller (ADF) regression:

$$\Delta \hat{u}_t = \gamma \hat{u}_{t-1} + \sum_{i=1}^{k} \pi_i \Delta \hat{u}_{t-i} + \varepsilon_t. \tag{2.4}$$

The rejection of a unit root points to a cointegration relationship.

Johansen's (1988) maximum eigenvalue test is a system-based test that allows for several cointegration relationships. Take the vector error correction model (VECM)[1]

$$\Delta \mathbf{z}_t = \mathbf{\Pi} \mathbf{z}_{t-1} + \sum_{i=1}^{k} \mathbf{\Gamma}_p \Delta \mathbf{z}_{t-p} + \mathbf{d}_t + \varepsilon_t. \tag{2.5}$$

blabla Johansen test statistic

Banerjee and Boswijk

[1]Due to practical reasons we omit the derivation of the VECM which is presumed to be known.

To combine the results from the underlying tests Bayer and Hanck draw upon Fisher's combined probability test (Fisher, 1932). It merges the tests using the formula

$$\tilde{\chi}^2_{\mathcal{I}} := -2 \sum_{i \in \mathcal{I}} \ln (p_i). \tag{2.6}$$

Let t_i be the i^{th} test statistic. If test i rejects for large values, take $\xi_i := t_i$. If test i rejects for small values, take $-\xi_i := t_i$. With $\Xi_i(x) := \Pr_{\mathcal{H}'}(\xi_i \geq x)$ the p-value of the i^{th} test is $p_i := \Xi_i(\xi_i)$.

Fisher (1932) shows that under the assumption of independence the null distribution of $\tilde{\chi}^2_{\mathcal{I}}$ follows a chi-squared distribution with $2\mathcal{I}$ degrees of freedom. If this assumption is violated the null distribution is less evident. Here, the latter case occurs, as the ξ_i are not independent. The $\tilde{\chi}^2_{\mathcal{I}}$, however, have well-defined asymptotic null distributions $F_{\mathcal{F}_{\mathcal{I}}}$, as $\tilde{\chi}^2_{\mathcal{I}} \to_d \mathcal{F}_{\mathcal{I}}$ under $\mathcal{H}_0$ if $T \to \infty$, with $\mathcal{F}_{\mathcal{I}}$ some random variable. It is therefore feasible to simulate the joint null distribution of the ξ_i to obtain the distribution $F_{\mathcal{F}_{\mathcal{I}}}$ of (2.6). The $F_{\mathcal{F}_{\mathcal{I}}}$ depend on which and how many tests are combined. The distributions of the ξ_i depend on $K-1$ and the deterministic case.

3 Simulation

In this section, we describe the simulation of the null distribution of the Bayer Hanck meta test. The objective is to obtain data for training machine learning algorithms on approximating the p-values of the aforementioned test. In consideration of the different forms of the meta test we generate six data sets. These vary according to the specific combinations of the underlying tests and also account for the above-mentioned restrictions on the deterministic parts of the model.

The following approach relies largely on previous work by Pesavento (2004). For calculating the Bayer Hanck test statistic we require the p-values of the underlying tests. For this, we simulate their null distributions. It can be shown that asymptotically these are functions of standard Brownian motions. Here, the latter are constructed by step functions using Gaussian random walk of size $N = 1000$. The number of repetitions is set to 1,000,000. Furthermore, we consider $\mathrm{R}^2 \in \{0, 0.05, 0.1, ..., 0.95\}$, the maximum number of lags $K = 11$ and $c = 0$[2] (c mal definieren).

[2]Since we solely aim at simulating the distribution of the null of no cointegration we

From the mass of test statistics we build the cumulative distribution function of each underlying test and calculate the respective p-values. These are inserted into (2.6) to eventually obtain the Bayer Hanck test statistics. Analogous to the previous approach, we deduce the associated null distribution and the p-values.

4 Models

We now use the generated data for training machine learning algorithms on predicting the approximated empirical CDF of the Bayer Hanck test. We work with the values of the test statistic and the number of lags k as predictors. As it is our objective to describe the null distribution with a less memory-intensive model we will only consider linear methods. For the same objective we compare the models according to their in-sample RMSE. The threat of overfitting is thus of no particular relevance here. For this reason, and to reduce computation time, we use no cross-validation.

As the empirical CDF is typically known to be curved in an S-shape we skip the classic linear regression in favor of a more flexible model. We stay with least squares regression, but try various combinations of polynomial functions and interaction terms of the aforementioned regressors. The search for the best model is carried out via brute-force.

4.1 Polynomial Regression

Polynomial Regression extends the classic linear regression model by fitting a polynomial equation of arbitrary order to the data. It thus takes the form

Gleichung hier einfügen, abstimmen mit Liste von Jens.

Here, we consider up to 10 degrees. Higher order polynomials produce very non-linear curves, but are prone to wiggly behaviour at the boundaries. This can be problematic, as

We estimate all linear models for each possible combination of the underlying tests and the specifications of the model deterministics.

5 Package

will not consider any further values of c here.

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A Appendices

Number	Functional form	Range of γ
1	$p = \mathrm{poly}(t, \gamma) + (1/k)$	$\gamma \in \mathbb{Z}\,[3, 10]$
2	$p = \mathrm{poly}(t, \gamma) + (1/k) + \mathrm{poly}(t, \gamma) * 1/k$	$\gamma \in \mathbb{Z}\,[3, 10]$
3	$p = \mathrm{poly}(t, \gamma) + \log(k) + \mathrm{poly}(k, \gamma) * \log(k)$	$\gamma \in \mathbb{Z}\,[3, 10]$
4	$p = \mathrm{poly}(t, \gamma) + k + (1/k)$	$\gamma \in \mathbb{Z}\,[3, 10]$
5	$p = \mathrm{poly}(\log(t), \gamma) + \log(k)$	$\gamma \in \mathbb{Z}\,[3, 10]$
6	$p = \mathrm{poly}(\log(t), \gamma) * \log(k)$	$\gamma \in \mathbb{Z}\,[3, 10]$
7	$p = \mathrm{poly}(\log(t), \gamma) + k$	$\gamma \in \mathbb{Z}\,[3, 10]$
8	$p = \mathrm{poly}(\log(t), \gamma) * k$	$\gamma \in \mathbb{Z}\,[3, 10]$
9	$p = \mathrm{poly}(\log(t), \gamma) * k + 1/k$	$\gamma \in \mathbb{Z}\,[3, 10]$
10	$p = \mathrm{poly}(\log(t), \gamma) * \log(k)$	$\gamma \in \mathbb{Z}\,[3, 10]$
11	$p = \mathrm{poly}(\log(t), \gamma) * \log(k) + 1/k$	$\gamma \in \mathbb{Z}\,[3, 10]$
12	$p = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), \gamma) * 1/k$	$\gamma \in \mathbb{Z}\,[3, 10]$
13	$p = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), \gamma) * \sqrt{k}$	$\gamma \in \mathbb{Z}\,[3, 10]$
14	$p = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), \gamma) * 1/k + \sqrt{k}$	$\gamma \in \mathbb{Z}\,[3, 10]$
15	$\mathrm{bc}(p) = \mathrm{poly}(\mathrm{bc}(t), \gamma) + \log(k)$	$\gamma \in \mathbb{Z}\,[3, 10]$
16	$\mathrm{bc}(p) = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k)$	$\gamma \in \mathbb{Z}\,[3, 10]$
17	$\mathrm{bc}(p) = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k) + 1/k$	$\gamma \in \mathbb{Z}\,[3, 10]$
18	$\mathrm{bc}(p) = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), \gamma) * 1/k$	$\gamma \in \mathbb{Z}\,[3, 10]$
19	$\mathrm{bc}(p) = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), \gamma) * \sqrt{k}$	$\gamma \in \mathbb{Z}\,[3, 10]$
20	$\log(p) = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k)$	$\gamma \in \mathbb{Z}\,[3, 10]$
21	$\log(p) = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k) + 1/k$	$\gamma \in \mathbb{Z}\,[3, 10]$
22	$\log(p) = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), \gamma) * \sqrt{k}$	$\gamma \in \mathbb{Z}\,[3, 10]$
23	$\log(p) = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), \gamma) * 1/k$	$\gamma \in \mathbb{Z}\,[3, 10]$
24	$\log(p) = \mathrm{poly}(\mathrm{bc}(t), \gamma) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), \gamma) * 1/k + \sqrt{k}$	$\gamma \in \mathbb{Z}\,[3, 10]$

Table A1: Description of all tested models....

Functional form	Full distribution RMSE	Full distribution cRMSE	Lower tail ($p < 0.2$) RMSE	Lower tail ($p < 0.2$) cRMSE
$\mathrm{bc}(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * \sqrt{k}$	$4.97 \cdot 10^{-4}$	$4.69 \cdot 10^{-4}$	$8.05 \cdot 10^{-4}$	$7.16 \cdot 10^{-4}$
$\mathrm{bc}(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k$	$5.39 \cdot 10^{-4}$	$5.11 \cdot 10^{-4}$	$8.54 \cdot 10^{-4}$	$7.61 \cdot 10^{-4}$
$p = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * \sqrt{k}$	$7.68 \cdot 10^{-4}$	$6.91 \cdot 10^{-4}$	$1.01 \cdot 10^{-3}$	$8.97 \cdot 10^{-4}$
$p = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k + \sqrt{k}$	$7.79 \cdot 10^{-4}$	$7.04 \cdot 10^{-4}$	$1.05 \cdot 10^{-3}$	$9.31 \cdot 10^{-4}$
$p = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k$	$7.82 \cdot 10^{-4}$	$7.07 \cdot 10^{-4}$	$1.06 \cdot 10^{-3}$	$9.41 \cdot 10^{-4}$

Table A2: The 5 best models for the combined Non-Cointegration test of Bayer and Hanck, where *all* underlying test are included and case 1.

Functional form	Full distribution RMSE	Full distribution cRMSE	Lower tail ($p < 0.2$) RMSE	Lower tail ($p < 0.2$) cRMSE
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k)$	$1.27 \cdot 10^{-3}$	$1.25 \cdot 10^{-3}$	$1.05 \cdot 10^{-3}$	$9.52 \cdot 10^{-4}$
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * \sqrt{k}$	$6.82 \cdot 10^{-4}$	$6.22 \cdot 10^{-4}$	$1.28 \cdot 10^{-3}$	$1.12 \cdot 10^{-3}$
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k$	$7.32 \cdot 10^{-4}$	$6.63 \cdot 10^{-4}$	$1.39 \cdot 10^{-3}$	$1.20 \cdot 10^{-3}$
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k + \sqrt{k}$	$8.38 \cdot 10^{-4}$	$7.78 \cdot 10^{-4}$	$1.48 \cdot 10^{-3}$	$1.31 \cdot 10^{-3}$
$\mathrm{bc}(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k$	$9.08 \cdot 10^{-4}$	$8.42 \cdot 10^{-4}$	$1.69 \cdot 10^{-3}$	$1.50 \cdot 10^{-3}$

Table A3: The 5 best models for the combined Non-Cointegration test of Bayer and Hanck, where *all* underlying test are included and case 2.

Functional form	Full distribution RMSE	Full distribution cRMSE	Lower tail ($p < 0.2$) RMSE	Lower tail ($p < 0.2$) cRMSE
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * \sqrt{k}$	$4.58 \cdot 10^{-4}$	$4.55 \cdot 10^{-4}$	$3.37 \cdot 10^{-4}$	$3.16 \cdot 10^{-4}$
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k$	$5.17 \cdot 10^{-4}$	$5.14 \cdot 10^{-4}$	$3.90 \cdot 10^{-4}$	$3.73 \cdot 10^{-4}$
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k)$	$1.04 \cdot 10^{-3}$	$1.04 \cdot 10^{-3}$	$6.76 \cdot 10^{-4}$	$6.50 \cdot 10^{-4}$
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k + \sqrt{k}$	$1.18 \cdot 10^{-3}$	$1.17 \cdot 10^{-3}$	$2.06 \cdot 10^{-3}$	$2.05 \cdot 10^{-3}$
$\mathrm{bc}(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k$	$1.16 \cdot 10^{-3}$	$1.06 \cdot 10^{-3}$	$2.08 \cdot 10^{-3}$	$1.80 \cdot 10^{-3}$

Table A4: The 5 best models for the combined Non-Cointegration test of Bayer and Hanck, where *all* underlying test are included and case 3.

Functional form	Full distribution RMSE	Full distribution cRMSE	Lower tail ($p < 0.2$) RMSE	Lower tail ($p < 0.2$) cRMSE
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k$	$4.75 \cdot 10^{-4}$	$4.44 \cdot 10^{-4}$	$7.81 \cdot 10^{-4}$	$6.84 \cdot 10^{-4}$
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k)$	$6.54 \cdot 10^{-4}$	$5.87 \cdot 10^{-4}$	$1.01 \cdot 10^{-3}$	$7.81 \cdot 10^{-4}$
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * \sqrt{k}$	$7.60 \cdot 10^{-4}$	$6.13 \cdot 10^{-4}$	$1.46 \cdot 10^{-3}$	$1.06 \cdot 10^{-3}$
$\log(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k + \sqrt{k}$	$7.64 \cdot 10^{-4}$	$7.45 \cdot 10^{-4}$	$1.29 \cdot 10^{-3}$	$1.23 \cdot 10^{-3}$
$\mathrm{bc}(p) = \mathrm{poly}(\mathrm{bc}(t), 10) * \log(k) + \mathrm{poly}(\mathrm{bc}(t), 10) * 1/k$	$1.01 \cdot 10^{-3}$	$9.17 \cdot 10^{-4}$	$1.89 \cdot 10^{-3}$	$1.65 \cdot 10^{-3}$

Table A5: The 5 best models for the combined Non-Cointegration test of Bayer and Hanck, where *EG-J* underlying test are included and case 1.

Functional form	Full distribution		Lower tail ($p < 0.2$)	
	RMSE	cRMSE	RMSE	cRMSE
$\log(p) = \text{poly}(\text{bc}(t), 10) * \log(k)$	$7.36 \cdot 10^{-4}$	$7.25 \cdot 10^{-4}$	$7.04 \cdot 10^{-4}$	$6.45 \cdot 10^{-4}$
$\log(p) = \text{poly}(\text{bc}(t), 10) * \log(k) + \text{poly}(\text{bc}(t), 10) * 1/k$	$5.53 \cdot 10^{-4}$	$5.12 \cdot 10^{-4}$	$9.75 \cdot 10^{-4}$	$8.56 \cdot 10^{-4}$
$\log(p) = \text{poly}(\text{bc}(t), 10) * \log(k) + \text{poly}(\text{bc}(t), 10) * \sqrt{k}$	$5.53 \cdot 10^{-4}$	$5.11 \cdot 10^{-4}$	$9.87 \cdot 10^{-4}$	$8.66 \cdot 10^{-4}$
$\log(p) = \text{poly}(\text{bc}(t), 10) * \log(k) + \text{poly}(\text{bc}(t), 10) * 1/k + \sqrt{k}$	$6.00 \cdot 10^{-4}$	$5.62 \cdot 10^{-4}$	$1.11 \cdot 10^{-3}$	$1.01 \cdot 10^{-3}$
$\text{bc}(p) = \text{poly}(\text{bc}(t), 10) * \log(k) + \text{poly}(\text{bc}(t), 10) * \sqrt{k}$	$1.05 \cdot 10^{-3}$	$9.54 \cdot 10^{-4}$	$2.00 \cdot 10^{-3}$	$1.75 \cdot 10^{-3}$

Table A6: The 5 best models for the combined Non-Cointegration test of Bayer and Hanck, where *EG-J* underlying test are included and case 2.

Functional form	Full distribution		Lower tail ($p < 0.2$)	
	RMSE	cRMSE	RMSE	cRMSE
$\log(p) = \text{poly}(\text{bc}(t), 10) * \log(k) + \text{poly}(\text{bc}(t), 10) * \sqrt{k}$	$3.85 \cdot 10^{-4}$	$3.73 \cdot 10^{-4}$	$5.03 \cdot 10^{-4}$	$4.58 \cdot 10^{-4}$
$\log(p) = \text{poly}(\text{bc}(t), 10) * \log(k)$	$7.55 \cdot 10^{-4}$	$7.54 \cdot 10^{-4}$	$4.85 \cdot 10^{-4}$	$4.70 \cdot 10^{-4}$
$\log(p) = \text{poly}(\text{bc}(t), 10) * \log(k) + \text{poly}(\text{bc}(t), 10) * 1/k$	$3.73 \cdot 10^{-4}$	$3.59 \cdot 10^{-4}$	$5.34 \cdot 10^{-4}$	$4.83 \cdot 10^{-4}$
$\log(p) = \text{poly}(\text{bc}(t), 10) * \log(k) + \text{poly}(\text{bc}(t), 10) * 1/k + \sqrt{k}$	$4.87 \cdot 10^{-4}$	$4.76 \cdot 10^{-4}$	$8.52 \cdot 10^{-4}$	$8.19 \cdot 10^{-4}$
$\text{bc}(p) = \text{poly}(\text{bc}(t), 10) * \log(k) + \text{poly}(\text{bc}(t), 10) * \sqrt{k}$	$1.02 \cdot 10^{-3}$	$9.35 \cdot 10^{-4}$	$1.94 \cdot 10^{-3}$	$1.70 \cdot 10^{-3}$

Table A7: The 5 best models for the combined Non-Cointegration test of Bayer and Hanck, where *EG-J* underlying test are included and case 3.

Eidesstattliche Versicherung

Ich versichere an Eides statt durch meine Unterschrift, dass ich die vorstehende Arbeit selbständig und ohne fremde Hilfe angefertigt und alle Stellen, die ich wörtlich oder annähernd wörtlich aus Veröffentlichungen entnommen habe, als solche kenntlich gemacht habe, mich auch keiner anderen als der angegebenen Literatur oder sonstiger Hilfsmittel bedient habe. Die Arbeit hat in dieser oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen.

Essen, den ____________________ ________________________________

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