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An Implementation of the Combined Non-Cointegration Test of Bayer and Hanck

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Contents

List of Figures **III**

List of Tables **V**

List of Abbreviations **VI**

1 Introduction **1**

2 The meta test by Bayer and Hanck **2**

3 Simulation **4**

4 Models **5**

4.1 Data Pre-Processing 5

4.2 Polynomial Regression 7

4.3 Least Absolute Shrinkage and Selection Operator (LASSO) . 7

4.4 Other Regression Models 8

5 Model Evaluation **9**

5.1 RMSE comparison 9

5.2 Correction for high values of the test statistic 12

6 Algorithmic implementation **14**

7 Conclusion **16**

References **VII**

Software-References **X**

A Appendices **XI**

A.1 Results for the p-approximation of the Bayer-Hanck Test with all underyling Tests XIII

A.1.1 Metrics of the 5 Best Models XIII

A.1.2 Metrics of all Models XIV

A.2 Results for the p-approximation of the Bayer-Hanck Test with Engle-Granger and Johansen as underlying tests XLI

A.2.1 Metrics of the 5 Best Models XLI

A.2.2 Metrics of all Models XLIV

List of Figures

1 Densities of the untransformed and Box-Cox-transformed test statistic for test types all and EJ, shown for case = 1 and $k = 1$. 6

2 Simulated p-values plotted against p-values approximated by final (corrected) models, exemplary for $k = 1$. 11

3 Simulated p-values plotted against p-values approximated by final (corrected) models, exemplary for $k = 1$, for the lower tail of the distribution. 12

4 Simulated p-values plotted against p-values approximated by final corrected models, exemplary for $k = 2$. Corrected (blue) and uncorrected (red) p-value predictions for all cases and $k = 2$, using EJ and All as underlying tests. 14

A1 Simulated against approximated p-values over the whole distribution for all cases and all underlying tests. LXXI

A2 Simulated vs. approximated p-values for the lower tail of the distribution for all cases and all underlying test. LXXII

A3 Simulated against approximated p-values over the whole distribution for all cases and EJ test type. LXXIII

A4 Simulated vs. approximated p-values for the lower tail of the distribution for all cases and EJ test type. LXXIV

A5 Corrected (blue) and uncorrected (red) p-value predictions for all cases and all underlying tests. LXXV

A6 Corrected (blue) and uncorrected (red) p-value predictions for all cases using Engle-Granger and Johansen as underlying tests. LXXVI

List of Tables

1 The final models, selected according to the lowest cRMSE for the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. . 10

2 The Lasso models, to check the final polynomial regression models for any redundancy. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. 10

A1 Description of all tested functional forms for polynomial regression. All functional forms were tested for a maximum polynomial degree from 3 to 13. The shorthand notation was used for the description. XII

A2 The five best models, based on the cRMSE for the lower tail of the distribution, for the first case (no constant, no trend) and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. XIII

A3 The five best models, based on the cRMSE for the lower tail of the distribution, for the second case (with constant, no trend) and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. . XIII

A4 The five best models, based on the cRMSE for the lower tail of the distribution, for the third case (with constant and trend) and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. XIV

A5 Performance of the models for the first case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. XIV

A6 Performance of the models for the second case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. XXIII

A7 Performance of the models for the third case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. XXXII

A8 The five best models, based on the cRMSE for the lower tail of the distribution, for the first case (no constant, no trend) with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. XLII

A9 The five best models, based on the cRMSE for the lower tail of the distribution, for the second case (with constant, no trend) with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. XLII

A10 The five best models, based on the cRMSE for the lower tail of the distribution, for the third case (with constant and trend) with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. XLIII

A11 Performance of the models for the first case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. . XLIV

A12 Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. . LIII

A13 Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. . LXII

List of Abbreviations

CDF	Cumulative Distribution Function	5
CRAN	Comprehensive R Archive Network	14
cRMSE	corrected Root Mean Squared Error	9
FCLT	Functional Central Limit Theorem	2
GAM	Generalized Additive Models	8
LASSO	Least Absolute Shrinkage and Selection Operator	I
OLS	Ordinary Least Squares	3
RMSE	Root Mean Squared Error	5
VECM	Vector Error Correction Model	3

1 Introduction

Meta tests have been shown to be a powerful tool when testing for the null of non-cointegration. The distribution of their test statistic, however, is mostly not available in closed form. The calculation of the critical values, let alone p-values, is therefore a cumbersome procedure, as one has to simulate sufficient values of the test statistic under the null hypothesis to approximate their distribution. When implementing those meta tests in econometric software packages, it is therefore necessary to include the full null distribution for each combination of the underlying tests. Many software packages' size limitations are therefore quickly exceeded.

One possible approach to this problem is to model the relationship between the p-values and the test statistic with a regression model. Instead of including the full null distribution in an econometric software package this model will be stored. This might reduce the size of the software package considerably.

In this paper we approximate the p-values of the meta test by Bayer and Hanck (2012) which tests for the null of non-cointegration with supervised machine learning algorithms. We train our models on simulated values of the test statistic and the corresponding p-values for various specifications of the aforementioned test. Subsequently, these models will be included in the software package `combcoint` for the statistical programming language R. We find that this approach indeed reduces the size of the package significantly. Section 2 explains the theoretical background of the meta test by Bayer and Hanck. We briefly introduce the underlying tests and describe the combination procedure of the meta test. Section 3 explains the simulation of the values of the test statistic and the calculation of the p-values. In Section 4 we describe the pre-processing of the data and the regression models used. Section 5 evaluates those models. For this, their predictive performances are compared by calculating several in-sample metrics. Furthermore, we discuss problems which arise when approximating distributions under certain assumptions and how to fix them. Finally, Section 6 outlines the implementation of the models in the aforementioned software package. Section 7 concludes.

2 The meta test by Bayer and Hanck

The choice as to which of the available cointegration tests to use is an issue in econometric time series analysis. Bayer and Hanck (2012) propose powerful meta tests which provide unambiguous test decisions. They combine several residual- and system-based tests in the manner of Fisher's (1932) Chi-squared test.

Bayer and Hanck build on previous work from Pesavento (2004), who considers the model

$$\Delta \boldsymbol{x}_t = \boldsymbol{\tau}_1 + \boldsymbol{v}_{1t} \tag{2.1}$$

$$y_t = (\mu_2 - \boldsymbol{\theta}'\boldsymbol{\mu}_1) + (\tau_2 - \boldsymbol{\theta}'\boldsymbol{\tau}_1)t + \boldsymbol{\theta}'\boldsymbol{x}_t + u_t, \tag{2.2}$$

$$\text{with } u_t = \rho u_{t-1} + v_{2t}. \tag{2.3}$$

(2.1) represents the regressor dynamics, while (2.2) describes the cointegrating relation. The observed sample $\mathbf{z}_0, ..., \mathbf{z}_T$ can be written as $\mathbf{z}_t = (\mathbf{x}_t', y_t)'$. The deterministic part of the model is described by restrictions on $\boldsymbol{\mu}_1'$, μ_2, $\boldsymbol{\tau}_1$ and τ_2. Consider $\boldsymbol{\tau} = [\boldsymbol{\tau}_1'\tau_2]'$. Then, these restrictions are (1) $\mu_2 - \boldsymbol{\theta}'\boldsymbol{\mu}_1$ and $\boldsymbol{\tau} = 0$ which translates to no deterministics, (2) $\boldsymbol{\tau} = 0$ which corresponds to a constant in the cointegrating vector, (3) $\tau_2 - \boldsymbol{\theta}'\boldsymbol{\tau}_1 = 0$, a constant plus trend. $\boldsymbol{v}_t = [\boldsymbol{v}_{1t}' v_{2t}]'$ with $\boldsymbol{\Omega}$ the long-run covariance matrix of $\boldsymbol{v}_t$. It can be shown that $\{\boldsymbol{v}_t\}$ satisfies an Functional Central Limit Theorem (FCLT), i.e. $T^{-1/2}\sum_{t=1}^{[\cdot T]} \boldsymbol{v}_t \Rightarrow \boldsymbol{\Omega}^{1/2}\boldsymbol{W}(\cdot)$, with $\boldsymbol{W}$ denoting a Wiener process. It is also assumed that the variables in $\mathbf{x}_t$ are not cointegrated. It further follows from (2.3) that the vector $\mathbf{z}_t$ is cointegrated if $|\rho| < 1$. Hence the null hypothesis of no cointegration can be formulated as $H_0 : \rho = 1$.

Furthermore, Pesavento introduces two other parameters. Firstly, R^2 measures the squared correlation between $\boldsymbol{v}_{1t}$ and v_{2t}. It can be interpreted as the influence of the right-hand side variables in (2.2). It ranks between zero and one. When there is no long-run correlation between those variables and the errors from the cointegration regression, R^2 equals zero. Secondly, the number of lags is approximated by a finite number p.

Bayer and Hanck's meta test enables the combination of up to four standalone tests. Namely, these are the tests of Engle and Granger (1987), Johansen (1988), Boswijk (1994) and Banerjee et al. (1998). For the sake of brevity we will not present a detailed derivation of the underlying tests.

Engle and Granger (1987) propose a two-step procedure to test the null hypothesis of no cointegration against the alternative of at least one cointe-

grating vector. First, the long-run relationship between y_t and $\mathbf{x}_t$ is estimated by least squares regression. The obtained residuals $\hat{u}_t$ are then tested for a unit root. For this, Engle and Granger suggest the use of the t-statistic t_γ^{ADF} in the Augmented Dickey-Fuller (ADF) regression:

$$\Delta\hat{u}_t = \gamma\hat{u}_{t-1} + \sum_{p=1}^{P-1} \nu_p \Delta\hat{u}_{t-p} + \varepsilon_t. \tag{2.4}$$

The rejection of a unit root points to a cointegration relationship.

Johansen's (1988) maximum eigenvalue test is a system-based test that allows testing for several cointegration relationships. Take the Vector Error Correction Model (VECM)

$$\Delta\mathbf{z}_t = \mathbf{\Pi}\mathbf{z}_{t-1} + \sum_{p=1}^{P-1} \mathbf{\Gamma}_p \Delta\mathbf{z}_{t-p} + \mathbf{d}_t + \varepsilon_t. \tag{2.5}$$

We base this test on the test statistic $\varrho_{\max}(h) = -T\ln(1-\hat{\pi}_1)$. $\hat{\pi}_1$ is the largest solution to $[\pi\mathbf{S}_{11} - \mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}] = 0$, with the $\mathbf{S}_{ij}$ being moment matrices of reduced rank regression residuals.

The third and fourth tests considered are error correction-based. Both estimate the equation

$$\Delta y_t = d_t + \boldsymbol{\pi}_{0x}'\Delta\mathbf{x}_t + \varphi_0 y_{t-1} + \boldsymbol{\varphi}_1'\mathbf{x}_{t-1} + \sum_{p=1}^{P}(\boldsymbol{\pi}_{px}'\Delta\mathbf{x}_{t-p} + \pi_{py}\Delta y_{t-p}) + \varepsilon_t \tag{2.6}$$

by Ordinary Least Squares (OLS). P is chosen so that the ε_t is approximately white noise. Banerjee et al. (1998) then test the null of non-cointegration by applying a t-test on φ_0, i.e. $\mathcal{H}_0 : \varphi_0 = 0$. Boswijk (1994) uses the Wald statistic for testing $\mathcal{H}_0 : (\varphi_0, \boldsymbol{\phi}_1')' = \mathbf{0}$.

To combine the results from the underlying tests Bayer and Hanck draw upon Fisher's combined probability test (Fisher, 1932). It merges the test decisions using the formula

$$\tilde{\chi}_{\mathcal{I}}^2 := -2\sum_{i\in\mathcal{I}} \ln(p_i), \tag{2.7}$$

where t_i is the ith test statistic. If test i rejects for large values, take $\xi_i := t_i$. If test i rejects for small values, take $-\xi_i := t_i$. With $\Xi_i(x) := \Pr_{\mathcal{H}_\prime}(\xi_i \geq x)$ the p-value of the ith test is $p_i := \Xi_i(\xi_i)$.

Fisher shows that under the assumption of independence the null distribution

of $\tilde{\chi}^2_{\mathcal{I}}$ follows a chi-squared distribution with $2\mathcal{I}$ degrees of freedom. If this assumption is violated the null distribution is less evident. Here, the latter case occurs, as the ξ_i are not independent. The $\tilde{\chi}^2_{\mathcal{I}}$, however, have well-defined asymptotic null distributions $F_{\mathcal{F}_{\mathcal{I}}}$, as $\tilde{\chi}^2_{\mathcal{I}} \to_d \mathcal{F}_{\mathcal{I}}$ under $\mathcal{H}_0$ if $T \to \infty$, with $\mathcal{F}_{\mathcal{I}}$ some random variable. It is therefore feasible to simulate the joint null distribution of the ξ_i to obtain the distribution $F_{\mathcal{F}_{\mathcal{I}}}$ of (2.7). The $F_{\mathcal{F}_{\mathcal{I}}}$ depend on number and type of the combined tests. The distributions of the ξ_i depend on $K-1$ and the deterministic case.

3 Simulation

We can simulate the null distributions of the underlying tests and compute their p-values. These results are used to derive the null distribution of the meta test by Bayer and Hanck. In this section, the simulation approach is explained, which generates a large number of test statistics of said test. A sufficient amount of data ought to be obtained to train machine learning algorithms on approximating the p-values of the test.

In the implementation phase of the package we plan on allowing the user to choose between two different combinations of the underlying tests. These will be a combination of the tests of Engle-Granger and Johansen (EJ) and a combination of all possible underlying tests (all). Thus, we calculate (2.7) twice, with a different selection of ξ_i. We account for the above-mentioned restrictions on the deterministic part of the model by slightly adjusting the underlying process in each case. In total, we generate three different data sets, each containing the test statistic once as a combination of the EJ tests and once for all tests.

The following approach relies largely on previous work by Pesavento (2004). It can be shown that the asymptotic null distributions of the underlying tests are functions of standard Brownian motions. We construct this by step functions using Gaussian random walk. For this, we sum up $N = 1000$ independent, standard normally-distributed random numbers. We further simulate an Ornstein-Uhlenbeck-Process. From these processes we calculate different terms to construct the null distributions of the underlying tests. As stated before, we correct the processes according to the three different restrictions on the model deterministics. For the first case, no trend type, the process needs no correction. For the second case, a constant trend, both processes need to be demeaned. For the last case, a constant trend and drift, both processes are adjusted for the mean and trend.

The number of repetitions is set to 1,000,000. Moreover, we consider $R^2 \in \{0, 0.05, 0.1, ..., 0.95\}$ and the number of regressors $\{k \in \mathbb{Z} : 1 \leq k \leq 11\}$. Pesavento further introduces the local-to-unity parameter $c := T(\rho - 1)$. For negative values of c the variables are cointegrated, while for $c = 0$ there is no cointegration. Since we solely aim at simulating the distribution under the null hypothesis of no cointegration we will not consider other values different from $c = 0$.

To calculate the $\tilde{\chi}^2_{\mathcal{I}}$ we require the p-values of the underlying tests. For this, we build the Cumulative Distribution Function (CDF) of each underlying test for the three different data sets and calculate the respective p-values. These are inserted into (2.7) to obtain the meta test statistic. Analogous to the previous approach, we deduce the associated null distribution and the p-values.

4 Models

We use the simulated data for training machine learning algorithms on predicting the approximated empirical CDF of the combined non-cointegration test. We want to approximate the CDF as a function of the test statistic $\tilde{\chi}^2_{\mathcal{I}}$ and the number of regressors k. As we seek solutions for describing the null distribution with a less memory-intensive model, we only consider linear models. Non-linear models typically take up too much memory to be suitable here. Moreover, we compare the models according to their in-sample Root Mean Squared Error (RMSE). This seems appropriate as we want to predict already known data. Overfitting therefore does not pose a threat. For this reason, and to reduce computation time, we use no cross-validation.

4.1 Data Pre-Processing

One approach for improving a model's predictive ability is to pre-process the training data. Some models react sensitively to certain characteristics of the predictor or response data. Those characteristics include, inter alia, distributional skewness and outliers. There exist several methods to lower their potentially bad impact on the model's performance.

Since we simulated the training data under the null of non-cointegration we expect the distribution of the test statistic to be right skewed. Figure 1 shows that it is also (right) long-tailed. If we train the regression model on

this data there can be difficulties when predicting from high values of the test statistic.

A way to deal with these issues are power transforms. A well-known family of these transformations to un-skew data is the Box-Cox transformation (Box & Cox, 1964). They aim at transforming the data so that it closely resembles the normal distribution. The exact transformation depends on the parameter λ, whose optimal value can be empirically estimated:

$$y^{(\lambda)} = \begin{cases} \frac{y^{\lambda}-1}{\lambda}, & \lambda \neq 0 \\ \log(y), & \lambda = 0 \end{cases} \tag{4.1}$$

Box and Cox (1964) developed these transformations for the dependent variable. Kuhn and Johnson (2013), however, report that it proves as effective for transforming individual regressors. We estimate λ for the values of the test statistics of the meta test and transform them according to (4.1). This forces their distribution into a more symmetric form, albeit still right-tailed (again, see Figure 1).

Figure 1: Densities of the untransformed and Box-Cox-transformed test statistic for test types all and EJ, shown for case = 1 and $k = 1$.

Since the response variable consists of the p-values, simulated under the null hypothesis, it follows a uniform distribution. Therefore, it is already symmetric. We still include a Box-Cox transformed and a logarithmised version of the response variable to see if it benefits the prediction. We also decompose the categorical variable k into dummy variables. Additionally,

we add the same variable as a numeric. Thus, transformations such as the logarithm can be performed.

4.2 Polynomial Regression

As mentioned, we restrict ourselves to linear models. The empirical CDF, which we aim to predict, is typically known to have a curved shape. Thus, we skip the simple linear regression model in favor of more flexible alternatives. We stay with least squares regression, but try various combinations of polynomial functions and interaction terms of the aforementioned regressors. The search for the best model is carried out via brute-force.

Polynomial Regression extends the classic linear regression model by fitting a polynomial equation of arbitrary order to the data. A polynomial regression with n degrees thus takes the form

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + ... + \beta_n x_i^n + \varepsilon_i, \tag{4.2}$$

where ε_i is the error term (James et al., 2013).

We calculate orthogonal polynomials of the test statistic of the combined non-cointegration test, considering up to 13 degrees. Due to the CDF's characteristic form, the minimum grade of the polynomials is 3. We also add different interaction terms with different forms of the regressor k, to better capture the curvilinear relationship between predictors and the response. Table A1 in Appendix A displays the forms of all considered regression models. Since there is no need to prevent overfitting we expect higher order polynomials to perform better, as they are more flexible. These polynomials, however, tend to show oscillation at the boundaries. This makes prediction for more extreme values of the test statistic a risky endeavour. We will address this issue in a later chapter.

4.3 LASSO

As mentioned, we expect the polynomial regression models to perform best with high order polynomials. Every polynomial added increases the complexity of the model. Table A1 shows that in some instances for each additional polynomial another set of interaction terms is also included. It is possible that some regressors (including the interaction terms) benefit the prediction, while others are redundant. Yet, it would be too cumbersome

to manually add and remove different regressors. Nevertheless, we do not want the models to be larger than necessary. Therefore, it may be necessary to perform variable selection and regularization. An approach for this is the LASSO. As the LASSO is sensitive to the scale unit the optimization is performed on the standardised variables.

The LASSO coefficients minimize

$$\hat{\tilde{\beta}}^{\text{lasso}} = \arg\min_{\tilde{\beta}} \sum_{i=1}^{n} \left(\tilde{y}_i - \tilde{\beta}_0 - \sum_{j=1}^{p} \tilde{x}_{ij} \tilde{\beta}_j \right)^2 + \Upsilon \sum_{j=1}^{p} \left| \tilde{\beta}_j \right|, \tag{4.3}$$

where the first term describes the residual sum of squares, subject to a term known as L1 penalty. Υ is a tuning parameter which defines the degree of regularisation. The L1 penalty shrinks the coefficients and, for Υ sufficiently large, can set them to zero. The value of Υ is data dependent and is usually estimated with cross-validation. (James et al., 2013)

We plan on using the LASSO on the best models from chapter 4.2. Thus, we can assess if said models include redundant regressors.

4.4 Other Regression Models

We considered various other regression models. For different reasons they were not too suitable for our particular case.

Especially non-linear methods should provide a good prediction. Generalized Additive Models (GAM) are another extension of simple linear regression. They replace each linear component in the latter with a (smooth) non-linear function. These basis functions are then summed together. This allows for a very flexible fit to the data. (James et al., 2013)

Tree-based models, especially used as base learners in ensemble methods, are also known to be very flexible. However, they tend to perform worse with a small amount of regressors. Thus, we directly discarded this approach in an early stage of the model estimation.

Eventually, these models have the major disadvantage that they take up far too much memory space compared to the polynomial regression models. A Random Forest built with `ranger`, for example, requires 350 MB. A simple GAM needs up to 800 MB. In comparison, a polynomial regression model only consumes 71.4 kB of memory. Given these limitations, we stick to our decision to solely work with linear regression models.

5 Model Evaluation

We estimate all models for two different combinations of the underlying tests. Namely, these are a combination of the Engle-Granger and Johansen test (EJ) and a combination of all four underlying tests (all). Furthermore, we estimate one model per specification of the model deterministics. Altogether, this results in a total of six different models.

5.1 RMSE comparison

To measure the performance of our regression models we calculate their in-sample RMSE. This is an indication of how far the residuals of the models are from zero, with lower values preferable. We calculate the RMSE for predictions on the full distribution, as well as predictions on the lower tail ($p \leq 0.2$). We pay specific attention to the latter, as it is crucial for the test decision of the meta test. We also add a "corrected" version of the RMSE, corrected Root Mean Squared Error (cRMSE), for both the full distribution and the lower tail. Here, we limited the predictions to lie within the interval $[0, 1]$, i.e. if the prediction lies outside the interval it is automatically set to its nearest interval boundary.

Table A5 - A7 in Appendix A list all variations of the RMSE for the calculated regression models for all underlying tests. Table A11 - A13 list all variations of the RMSE for EJ as underlying tests. We select the best models according to the cRMSE. This seems to be the most appropriate metric, as we plan on incorporating aforesaid correction of the prediction in the final models. Furthermore, a good prediction on the lower tail of the distribution is of higher relevance. Table A2 - A4 list the five best models, as measured by the cRMSE, for each case and all underlying tests. Table A8 - A10 list the same, but for EJ as underlying tests. For both test types and all cases it shows that a combination of high order polynomials, dummy variables and the associated interaction terms achieves superior predictive performance compared to simpler models. For all considered types of the RMSE the best models require a polynomial of minimum grade 12. This is no surprise, considering we are optimising an in-sample fit.

Table 1 lists the final models for each case and test type. For purposes of presentation we abbreviate the description of the models. The response variable is referred to as p, with $\mathrm{bc}(p)$ and $\log(p)$ standing for its Box-Cox-transformed and logarithmised forms, respectively. t refers to the value of

Table 1: The final models, selected according to the lowest cRMSE for the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

			Full Distribution		**Lower Tail** ($p \leq 0.2$)	
Type	Case	Model	RMSE	cRMSE	RMSE	cRMSE
all	1	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.79e-04	1.73e-04	1.73e-04	1.71e-04
all	2	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.05e-04	2.02e-04	2.15e-04	2.11e-04
all	3	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.92e-04	1.86e-04	2.02e-04	1.95e-04
EJ	1	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.40e-04	1.40e-04	1.48e-04	1.48e-04
EJ	2	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.57e-04	1.56e-04	1.64e-04	1.63e-04
EJ	3	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.45e-04	1.45e-04	1.53e-04	1.52e-04

the test statistic, while k describes the eponymous categorical variable. The latter's dummy variable form is abbreviated with k_d.

The forms of the models share many common features, mostly using the maximum polynomial degree available. One might say this represents a corner solution. However, since we are optimising the models on the in-sample RMSE, we could increase the degree of the polynomials endlessly, to improve the fit. Since it can be shown that we are already tweaking on the fifth decimal place, we will not pursue this procedure any further.

As announced, we use the LASSO to check if the final models contain any redundancy. As before, we skip cross-validation and use an automatically generated sequence for Υ. The default are 100 values for Υ. The program, however, stops early if the percent (of null) deviance explained does not change sufficiently for the next Υ. (Friedman et al., 2010)

Table 2: The Lasso models, to check the final polynomial regression models for any redundancy. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

			Full Distribution		**Lower Tail** ($p \leq 0.2$)	
Type	Case	Model	RMSE	cRMSE	RMSE	cRMSE
all	1	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	8.76e-03	8.74e-03	9.39e-03	9.38e-03
all	2	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	9.72e-03	9.69e-03	1.07e-02	1.06e-02
all	3	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	9.78e-03	9.72e-03	1.07e-02	1.07e-02
EJ	1	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	9.59e-03	9.56e-03	1.04e-02	1.04e-02
EJ	2	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	9.76e-03	9.73e-03	1.06e-02	1.06e-02
EJ	3	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	9.84e-03	9.81e-03	1.07e-02	1.07e-02

Here, for all models the actual number of Υ is less than 100. For our purposes

it is sufficient to choose the lowest Υ out of that sequence, ergo the model with the least shrinkage. The number of nonzero coefficients ranges between 47 to 60 for the models with the test type all and between 27 to 34 for the models with test type EJ. In both instances more than half of all regressors of the final polynomial regression models were not considered. Table 2 lists the predictive performance of the LASSO models. Clearly all variations of the RMSE are much higher than for the polynomial regression models. This supports the idea that more complex models are indeed more appropriate in this case. We therefore drop the LASSO models and continue to use our final polynomial regression models.

To assess model performance graphically, we plot the simulated p-values from chapter 3 against the (corrected) p-values approximated by the final models. If these values are highly correlated, they are scattered symmetrically around a 45 degree diagonal passing through the zero point. The vertical distance between any point and the diagonal is the prediction error for that point. Figure 2 shows this plot for all cases and $k = 1$. The models appear very accurate, there seems to be a high correlation between the predicted and the actual values. Figure A1 and A3 in Appendix A show these plots for all k.

Figure 2: Simulated p-values plotted against p-values approximated by final (corrected) models, exemplary for $k = 1$.

Looking at the same plot for the lower tail of the distribution (with $p \leq 0.2$), the models show high reliability even in the crucial share of the p-values (see

Figure 3). Figure A2 and A4 in Appendix A show these plots for all k.

Figure 3: Simulated p-values plotted against p-values approximated by final (corrected) models, exemplary for $k = 1$, for the lower tail of the distribution.

5.2 Correction for high values of the test statistic

As described in chapter 3 the data set used for training the models was simulated under the null hypothesis of no cointegration. For this reason most values of the test statistic are comparatively small. Even after its transformation the distribution of the test statistic has a longer right tail. When using the models within a software package, as originally intended, it is likely that they will also face input values located on the far right of the distribution. It cannot be ruled out that the models might fail to make sensible predictions for such values of the test statistic.

In order to test this, we generate a sequence from 1 to 100 in single steps, representing possible values of the test statistic. On this basis, we predict p-values with all six final models for all $k \in [1, 11]$. We include the correction to limit the predictions on the interval $[0, 1]$, as described in chapter 5.1.

Figure A5 in Appendix A shows the prediction of the models for all underlying tests on said sequence. In the majority of cases the corrected models perform well, with the prediction curve taking the expected shape. In two cases, when $k = 2$ and $k = 3$ in the model with no deterministics (case $= 1$), the predicted values rise again, taking values not equal to zero.

Figure A6 in the Appendix shows the same behaviour for the data with Engle-Granger and Johansen as underlying tests for all combinations of cases and k. Above a certain value of the test statistic the predicted values sharply increase, tending towards 1. It should be noted that this upper boundary is enforced by our built-in correction for predicted values outside the interval [0, 1]. Without this intervention the predicted values would probably rise even further. If we predict on an extended sequence with no correction, the prediction line most likely oscillates above a certain value. The graphic for all k can be seen in Figure A6 in the Appendix.

There can be several reasons why the models' prediction behaves this way. Oscillation at the edges of an interval is a common problem in polynomial interpolation, especially when using polynomials of high order. Additionally, the distribution of the test statistic in the training data may have made matters worse, as the models were trained on very few high values of the test statistic. It must also be considered that we chose our models according to their predictive performance on the lower tail of the distribution of the test statistic, possibly neglecting the predictive performance on the upper tail. If this incident is not rectified the models will be unable to provide reliable test decisions, as they tend to falsely not reject the null hypothesis at high values of the test statistic. The approach is therefore prone to type II errors.

From this it appears that there is a need for either more reliable models or further correction of the prediction from the existing models. One possible solution can be the re-estimation of the models using regression splines. In theory, those lead to similar results while being less prone to oscillation at the tails. However, with regression splines there is still no guarantee they will show the desired behaviour at said tails. Another approach is to determine a critical value of the test statistic whereby every exceeding value is automatically assigned a low p-value. Due to the former approach being very time-consuming we decide to try the last one.

Figure 4: Simulated p-values plotted against p-values approximated by final corrected models, exemplary for $k = 2$. Corrected (blue) and uncorrected (red) p-value predictions for all cases and $k = 2$, using EJ and All as underlying tests.

We define the critical value by taking the value of the test statistic with the lowest p-value. We do this for all combinations of the test type, cases and ks. The resulting values range between 36 and 44 for the EJ combination and 55 and 80 for all combinations respectively. Graphically, it should be obvious that these values are supposed to have rather low p-values. Thus, we can be confident that our approach suffices for our purposes without any unintended side-effects. Therefore, the problem of erratic behaviour for higher values of the test statistic is avoided.

6 Algorithmic implementation

Since the corrected models seem to provide a reliable prediction for the p-values we will include them in the pre-existing R-package `bayerhanck`. The package is currently hosted on GitHub and can be installed directly. Up to now the release on Comprehensive R Archive Network (CRAN) has not been possible as packages must not exceed a maximum size of 5 MB. The package, however, has a size of 12.2 MB, far exceeding the specified limit. A large part of this can be attributed to the attached data, namely

the critical values and the full null distributions. We plan on replacing these with our models to reduce the size of the package significantly.

We will not go into further detail on the general implementation of the `bayerhanck` package, e.g. the calculation of the underlying tests. Instead, we solely focus on the implementation of the polynomial regression models and their corrections. First, we create a nested `tibble`, a more efficient version of R's traditional `data.frame`, called `models`. It contains the final (uncorrected) models, the λ for the Box-Cox transformation of both the test statistic and the predicted p-value, as well as the critical values for the correction of the prediction of high values of the test statistic (see chapter 5.2). All objects and values can be selected according to their test- and trendtype.

In the package the meta test by Bayer and Hanck is performed by calling the function `bayerhanck()`. Here, the user may specify the test type, the number of lags p and the deterministic component. The function then calculates the test statistics of the underlying tests, EJ or all, and their p-values. These are then aggregated according to (2.7). Based on the test statistic a decision is taken whether to reject or accept the null hypothesis. Formerly, this relied heavily on the attached critical values and null distributions. As of now, `bayerhanck()` calls the internal function `get_p_value()`, which approximates the p-value with our polynomial regression models. Firstly, it accesses `models` to obtain the critical value from chapter 5.2, given the configuration in the function arguments of `bayerhanck()`. It then differentiates if the calculated test statistic lies beneath or above this critical value. In the latter case it skips the following steps and directly outputs a p-value of $1e-12$. If the test statistic lies beneath the critical value the p-value will be predicted with a polynomial regression model. For this, the value of the test statistic is transformed according to 4.1 with its associated λ stored in `models`. The corresponding model is selected to perform the estimation of the p-value, based on the transformed test statistic and k as regressors. If the response variable underwent a Box-Cox transformation in the initial fitting of the model, the newly predicted response is transformed back as a next step. Lastly, the algorithm forces the estimated p-value within the theoretical boundaries of 0 and 1. If a p-value ≤ 0 is predicted, it is corrected to $1e-12$. For values > 1, it is set to $9.9999e-1$. The function `bayerhanck()` then outputs the value of the test statistic of the meta test by Bayer and Hanck, accompanied by its p-value.

Eventually, this approach succeeded in reducing the package size significantly.

The implementation of the polynomial regression models for predicting the p-values reduced the memory space required by the package to less than 5 MB.

7 Conclusion

In this paper we fit machine learning models to predict the p-values of the meta tests of Bayer and Hanck, which tests for the null of non-cointegration. This was motivated by the fact that the test statistic of said test is not available in closed form. Therefore, one has to rely on simulation to obtain the distribution of the test statistic to eventually calculate the p-values. For these reasons, the prior implementation of the said test into a software package widely exceeded the given size limitations. It was our objective to reduce the size of this software package by including the machine learning models instead of the full null distributions for each combination of the underlying tests and deterministic case.

First, we gave a brief introduction into the theoretical background of the meta test, as well as it's underlying tests. We further described our simulation approach to simulate the distribution of the test statistic, which is later used as the training data. Due to the distribution being right-skewed, we used a Box-Cox transformation prior to fitting the models. We also used various transformations on the response variable, creating multiple versions of it.

After creating the data we used it to train the machine learning algorithms. Since we aim at describing the null distribution with a less memory-intensive approach, we only considered linear models. Still, the CDF is known to have a curved shape. To make the models more flexible we included various interaction terms and polynomials with up to 13 degrees. For evaluating the models we calculated several in-sample metrics. Eventually, we decided to choose the models that minimize the cRMSE over the lower tail of the distribution, corrected for values ranging between 0 and 1. As expected, the most complex models performed best. To check for redundancy in those final models, we used a LASSO on them. Additionally, we (again) corrected the final models to suppress oscillation at the boundaries. Finally, these models were included in the pre-existing software package `bayerhanck` for the statistical programming language R.

Our approach successfully reduced the size of the aforementioned package significantly and therefore it is possible to submit the package to CRAN,

which is a future objective.

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A Appendices

Table A1 list the different functional forms of the polynomial regression we tested. In total we investigated 21 different forms and for each of these forms we investigated the polynomial in the range from 3 to 13. As equations with many polynomials are getting very long we will use a short-hand notation. For example the first equation in Table A1 for a polynomial of 3 is in short-hand notation

$$p = c + \text{poly}\left(\text{bc}(t), 3\right) \tag{A.1}$$

and represents

$$p = c + \gamma_{1,1}t + \gamma_{1,2}t^2 + \gamma_{1,1}t^3. \tag{A.2}$$

Table A1: Description of all tested functional forms for polynomial regression. All functional forms were tested for a maximum polynomial degree from 3 to 13. The shorthand notation was used for the description.

Number	Functional form	Range of γ
1	$p = c + \text{poly}\left(\text{bc}(t), \gamma\right)$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
2	$p = c + \text{poly}\left(\text{bc}(t), \gamma\right) + k$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
3	$p = c + \text{poly}\left(\text{bc}(t), \gamma\right) * k$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
4	$p = c + \text{poly}\left(\text{bc}(t), \gamma\right) + \log(k)$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
5	$p = c + \text{poly}\left(\text{bc}(t), \gamma\right) * \log(k)$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
6	$p = c + \text{poly}\left(\text{bc}(t), \gamma\right) + k_d$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
7	$p = c + \text{poly}\left(\text{bc}(t), \gamma\right) * k_d$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
8	$\log(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right)$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
9	$\log(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) + k$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
10	$\log(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) * k$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
11	$\log(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) + \log(k)$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
12	$\log(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) * \log(k)$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
13	$\log(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) + k_d$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
14	$\log(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) * k_d$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
15	$\text{bc}(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right)$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
16	$\text{bc}(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) + k$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
17	$\text{bc}(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) * k$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
18	$\text{bc}(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) + \log(k)$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
19	$\text{bc}(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) * \log(k)$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
20	$\text{bc}(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) + k_d$	$\gamma \in \mathbb{Z}\left[3, 13\right]$
21	$\text{bc}(p) = c + \text{poly}\left(\text{bc}(t), \gamma\right) * k_d$	$\gamma \in \mathbb{Z}\left[3, 13\right]$

A.1 Results for the p-approximation of the Bayer-Hanck Test with all underyling Tests

A.1.1 Metrics of the 5 Best Models

Table A2: The five best models, based on the cRMSE for the lower tail of the distribution, for the first case (no constant, no trend) and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.79e-04	1.73e-04	1.73e-04	1.71e-04
2	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.76e-04	1.74e-04	1.88e-04	1.86e-04
3	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.00e-04	1.95e-04	2.10e-04	2.05e-04
4	$p = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.40e-04	2.27e-04	2.28e-04	2.18e-04
5	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	2.16e-04	2.09e-04	2.28e-04	2.19e-04

Table A3: The five best models, based on the cRMSE for the lower tail of the distribution, for the second case (with constant, no trend) and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.05e-04	2.02e-04	2.15e-04	2.11e-04
2	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	2.12e-04	2.04e-04	2.26e-04	2.17e-04
3	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	2.83e-04	2.68e-04	2.80e-04	2.66e-04
4	$p = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	3.68e-04	3.41e-04	2.87e-04	2.83e-04
5	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	3.70e-04	3.37e-04	4.10e-04	3.73e-04

Table A4: The five best models, based on the cRMSE for the lower tail of the distribution, for the third case (with constant and trend) and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.92e-04	1.86e-04	2.02e-04	1.95e-04
2	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.57e-04	2.39e-04	2.76e-04	2.56e-04
3	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	3.47e-04	3.22e-04	3.17e-04	3.00e-04
4	$p = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	3.85e-04	3.52e-04	3.11e-04	3.00e-04
5	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	3.41e-04	3.06e-04	3.77e-04	3.37e-04

A.1.2 Metrics of all Models

Table A5: Performance of the models for the first case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$p = c + \text{poly}\,(\text{bc}(t), 3)$	3.21e-02	2.38e-02	2.51e-02	2.45e-02
2	$p = c + \text{poly}\,(\text{bc}(t), 4)$	2.48e-02	2.40e-02	2.59e-02	2.55e-02
3	$p = c + \text{poly}\,(\text{bc}(t), 5)$	2.23e-02	2.16e-02	2.15e-02	2.15e-02
4	$p = c + \text{poly}\,(\text{bc}(t), 6)$	1.92e-02	1.87e-02	1.92e-02	1.91e-02
5	$p = c + \text{poly}\,(\text{bc}(t), 7)$	1.82e-02	1.78e-02	1.95e-02	1.90e-02
6	$p = c + \text{poly}\,(\text{bc}(t), 8)$	1.68e-02	1.67e-02	1.81e-02	1.81e-02
7	$p = c + \text{poly}\,(\text{bc}(t), 9)$	1.67e-02	1.66e-02	1.82e-02	1.81e-02
8	$p = c + \text{poly}\,(\text{bc}(t), 10)$	1.66e-02	1.66e-02	1.81e-02	1.81e-02
9	$p = c + \text{poly}\,(\text{bc}(t), 11)$	1.65e-02	1.65e-02	1.80e-02	1.80e-02
10	$p = c + \text{poly}\,(\text{bc}(t), 12)$	1.65e-02	1.65e-02	1.80e-02	1.80e-02
11	$p = c + \text{poly}\,(\text{bc}(t), 13)$	1.65e-02	1.65e-02	1.80e-02	1.80e-02
12	$p = c + \text{poly}\,(\text{bc}(t), 3) + k$	3.04e-02	2.11e-02	2.07e-02	1.98e-02

Table A5: Performance of the models for the first case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
13	$p = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.25e-02	2.14e-02	2.16e-02	2.10e-02
14	$p = c + \text{poly}\,(\text{bc}(t), 5) + k$	1.97e-02	1.86e-02	1.60e-02	1.58e-02
15	$p = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.60e-02	1.53e-02	1.27e-02	1.26e-02
16	$p = c + \text{poly}\,(\text{bc}(t), 7) + k$	1.49e-02	1.42e-02	1.31e-02	1.24e-02
17	$p = c + \text{poly}\,(\text{bc}(t), 8) + k$	1.30e-02	1.29e-02	1.10e-02	1.09e-02
18	$p = c + \text{poly}\,(\text{bc}(t), 9) + k$	1.29e-02	1.28e-02	1.10e-02	1.10e-02
19	$p = c + \text{poly}\,(\text{bc}(t), 10) + k$	1.28e-02	1.28e-02	1.09e-02	1.09e-02
20	$p = c + \text{poly}\,(\text{bc}(t), 11) + k$	1.27e-02	1.26e-02	1.08e-02	1.08e-02
21	$p = c + \text{poly}\,(\text{bc}(t), 12) + k$	1.27e-02	1.26e-02	1.08e-02	1.08e-02
22	$p = c + \text{poly}\,(\text{bc}(t), 13) + k$	1.27e-02	1.26e-02	1.08e-02	1.08e-02
23	$p = c + \text{poly}\,(\text{bc}(t), 3) * k$	2.77e-02	1.74e-02	1.82e-02	1.72e-02
24	$p = c + \text{poly}\,(\text{bc}(t), 4) * k$	1.85e-02	1.74e-02	1.89e-02	1.82e-02
25	$p = c + \text{poly}\,(\text{bc}(t), 5) * k$	1.42e-02	1.39e-02	1.19e-02	1.18e-02
26	$p = c + \text{poly}\,(\text{bc}(t), 6) * k$	8.65e-03	7.68e-03	8.52e-03	7.61e-03
27	$p = c + \text{poly}\,(\text{bc}(t), 7) * k$	6.90e-03	6.06e-03	6.51e-03	6.22e-03
28	$p = c + \text{poly}\,(\text{bc}(t), 8) * k$	5.41e-03	5.21e-03	5.60e-03	5.55e-03
29	$p = c + \text{poly}\,(\text{bc}(t), 9) * k$	5.23e-03	5.10e-03	5.55e-03	5.49e-03
30	$p = c + \text{poly}\,(\text{bc}(t), 10) * k$	4.81e-03	4.79e-03	5.26e-03	5.25e-03
31	$p = c + \text{poly}\,(\text{bc}(t), 11) * k$	4.79e-03	4.78e-03	5.24e-03	5.23e-03
32	$p = c + \text{poly}\,(\text{bc}(t), 12) * k$	4.76e-03	4.75e-03	5.22e-03	5.22e-03
33	$p = c + \text{poly}\,(\text{bc}(t), 13) * k$	4.75e-03	4.75e-03	5.22e-03	5.22e-03
34	$p = c + \text{poly}\,(\text{bc}(t), 3) + \log(k)$	3.03e-02	2.07e-02	2.02e-02	1.93e-02
35	$p = c + \text{poly}\,(\text{bc}(t), 4) + \log(k)$	2.23e-02	2.10e-02	2.11e-02	2.05e-02
36	$p = c + \text{poly}\,(\text{bc}(t), 5) + \log(k)$	1.94e-02	1.82e-02	1.54e-02	1.52e-02
37	$p = c + \text{poly}\,(\text{bc}(t), 6) + \log(k)$	1.56e-02	1.48e-02	1.19e-02	1.18e-02
38	$p = c + \text{poly}\,(\text{bc}(t), 7) + \log(k)$	1.45e-02	1.38e-02	1.23e-02	1.15e-02

Table A5: Performance of the models for the first case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
39	$p = c + \text{poly}\,(\text{bc}(t), 8) + \log(k)$	1.26e-02	1.24e-02	1.00e-02	1.00e-02
40	$p = c + \text{poly}\,(\text{bc}(t), 9) + \log(k)$	1.25e-02	1.23e-02	1.01e-02	1.00e-02
41	$p = c + \text{poly}\,(\text{bc}(t), 10) + \log(k)$	1.24e-02	1.23e-02	9.96e-03	9.93e-03
42	$p = c + \text{poly}\,(\text{bc}(t), 11) + \log(k)$	1.22e-02	1.21e-02	9.88e-03	9.84e-03
43	$p = c + \text{poly}\,(\text{bc}(t), 12) + \log(k)$	1.22e-02	1.21e-02	9.85e-03	9.83e-03
44	$p = c + \text{poly}\,(\text{bc}(t), 13) + \log(k)$	1.22e-02	1.21e-02	9.85e-03	9.83e-03
45	$p = c + \text{poly}\,(\text{bc}(t), 3) * \log(k)$	2.74e-02	1.69e-02	1.76e-02	1.66e-02
46	$p = c + \text{poly}\,(\text{bc}(t), 4) * \log(k)$	1.83e-02	1.70e-02	1.85e-02	1.77e-02
47	$p = c + \text{poly}\,(\text{bc}(t), 5) * \log(k)$	1.33e-02	1.31e-02	1.05e-02	1.05e-02
48	$p = c + \text{poly}\,(\text{bc}(t), 6) * \log(k)$	6.94e-03	5.74e-03	6.79e-03	5.48e-03
49	$p = c + \text{poly}\,(\text{bc}(t), 7) * \log(k)$	5.18e-03	3.91e-03	3.90e-03	3.47e-03
50	$p = c + \text{poly}\,(\text{bc}(t), 8) * \log(k)$	2.70e-03	2.29e-03	2.18e-03	2.04e-03
51	$p = c + \text{poly}\,(\text{bc}(t), 9) * \log(k)$	2.40e-03	2.08e-03	2.08e-03	1.90e-03
52	$p = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	1.03e-03	9.71e-04	9.22e-04	8.74e-04
53	$p = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	1.03e-03	9.68e-04	9.34e-04	8.82e-04
54	$p = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	8.29e-04	8.02e-04	7.39e-04	7.35e-04
55	$p = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	7.71e-04	7.63e-04	7.37e-04	7.31e-04
56	$p = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	3.03e-02	2.07e-02	2.02e-02	1.93e-02
57	$p = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	2.23e-02	2.10e-02	2.11e-02	2.05e-02
58	$p = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	1.94e-02	1.82e-02	1.54e-02	1.52e-02
59	$p = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	1.56e-02	1.48e-02	1.18e-02	1.18e-02
60	$p = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	1.45e-02	1.38e-02	1.23e-02	1.15e-02
61	$p = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	1.26e-02	1.24e-02	1.00e-02	1.00e-02
62	$p = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	1.25e-02	1.23e-02	1.01e-02	1.00e-02
63	$p = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	1.24e-02	1.23e-02	9.95e-03	9.92e-03
64	$p = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	1.22e-02	1.21e-02	9.87e-03	9.83e-03

Table A5: Performance of the models for the first case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
65	$p = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	1.22e-02	1.21e-02	9.85e-03	9.82e-03
66	$p = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	1.22e-02	1.21e-02	9.84e-03	9.82e-03
67	$p = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	2.73e-02	1.69e-02	1.76e-02	1.65e-02
68	$p = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	1.79e-02	1.67e-02	1.82e-02	1.74e-02
69	$p = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	1.31e-02	1.29e-02	1.04e-02	1.04e-02
70	$p = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	6.23e-03	5.18e-03	6.20e-03	4.95e-03
71	$p = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	4.52e-03	3.50e-03	3.43e-03	3.07e-03
72	$p = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	2.28e-03	1.90e-03	1.80e-03	1.66e-03
73	$p = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	2.01e-03	1.71e-03	1.73e-03	1.57e-03
74	$p = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	6.70e-04	6.04e-04	5.69e-04	5.19e-04
75	$p = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	5.22e-04	4.65e-04	4.32e-04	3.90e-04
76	$p = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.40e-04	2.27e-04	2.28e-04	2.18e-04
77	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.79e-04	1.73e-04	1.73e-04	1.71e-04
78	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3)$	2.77e-02	1.93e-02	3.07e-02	2.11e-02
79	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4)$	2.13e-02	2.13e-02	2.34e-02	2.34e-02
80	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5)$	1.81e-02	1.70e-02	1.99e-02	1.86e-02
81	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6)$	1.76e-02	1.69e-02	1.93e-02	1.84e-02
82	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7)$	1.71e-02	1.70e-02	1.87e-02	1.85e-02
83	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8)$	4.36e-02	1.72e-02	4.86e-02	1.88e-02
84	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9)$	2.18e-02	1.97e-02	2.40e-02	2.16e-02
85	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10)$	1.77e+04	2.00e-02	1.98e+04	2.20e-02
86	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11)$	1.73e-02	1.70e-02	1.90e-02	1.86e-02
87	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12)$	1.66e-02	1.66e-02	1.81e-02	1.81e-02
88	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13)$	5.36e+00	1.75e-02	6.00e+00	1.91e-02
89	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k$	3.04e-02	2.30e-02	3.38e-02	2.54e-02
90	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.43e-02	2.43e-02	2.69e-02	2.69e-02

Table A5: Performance of the models for the first case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
91	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k$	2.15e-02	2.06e-02	2.38e-02	2.27e-02
92	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k$	2.11e-02	2.05e-02	2.33e-02	2.26e-02
93	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k$	2.06e-02	2.05e-02	2.28e-02	2.26e-02
94	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k$	4.54e-02	2.08e-02	5.06e-02	2.29e-02
95	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k$	2.48e-02	2.29e-02	2.74e-02	2.53e-02
96	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k$	1.78e+04	2.31e-02	1.99e+04	2.56e-02
97	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k$	2.09e-02	2.06e-02	2.31e-02	2.28e-02
98	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k$	2.03e-02	2.03e-02	2.24e-02	2.24e-02
99	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k$	5.37e+00	2.10e-02	6.00e+00	2.32e-02
100	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k$	2.85e-02	1.37e-02	3.18e-02	1.53e-02
101	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k$	1.13e-02	1.13e-02	1.26e-02	1.26e-02
102	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * k$	7.47e-03	5.87e-03	8.30e-03	6.50e-03
103	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * k$	8.95e-03	8.11e-03	9.97e-03	9.02e-03
104	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * k$	1.97e+05	1.67e-02	2.20e+05	1.86e-02
105	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * k$	1.59e-02	1.20e-02	1.78e-02	1.34e-02
106	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * k$	1.04e-01	6.10e-03	1.16e-01	6.78e-03
107	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * k$	5.25e-03	5.12e-03	5.81e-03	5.67e-03
108	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * k$	6.85e-03	5.94e-03	7.61e-03	6.59e-03
109	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * k$	1.64e+00	7.03e-03	1.83e+00	7.80e-03
110	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k$	2.36e+02	5.27e-03	6.22e-03	5.82e-03
111	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + \log(k)$	3.07e-02	2.33e-02	3.41e-02	2.58e-02
112	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + \log(k)$	2.46e-02	2.45e-02	2.72e-02	2.72e-02
113	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + \log(k)$	2.18e-02	2.09e-02	2.41e-02	2.31e-02
114	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + \log(k)$	2.14e-02	2.08e-02	2.37e-02	2.30e-02
115	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + \log(k)$	2.10e-02	2.08e-02	2.31e-02	2.30e-02
116	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + \log(k)$	4.58e-02	2.11e-02	5.11e-02	2.33e-02

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	Model	RMSE	cRMSE	RMSE	cRMSE
117	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + \log(k)$	2.50e-02	2.32e-02	2.77e-02	2.57e-02
118	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + \log(k)$	1.78e+04	2.34e-02	1.98e+04	2.59e-02
119	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + \log(k)$	2.12e-02	2.09e-02	2.34e-02	2.31e-02
120	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + \log(k)$	2.06e-02	2.06e-02	2.27e-02	2.27e-02
121	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + \log(k)$	5.33e+00	2.13e-02	5.96e+00	2.35e-02
122	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * \log(k)$	2.88e-02	1.32e-02	3.21e-02	1.47e-02
123	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * \log(k)$	9.50e-03	9.49e-03	1.06e-02	1.06e-02
124	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * \log(k)$	7.11e-03	4.01e-03	7.91e-03	4.42e-03
125	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * \log(k)$	7.80e-03	6.89e-03	8.68e-03	7.66e-03
126	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * \log(k)$	2.44e+03	1.62e-02	2.73e+03	1.81e-02
127	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * \log(k)$	1.32e-02	9.75e-03	1.47e-02	1.08e-02
128	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * \log(k)$	1.15e-02	2.94e-03	1.29e-02	3.22e-03
129	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	3.82e-03	1.96e-03	4.22e-03	2.11e-03
130	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	8.76e-03	5.56e-03	9.75e-03	6.16e-03
131	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	2.22e+01	6.42e-03	2.48e+01	7.13e-03
132	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	3.48e+00	2.21e-03	3.48e-03	2.34e-03
133	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	3.07e-02	2.33e-02	3.41e-02	2.58e-02
134	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	2.46e-02	2.45e-02	2.72e-02	2.72e-02
135	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	2.18e-02	2.09e-02	2.41e-02	2.31e-02
136	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	2.14e-02	2.08e-02	2.37e-02	2.30e-02
137	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	2.09e-02	2.08e-02	2.31e-02	2.30e-02
138	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	4.58e-02	2.11e-02	5.10e-02	2.33e-02
139	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	2.50e-02	2.32e-02	2.77e-02	2.57e-02
140	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	1.78e+04	2.34e-02	1.98e+04	2.59e-02
141	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	2.12e-02	2.09e-02	2.34e-02	2.31e-02
142	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	2.06e-02	2.06e-02	2.27e-02	2.27e-02

Table A5: Performance of the models for the first case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
143	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	5.34e+00	2.13e-02	5.97e+00	2.35e-02
144	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	2.82e-02	1.32e-02	3.15e-02	1.47e-02
145	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	9.75e-03	9.72e-03	1.09e-02	1.08e-02
146	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	5.96e-03	3.23e-03	6.65e-03	3.59e-03
147	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	8.80e-03	7.54e-03	9.82e-03	8.42e-03
148	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	4.24e+05	1.91e-02	4.74e+05	2.12e-02
149	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	2.82e-02	1.74e-02	2.78e-02	1.94e-02
150	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	3.17e+01	1.23e-02	3.54e+01	1.36e-02
151	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	1.68e-02	7.51e-03	1.87e-02	8.35e-03
152	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	8.20e-03	2.40e-03	9.17e-03	2.68e-03
153	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	7.66e-04	6.42e-04	8.52e-04	7.13e-04
154	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	3.38e-04	3.05e-04	3.72e-04	3.34e-04
155	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3)$	4.22e-02	2.28e-02	2.47e-02	2.44e-02
156	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4)$	2.68e-02	2.53e-02	2.77e-02	2.75e-02
157	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5)$	1.87e-02	1.82e-02	1.95e-02	1.95e-02
158	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 6)$	1.75e-02	1.72e-02	1.90e-02	1.86e-02
159	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 7)$	1.76e-02	1.72e-02	1.91e-02	1.87e-02
160	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 8)$	1.66e-02	1.66e-02	1.81e-02	1.81e-02
161	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 9)$	1.66e-02	1.66e-02	1.81e-02	1.81e-02
162	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 10)$	1.65e-02	1.65e-02	1.80e-02	1.80e-02
163	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 11)$	1.65e-02	1.65e-02	1.80e-02	1.80e-02
164	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12)$	1.65e-02	1.65e-02	1.80e-02	1.80e-02
165	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13)$	1.65e-02	1.65e-02	1.80e-02	1.80e-02
166	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3) + k$	4.09e-02	1.95e-02	2.04e-02	1.99e-02
167	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.42e-02	2.24e-02	2.40e-02	2.35e-02
168	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5) + k$	1.46e-02	1.38e-02	1.34e-02	1.34e-02

Table A5: Performance of the models for the first case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
169	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + k$	1.30e-02	1.25e-02	1.27e-02	1.21e-02
170	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + k$	1.30e-02	1.25e-02	1.28e-02	1.21e-02
171	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + k$	1.16e-02	1.16e-02	1.12e-02	1.12e-02
172	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + k$	1.16e-02	1.16e-02	1.12e-02	1.12e-02
173	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + k$	1.15e-02	1.15e-02	1.11e-02	1.11e-02
174	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + k$	1.15e-02	1.15e-02	1.11e-02	1.11e-02
175	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + k$	1.15e-02	1.15e-02	1.11e-02	1.11e-02
176	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + k$	1.15e-02	1.15e-02	1.11e-02	1.11e-02
177	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * k$	3.74e-02	1.56e-02	1.71e-02	1.65e-02
178	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * k$	2.05e-02	1.89e-02	2.09e-02	2.04e-02
179	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * k$	8.16e-03	7.98e-03	8.24e-03	8.18e-03
180	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * k$	7.62e-03	6.52e-03	8.27e-03	7.01e-03
181	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * k$	5.59e-03	5.27e-03	5.88e-03	5.71e-03
182	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * k$	5.09e-03	5.02e-03	5.60e-03	5.52e-03
183	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * k$	4.86e-03	4.85e-03	5.36e-03	5.34e-03
184	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * k$	4.82e-03	4.81e-03	5.31e-03	5.30e-03
185	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k$	4.80e-03	4.80e-03	5.28e-03	5.28e-03
186	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k$	4.79e-03	4.79e-03	5.28e-03	5.28e-03
187	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k$	4.79e-03	4.79e-03	5.28e-03	5.28e-03
188	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + \log(k)$	4.08e-02	1.91e-02	1.99e-02	1.94e-02
189	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + \log(k)$	2.39e-02	2.21e-02	2.36e-02	2.31e-02
190	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + \log(k)$	1.41e-02	1.33e-02	1.27e-02	1.26e-02
191	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + \log(k)$	1.24e-02	1.19e-02	1.19e-02	1.13e-02
192	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + \log(k)$	1.25e-02	1.19e-02	1.20e-02	1.13e-02
193	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + \log(k)$	1.10e-02	1.10e-02	1.04e-02	1.03e-02
194	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + \log(k)$	1.10e-02	1.09e-02	1.03e-02	1.03e-02

Table A5: Performance of the models for the first case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
195	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + \log(k)$	1.09e-02	1.09e-02	1.02e-02	1.02e-02
196	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + \log(k)$	1.09e-02	1.09e-02	1.02e-02	1.02e-02
197	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + \log(k)$	1.09e-02	1.09e-02	1.02e-02	1.02e-02
198	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + \log(k)$	1.09e-02	1.09e-02	1.02e-02	1.02e-02
199	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * \log(k)$	3.67e-02	1.51e-02	1.66e-02	1.59e-02
200	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * \log(k)$	2.03e-02	1.84e-02	2.04e-02	1.99e-02
201	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * \log(k)$	6.30e-03	6.22e-03	6.13e-03	6.04e-03
202	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * \log(k)$	6.15e-03	4.61e-03	6.63e-03	4.82e-03
203	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * \log(k)$	2.93e-03	2.14e-03	2.31e-03	2.06e-03
204	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * \log(k)$	1.95e-03	1.76e-03	2.07e-03	1.85e-03
205	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * \log(k)$	1.11e-03	1.05e-03	1.11e-03	1.04e-03
206	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * \log(k)$	8.95e-04	8.39e-04	8.76e-04	8.04e-04
207	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * \log(k)$	7.74e-04	7.62e-04	7.18e-04	7.01e-04
208	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * \log(k)$	7.22e-04	7.19e-04	6.67e-04	6.64e-04
209	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * \log(k)$	7.12e-04	7.11e-04	6.54e-04	6.53e-04
210	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + k_d$	4.08e-02	1.91e-02	1.99e-02	1.94e-02
211	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + k_d$	2.39e-02	2.21e-02	2.36e-02	2.31e-02
212	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + k_d$	1.41e-02	1.33e-02	1.27e-02	1.26e-02
213	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + k_d$	1.24e-02	1.19e-02	1.19e-02	1.13e-02
214	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + k_d$	1.25e-02	1.19e-02	1.20e-02	1.13e-02
215	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + k_d$	1.10e-02	1.10e-02	1.04e-02	1.03e-02
216	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + k_d$	1.10e-02	1.09e-02	1.03e-02	1.03e-02
217	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + k_d$	1.09e-02	1.09e-02	1.02e-02	1.02e-02
218	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + k_d$	1.09e-02	1.09e-02	1.02e-02	1.02e-02
219	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + k_d$	1.09e-02	1.09e-02	1.02e-02	1.02e-02
220	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + k_d$	1.09e-02	1.09e-02	1.02e-02	1.02e-02

Table A5: Performance of the models for the first case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
221	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * k_d$	3.67e-02	1.50e-02	1.64e-02	1.58e-02
222	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * k_d$	2.00e-02	1.82e-02	2.01e-02	1.96e-02
223	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * k_d$	6.14e-03	6.09e-03	6.00e-03	5.93e-03
224	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * k_d$	5.75e-03	4.29e-03	6.20e-03	4.49e-03
225	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * k_d$	2.24e-03	1.83e-03	1.96e-03	1.74e-03
226	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * k_d$	1.54e-03	1.36e-03	1.67e-03	1.45e-03
227	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * k_d$	7.92e-04	7.23e-04	8.39e-04	7.60e-04
228	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * k_d$	4.80e-04	4.28e-04	5.13e-04	4.52e-04
229	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k_d$	2.16e-04	2.09e-04	2.28e-04	2.19e-04
230	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k_d$	2.00e-04	1.95e-04	2.10e-04	2.05e-04
231	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k_d$	1.76e-04	1.74e-04	1.88e-04	1.86e-04

Table A6: Performance of the models for the second case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$p = c + \mathrm{poly}\,(\mathrm{bc}(t), 3)$	3.13e-02	2.37e-02	2.52e-02	2.44e-02
2	$p = c + \mathrm{poly}\,(\mathrm{bc}(t), 4)$	2.60e-02	2.47e-02	2.71e-02	2.63e-02
3	$p = c + \mathrm{poly}\,(\mathrm{bc}(t), 5)$	2.18e-02	2.15e-02	2.07e-02	2.06e-02
4	$p = c + \mathrm{poly}\,(\mathrm{bc}(t), 6)$	1.66e-02	1.62e-02	1.80e-02	1.76e-02
5	$p = c + \mathrm{poly}\,(\mathrm{bc}(t), 7)$	1.62e-02	1.58e-02	1.72e-02	1.71e-02
6	$p = c + \mathrm{poly}\,(\mathrm{bc}(t), 8)$	1.53e-02	1.52e-02	1.66e-02	1.66e-02
7	$p = c + \mathrm{poly}\,(\mathrm{bc}(t), 9)$	1.52e-02	1.51e-02	1.66e-02	1.66e-02

Table A6: Performance of the models for the second case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
8	$p = c + \text{poly}\,(\text{bc}(t), 10)$	1.50e-02	1.50e-02	1.64e-02	1.64e-02
9	$p = c + \text{poly}\,(\text{bc}(t), 11)$	1.50e-02	1.49e-02	1.64e-02	1.64e-02
10	$p = c + \text{poly}\,(\text{bc}(t), 12)$	1.49e-02	1.49e-02	1.64e-02	1.64e-02
11	$p = c + \text{poly}\,(\text{bc}(t), 13)$	1.49e-02	1.49e-02	1.64e-02	1.64e-02
12	$p = c + \text{poly}\,(\text{bc}(t), 3) + k$	2.98e-02	2.14e-02	2.17e-02	2.07e-02
13	$p = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.42e-02	2.26e-02	2.39e-02	2.29e-02
14	$p = c + \text{poly}\,(\text{bc}(t), 5) + k$	1.95e-02	1.89e-02	1.62e-02	1.59e-02
15	$p = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.35e-02	1.30e-02	1.26e-02	1.19e-02
16	$p = c + \text{poly}\,(\text{bc}(t), 7) + k$	1.30e-02	1.24e-02	1.13e-02	1.11e-02
17	$p = c + \text{poly}\,(\text{bc}(t), 8) + k$	1.18e-02	1.17e-02	1.04e-02	1.04e-02
18	$p = c + \text{poly}\,(\text{bc}(t), 9) + k$	1.17e-02	1.16e-02	1.05e-02	1.04e-02
19	$p = c + \text{poly}\,(\text{bc}(t), 10) + k$	1.14e-02	1.14e-02	1.02e-02	1.01e-02
20	$p = c + \text{poly}\,(\text{bc}(t), 11) + k$	1.14e-02	1.13e-02	1.02e-02	1.01e-02
21	$p = c + \text{poly}\,(\text{bc}(t), 12) + k$	1.14e-02	1.13e-02	1.01e-02	1.01e-02
22	$p = c + \text{poly}\,(\text{bc}(t), 13) + k$	1.13e-02	1.13e-02	1.01e-02	1.01e-02
23	$p = c + \text{poly}\,(\text{bc}(t), 3) * k$	2.79e-02	1.89e-02	1.98e-02	1.87e-02
24	$p = c + \text{poly}\,(\text{bc}(t), 4) * k$	2.12e-02	1.97e-02	2.17e-02	2.07e-02
25	$p = c + \text{poly}\,(\text{bc}(t), 5) * k$	1.62e-02	1.59e-02	1.38e-02	1.36e-02
26	$p = c + \text{poly}\,(\text{bc}(t), 6) * k$	8.99e-03	8.22e-03	9.48e-03	8.65e-03
27	$p = c + \text{poly}\,(\text{bc}(t), 7) * k$	8.27e-03	7.42e-03	7.97e-03	7.69e-03
28	$p = c + \text{poly}\,(\text{bc}(t), 8) * k$	6.57e-03	6.34e-03	6.74e-03	6.71e-03
29	$p = c + \text{poly}\,(\text{bc}(t), 9) * k$	6.31e-03	6.19e-03	6.81e-03	6.74e-03
30	$p = c + \text{poly}\,(\text{bc}(t), 10) * k$	5.84e-03	5.82e-03	6.37e-03	6.36e-03
31	$p = c + \text{poly}\,(\text{bc}(t), 11) * k$	5.80e-03	5.78e-03	6.37e-03	6.36e-03
32	$p = c + \text{poly}\,(\text{bc}(t), 12) * k$	5.77e-03	5.76e-03	6.34e-03	6.34e-03
33	$p = c + \text{poly}\,(\text{bc}(t), 13) * k$	5.74e-03	5.74e-03	6.32e-03	6.32e-03

Table A6: Performance of the models for the second case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
34	$p = c + \text{poly}\,(\text{bc}(t), 3) + \log(k)$	2.96e-02	2.11e-02	2.11e-02	2.01e-02
35	$p = c + \text{poly}\,(\text{bc}(t), 4) + \log(k)$	2.39e-02	2.22e-02	2.34e-02	2.24e-02
36	$p = c + \text{poly}\,(\text{bc}(t), 5) + \log(k)$	1.92e-02	1.85e-02	1.54e-02	1.51e-02
37	$p = c + \text{poly}\,(\text{bc}(t), 6) + \log(k)$	1.30e-02	1.24e-02	1.16e-02	1.09e-02
38	$p = c + \text{poly}\,(\text{bc}(t), 7) + \log(k)$	1.25e-02	1.19e-02	1.03e-02	1.00e-02
39	$p = c + \text{poly}\,(\text{bc}(t), 8) + \log(k)$	1.13e-02	1.11e-02	9.22e-03	9.19e-03
40	$p = c + \text{poly}\,(\text{bc}(t), 9) + \log(k)$	1.11e-02	1.10e-02	9.31e-03	9.22e-03
41	$p = c + \text{poly}\,(\text{bc}(t), 10) + \log(k)$	1.08e-02	1.08e-02	8.94e-03	8.91e-03
42	$p = c + \text{poly}\,(\text{bc}(t), 11) + \log(k)$	1.08e-02	1.07e-02	8.95e-03	8.90e-03
43	$p = c + \text{poly}\,(\text{bc}(t), 12) + \log(k)$	1.08e-02	1.07e-02	8.93e-03	8.89e-03
44	$p = c + \text{poly}\,(\text{bc}(t), 13) + \log(k)$	1.08e-02	1.07e-02	8.91e-03	8.88e-03
45	$p = c + \text{poly}\,(\text{bc}(t), 3) * \log(k)$	2.73e-02	1.80e-02	1.88e-02	1.76e-02
46	$p = c + \text{poly}\,(\text{bc}(t), 4) * \log(k)$	2.03e-02	1.88e-02	2.08e-02	1.97e-02
47	$p = c + \text{poly}\,(\text{bc}(t), 5) * \log(k)$	1.52e-02	1.49e-02	1.24e-02	1.21e-02
48	$p = c + \text{poly}\,(\text{bc}(t), 6) * \log(k)$	7.13e-03	6.13e-03	7.32e-03	6.20e-03
49	$p = c + \text{poly}\,(\text{bc}(t), 7) * \log(k)$	6.15e-03	4.96e-03	5.17e-03	4.74e-03
50	$p = c + \text{poly}\,(\text{bc}(t), 8) * \log(k)$	3.61e-03	3.17e-03	3.01e-03	2.95e-03
51	$p = c + \text{poly}\,(\text{bc}(t), 9) * \log(k)$	3.17e-03	2.91e-03	3.19e-03	3.01e-03
52	$p = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	2.05e-03	2.00e-03	2.07e-03	2.06e-03
53	$p = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	1.93e-03	1.88e-03	2.08e-03	2.04e-03
54	$p = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	1.87e-03	1.83e-03	1.99e-03	1.98e-03
55	$p = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	1.76e-03	1.75e-03	1.93e-03	1.93e-03
56	$p = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	2.96e-02	2.11e-02	2.11e-02	2.01e-02
57	$p = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	2.39e-02	2.22e-02	2.34e-02	2.24e-02
58	$p = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	1.92e-02	1.85e-02	1.54e-02	1.51e-02
59	$p = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	1.30e-02	1.24e-02	1.16e-02	1.09e-02

Table A6: Performance of the models for the second case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
60	$p = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	1.25e-02	1.18e-02	1.03e-02	1.00e-02
61	$p = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	1.13e-02	1.11e-02	9.21e-03	9.18e-03
62	$p = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	1.11e-02	1.10e-02	9.29e-03	9.21e-03
63	$p = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	1.08e-02	1.07e-02	8.93e-03	8.90e-03
64	$p = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	1.08e-02	1.07e-02	8.93e-03	8.89e-03
65	$p = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	1.08e-02	1.07e-02	8.91e-03	8.88e-03
66	$p = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	1.08e-02	1.07e-02	8.89e-03	8.87e-03
67	$p = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	2.73e-02	1.79e-02	1.87e-02	1.76e-02
68	$p = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	2.03e-02	1.87e-02	2.07e-02	1.96e-02
69	$p = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	1.51e-02	1.47e-02	1.22e-02	1.20e-02
70	$p = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	6.87e-03	5.86e-03	7.04e-03	5.88e-03
71	$p = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	5.73e-03	4.63e-03	4.68e-03	4.24e-03
72	$p = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	2.95e-03	2.50e-03	2.23e-03	2.13e-03
73	$p = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	2.57e-03	2.29e-03	2.46e-03	2.27e-03
74	$p = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	1.06e-03	9.77e-04	7.74e-04	7.45e-04
75	$p = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	8.12e-04	7.20e-04	7.88e-04	6.93e-04
76	$p = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	3.68e-04	3.41e-04	2.87e-04	2.83e-04
77	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	2.83e-04	2.68e-04	2.80e-04	2.66e-04
78	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3)$	2.47e-02	1.78e-02	2.74e-02	1.96e-02
79	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4)$	2.45e-02	2.42e-02	2.72e-02	2.68e-02
80	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5)$	1.88e-02	1.60e-02	2.08e-02	1.77e-02
81	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6)$	1.66e-02	1.63e-02	1.83e-02	1.79e-02
82	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7)$	1.44e+00	2.21e-02	1.62e+00	2.45e-02
83	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8)$	2.01e-02	1.83e-02	2.23e-02	2.02e-02
84	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9)$	1.63e-02	1.51e-02	1.79e-02	1.67e-02
85	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10)$	1.66e-02	1.52e-02	1.83e-02	1.67e-02

Table A6: Performance of the models for the second case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
86	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11)$	1.90e-02	1.77e-02	2.09e-02	1.95e-02
87	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12)$	3.56e+15	1.72e-02	5.59e+01	1.90e-02
88	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13)$	3.77e+20	1.53e-02	1.71e-02	1.68e-02
89	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k$	2.64e-02	2.01e-02	2.93e-02	2.22e-02
90	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.61e-02	2.57e-02	2.90e-02	2.85e-02
91	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k$	2.06e-02	1.81e-02	2.28e-02	2.00e-02
92	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.86e-02	1.82e-02	2.06e-02	2.02e-02
93	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k$	1.44e+00	2.37e-02	1.61e+00	2.63e-02
94	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k$	2.18e-02	2.02e-02	2.42e-02	2.24e-02
95	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k$	1.83e-02	1.73e-02	2.03e-02	1.92e-02
96	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k$	1.86e-02	1.74e-02	2.06e-02	1.92e-02
97	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k$	2.08e-02	1.96e-02	2.30e-02	2.17e-02
98	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k$	3.53e+15	1.91e-02	5.59e+01	2.12e-02
99	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k$	3.74e+20	1.75e-02	1.96e-02	1.93e-02
100	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k$	2.54e-02	1.34e-02	2.83e-02	1.48e-02
101	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k$	1.82e-02	1.75e-02	2.02e-02	1.95e-02
102	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * k$	7.96e-03	6.50e-03	8.85e-03	7.19e-03
103	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * k$	9.83e-03	9.02e-03	1.09e-02	1.00e-02
104	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * k$	2.63e+03	2.11e-02	2.94e+03	2.35e-02
105	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * k$	1.84e+03	1.50e-02	2.10e-02	1.67e-02
106	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * k$	6.81e+08	6.94e-03	5.59e-02	7.68e-03
107	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * k$	4.39e+08	6.28e-03	7.19e-03	6.95e-03
108	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * k$	9.29e+03	7.94e-03	2.88e-01	8.79e-03
109	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * k$	2.80e+06	7.40e-03	2.23e-01	8.20e-03
110	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k$	3.80e+10	6.16e-03	6.94e-03	6.80e-03
111	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + \log(k)$	2.67e-02	2.04e-02	2.97e-02	2.26e-02

Table A6: Performance of the models for the second case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
112	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + \log(k)$	2.64e-02	2.60e-02	2.93e-02	2.89e-02
113	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + \log(k)$	2.09e-02	1.85e-02	2.32e-02	2.05e-02
114	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + \log(k)$	1.89e-02	1.86e-02	2.10e-02	2.06e-02
115	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + \log(k)$	1.44e+00	2.40e-02	1.60e+00	2.66e-02
116	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + \log(k)$	2.22e-02	2.05e-02	2.46e-02	2.27e-02
117	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + \log(k)$	1.87e-02	1.78e-02	2.07e-02	1.96e-02
118	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + \log(k)$	1.90e-02	1.78e-02	2.11e-02	1.97e-02
119	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + \log(k)$	2.11e-02	1.99e-02	2.34e-02	2.21e-02
120	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + \log(k)$	3.54e+15	1.95e-02	5.61e+01	2.16e-02
121	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + \log(k)$	3.75e+20	1.79e-02	2.01e-02	1.98e-02
122	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * \log(k)$	2.46e-02	1.21e-02	2.75e-02	1.34e-02
123	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * \log(k)$	1.68e-02	1.63e-02	1.88e-02	1.82e-02
124	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * \log(k)$	6.05e-03	3.39e-03	6.75e-03	3.76e-03
125	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * \log(k)$	8.44e-03	7.39e-03	9.42e-03	8.23e-03
126	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * \log(k)$	1.65e+03	2.07e-02	1.85e+03	2.31e-02
127	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * \log(k)$	1.71e+02	1.46e-02	2.13e-02	1.63e-02
128	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * \log(k)$	1.34e+08	4.56e-03	5.61e-02	5.05e-03
129	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	8.47e+07	2.58e-03	3.04e-03	2.82e-03
130	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	1.02e+05	3.56e-03	5.15e-02	3.93e-03
131	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	9.07e+02	3.16e-03	7.47e-03	3.47e-03
132	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	2.34e+05	2.20e-03	2.47e-03	2.38e-03
133	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	2.67e-02	2.05e-02	2.97e-02	2.27e-02
134	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	2.64e-02	2.60e-02	2.94e-02	2.89e-02
135	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	2.10e-02	1.85e-02	2.33e-02	2.05e-02
136	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	1.90e-02	1.86e-02	2.10e-02	2.06e-02
137	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	1.43e+00	2.40e-02	1.60e+00	2.66e-02

Table A6: Performance of the models for the second case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
138	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	2.22e-02	2.05e-02	2.47e-02	2.28e-02
139	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	1.88e-02	1.78e-02	2.08e-02	1.97e-02
140	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	1.91e-02	1.78e-02	2.11e-02	1.97e-02
141	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	2.12e-02	2.00e-02	2.34e-02	2.21e-02
142	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	3.54e+15	1.96e-02	5.62e+01	2.17e-02
143	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	3.75e+20	1.79e-02	2.01e-02	1.98e-02
144	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	2.48e-02	1.21e-02	2.77e-02	1.34e-02
145	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	1.68e-02	1.62e-02	1.87e-02	1.81e-02
146	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	5.90e-03	2.84e-03	6.59e-03	3.16e-03
147	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	8.66e-03	7.40e-03	9.67e-03	8.26e-03
148	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	1.00e+04	2.09e-02	1.12e+04	2.33e-02
149	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	2.95e-02	1.99e-02	3.04e-02	2.22e-02
150	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	5.86e+00	1.22e-02	6.54e+00	1.35e-02
151	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	1.52e-02	6.29e-03	8.53e-03	7.00e-03
152	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	2.73e-03	1.87e-03	3.05e-03	2.09e-03
153	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	6.90e-04	6.23e-04	7.67e-04	6.92e-04
154	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	3.70e-04	3.37e-04	4.10e-04	3.73e-04
155	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3)$	3.85e-02	2.32e-02	2.58e-02	2.50e-02
156	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4)$	2.86e-02	2.55e-02	2.86e-02	2.77e-02
157	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5)$	1.69e-02	1.69e-02	1.81e-02	1.81e-02
158	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 6)$	1.66e-02	1.60e-02	1.82e-02	1.76e-02
159	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 7)$	1.55e-02	1.52e-02	1.67e-02	1.67e-02
160	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 8)$	1.50e-02	1.50e-02	1.65e-02	1.65e-02
161	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 9)$	1.50e-02	1.50e-02	1.65e-02	1.65e-02
162	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 10)$	1.49e-02	1.49e-02	1.64e-02	1.64e-02
163	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 11)$	1.49e-02	1.49e-02	1.64e-02	1.64e-02

Table A6: Performance of the models for the second case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
164	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12)$	1.49e-02	1.49e-02	1.64e-02	1.64e-02
165	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13)$	1.49e-02	1.49e-02	1.64e-02	1.64e-02
166	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + k$	3.72e-02	2.06e-02	2.25e-02	2.16e-02
167	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + k$	2.66e-02	2.32e-02	2.57e-02	2.47e-02
168	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + k$	1.29e-02	1.29e-02	1.28e-02	1.27e-02
169	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + k$	1.26e-02	1.18e-02	1.29e-02	1.20e-02
170	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + k$	1.11e-02	1.07e-02	1.07e-02	1.07e-02
171	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + k$	1.05e-02	1.04e-02	1.04e-02	1.04e-02
172	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + k$	1.04e-02	1.04e-02	1.04e-02	1.04e-02
173	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + k$	1.03e-02	1.03e-02	1.03e-02	1.02e-02
174	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + k$	1.03e-02	1.03e-02	1.03e-02	1.02e-02
175	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + k$	1.03e-02	1.03e-02	1.02e-02	1.02e-02
176	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + k$	1.03e-02	1.03e-02	1.02e-02	1.02e-02
177	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * k$	3.55e-02	1.77e-02	1.99e-02	1.90e-02
178	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * k$	2.37e-02	2.06e-02	2.34e-02	2.24e-02
179	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * k$	9.43e-03	9.42e-03	9.76e-03	9.75e-03
180	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * k$	8.81e-03	7.82e-03	9.67e-03	8.54e-03
181	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * k$	6.74e-03	6.40e-03	7.08e-03	6.97e-03
182	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * k$	6.03e-03	5.99e-03	6.65e-03	6.60e-03
183	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * k$	5.98e-03	5.95e-03	6.59e-03	6.56e-03
184	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * k$	5.79e-03	5.79e-03	6.39e-03	6.38e-03
185	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k$	5.79e-03	5.78e-03	6.39e-03	6.38e-03
186	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k$	5.77e-03	5.77e-03	6.37e-03	6.37e-03
187	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k$	5.77e-03	5.77e-03	6.37e-03	6.36e-03
188	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + \log(k)$	3.70e-02	2.01e-02	2.20e-02	2.10e-02
189	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + \log(k)$	2.63e-02	2.28e-02	2.52e-02	2.42e-02

Table A6: Performance of the models for the second case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
190	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + \log(k)$	1.23e-02	1.22e-02	1.18e-02	1.18e-02
191	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + \log(k)$	1.19e-02	1.11e-02	1.20e-02	1.10e-02
192	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + \log(k)$	1.03e-02	9.89e-03	9.62e-03	9.52e-03
193	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + \log(k)$	9.63e-03	9.58e-03	9.27e-03	9.20e-03
194	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + \log(k)$	9.60e-03	9.56e-03	9.23e-03	9.17e-03
195	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + \log(k)$	9.47e-03	9.44e-03	9.07e-03	9.03e-03
196	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + \log(k)$	9.47e-03	9.44e-03	9.07e-03	9.03e-03
197	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + \log(k)$	9.46e-03	9.43e-03	9.05e-03	9.02e-03
198	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + \log(k)$	9.45e-03	9.43e-03	9.05e-03	9.02e-03
199	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * \log(k)$	3.52e-02	1.67e-02	1.89e-02	1.78e-02
200	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * \log(k)$	2.29e-02	1.98e-02	2.25e-02	2.14e-02
201	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * \log(k)$	7.62e-03	7.61e-03	7.61e-03	7.59e-03
202	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * \log(k)$	6.88e-03	5.54e-03	7.53e-03	5.99e-03
203	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * \log(k)$	3.86e-03	3.22e-03	3.60e-03	3.39e-03
204	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * \log(k)$	2.50e-03	2.39e-03	2.75e-03	2.63e-03
205	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * \log(k)$	2.35e-03	2.27e-03	2.58e-03	2.49e-03
206	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * \log(k)$	1.84e-03	1.82e-03	2.03e-03	2.01e-03
207	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * \log(k)$	1.84e-03	1.82e-03	2.02e-03	2.00e-03
208	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * \log(k)$	1.78e-03	1.78e-03	1.97e-03	1.97e-03
209	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * \log(k)$	1.77e-03	1.77e-03	1.96e-03	1.96e-03
210	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + k_d$	3.70e-02	2.01e-02	2.20e-02	2.10e-02
211	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + k_d$	2.63e-02	2.28e-02	2.52e-02	2.42e-02
212	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + k_d$	1.23e-02	1.22e-02	1.18e-02	1.18e-02
213	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + k_d$	1.19e-02	1.11e-02	1.20e-02	1.10e-02
214	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + k_d$	1.03e-02	9.88e-03	9.61e-03	9.51e-03
215	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + k_d$	9.62e-03	9.57e-03	9.26e-03	9.19e-03

Table A6: Performance of the models for the second case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
216	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + k_d$	9.59e-03	9.55e-03	9.22e-03	9.16e-03
217	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + k_d$	9.46e-03	9.43e-03	9.05e-03	9.02e-03
218	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + k_d$	9.46e-03	9.43e-03	9.06e-03	9.02e-03
219	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + k_d$	9.45e-03	9.42e-03	9.04e-03	9.01e-03
220	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + k_d$	9.44e-03	9.42e-03	9.04e-03	9.01e-03
221	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * k_d$	3.51e-02	1.66e-02	1.88e-02	1.77e-02
222	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * k_d$	2.28e-02	1.97e-02	2.24e-02	2.14e-02
223	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * k_d$	7.41e-03	7.39e-03	7.35e-03	7.34e-03
224	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * k_d$	6.62e-03	5.24e-03	7.24e-03	5.65e-03
225	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * k_d$	3.12e-03	2.63e-03	2.93e-03	2.67e-03
226	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * k_d$	1.75e-03	1.59e-03	1.92e-03	1.73e-03
227	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * k_d$	1.53e-03	1.41e-03	1.67e-03	1.53e-03
228	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * k_d$	5.04e-04	4.55e-04	5.44e-04	4.88e-04
229	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k_d$	4.89e-04	4.39e-04	5.30e-04	4.72e-04
230	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k_d$	2.05e-04	2.02e-04	2.15e-04	2.11e-04
231	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k_d$	2.12e-04	2.04e-04	2.26e-04	2.17e-04

Table A7: Performance of the models for the third case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$p = c + \mathrm{poly}\,(\mathrm{bc}(t), 3)$	3.18e-02	2.47e-02	2.62e-02	2.52e-02
2	$p = c + \mathrm{poly}\,(\mathrm{bc}(t), 4)$	2.76e-02	2.58e-02	2.84e-02	2.73e-02

Table A7: Performance of the models for the third case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
3	$p = c + \text{poly}\,(\text{bc}(t), 5)$	2.34e-02	2.25e-02	2.17e-02	2.15e-02
4	$p = c + \text{poly}\,(\text{bc}(t), 6)$	1.88e-02	1.79e-02	1.86e-02	1.84e-02
5	$p = c + \text{poly}\,(\text{bc}(t), 7)$	1.81e-02	1.75e-02	1.92e-02	1.85e-02
6	$p = c + \text{poly}\,(\text{bc}(t), 8)$	1.56e-02	1.54e-02	1.67e-02	1.67e-02
7	$p = c + \text{poly}\,(\text{bc}(t), 9)$	1.54e-02	1.53e-02	1.68e-02	1.67e-02
8	$p = c + \text{poly}\,(\text{bc}(t), 10)$	1.52e-02	1.51e-02	1.66e-02	1.65e-02
9	$p = c + \text{poly}\,(\text{bc}(t), 11)$	1.50e-02	1.50e-02	1.64e-02	1.64e-02
10	$p = c + \text{poly}\,(\text{bc}(t), 12)$	1.50e-02	1.49e-02	1.64e-02	1.64e-02
11	$p = c + \text{poly}\,(\text{bc}(t), 13)$	1.49e-02	1.49e-02	1.64e-02	1.64e-02
12	$p = c + \text{poly}\,(\text{bc}(t), 3) + k$	3.05e-02	2.27e-02	2.30e-02	2.20e-02
13	$p = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.60e-02	2.40e-02	2.56e-02	2.44e-02
14	$p = c + \text{poly}\,(\text{bc}(t), 5) + k$	2.15e-02	2.03e-02	1.78e-02	1.75e-02
15	$p = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.63e-02	1.52e-02	1.38e-02	1.35e-02
16	$p = c + \text{poly}\,(\text{bc}(t), 7) + k$	1.56e-02	1.48e-02	1.46e-02	1.37e-02
17	$p = c + \text{poly}\,(\text{bc}(t), 8) + k$	1.26e-02	1.23e-02	1.12e-02	1.11e-02
18	$p = c + \text{poly}\,(\text{bc}(t), 9) + k$	1.23e-02	1.22e-02	1.13e-02	1.12e-02
19	$p = c + \text{poly}\,(\text{bc}(t), 10) + k$	1.21e-02	1.20e-02	1.10e-02	1.09e-02
20	$p = c + \text{poly}\,(\text{bc}(t), 11) + k$	1.18e-02	1.17e-02	1.08e-02	1.07e-02
21	$p = c + \text{poly}\,(\text{bc}(t), 12) + k$	1.17e-02	1.17e-02	1.07e-02	1.07e-02
22	$p = c + \text{poly}\,(\text{bc}(t), 13) + k$	1.17e-02	1.17e-02	1.07e-02	1.07e-02
23	$p = c + \text{poly}\,(\text{bc}(t), 3) * k$	2.82e-02	1.97e-02	2.06e-02	1.94e-02
24	$p = c + \text{poly}\,(\text{bc}(t), 4) * k$	2.26e-02	2.07e-02	2.29e-02	2.17e-02
25	$p = c + \text{poly}\,(\text{bc}(t), 5) * k$	1.79e-02	1.71e-02	1.50e-02	1.47e-02
26	$p = c + \text{poly}\,(\text{bc}(t), 6) * k$	1.09e-02	9.78e-03	1.05e-02	9.74e-03
27	$p = c + \text{poly}\,(\text{bc}(t), 7) * k$	9.84e-03	8.83e-03	9.58e-03	9.00e-03
28	$p = c + \text{poly}\,(\text{bc}(t), 8) * k$	7.14e-03	6.84e-03	7.19e-03	7.14e-03

Table A7: Performance of the models for the third case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
29	$p = c + \text{poly}\,(\text{bc}(t), 9) * k$	6.86e-03	6.66e-03	7.30e-03	7.17e-03
30	$p = c + \text{poly}\,(\text{bc}(t), 10) * k$	6.22e-03	6.15e-03	6.70e-03	6.68e-03
31	$p = c + \text{poly}\,(\text{bc}(t), 11) * k$	6.08e-03	6.06e-03	6.66e-03	6.64e-03
32	$p = c + \text{poly}\,(\text{bc}(t), 12) * k$	6.04e-03	6.03e-03	6.61e-03	6.60e-03
33	$p = c + \text{poly}\,(\text{bc}(t), 13) * k$	5.99e-03	5.99e-03	6.59e-03	6.58e-03
34	$p = c + \text{poly}\,(\text{bc}(t), 3) + \log(k)$	3.03e-02	2.24e-02	2.25e-02	2.14e-02
35	$p = c + \text{poly}\,(\text{bc}(t), 4) + \log(k)$	2.58e-02	2.36e-02	2.52e-02	2.39e-02
36	$p = c + \text{poly}\,(\text{bc}(t), 5) + \log(k)$	2.12e-02	1.99e-02	1.71e-02	1.68e-02
37	$p = c + \text{poly}\,(\text{bc}(t), 6) + \log(k)$	1.59e-02	1.47e-02	1.29e-02	1.26e-02
38	$p = c + \text{poly}\,(\text{bc}(t), 7) + \log(k)$	1.51e-02	1.42e-02	1.37e-02	1.28e-02
39	$p = c + \text{poly}\,(\text{bc}(t), 8) + \log(k)$	1.20e-02	1.17e-02	1.00e-02	9.97e-03
40	$p = c + \text{poly}\,(\text{bc}(t), 9) + \log(k)$	1.17e-02	1.16e-02	1.01e-02	1.00e-02
41	$p = c + \text{poly}\,(\text{bc}(t), 10) + \log(k)$	1.15e-02	1.14e-02	9.76e-03	9.71e-03
42	$p = c + \text{poly}\,(\text{bc}(t), 11) + \log(k)$	1.12e-02	1.11e-02	9.57e-03	9.51e-03
43	$p = c + \text{poly}\,(\text{bc}(t), 12) + \log(k)$	1.11e-02	1.11e-02	9.52e-03	9.49e-03
44	$p = c + \text{poly}\,(\text{bc}(t), 13) + \log(k)$	1.11e-02	1.10e-02	9.48e-03	9.46e-03
45	$p = c + \text{poly}\,(\text{bc}(t), 3) * \log(k)$	2.76e-02	1.87e-02	1.94e-02	1.82e-02
46	$p = c + \text{poly}\,(\text{bc}(t), 4) * \log(k)$	2.16e-02	1.97e-02	2.19e-02	2.06e-02
47	$p = c + \text{poly}\,(\text{bc}(t), 5) * \log(k)$	1.69e-02	1.61e-02	1.36e-02	1.33e-02
48	$p = c + \text{poly}\,(\text{bc}(t), 6) * \log(k)$	9.39e-03	7.97e-03	8.33e-03	7.42e-03
49	$p = c + \text{poly}\,(\text{bc}(t), 7) * \log(k)$	8.05e-03	6.84e-03	7.35e-03	6.49e-03
50	$p = c + \text{poly}\,(\text{bc}(t), 8) * \log(k)$	4.26e-03	3.73e-03	3.47e-03	3.37e-03
51	$p = c + \text{poly}\,(\text{bc}(t), 9) * \log(k)$	3.75e-03	3.38e-03	3.66e-03	3.40e-03
52	$p = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	2.35e-03	2.17e-03	2.22e-03	2.18e-03
53	$p = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	2.04e-03	1.97e-03	2.14e-03	2.08e-03
54	$p = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	1.91e-03	1.87e-03	1.99e-03	1.97e-03

Table A7: Performance of the models for the third case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
55	$p = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	1.74e-03	1.73e-03	1.91e-03	1.91e-03
56	$p = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	3.02e-02	2.23e-02	2.25e-02	2.14e-02
57	$p = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	2.58e-02	2.36e-02	2.51e-02	2.38e-02
58	$p = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	2.12e-02	1.98e-02	1.71e-02	1.68e-02
59	$p = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	1.59e-02	1.47e-02	1.28e-02	1.25e-02
60	$p = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	1.51e-02	1.42e-02	1.37e-02	1.27e-02
61	$p = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	1.20e-02	1.17e-02	9.97e-03	9.94e-03
62	$p = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	1.17e-02	1.15e-02	1.01e-02	9.98e-03
63	$p = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	1.15e-02	1.13e-02	9.72e-03	9.68e-03
64	$p = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	1.11e-02	1.11e-02	9.53e-03	9.47e-03
65	$p = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	1.11e-02	1.10e-02	9.48e-03	9.45e-03
66	$p = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	1.11e-02	1.10e-02	9.44e-03	9.42e-03
67	$p = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	2.75e-02	1.87e-02	1.94e-02	1.82e-02
68	$p = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	2.15e-02	1.97e-02	2.18e-02	2.05e-02
69	$p = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	1.64e-02	1.58e-02	1.32e-02	1.29e-02
70	$p = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	7.94e-03	6.79e-03	8.13e-03	6.78e-03
71	$p = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	6.45e-03	5.21e-03	5.12e-03	4.69e-03
72	$p = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	3.46e-03	2.96e-03	2.69e-03	2.55e-03
73	$p = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	3.18e-03	2.83e-03	2.94e-03	2.72e-03
74	$p = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	1.17e-03	1.07e-03	8.90e-04	8.32e-04
75	$p = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	1.05e-03	9.38e-04	9.59e-04	8.50e-04
76	$p = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	3.85e-04	3.52e-04	3.11e-04	3.00e-04
77	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	3.47e-04	3.22e-04	3.17e-04	3.00e-04
78	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3)$	2.42e-02	1.78e-02	2.68e-02	1.96e-02
79	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4)$	2.90e-02	2.79e-02	3.22e-02	3.10e-02
80	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5)$	2.11e-02	1.67e-02	2.34e-02	1.84e-02

Table A7: Performance of the models for the third case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
81	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6)$	1.81e-02	1.61e-02	2.00e-02	1.77e-02
82	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7)$	1.55e-02	1.53e-02	1.71e-02	1.69e-02
83	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8)$	1.79e-02	1.56e-02	1.98e-02	1.72e-02
84	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9)$	2.29e-02	1.99e-02	2.54e-02	2.20e-02
85	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10)$	7.23e+02	2.18e-02	8.09e+02	2.42e-02
86	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11)$	1.70e-02	1.63e-02	1.88e-02	1.80e-02
87	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12)$	1.51e-02	1.50e-02	1.66e-02	1.65e-02
88	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13)$	1.51e-02	1.50e-02	1.66e-02	1.65e-02
89	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k$	2.55e-02	1.95e-02	2.82e-02	2.15e-02
90	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k$	3.00e-02	2.88e-02	3.33e-02	3.20e-02
91	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k$	2.23e-02	1.82e-02	2.48e-02	2.01e-02
92	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.95e-02	1.76e-02	2.16e-02	1.95e-02
93	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k$	1.71e-02	1.69e-02	1.89e-02	1.87e-02
94	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k$	1.93e-02	1.72e-02	2.14e-02	1.90e-02
95	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k$	2.40e-02	2.11e-02	2.66e-02	2.34e-02
96	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k$	7.22e+02	2.30e-02	8.07e+02	2.55e-02
97	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k$	1.85e-02	1.78e-02	2.05e-02	1.97e-02
98	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k$	1.67e-02	1.67e-02	1.85e-02	1.84e-02
99	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k$	1.68e-02	1.67e-02	1.85e-02	1.84e-02
100	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k$	2.28e-02	1.25e-02	2.54e-02	1.37e-02
101	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k$	2.40e-02	2.29e-02	2.68e-02	2.55e-02
102	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * k$	1.53e-02	8.52e-03	1.71e-02	9.46e-03
103	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * k$	1.26e-02	9.16e-03	1.40e-02	1.02e-02
104	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * k$	1.26e+04	1.83e-02	1.41e+04	2.04e-02
105	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * k$	2.08e-02	1.41e-02	2.32e-02	1.57e-02
106	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * k$	3.10e-01	9.08e-03	3.46e-01	1.01e-02

Table A7: Performance of the models for the third case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
107	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * k$	5.90e+00	1.21e-02	6.59e+00	1.34e-02
108	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * k$	6.00e+03	9.09e-03	1.16e-02	1.01e-02
109	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * k$	3.44e+08	7.68e-03	1.10e-01	8.49e-03
110	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k$	1.56e+09	6.78e-03	7.98e-03	7.48e-03
111	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + \log(k)$	2.57e-02	1.97e-02	2.85e-02	2.18e-02
112	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + \log(k)$	3.02e-02	2.90e-02	3.36e-02	3.23e-02
113	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + \log(k)$	2.26e-02	1.85e-02	2.51e-02	2.05e-02
114	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + \log(k)$	1.98e-02	1.80e-02	2.20e-02	1.99e-02
115	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + \log(k)$	1.75e-02	1.73e-02	1.93e-02	1.91e-02
116	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + \log(k)$	1.96e-02	1.75e-02	2.17e-02	1.94e-02
117	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + \log(k)$	2.42e-02	2.14e-02	2.69e-02	2.37e-02
118	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + \log(k)$	7.20e+02	2.33e-02	8.05e+02	2.58e-02
119	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + \log(k)$	1.88e-02	1.82e-02	2.09e-02	2.01e-02
120	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + \log(k)$	1.71e-02	1.70e-02	1.89e-02	1.88e-02
121	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + \log(k)$	1.71e-02	1.70e-02	1.89e-02	1.88e-02
122	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * \log(k)$	2.22e-02	1.11e-02	2.48e-02	1.23e-02
123	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * \log(k)$	2.31e-02	2.19e-02	2.57e-02	2.44e-02
124	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * \log(k)$	1.50e-02	6.28e-03	1.67e-02	7.01e-03
125	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * \log(k)$	1.31e-02	6.77e-03	1.46e-02	7.55e-03
126	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * \log(k)$	3.55e+02	1.70e-02	3.97e+02	1.90e-02
127	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * \log(k)$	2.23e-01	1.38e-02	2.50e-01	1.54e-02
128	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * \log(k)$	2.95e-02	6.62e-03	3.30e-02	7.38e-03
129	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	1.44e+01	1.09e-02	1.61e+01	1.22e-02
130	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	2.29e+10	9.32e-03	1.37e-02	1.04e-02
131	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	4.09e+18	5.92e-03	2.53e-01	6.59e-03
132	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	2.11e+19	3.64e-03	5.14e-03	4.01e-03

Table A7: Performance of the models for the third case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
133	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	2.57e-02	1.98e-02	2.85e-02	2.18e-02
134	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	3.02e-02	2.91e-02	3.36e-02	3.23e-02
135	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	2.26e-02	1.85e-02	2.51e-02	2.05e-02
136	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	1.99e-02	1.80e-02	2.20e-02	1.99e-02
137	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	1.75e-02	1.73e-02	1.94e-02	1.91e-02
138	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	1.96e-02	1.76e-02	2.17e-02	1.94e-02
139	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	2.43e-02	2.14e-02	2.69e-02	2.38e-02
140	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	7.20e+02	2.33e-02	8.05e+02	2.59e-02
141	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	1.89e-02	1.82e-02	2.09e-02	2.01e-02
142	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	1.71e-02	1.71e-02	1.90e-02	1.89e-02
143	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	1.72e-02	1.71e-02	1.90e-02	1.89e-02
144	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	2.30e-02	1.12e-02	2.56e-02	1.23e-02
145	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	2.29e-02	2.18e-02	2.55e-02	2.43e-02
146	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	1.65e-02	6.26e-03	1.84e-02	7.00e-03
147	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	7.48e-03	6.43e-03	8.35e-03	7.18e-03
148	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	3.01e+02	2.00e-02	3.36e+02	2.23e-02
149	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	2.66e-02	1.91e-02	2.96e-02	2.13e-02
150	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	4.06e+00	1.04e-02	4.53e+00	1.16e-02
151	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	5.43e-03	4.43e-03	6.06e-03	4.93e-03
152	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	2.30e-03	1.31e-03	2.57e-03	1.46e-03
153	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	6.01e-04	5.37e-04	6.68e-04	5.95e-04
154	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	3.41e-04	3.06e-04	3.77e-04	3.37e-04
155	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3)$	3.98e-02	2.43e-02	2.72e-02	2.63e-02
156	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4)$	3.07e-02	2.63e-02	2.97e-02	2.86e-02
157	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5)$	1.81e-02	1.77e-02	1.89e-02	1.89e-02
158	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 6)$	1.72e-02	1.65e-02	1.87e-02	1.80e-02

Table A7: Performance of the models for the third case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
159	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 7)$	1.71e-02	1.64e-02	1.85e-02	1.79e-02
160	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 8)$	1.51e-02	1.51e-02	1.66e-02	1.66e-02
161	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 9)$	1.51e-02	1.51e-02	1.66e-02	1.66e-02
162	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 10)$	1.50e-02	1.50e-02	1.64e-02	1.64e-02
163	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 11)$	1.50e-02	1.49e-02	1.64e-02	1.64e-02
164	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12)$	1.49e-02	1.49e-02	1.64e-02	1.64e-02
165	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13)$	1.49e-02	1.49e-02	1.64e-02	1.64e-02
166	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3) + k$	3.86e-02	2.21e-02	2.43e-02	2.32e-02
167	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.90e-02	2.42e-02	2.70e-02	2.58e-02
168	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5) + k$	1.48e-02	1.43e-02	1.43e-02	1.43e-02
169	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.37e-02	1.28e-02	1.40e-02	1.31e-02
170	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 7) + k$	1.35e-02	1.27e-02	1.38e-02	1.30e-02
171	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 8) + k$	1.10e-02	1.09e-02	1.11e-02	1.10e-02
172	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 9) + k$	1.10e-02	1.09e-02	1.11e-02	1.10e-02
173	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 10) + k$	1.08e-02	1.07e-02	1.08e-02	1.08e-02
174	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 11) + k$	1.08e-02	1.07e-02	1.08e-02	1.08e-02
175	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12) + k$	1.07e-02	1.07e-02	1.08e-02	1.07e-02
176	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) + k$	1.07e-02	1.07e-02	1.08e-02	1.07e-02
177	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3) * k$	3.67e-02	1.89e-02	2.13e-02	2.02e-02
178	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4) * k$	2.57e-02	2.15e-02	2.45e-02	2.33e-02
179	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5) * k$	1.08e-02	1.06e-02	1.08e-02	1.08e-02
180	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 6) * k$	9.89e-03	8.70e-03	1.08e-02	9.41e-03
181	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 7) * k$	8.31e-03	7.54e-03	8.63e-03	8.15e-03
182	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 8) * k$	6.44e-03	6.37e-03	7.08e-03	7.00e-03
183	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 9) * k$	6.34e-03	6.29e-03	6.97e-03	6.91e-03
184	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 10) * k$	6.09e-03	6.07e-03	6.69e-03	6.67e-03

Table A7: Performance of the models for the third case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
185	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k$	6.05e-03	6.04e-03	6.66e-03	6.65e-03
186	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k$	6.02e-03	6.02e-03	6.63e-03	6.63e-03
187	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k$	6.02e-03	6.02e-03	6.63e-03	6.63e-03
188	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + \log(k)$	3.84e-02	2.17e-02	2.38e-02	2.27e-02
189	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + \log(k)$	2.87e-02	2.39e-02	2.66e-02	2.53e-02
190	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + \log(k)$	1.42e-02	1.37e-02	1.34e-02	1.34e-02
191	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + \log(k)$	1.30e-02	1.21e-02	1.31e-02	1.21e-02
192	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + \log(k)$	1.29e-02	1.20e-02	1.29e-02	1.20e-02
193	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + \log(k)$	1.02e-02	1.01e-02	9.96e-03	9.87e-03
194	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + \log(k)$	1.02e-02	1.01e-02	9.93e-03	9.85e-03
195	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + \log(k)$	9.91e-03	9.88e-03	9.62e-03	9.59e-03
196	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + \log(k)$	9.89e-03	9.84e-03	9.63e-03	9.57e-03
197	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + \log(k)$	9.82e-03	9.80e-03	9.55e-03	9.52e-03
198	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + \log(k)$	9.82e-03	9.80e-03	9.55e-03	9.52e-03
199	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * \log(k)$	3.63e-02	1.78e-02	2.01e-02	1.90e-02
200	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * \log(k)$	2.47e-02	2.06e-02	2.35e-02	2.22e-02
201	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * \log(k)$	9.27e-03	8.83e-03	8.79e-03	8.78e-03
202	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * \log(k)$	7.96e-03	6.44e-03	8.61e-03	6.87e-03
203	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * \log(k)$	5.85e-03	4.96e-03	6.01e-03	5.22e-03
204	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * \log(k)$	2.89e-03	2.72e-03	3.17e-03	2.98e-03
205	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * \log(k)$	2.62e-03	2.51e-03	2.87e-03	2.74e-03
206	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * \log(k)$	1.95e-03	1.90e-03	2.13e-03	2.07e-03
207	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * \log(k)$	1.86e-03	1.82e-03	2.05e-03	2.01e-03
208	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * \log(k)$	1.76e-03	1.76e-03	1.95e-03	1.94e-03
209	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * \log(k)$	1.75e-03	1.75e-03	1.94e-03	1.94e-03
210	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + k_d$	3.83e-02	2.16e-02	2.38e-02	2.27e-02

Table A7: Performance of the models for the third case and all underlying tests included. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
211	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + k_d$	2.87e-02	2.38e-02	2.66e-02	2.53e-02
212	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + k_d$	1.42e-02	1.37e-02	1.34e-02	1.34e-02
213	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + k_d$	1.30e-02	1.21e-02	1.31e-02	1.20e-02
214	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + k_d$	1.28e-02	1.20e-02	1.28e-02	1.19e-02
215	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + k_d$	1.01e-02	1.01e-02	9.92e-03	9.83e-03
216	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + k_d$	1.01e-02	1.01e-02	9.90e-03	9.81e-03
217	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + k_d$	9.88e-03	9.85e-03	9.59e-03	9.56e-03
218	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + k_d$	9.87e-03	9.81e-03	9.60e-03	9.53e-03
219	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + k_d$	9.80e-03	9.78e-03	9.51e-03	9.49e-03
220	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + k_d$	9.79e-03	9.78e-03	9.51e-03	9.49e-03
221	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * k_d$	3.59e-02	1.77e-02	2.00e-02	1.89e-02
222	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * k_d$	2.46e-02	2.04e-02	2.33e-02	2.21e-02
223	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * k_d$	8.33e-03	8.31e-03	8.17e-03	8.15e-03
224	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * k_d$	7.91e-03	6.20e-03	8.63e-03	6.65e-03
225	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * k_d$	3.56e-03	2.94e-03	3.19e-03	2.96e-03
226	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * k_d$	2.31e-03	2.08e-03	2.52e-03	2.26e-03
227	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * k_d$	1.81e-03	1.68e-03	1.95e-03	1.79e-03
228	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * k_d$	7.62e-04	6.42e-04	8.35e-04	6.98e-04
229	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k_d$	5.27e-04	4.81e-04	5.60e-04	5.06e-04
230	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k_d$	2.57e-04	2.39e-04	2.76e-04	2.56e-04
231	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k_d$	1.92e-04	1.86e-04	2.02e-04	1.95e-04

A.2 Results for the p-approximation of the Bayer-Hanck Test with Engle-Granger and Johansen as underlying tests

A.2.1 Metrics of the 5 Best Models

Table A8: The five best models, based on the cRMSE for the lower tail of the distribution, for the first case (no constant, no trend) with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.40e-04	1.40e-04	1.48e-04	1.48e-04
2	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	2.01e-04	1.93e-04	1.84e-04	1.82e-04
3	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	1.92e-04	1.86e-04	2.09e-04	2.01e-04
4	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	1.93e-04	1.91e-04	2.06e-04	2.03e-04
5	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	2.12e-04	2.03e-04	2.30e-04	2.20e-04

Table A9: The five best models, based on the cRMSE for the lower tail of the distribution, for the second case (with constant, no trend) with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.57e-04	1.56e-04	1.64e-04	1.63e-04
2	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	2.11e-04	2.02e-04	1.87e-04	1.84e-04
3	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.10e-04	2.00e-04	2.27e-04	2.16e-04
4	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	2.10e-04	2.07e-04	2.19e-04	2.16e-04
5	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.11e-04	2.04e-04	2.25e-04	2.17e-04

Table A10: The five best models, based on the cRMSE for the lower tail of the distribution, for the third case (with constant and trend) with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k_d$	1.45e-04	1.45e-04	1.53e-04	1.52e-04
2	$p = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k_d$	1.88e-04	1.81e-04	1.70e-04	1.68e-04
3	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k_d$	1.84e-04	1.78e-04	1.93e-04	1.86e-04
4	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k_d$	1.85e-04	1.83e-04	1.90e-04	1.88e-04
5	$p = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k_d$	2.30e-04	2.19e-04	2.28e-04	2.17e-04

A.2.2 Metrics of all Models

Table A11: Performance of the models for the first case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$p = c + \text{poly}\left(\text{bc}(t), 3\right)$	3.40e-02	2.23e-02	2.37e-02	2.25e-02
2	$p = c + \text{poly}\left(\text{bc}(t), 4\right)$	2.22e-02	2.13e-02	2.29e-02	2.23e-02
3	$p = c + \text{poly}\left(\text{bc}(t), 5\right)$	1.97e-02	1.92e-02	1.77e-02	1.76e-02
4	$p = c + \text{poly}\left(\text{bc}(t), 6\right)$	1.23e-02	1.18e-02	1.31e-02	1.26e-02
5	$p = c + \text{poly}\left(\text{bc}(t), 7\right)$	1.21e-02	1.15e-02	1.25e-02	1.23e-02
6	$p = c + \text{poly}\left(\text{bc}(t), 8\right)$	1.12e-02	1.10e-02	1.18e-02	1.17e-02
7	$p = c + \text{poly}\left(\text{bc}(t), 9\right)$	1.07e-02	1.06e-02	1.17e-02	1.16e-02
8	$p = c + \text{poly}\left(\text{bc}(t), 10\right)$	1.05e-02	1.05e-02	1.14e-02	1.14e-02
9	$p = c + \text{poly}\left(\text{bc}(t), 11\right)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
10	$p = c + \text{poly}\left(\text{bc}(t), 12\right)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
11	$p = c + \text{poly}\left(\text{bc}(t), 13\right)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
12	$p = c + \text{poly}\left(\text{bc}(t), 3\right) + k$	3.34e-02	2.11e-02	2.20e-02	2.06e-02
13	$p = c + \text{poly}\left(\text{bc}(t), 4\right) + k$	2.11e-02	2.01e-02	2.10e-02	2.04e-02
14	$p = c + \text{poly}\left(\text{bc}(t), 5\right) + k$	1.85e-02	1.78e-02	1.53e-02	1.50e-02
15	$p = c + \text{poly}\left(\text{bc}(t), 6\right) + k$	1.03e-02	9.67e-03	9.52e-03	8.87e-03
16	$p = c + \text{poly}\left(\text{bc}(t), 7\right) + k$	1.01e-02	9.38e-03	8.75e-03	8.40e-03
17	$p = c + \text{poly}\left(\text{bc}(t), 8\right) + k$	8.96e-03	8.64e-03	7.57e-03	7.54e-03
18	$p = c + \text{poly}\left(\text{bc}(t), 9\right) + k$	8.32e-03	8.24e-03	7.47e-03	7.39e-03
19	$p = c + \text{poly}\left(\text{bc}(t), 10\right) + k$	8.03e-03	7.99e-03	7.08e-03	7.07e-03
20	$p = c + \text{poly}\left(\text{bc}(t), 11\right) + k$	7.95e-03	7.92e-03	7.07e-03	7.04e-03
21	$p = c + \text{poly}\left(\text{bc}(t), 12\right) + k$	7.93e-03	7.90e-03	7.04e-03	7.02e-03
22	$p = c + \text{poly}\left(\text{bc}(t), 13\right) + k$	7.91e-03	7.89e-03	7.03e-03	7.01e-03
23	$p = c + \text{poly}\left(\text{bc}(t), 3\right) * k$	3.27e-02	2.00e-02	2.14e-02	1.99e-02

Table A11: Performance of the models for the first case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
24	$p = c + \text{poly}\,(\text{bc}(t), 4) * k$	2.00e-02	1.89e-02	2.03e-02	1.96e-02
25	$p = c + \text{poly}\,(\text{bc}(t), 5) * k$	1.68e-02	1.62e-02	1.39e-02	1.37e-02
26	$p = c + \text{poly}\,(\text{bc}(t), 6) * k$	7.47e-03	6.63e-03	7.55e-03	6.71e-03
27	$p = c + \text{poly}\,(\text{bc}(t), 7) * k$	7.03e-03	6.06e-03	6.42e-03	5.96e-03
28	$p = c + \text{poly}\,(\text{bc}(t), 8) * k$	5.40e-03	4.88e-03	4.79e-03	4.72e-03
29	$p = c + \text{poly}\,(\text{bc}(t), 9) * k$	4.34e-03	4.21e-03	4.61e-03	4.51e-03
30	$p = c + \text{poly}\,(\text{bc}(t), 10) * k$	3.76e-03	3.72e-03	3.97e-03	3.97e-03
31	$p = c + \text{poly}\,(\text{bc}(t), 11) * k$	3.61e-03	3.59e-03	3.95e-03	3.93e-03
32	$p = c + \text{poly}\,(\text{bc}(t), 12) * k$	3.57e-03	3.55e-03	3.89e-03	3.89e-03
33	$p = c + \text{poly}\,(\text{bc}(t), 13) * k$	3.53e-03	3.52e-03	3.88e-03	3.87e-03
34	$p = c + \text{poly}\,(\text{bc}(t), 3) + \log(k)$	3.33e-02	2.10e-02	2.18e-02	2.04e-02
35	$p = c + \text{poly}\,(\text{bc}(t), 4) + \log(k)$	2.10e-02	1.99e-02	2.07e-02	2.01e-02
36	$p = c + \text{poly}\,(\text{bc}(t), 5) + \log(k)$	1.83e-02	1.76e-02	1.49e-02	1.47e-02
37	$p = c + \text{poly}\,(\text{bc}(t), 6) + \log(k)$	1.00e-02	9.38e-03	9.00e-03	8.32e-03
38	$p = c + \text{poly}\,(\text{bc}(t), 7) + \log(k)$	9.78e-03	9.09e-03	8.19e-03	7.82e-03
39	$p = c + \text{poly}\,(\text{bc}(t), 8) + \log(k)$	8.66e-03	8.33e-03	6.93e-03	6.89e-03
40	$p = c + \text{poly}\,(\text{bc}(t), 9) + \log(k)$	7.99e-03	7.91e-03	6.81e-03	6.73e-03
41	$p = c + \text{poly}\,(\text{bc}(t), 10) + \log(k)$	7.69e-03	7.65e-03	6.39e-03	6.37e-03
42	$p = c + \text{poly}\,(\text{bc}(t), 11) + \log(k)$	7.61e-03	7.57e-03	6.37e-03	6.34e-03
43	$p = c + \text{poly}\,(\text{bc}(t), 12) + \log(k)$	7.59e-03	7.55e-03	6.34e-03	6.32e-03
44	$p = c + \text{poly}\,(\text{bc}(t), 13) + \log(k)$	7.57e-03	7.54e-03	6.32e-03	6.31e-03
45	$p = c + \text{poly}\,(\text{bc}(t), 3) * \log(k)$	3.25e-02	1.98e-02	2.11e-02	1.96e-02
46	$p = c + \text{poly}\,(\text{bc}(t), 4) * \log(k)$	1.97e-02	1.86e-02	2.00e-02	1.92e-02
47	$p = c + \text{poly}\,(\text{bc}(t), 5) * \log(k)$	1.64e-02	1.59e-02	1.34e-02	1.32e-02
48	$p = c + \text{poly}\,(\text{bc}(t), 6) * \log(k)$	6.65e-03	5.68e-03	6.55e-03	5.55e-03
49	$p = c + \text{poly}\,(\text{bc}(t), 7) * \log(k)$	6.11e-03	4.95e-03	5.14e-03	4.57e-03

Table A11: Performance of the models for the first case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
50	$p = c + \text{poly}\,(\text{bc}(t), 8) * \log(k)$	4.13e-03	3.42e-03	2.88e-03	2.77e-03
51	$p = c + \text{poly}\,(\text{bc}(t), 9) * \log(k)$	2.62e-03	2.39e-03	2.59e-03	2.40e-03
52	$p = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	1.43e-03	1.32e-03	1.07e-03	1.05e-03
53	$p = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	9.91e-04	9.04e-04	9.96e-04	9.13e-04
54	$p = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	8.17e-04	7.53e-04	7.30e-04	7.15e-04
55	$p = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	5.98e-04	5.84e-04	6.22e-04	6.19e-04
56	$p = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	3.33e-02	2.10e-02	2.18e-02	2.04e-02
57	$p = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	2.10e-02	1.99e-02	2.07e-02	2.01e-02
58	$p = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	1.83e-02	1.76e-02	1.49e-02	1.47e-02
59	$p = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	1.00e-02	9.38e-03	9.00e-03	8.32e-03
60	$p = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	9.78e-03	9.09e-03	8.19e-03	7.82e-03
61	$p = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	8.66e-03	8.32e-03	6.92e-03	6.88e-03
62	$p = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	7.99e-03	7.91e-03	6.81e-03	6.72e-03
63	$p = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	7.69e-03	7.65e-03	6.38e-03	6.37e-03
64	$p = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	7.61e-03	7.57e-03	6.37e-03	6.34e-03
65	$p = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	7.59e-03	7.55e-03	6.33e-03	6.32e-03
66	$p = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	7.57e-03	7.53e-03	6.32e-03	6.31e-03
67	$p = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	3.25e-02	1.98e-02	2.11e-02	1.95e-02
68	$p = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	1.97e-02	1.86e-02	2.00e-02	1.92e-02
69	$p = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	1.64e-02	1.58e-02	1.34e-02	1.31e-02
70	$p = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	6.48e-03	5.57e-03	6.54e-03	5.51e-03
71	$p = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	5.34e-03	4.50e-03	4.25e-03	3.90e-03
72	$p = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	2.12e-03	1.96e-03	2.13e-03	1.97e-03
73	$p = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	1.71e-03	1.56e-03	1.33e-03	1.27e-03
74	$p = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	6.93e-04	6.12e-04	7.01e-04	6.18e-04
75	$p = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	5.28e-04	4.68e-04	4.19e-04	4.00e-04

Table A11: Performance of the models for the first case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
76	$p = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.52e-04	2.41e-04	2.62e-04	2.50e-04
77	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	2.01e-04	1.93e-04	1.84e-04	1.82e-04
78	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3)$	3.62e-02	1.88e-02	4.03e-02	2.07e-02
79	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4)$	2.26e-02	2.25e-02	2.51e-02	2.50e-02
80	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5)$	1.39e-02	1.17e-02	1.54e-02	1.30e-02
81	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
82	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7)$	1.06e-02	1.05e-02	1.16e-02	1.16e-02
83	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
84	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
85	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
86	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
87	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
88	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
89	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k$	3.66e-02	1.97e-02	4.08e-02	2.18e-02
90	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.30e-02	2.29e-02	2.56e-02	2.55e-02
91	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k$	1.46e-02	1.26e-02	1.62e-02	1.40e-02
92	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.14e-02	1.14e-02	1.26e-02	1.26e-02
93	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k$	1.15e-02	1.15e-02	1.28e-02	1.27e-02
94	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k$	1.14e-02	1.14e-02	1.26e-02	1.26e-02
95	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k$	1.14e-02	1.14e-02	1.25e-02	1.25e-02
96	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k$	1.14e-02	1.14e-02	1.25e-02	1.25e-02
97	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k$	1.14e-02	1.14e-02	1.25e-02	1.25e-02
98	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k$	1.14e-02	1.14e-02	1.25e-02	1.25e-02
99	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k$	1.14e-02	1.14e-02	1.25e-02	1.25e-02
100	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k$	3.63e-02	1.66e-02	4.05e-02	1.83e-02
101	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k$	1.89e-02	1.88e-02	2.10e-02	2.10e-02

Table A11: Performance of the models for the first case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
102	$\log(p) = c + \text{poly}\left(\text{bc}(t), 5\right) * k$	8.32e-03	5.79e-03	9.28e-03	6.43e-03
103	$\log(p) = c + \text{poly}\left(\text{bc}(t), 6\right) * k$	3.65e-03	3.63e-03	4.03e-03	4.01e-03
104	$\log(p) = c + \text{poly}\left(\text{bc}(t), 7\right) * k$	3.94e-03	3.89e-03	4.36e-03	4.30e-03
105	$\log(p) = c + \text{poly}\left(\text{bc}(t), 8\right) * k$	3.70e-03	3.65e-03	4.09e-03	4.03e-03
106	$\log(p) = c + \text{poly}\left(\text{bc}(t), 9\right) * k$	3.60e-03	3.59e-03	3.97e-03	3.97e-03
107	$\log(p) = c + \text{poly}\left(\text{bc}(t), 10\right) * k$	3.59e-03	3.59e-03	3.96e-03	3.96e-03
108	$\log(p) = c + \text{poly}\left(\text{bc}(t), 11\right) * k$	3.59e-03	3.59e-03	3.96e-03	3.96e-03
109	$\log(p) = c + \text{poly}\left(\text{bc}(t), 12\right) * k$	3.59e-03	3.59e-03	3.96e-03	3.96e-03
110	$\log(p) = c + \text{poly}\left(\text{bc}(t), 13\right) * k$	3.59e-03	3.59e-03	3.96e-03	3.96e-03
111	$\log(p) = c + \text{poly}\left(\text{bc}(t), 3\right) + \log(k)$	3.67e-02	1.98e-02	4.09e-02	2.19e-02
112	$\log(p) = c + \text{poly}\left(\text{bc}(t), 4\right) + \log(k)$	2.30e-02	2.29e-02	2.56e-02	2.55e-02
113	$\log(p) = c + \text{poly}\left(\text{bc}(t), 5\right) + \log(k)$	1.47e-02	1.28e-02	1.64e-02	1.41e-02
114	$\log(p) = c + \text{poly}\left(\text{bc}(t), 6\right) + \log(k)$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
115	$\log(p) = c + \text{poly}\left(\text{bc}(t), 7\right) + \log(k)$	1.17e-02	1.16e-02	1.29e-02	1.29e-02
116	$\log(p) = c + \text{poly}\left(\text{bc}(t), 8\right) + \log(k)$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
117	$\log(p) = c + \text{poly}\left(\text{bc}(t), 9\right) + \log(k)$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
118	$\log(p) = c + \text{poly}\left(\text{bc}(t), 10\right) + \log(k)$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
119	$\log(p) = c + \text{poly}\left(\text{bc}(t), 11\right) + \log(k)$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
120	$\log(p) = c + \text{poly}\left(\text{bc}(t), 12\right) + \log(k)$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
121	$\log(p) = c + \text{poly}\left(\text{bc}(t), 13\right) + \log(k)$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
122	$\log(p) = c + \text{poly}\left(\text{bc}(t), 3\right) * \log(k)$	3.61e-02	1.62e-02	4.03e-02	1.79e-02
123	$\log(p) = c + \text{poly}\left(\text{bc}(t), 4\right) * \log(k)$	1.84e-02	1.83e-02	2.05e-02	2.05e-02
124	$\log(p) = c + \text{poly}\left(\text{bc}(t), 5\right) * \log(k)$	7.76e-03	4.70e-03	8.66e-03	5.24e-03
125	$\log(p) = c + \text{poly}\left(\text{bc}(t), 6\right) * \log(k)$	8.47e-04	7.92e-04	9.34e-04	8.72e-04
126	$\log(p) = c + \text{poly}\left(\text{bc}(t), 7\right) * \log(k)$	1.75e-03	1.63e-03	1.95e-03	1.81e-03
127	$\log(p) = c + \text{poly}\left(\text{bc}(t), 8\right) * \log(k)$	1.17e-03	9.43e-04	1.30e-03	1.04e-03

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	Model	RMSE	cRMSE	RMSE	cRMSE
128	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * \log(k)$	6.42e-04	6.32e-04	7.02e-04	6.91e-04
129	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	5.81e-04	5.81e-04	6.33e-04	6.33e-04
130	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	5.79e-04	5.79e-04	6.31e-04	6.31e-04
131	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	5.79e-04	5.79e-04	6.31e-04	6.31e-04
132	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	5.80e-04	5.80e-04	6.32e-04	6.32e-04
133	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	3.67e-02	1.98e-02	4.09e-02	2.19e-02
134	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	2.30e-02	2.29e-02	2.56e-02	2.55e-02
135	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	1.47e-02	1.28e-02	1.64e-02	1.41e-02
136	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
137	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	1.17e-02	1.16e-02	1.29e-02	1.29e-02
138	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
139	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
140	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
141	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
142	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
143	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	1.15e-02	1.15e-02	1.27e-02	1.27e-02
144	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	3.60e-02	1.62e-02	4.02e-02	1.80e-02
145	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	1.84e-02	1.83e-02	2.06e-02	2.04e-02
146	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	7.67e-03	4.63e-03	8.57e-03	5.16e-03
147	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	7.27e-04	6.81e-04	8.09e-04	7.58e-04
148	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	3.84e-04	3.63e-04	4.25e-04	4.01e-04
149	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	2.93e-04	2.73e-04	3.22e-04	2.99e-04
150	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	2.12e-04	2.03e-04	2.30e-04	2.20e-04
151	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	2.55e-04	2.34e-04	2.80e-04	2.56e-04
152	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	2.78e-04	2.55e-04	3.05e-04	2.80e-04
153	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.88e-04	2.59e-04	3.17e-04	2.84e-04

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	Model	RMSE	cRMSE	RMSE	cRMSE
154	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	2.58e-04	2.40e-04	2.84e-04	2.63e-04
155	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3)$	5.55e-02	2.01e-02	2.21e-02	2.12e-02
156	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4)$	2.48e-02	2.34e-02	2.61e-02	2.55e-02
157	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5)$	1.38e-02	1.38e-02	1.43e-02	1.43e-02
158	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 6)$	1.24e-02	1.17e-02	1.36e-02	1.28e-02
159	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 7)$	1.13e-02	1.08e-02	1.20e-02	1.18e-02
160	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 8)$	1.05e-02	1.05e-02	1.15e-02	1.15e-02
161	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 9)$	1.06e-02	1.05e-02	1.16e-02	1.16e-02
162	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 10)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
163	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 11)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
164	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
165	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13)$	1.04e-02	1.04e-02	1.14e-02	1.14e-02
166	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3) + k$	5.52e-02	1.86e-02	2.03e-02	1.92e-02
167	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.36e-02	2.22e-02	2.45e-02	2.38e-02
168	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5) + k$	1.15e-02	1.15e-02	1.12e-02	1.11e-02
169	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 6) + k$	9.95e-03	9.03e-03	1.03e-02	9.21e-03
170	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 7) + k$	8.46e-03	7.82e-03	7.97e-03	7.80e-03
171	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 8) + k$	7.38e-03	7.35e-03	7.34e-03	7.30e-03
172	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 9) + k$	7.45e-03	7.39e-03	7.42e-03	7.36e-03
173	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 10) + k$	7.21e-03	7.19e-03	7.14e-03	7.12e-03
174	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 11) + k$	7.21e-03	7.19e-03	7.15e-03	7.12e-03
175	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12) + k$	7.19e-03	7.18e-03	7.13e-03	7.11e-03
176	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) + k$	7.19e-03	7.18e-03	7.13e-03	7.11e-03
177	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3) * k$	5.41e-02	1.74e-02	1.94e-02	1.82e-02
178	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4) * k$	2.26e-02	2.11e-02	2.37e-02	2.28e-02
179	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5) * k$	9.50e-03	9.49e-03	9.29e-03	9.29e-03

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	Model	RMSE	cRMSE	RMSE	cRMSE
180	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * k$	7.60e-03	6.42e-03	8.28e-03	6.92e-03
181	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * k$	5.71e-03	4.61e-03	5.22e-03	4.96e-03
182	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * k$	3.91e-03	3.87e-03	4.28e-03	4.23e-03
183	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * k$	3.97e-03	3.91e-03	4.36e-03	4.29e-03
184	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * k$	3.57e-03	3.56e-03	3.93e-03	3.92e-03
185	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k$	3.57e-03	3.56e-03	3.93e-03	3.92e-03
186	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k$	3.54e-03	3.54e-03	3.90e-03	3.90e-03
187	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k$	3.54e-03	3.54e-03	3.90e-03	3.90e-03
188	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + \log(k)$	5.52e-02	1.84e-02	2.00e-02	1.89e-02
189	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + \log(k)$	2.34e-02	2.20e-02	2.43e-02	2.35e-02
190	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + \log(k)$	1.12e-02	1.12e-02	1.07e-02	1.07e-02
191	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + \log(k)$	9.61e-03	8.66e-03	9.86e-03	8.69e-03
192	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + \log(k)$	8.06e-03	7.39e-03	7.37e-03	7.18e-03
193	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + \log(k)$	6.92e-03	6.89e-03	6.68e-03	6.64e-03
194	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + \log(k)$	6.99e-03	6.93e-03	6.78e-03	6.70e-03
195	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + \log(k)$	6.73e-03	6.72e-03	6.47e-03	6.45e-03
196	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + \log(k)$	6.74e-03	6.72e-03	6.47e-03	6.45e-03
197	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + \log(k)$	6.72e-03	6.71e-03	6.45e-03	6.44e-03
198	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + \log(k)$	6.72e-03	6.71e-03	6.45e-03	6.44e-03
199	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * \log(k)$	5.40e-02	1.71e-02	1.91e-02	1.78e-02
200	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * \log(k)$	2.24e-02	2.08e-02	2.34e-02	2.25e-02
201	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * \log(k)$	8.83e-03	8.82e-03	8.44e-03	8.44e-03
202	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * \log(k)$	6.78e-03	5.42e-03	7.37e-03	5.79e-03
203	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * \log(k)$	4.46e-03	3.00e-03	3.50e-03	3.11e-03
204	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * \log(k)$	1.76e-03	1.65e-03	1.87e-03	1.75e-03
205	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * \log(k)$	1.90e-03	1.76e-03	2.05e-03	1.89e-03

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	Model	RMSE	cRMSE	RMSE	cRMSE
206	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * \log(k)$	7.25e-04	6.94e-04	7.88e-04	7.52e-04
207	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * \log(k)$	7.32e-04	6.95e-04	7.98e-04	7.56e-04
208	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * \log(k)$	5.72e-04	5.71e-04	6.21e-04	6.20e-04
209	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * \log(k)$	5.68e-04	5.64e-04	6.25e-04	6.20e-04
210	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + k_d$	5.52e-02	1.84e-02	2.00e-02	1.89e-02
211	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + k_d$	2.34e-02	2.20e-02	2.43e-02	2.35e-02
212	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + k_d$	1.12e-02	1.12e-02	1.07e-02	1.07e-02
213	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + k_d$	9.61e-03	8.66e-03	9.86e-03	8.69e-03
214	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + k_d$	8.06e-03	7.38e-03	7.37e-03	7.18e-03
215	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + k_d$	6.92e-03	6.89e-03	6.68e-03	6.64e-03
216	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + k_d$	6.99e-03	6.93e-03	6.77e-03	6.70e-03
217	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + k_d$	6.73e-03	6.72e-03	6.46e-03	6.45e-03
218	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + k_d$	6.74e-03	6.72e-03	6.47e-03	6.45e-03
219	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + k_d$	6.72e-03	6.71e-03	6.45e-03	6.44e-03
220	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + k_d$	6.72e-03	6.71e-03	6.45e-03	6.44e-03
221	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * k_d$	5.39e-02	1.71e-02	1.90e-02	1.78e-02
222	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * k_d$	2.24e-02	2.08e-02	2.34e-02	2.25e-02
223	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * k_d$	8.81e-03	8.81e-03	8.43e-03	8.43e-03
224	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * k_d$	6.76e-03	5.39e-03	7.36e-03	5.77e-03
225	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * k_d$	2.14e-03	2.01e-03	2.23e-03	2.09e-03
226	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * k_d$	1.87e-03	1.70e-03	2.04e-03	1.85e-03
227	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * k_d$	5.81e-04	5.65e-04	5.89e-04	5.69e-04
228	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * k_d$	5.40e-04	4.75e-04	5.91e-04	5.17e-04
229	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k_d$	1.93e-04	1.91e-04	2.06e-04	2.03e-04
230	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k_d$	1.92e-04	1.86e-04	2.09e-04	2.01e-04
231	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k_d$	1.40e-04	1.40e-04	1.48e-04	1.48e-04

Table A12: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$p = c + \text{poly}\,(\text{bc}(t), 3)$	3.46e-02	2.29e-02	2.44e-02	2.32e-02
2	$p = c + \text{poly}\,(\text{bc}(t), 4)$	2.28e-02	2.19e-02	2.35e-02	2.30e-02
3	$p = c + \text{poly}\,(\text{bc}(t), 5)$	2.03e-02	1.98e-02	1.85e-02	1.83e-02
4	$p = c + \text{poly}\,(\text{bc}(t), 6)$	1.31e-02	1.26e-02	1.39e-02	1.35e-02
5	$p = c + \text{poly}\,(\text{bc}(t), 7)$	1.31e-02	1.25e-02	1.37e-02	1.34e-02
6	$p = c + \text{poly}\,(\text{bc}(t), 8)$	1.21e-02	1.18e-02	1.27e-02	1.27e-02
7	$p = c + \text{poly}\,(\text{bc}(t), 9)$	1.15e-02	1.15e-02	1.26e-02	1.25e-02
8	$p = c + \text{poly}\,(\text{bc}(t), 10)$	1.13e-02	1.13e-02	1.24e-02	1.24e-02
9	$p = c + \text{poly}\,(\text{bc}(t), 11)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
10	$p = c + \text{poly}\,(\text{bc}(t), 12)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
11	$p = c + \text{poly}\,(\text{bc}(t), 13)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
12	$p = c + \text{poly}\,(\text{bc}(t), 3) + k$	3.39e-02	2.16e-02	2.25e-02	2.11e-02
13	$p = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.16e-02	2.06e-02	2.15e-02	2.08e-02
14	$p = c + \text{poly}\,(\text{bc}(t), 5) + k$	1.90e-02	1.83e-02	1.58e-02	1.56e-02
15	$p = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.10e-02	1.03e-02	1.01e-02	9.49e-03
16	$p = c + \text{poly}\,(\text{bc}(t), 7) + k$	1.10e-02	1.03e-02	9.84e-03	9.36e-03
17	$p = c + \text{poly}\,(\text{bc}(t), 8) + k$	9.71e-03	9.38e-03	8.33e-03	8.28e-03
18	$p = c + \text{poly}\,(\text{bc}(t), 9) + k$	9.00e-03	8.92e-03	8.17e-03	8.08e-03
19	$p = c + \text{poly}\,(\text{bc}(t), 10) + k$	8.74e-03	8.70e-03	7.80e-03	7.78e-03
20	$p = c + \text{poly}\,(\text{bc}(t), 11) + k$	8.66e-03	8.62e-03	7.79e-03	7.75e-03
21	$p = c + \text{poly}\,(\text{bc}(t), 12) + k$	8.65e-03	8.62e-03	7.77e-03	7.74e-03
22	$p = c + \text{poly}\,(\text{bc}(t), 13) + k$	8.61e-03	8.59e-03	7.74e-03	7.73e-03
23	$p = c + \text{poly}\,(\text{bc}(t), 3) * k$	3.30e-02	2.03e-02	2.17e-02	2.02e-02
24	$p = c + \text{poly}\,(\text{bc}(t), 4) * k$	2.02e-02	1.91e-02	2.05e-02	1.98e-02

Table A12: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
25	$p = c + \text{poly}\,(\text{bc}(t), 5) * k$	1.71e-02	1.66e-02	1.43e-02	1.41e-02
26	$p = c + \text{poly}\,(\text{bc}(t), 6) * k$	7.91e-03	7.02e-03	7.89e-03	7.14e-03
27	$p = c + \text{poly}\,(\text{bc}(t), 7) * k$	7.70e-03	6.70e-03	7.28e-03	6.75e-03
28	$p = c + \text{poly}\,(\text{bc}(t), 8) * k$	5.99e-03	5.48e-03	5.48e-03	5.41e-03
29	$p = c + \text{poly}\,(\text{bc}(t), 9) * k$	4.90e-03	4.77e-03	5.24e-03	5.14e-03
30	$p = c + \text{poly}\,(\text{bc}(t), 10) * k$	4.41e-03	4.38e-03	4.69e-03	4.69e-03
31	$p = c + \text{poly}\,(\text{bc}(t), 11) * k$	4.27e-03	4.24e-03	4.67e-03	4.65e-03
32	$p = c + \text{poly}\,(\text{bc}(t), 12) * k$	4.25e-03	4.23e-03	4.64e-03	4.63e-03
33	$p = c + \text{poly}\,(\text{bc}(t), 13) * k$	4.19e-03	4.18e-03	4.60e-03	4.60e-03
34	$p = c + \text{poly}\,(\text{bc}(t), 3) + \log(k)$	3.38e-02	2.14e-02	2.22e-02	2.08e-02
35	$p = c + \text{poly}\,(\text{bc}(t), 4) + \log(k)$	2.15e-02	2.04e-02	2.11e-02	2.05e-02
36	$p = c + \text{poly}\,(\text{bc}(t), 5) + \log(k)$	1.88e-02	1.81e-02	1.54e-02	1.51e-02
37	$p = c + \text{poly}\,(\text{bc}(t), 6) + \log(k)$	1.07e-02	9.96e-03	9.40e-03	8.78e-03
38	$p = c + \text{poly}\,(\text{bc}(t), 7) + \log(k)$	1.07e-02	9.88e-03	9.15e-03	8.64e-03
39	$p = c + \text{poly}\,(\text{bc}(t), 8) + \log(k)$	9.33e-03	8.98e-03	7.51e-03	7.46e-03
40	$p = c + \text{poly}\,(\text{bc}(t), 9) + \log(k)$	8.59e-03	8.50e-03	7.33e-03	7.23e-03
41	$p = c + \text{poly}\,(\text{bc}(t), 10) + \log(k)$	8.31e-03	8.26e-03	6.93e-03	6.91e-03
42	$p = c + \text{poly}\,(\text{bc}(t), 11) + \log(k)$	8.23e-03	8.18e-03	6.91e-03	6.87e-03
43	$p = c + \text{poly}\,(\text{bc}(t), 12) + \log(k)$	8.22e-03	8.17e-03	6.89e-03	6.86e-03
44	$p = c + \text{poly}\,(\text{bc}(t), 13) + \log(k)$	8.18e-03	8.14e-03	6.86e-03	6.85e-03
45	$p = c + \text{poly}\,(\text{bc}(t), 3) * \log(k)$	3.28e-02	1.99e-02	2.12e-02	1.97e-02
46	$p = c + \text{poly}\,(\text{bc}(t), 4) * \log(k)$	1.97e-02	1.87e-02	2.00e-02	1.93e-02
47	$p = c + \text{poly}\,(\text{bc}(t), 5) * \log(k)$	1.66e-02	1.60e-02	1.36e-02	1.34e-02
48	$p = c + \text{poly}\,(\text{bc}(t), 6) * \log(k)$	6.81e-03	5.73e-03	6.50e-03	5.57e-03
49	$p = c + \text{poly}\,(\text{bc}(t), 7) * \log(k)$	6.66e-03	5.42e-03	5.90e-03	5.17e-03
50	$p = c + \text{poly}\,(\text{bc}(t), 8) * \log(k)$	4.42e-03	3.69e-03	3.17e-03	3.06e-03

Table A12: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
51	$p = c + \text{poly}\,(\text{bc}(t), 9) * \log(k)$	2.74e-03	2.51e-03	2.75e-03	2.54e-03
52	$p = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	1.73e-03	1.62e-03	1.43e-03	1.41e-03
53	$p = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	1.30e-03	1.22e-03	1.34e-03	1.27e-03
54	$p = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	1.26e-03	1.18e-03	1.24e-03	1.21e-03
55	$p = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	1.01e-03	9.94e-04	1.08e-03	1.08e-03
56	$p = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	3.38e-02	2.14e-02	2.22e-02	2.08e-02
57	$p = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	2.15e-02	2.04e-02	2.11e-02	2.05e-02
58	$p = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	1.88e-02	1.81e-02	1.54e-02	1.51e-02
59	$p = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	1.07e-02	9.96e-03	9.39e-03	8.78e-03
60	$p = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	1.07e-02	9.87e-03	9.14e-03	8.63e-03
61	$p = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	9.32e-03	8.98e-03	7.50e-03	7.45e-03
62	$p = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	8.59e-03	8.49e-03	7.32e-03	7.22e-03
63	$p = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	8.31e-03	8.26e-03	6.92e-03	6.90e-03
64	$p = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	8.22e-03	8.17e-03	6.90e-03	6.86e-03
65	$p = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	8.22e-03	8.17e-03	6.88e-03	6.85e-03
66	$p = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	8.18e-03	8.14e-03	6.85e-03	6.84e-03
67	$p = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	3.27e-02	1.99e-02	2.12e-02	1.97e-02
68	$p = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	1.97e-02	1.86e-02	2.00e-02	1.92e-02
69	$p = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	1.66e-02	1.60e-02	1.36e-02	1.33e-02
70	$p = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	6.43e-03	5.53e-03	6.47e-03	5.44e-03
71	$p = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	5.48e-03	4.61e-03	4.39e-03	4.01e-03
72	$p = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	2.10e-03	1.93e-03	2.08e-03	1.92e-03
73	$p = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	1.79e-03	1.63e-03	1.40e-03	1.33e-03
74	$p = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	6.92e-04	6.11e-04	6.83e-04	6.00e-04
75	$p = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	5.61e-04	4.93e-04	4.35e-04	4.11e-04
76	$p = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.61e-04	2.50e-04	2.61e-04	2.49e-04

Table A12: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
77	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	2.11e-04	2.02e-04	1.87e-04	1.84e-04
78	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3)$	3.58e-02	1.91e-02	3.99e-02	2.11e-02
79	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4)$	2.33e-02	2.32e-02	2.59e-02	2.58e-02
80	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5)$	1.46e-02	1.26e-02	1.62e-02	1.39e-02
81	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6)$	1.13e-02	1.12e-02	1.24e-02	1.24e-02
82	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7)$	1.17e-02	1.16e-02	1.28e-02	1.27e-02
83	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
84	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
85	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
86	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
87	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
88	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
89	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k$	3.64e-02	2.01e-02	4.05e-02	2.22e-02
90	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.38e-02	2.37e-02	2.65e-02	2.64e-02
91	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k$	1.55e-02	1.35e-02	1.72e-02	1.49e-02
92	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.23e-02	1.23e-02	1.36e-02	1.36e-02
93	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k$	1.27e-02	1.26e-02	1.40e-02	1.39e-02
94	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k$	1.23e-02	1.23e-02	1.36e-02	1.36e-02
95	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k$	1.23e-02	1.23e-02	1.35e-02	1.35e-02
96	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k$	1.23e-02	1.23e-02	1.35e-02	1.35e-02
97	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k$	1.23e-02	1.23e-02	1.35e-02	1.35e-02
98	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k$	1.23e-02	1.23e-02	1.35e-02	1.35e-02
99	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k$	1.23e-02	1.23e-02	1.35e-02	1.35e-02
100	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k$	3.60e-02	1.66e-02	4.01e-02	1.84e-02
101	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k$	1.92e-02	1.91e-02	2.14e-02	2.13e-02
102	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * k$	8.60e-03	6.23e-03	9.58e-03	6.92e-03

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	Model	RMSE	cRMSE	RMSE	cRMSE
103	$\log(p) = c + \text{poly}\left(\text{bc}(t), 6\right) * k$	4.39e-03	4.33e-03	4.85e-03	4.78e-03
104	$\log(p) = c + \text{poly}\left(\text{bc}(t), 7\right) * k$	4.86e-03	4.76e-03	5.37e-03	5.26e-03
105	$\log(p) = c + \text{poly}\left(\text{bc}(t), 8\right) * k$	4.36e-03	4.31e-03	4.81e-03	4.75e-03
106	$\log(p) = c + \text{poly}\left(\text{bc}(t), 9\right) * k$	4.28e-03	4.27e-03	4.72e-03	4.71e-03
107	$\log(p) = c + \text{poly}\left(\text{bc}(t), 10\right) * k$	4.26e-03	4.26e-03	4.70e-03	4.70e-03
108	$\log(p) = c + \text{poly}\left(\text{bc}(t), 11\right) * k$	4.25e-03	4.25e-03	4.69e-03	4.69e-03
109	$\log(p) = c + \text{poly}\left(\text{bc}(t), 12\right) * k$	4.25e-03	4.25e-03	4.69e-03	4.69e-03
110	$\log(p) = c + \text{poly}\left(\text{bc}(t), 13\right) * k$	4.25e-03	4.25e-03	4.69e-03	4.69e-03
111	$\log(p) = c + \text{poly}\left(\text{bc}(t), 3\right) + \log(k)$	3.65e-02	2.03e-02	4.06e-02	2.24e-02
112	$\log(p) = c + \text{poly}\left(\text{bc}(t), 4\right) + \log(k)$	2.39e-02	2.38e-02	2.66e-02	2.65e-02
113	$\log(p) = c + \text{poly}\left(\text{bc}(t), 5\right) + \log(k)$	1.56e-02	1.37e-02	1.73e-02	1.51e-02
114	$\log(p) = c + \text{poly}\left(\text{bc}(t), 6\right) + \log(k)$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
115	$\log(p) = c + \text{poly}\left(\text{bc}(t), 7\right) + \log(k)$	1.29e-02	1.28e-02	1.42e-02	1.41e-02
116	$\log(p) = c + \text{poly}\left(\text{bc}(t), 8\right) + \log(k)$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
117	$\log(p) = c + \text{poly}\left(\text{bc}(t), 9\right) + \log(k)$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
118	$\log(p) = c + \text{poly}\left(\text{bc}(t), 10\right) + \log(k)$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
119	$\log(p) = c + \text{poly}\left(\text{bc}(t), 11\right) + \log(k)$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
120	$\log(p) = c + \text{poly}\left(\text{bc}(t), 12\right) + \log(k)$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
121	$\log(p) = c + \text{poly}\left(\text{bc}(t), 13\right) + \log(k)$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
122	$\log(p) = c + \text{poly}\left(\text{bc}(t), 3\right) * \log(k)$	3.58e-02	1.61e-02	3.99e-02	1.78e-02
123	$\log(p) = c + \text{poly}\left(\text{bc}(t), 4\right) * \log(k)$	1.86e-02	1.86e-02	2.08e-02	2.07e-02
124	$\log(p) = c + \text{poly}\left(\text{bc}(t), 5\right) * \log(k)$	7.74e-03	4.76e-03	8.65e-03	5.31e-03
125	$\log(p) = c + \text{poly}\left(\text{bc}(t), 6\right) * \log(k)$	1.36e-03	1.24e-03	1.51e-03	1.37e-03
126	$\log(p) = c + \text{poly}\left(\text{bc}(t), 7\right) * \log(k)$	2.69e-03	2.47e-03	3.00e-03	2.75e-03
127	$\log(p) = c + \text{poly}\left(\text{bc}(t), 8\right) * \log(k)$	1.47e-03	1.27e-03	1.64e-03	1.41e-03
128	$\log(p) = c + \text{poly}\left(\text{bc}(t), 9\right) * \log(k)$	1.12e-03	1.10e-03	1.24e-03	1.22e-03

Table A12: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
129	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	1.06e-03	1.05e-03	1.17e-03	1.16e-03
130	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	1.03e-03	1.03e-03	1.14e-03	1.14e-03
131	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	1.04e-03	1.03e-03	1.15e-03	1.15e-03
132	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	1.03e-03	1.03e-03	1.14e-03	1.14e-03
133	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	3.65e-02	2.03e-02	4.06e-02	2.24e-02
134	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	2.39e-02	2.38e-02	2.66e-02	2.65e-02
135	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	1.56e-02	1.37e-02	1.73e-02	1.52e-02
136	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
137	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	1.29e-02	1.28e-02	1.42e-02	1.41e-02
138	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
139	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
140	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
141	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
142	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
143	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	1.25e-02	1.25e-02	1.38e-02	1.38e-02
144	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	3.58e-02	1.61e-02	4.00e-02	1.78e-02
145	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	1.87e-02	1.86e-02	2.08e-02	2.07e-02
146	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	7.69e-03	4.63e-03	8.59e-03	5.17e-03
147	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	6.99e-04	6.53e-04	7.79e-04	7.27e-04
148	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	3.41e-04	3.26e-04	3.75e-04	3.59e-04
149	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	4.17e-04	3.88e-04	4.62e-04	4.29e-04
150	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	4.18e-04	3.45e-04	4.63e-04	3.80e-04
151	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	3.38e-04	3.04e-04	3.72e-04	3.33e-04
152	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	2.42e-04	2.32e-04	2.64e-04	2.52e-04
153	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.10e-04	2.00e-04	2.27e-04	2.16e-04
154	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	2.73e-04	2.46e-04	3.00e-04	2.68e-04

Table A12: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
155	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3)$	5.63e-02	2.07e-02	2.28e-02	2.19e-02
156	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4)$	2.53e-02	2.40e-02	2.67e-02	2.61e-02
157	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5)$	1.45e-02	1.45e-02	1.51e-02	1.51e-02
158	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6)$	1.32e-02	1.25e-02	1.44e-02	1.37e-02
159	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7)$	1.27e-02	1.19e-02	1.33e-02	1.30e-02
160	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8)$	1.13e-02	1.13e-02	1.24e-02	1.24e-02
161	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9)$	1.14e-02	1.14e-02	1.25e-02	1.25e-02
162	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
163	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
164	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
165	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13)$	1.12e-02	1.12e-02	1.23e-02	1.23e-02
166	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + k$	5.59e-02	1.91e-02	2.08e-02	1.97e-02
167	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + k$	2.40e-02	2.26e-02	2.50e-02	2.42e-02
168	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + k$	1.22e-02	1.21e-02	1.18e-02	1.17e-02
169	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + k$	1.05e-02	9.62e-03	1.08e-02	9.80e-03
170	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + k$	9.88e-03	8.80e-03	9.24e-03	8.87e-03
171	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + k$	8.03e-03	8.00e-03	8.02e-03	7.99e-03
172	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + k$	8.10e-03	8.05e-03	8.12e-03	8.05e-03
173	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + k$	7.87e-03	7.85e-03	7.85e-03	7.82e-03
174	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + k$	7.87e-03	7.85e-03	7.85e-03	7.82e-03
175	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + k$	7.85e-03	7.84e-03	7.83e-03	7.81e-03
176	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + k$	7.85e-03	7.84e-03	7.83e-03	7.81e-03
177	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * k$	5.48e-02	1.77e-02	1.96e-02	1.84e-02
178	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * k$	2.28e-02	2.13e-02	2.39e-02	2.31e-02
179	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * k$	9.94e-03	9.93e-03	9.77e-03	9.77e-03
180	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * k$	7.94e-03	6.83e-03	8.64e-03	7.37e-03

Table A12: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
181	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * k$	7.15e-03	5.48e-03	6.32e-03	5.92e-03
182	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * k$	4.52e-03	4.48e-03	4.94e-03	4.91e-03
183	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * k$	4.59e-03	4.54e-03	5.04e-03	4.98e-03
184	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * k$	4.23e-03	4.22e-03	4.65e-03	4.65e-03
185	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k$	4.24e-03	4.23e-03	4.66e-03	4.65e-03
186	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k$	4.21e-03	4.21e-03	4.63e-03	4.63e-03
187	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k$	4.20e-03	4.20e-03	4.63e-03	4.63e-03
188	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + \log(k)$	5.59e-02	1.88e-02	2.05e-02	1.94e-02
189	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + \log(k)$	2.38e-02	2.24e-02	2.47e-02	2.40e-02
190	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + \log(k)$	1.18e-02	1.17e-02	1.12e-02	1.12e-02
191	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + \log(k)$	1.00e-02	9.13e-03	1.02e-02	9.13e-03
192	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + \log(k)$	9.40e-03	8.27e-03	8.52e-03	8.12e-03
193	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + \log(k)$	7.44e-03	7.41e-03	7.19e-03	7.15e-03
194	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + \log(k)$	7.52e-03	7.46e-03	7.29e-03	7.22e-03
195	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + \log(k)$	7.26e-03	7.24e-03	6.99e-03	6.96e-03
196	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + \log(k)$	7.27e-03	7.24e-03	7.00e-03	6.97e-03
197	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + \log(k)$	7.25e-03	7.24e-03	6.97e-03	6.96e-03
198	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + \log(k)$	7.25e-03	7.24e-03	6.97e-03	6.96e-03
199	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * \log(k)$	5.47e-02	1.72e-02	1.91e-02	1.79e-02
200	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * \log(k)$	2.24e-02	2.09e-02	2.35e-02	2.26e-02
201	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * \log(k)$	9.06e-03	9.04e-03	8.66e-03	8.66e-03
202	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * \log(k)$	6.82e-03	5.48e-03	7.39e-03	5.84e-03
203	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * \log(k)$	6.04e-03	3.78e-03	4.62e-03	3.99e-03
204	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * \log(k)$	1.93e-03	1.84e-03	2.05e-03	1.96e-03
205	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * \log(k)$	2.11e-03	1.98e-03	2.29e-03	2.14e-03
206	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * \log(k)$	1.10e-03	1.08e-03	1.20e-03	1.18e-03

Table A12: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
207	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * \log(k)$	1.13e-03	1.09e-03	1.24e-03	1.19e-03
208	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * \log(k)$	1.02e-03	1.02e-03	1.12e-03	1.12e-03
209	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * \log(k)$	9.99e-04	9.96e-04	1.11e-03	1.10e-03
210	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + k_d$	5.58e-02	1.88e-02	2.05e-02	1.94e-02
211	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + k_d$	2.38e-02	2.24e-02	2.47e-02	2.39e-02
212	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + k_d$	1.18e-02	1.17e-02	1.12e-02	1.12e-02
213	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + k_d$	1.00e-02	9.13e-03	1.02e-02	9.12e-03
214	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + k_d$	9.39e-03	8.26e-03	8.51e-03	8.11e-03
215	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + k_d$	7.43e-03	7.40e-03	7.18e-03	7.14e-03
216	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + k_d$	7.51e-03	7.45e-03	7.28e-03	7.21e-03
217	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + k_d$	7.25e-03	7.24e-03	6.98e-03	6.95e-03
218	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + k_d$	7.26e-03	7.24e-03	6.99e-03	6.96e-03
219	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + k_d$	7.24e-03	7.23e-03	6.96e-03	6.95e-03
220	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + k_d$	7.24e-03	7.23e-03	6.96e-03	6.95e-03
221	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * k_d$	5.46e-02	1.72e-02	1.91e-02	1.79e-02
222	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * k_d$	2.24e-02	2.09e-02	2.35e-02	2.26e-02
223	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * k_d$	9.00e-03	8.98e-03	8.58e-03	8.58e-03
224	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * k_d$	6.78e-03	5.39e-03	7.38e-03	5.77e-03
225	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * k_d$	2.25e-03	2.10e-03	2.33e-03	2.18e-03
226	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * k_d$	1.88e-03	1.71e-03	2.05e-03	1.86e-03
227	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * k_d$	6.36e-04	6.18e-04	6.31e-04	6.07e-04
228	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * k_d$	5.67e-04	4.95e-04	6.15e-04	5.33e-04
229	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k_d$	2.10e-04	2.07e-04	2.19e-04	2.16e-04
230	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k_d$	2.11e-04	2.04e-04	2.25e-04	2.17e-04
231	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k_d$	1.57e-04	1.56e-04	1.64e-04	1.63e-04

Table A13: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1.

		Full Distribution		**Lower Tail** ($p \leq 0.2$)	
	Model	RMSE	cRMSE	RMSE	cRMSE
1	$p = c + \text{poly}\,(\text{bc}(t), 3)$	3.49e-02	2.31e-02	2.47e-02	2.35e-02
2	$p = c + \text{poly}\,(\text{bc}(t), 4)$	2.30e-02	2.22e-02	2.38e-02	2.32e-02
3	$p = c + \text{poly}\,(\text{bc}(t), 5)$	2.05e-02	2.00e-02	1.88e-02	1.86e-02
4	$p = c + \text{poly}\,(\text{bc}(t), 6)$	1.33e-02	1.28e-02	1.42e-02	1.38e-02
5	$p = c + \text{poly}\,(\text{bc}(t), 7)$	1.31e-02	1.26e-02	1.37e-02	1.35e-02
6	$p = c + \text{poly}\,(\text{bc}(t), 8)$	1.23e-02	1.20e-02	1.29e-02	1.29e-02
7	$p = c + \text{poly}\,(\text{bc}(t), 9)$	1.18e-02	1.17e-02	1.29e-02	1.28e-02
8	$p = c + \text{poly}\,(\text{bc}(t), 10)$	1.16e-02	1.16e-02	1.27e-02	1.26e-02
9	$p = c + \text{poly}\,(\text{bc}(t), 11)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
10	$p = c + \text{poly}\,(\text{bc}(t), 12)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
11	$p = c + \text{poly}\,(\text{bc}(t), 13)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
12	$p = c + \text{poly}\,(\text{bc}(t), 3) + k$	3.42e-02	2.18e-02	2.28e-02	2.14e-02
13	$p = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.18e-02	2.08e-02	2.17e-02	2.10e-02
14	$p = c + \text{poly}\,(\text{bc}(t), 5) + k$	1.92e-02	1.85e-02	1.60e-02	1.58e-02
15	$p = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.11e-02	1.05e-02	1.03e-02	9.70e-03
16	$p = c + \text{poly}\,(\text{bc}(t), 7) + k$	1.08e-02	1.02e-02	9.53e-03	9.23e-03
17	$p = c + \text{poly}\,(\text{bc}(t), 8) + k$	9.86e-03	9.53e-03	8.48e-03	8.45e-03
18	$p = c + \text{poly}\,(\text{bc}(t), 9) + k$	9.23e-03	9.15e-03	8.38e-03	8.30e-03
19	$p = c + \text{poly}\,(\text{bc}(t), 10) + k$	8.97e-03	8.93e-03	8.03e-03	8.01e-03
20	$p = c + \text{poly}\,(\text{bc}(t), 11) + k$	8.89e-03	8.85e-03	8.01e-03	7.98e-03
21	$p = c + \text{poly}\,(\text{bc}(t), 12) + k$	8.88e-03	8.84e-03	7.99e-03	7.97e-03
22	$p = c + \text{poly}\,(\text{bc}(t), 13) + k$	8.85e-03	8.83e-03	7.97e-03	7.96e-03
23	$p = c + \text{poly}\,(\text{bc}(t), 3) * k$	3.33e-02	2.04e-02	2.18e-02	2.03e-02
24	$p = c + \text{poly}\,(\text{bc}(t), 4) * k$	2.01e-02	1.91e-02	2.05e-02	1.98e-02

Table A13: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
25	$p = c + \text{poly}\,(\text{bc}(t), 5) * k$	1.72e-02	1.67e-02	1.45e-02	1.42e-02
26	$p = c + \text{poly}\,(\text{bc}(t), 6) * k$	7.74e-03	6.98e-03	7.95e-03	7.16e-03
27	$p = c + \text{poly}\,(\text{bc}(t), 7) * k$	7.29e-03	6.46e-03	6.77e-03	6.41e-03
28	$p = c + \text{poly}\,(\text{bc}(t), 8) * k$	5.91e-03	5.40e-03	5.44e-03	5.39e-03
29	$p = c + \text{poly}\,(\text{bc}(t), 9) * k$	4.96e-03	4.85e-03	5.32e-03	5.24e-03
30	$p = c + \text{poly}\,(\text{bc}(t), 10) * k$	4.51e-03	4.47e-03	4.81e-03	4.80e-03
31	$p = c + \text{poly}\,(\text{bc}(t), 11) * k$	4.36e-03	4.34e-03	4.78e-03	4.76e-03
32	$p = c + \text{poly}\,(\text{bc}(t), 12) * k$	4.31e-03	4.30e-03	4.72e-03	4.72e-03
33	$p = c + \text{poly}\,(\text{bc}(t), 13) * k$	4.29e-03	4.29e-03	4.72e-03	4.72e-03
34	$p = c + \text{poly}\,(\text{bc}(t), 3) + \log(k)$	3.41e-02	2.16e-02	2.25e-02	2.11e-02
35	$p = c + \text{poly}\,(\text{bc}(t), 4) + \log(k)$	2.17e-02	2.06e-02	2.14e-02	2.07e-02
36	$p = c + \text{poly}\,(\text{bc}(t), 5) + \log(k)$	1.90e-02	1.83e-02	1.56e-02	1.53e-02
37	$p = c + \text{poly}\,(\text{bc}(t), 6) + \log(k)$	1.07e-02	1.01e-02	9.66e-03	8.99e-03
38	$p = c + \text{poly}\,(\text{bc}(t), 7) + \log(k)$	1.05e-02	9.85e-03	8.81e-03	8.47e-03
39	$p = c + \text{poly}\,(\text{bc}(t), 8) + \log(k)$	9.48e-03	9.13e-03	7.66e-03	7.62e-03
40	$p = c + \text{poly}\,(\text{bc}(t), 9) + \log(k)$	8.82e-03	8.73e-03	7.54e-03	7.46e-03
41	$p = c + \text{poly}\,(\text{bc}(t), 10) + \log(k)$	8.55e-03	8.50e-03	7.16e-03	7.13e-03
42	$p = c + \text{poly}\,(\text{bc}(t), 11) + \log(k)$	8.46e-03	8.41e-03	7.13e-03	7.10e-03
43	$p = c + \text{poly}\,(\text{bc}(t), 12) + \log(k)$	8.45e-03	8.40e-03	7.11e-03	7.09e-03
44	$p = c + \text{poly}\,(\text{bc}(t), 13) + \log(k)$	8.42e-03	8.38e-03	7.09e-03	7.08e-03
45	$p = c + \text{poly}\,(\text{bc}(t), 3) * \log(k)$	3.30e-02	2.00e-02	2.13e-02	1.98e-02
46	$p = c + \text{poly}\,(\text{bc}(t), 4) * \log(k)$	1.96e-02	1.86e-02	1.99e-02	1.92e-02
47	$p = c + \text{poly}\,(\text{bc}(t), 5) * \log(k)$	1.67e-02	1.61e-02	1.37e-02	1.35e-02
48	$p = c + \text{poly}\,(\text{bc}(t), 6) * \log(k)$	6.51e-03	5.59e-03	6.50e-03	5.50e-03
49	$p = c + \text{poly}\,(\text{bc}(t), 7) * \log(k)$	5.97e-03	4.94e-03	4.99e-03	4.49e-03
50	$p = c + \text{poly}\,(\text{bc}(t), 8) * \log(k)$	4.15e-03	3.39e-03	2.89e-03	2.82e-03

Table A13: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
51	$p = c + \text{poly}\,(\text{bc}(t), 9) * \log(k)$	2.71e-03	2.51e-03	2.73e-03	2.56e-03
52	$p = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	1.75e-03	1.64e-03	1.49e-03	1.47e-03
53	$p = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	1.31e-03	1.26e-03	1.39e-03	1.33e-03
54	$p = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	1.18e-03	1.14e-03	1.21e-03	1.21e-03
55	$p = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	1.08e-03	1.07e-03	1.17e-03	1.17e-03
56	$p = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	3.41e-02	2.16e-02	2.24e-02	2.10e-02
57	$p = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	2.17e-02	2.06e-02	2.14e-02	2.07e-02
58	$p = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	1.90e-02	1.83e-02	1.56e-02	1.53e-02
59	$p = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	1.07e-02	1.01e-02	9.65e-03	8.98e-03
60	$p = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	1.05e-02	9.85e-03	8.80e-03	8.47e-03
61	$p = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	9.47e-03	9.13e-03	7.65e-03	7.61e-03
62	$p = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	8.81e-03	8.72e-03	7.53e-03	7.45e-03
63	$p = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	8.54e-03	8.49e-03	7.15e-03	7.12e-03
64	$p = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	8.46e-03	8.41e-03	7.12e-03	7.09e-03
65	$p = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	8.45e-03	8.40e-03	7.10e-03	7.08e-03
66	$p = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	8.42e-03	8.38e-03	7.08e-03	7.07e-03
67	$p = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	3.30e-02	2.00e-02	2.13e-02	1.98e-02
68	$p = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	1.96e-02	1.86e-02	1.99e-02	1.92e-02
69	$p = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	1.67e-02	1.61e-02	1.37e-02	1.34e-02
70	$p = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	6.36e-03	5.45e-03	6.41e-03	5.37e-03
71	$p = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	5.47e-03	4.60e-03	4.39e-03	4.00e-03
72	$p = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	2.05e-03	1.88e-03	2.03e-03	1.87e-03
73	$p = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	1.76e-03	1.61e-03	1.38e-03	1.30e-03
74	$p = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	6.46e-04	5.71e-04	6.40e-04	5.59e-04
75	$p = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	5.26e-04	4.66e-04	4.06e-04	3.82e-04
76	$p = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.30e-04	2.19e-04	2.28e-04	2.17e-04

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	Model	RMSE	cRMSE	RMSE	cRMSE
77	$p = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	1.88e-04	1.81e-04	1.70e-04	1.68e-04
78	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3)$	3.58e-02	1.93e-02	3.98e-02	2.12e-02
79	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4)$	2.35e-02	2.34e-02	2.61e-02	2.60e-02
80	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5)$	1.49e-02	1.28e-02	1.65e-02	1.41e-02
81	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
82	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7)$	1.16e-02	1.16e-02	1.28e-02	1.28e-02
83	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
84	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
85	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
86	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
87	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
88	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
89	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k$	3.63e-02	2.03e-02	4.04e-02	2.24e-02
90	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.41e-02	2.40e-02	2.68e-02	2.67e-02
91	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k$	1.58e-02	1.38e-02	1.75e-02	1.53e-02
92	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.26e-02	1.26e-02	1.40e-02	1.40e-02
93	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k$	1.28e-02	1.27e-02	1.41e-02	1.41e-02
94	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k$	1.26e-02	1.26e-02	1.39e-02	1.39e-02
95	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k$	1.26e-02	1.26e-02	1.39e-02	1.39e-02
96	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k$	1.26e-02	1.26e-02	1.39e-02	1.39e-02
97	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k$	1.26e-02	1.26e-02	1.39e-02	1.39e-02
98	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k$	1.26e-02	1.26e-02	1.39e-02	1.39e-02
99	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k$	1.26e-02	1.26e-02	1.39e-02	1.39e-02
100	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k$	3.56e-02	1.66e-02	3.97e-02	1.83e-02
101	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k$	1.94e-02	1.93e-02	2.16e-02	2.15e-02
102	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * k$	8.70e-03	6.31e-03	9.69e-03	7.00e-03

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	Model	RMSE	cRMSE	RMSE	cRMSE
103	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * k$	4.43e-03	4.40e-03	4.88e-03	4.86e-03
104	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * k$	4.51e-03	4.49e-03	4.98e-03	4.96e-03
105	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * k$	4.39e-03	4.38e-03	4.85e-03	4.83e-03
106	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * k$	4.36e-03	4.36e-03	4.81e-03	4.81e-03
107	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * k$	4.37e-03	4.36e-03	4.82e-03	4.81e-03
108	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * k$	4.37e-03	4.37e-03	4.82e-03	4.82e-03
109	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * k$	4.37e-03	4.36e-03	4.82e-03	4.81e-03
110	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k$	4.37e-03	4.36e-03	4.82e-03	4.81e-03
111	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + \log(k)$	3.64e-02	2.04e-02	4.05e-02	2.26e-02
112	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + \log(k)$	2.42e-02	2.41e-02	2.69e-02	2.68e-02
113	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + \log(k)$	1.59e-02	1.40e-02	1.77e-02	1.55e-02
114	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + \log(k)$	1.29e-02	1.29e-02	1.42e-02	1.42e-02
115	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + \log(k)$	1.30e-02	1.30e-02	1.43e-02	1.43e-02
116	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + \log(k)$	1.28e-02	1.28e-02	1.42e-02	1.42e-02
117	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + \log(k)$	1.28e-02	1.28e-02	1.42e-02	1.42e-02
118	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + \log(k)$	1.28e-02	1.28e-02	1.42e-02	1.42e-02
119	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + \log(k)$	1.28e-02	1.28e-02	1.42e-02	1.42e-02
120	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + \log(k)$	1.28e-02	1.28e-02	1.42e-02	1.42e-02
121	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + \log(k)$	1.28e-02	1.28e-02	1.42e-02	1.42e-02
122	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * \log(k)$	3.54e-02	1.61e-02	3.95e-02	1.78e-02
123	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * \log(k)$	1.88e-02	1.87e-02	2.09e-02	2.08e-02
124	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * \log(k)$	7.78e-03	4.76e-03	8.69e-03	5.31e-03
125	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * \log(k)$	1.28e-03	1.23e-03	1.42e-03	1.37e-03
126	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * \log(k)$	1.57e-03	1.51e-03	1.74e-03	1.68e-03
127	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * \log(k)$	1.26e-03	1.20e-03	1.40e-03	1.33e-03
128	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * \log(k)$	1.13e-03	1.12e-03	1.25e-03	1.24e-03

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	Model	RMSE	cRMSE	RMSE	cRMSE
129	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * \log(k)$	1.13e-03	1.12e-03	1.25e-03	1.24e-03
130	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * \log(k)$	1.14e-03	1.14e-03	1.26e-03	1.26e-03
131	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * \log(k)$	1.13e-03	1.13e-03	1.25e-03	1.25e-03
132	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * \log(k)$	1.13e-03	1.13e-03	1.25e-03	1.25e-03
133	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) + k_d$	3.64e-02	2.04e-02	4.05e-02	2.26e-02
134	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) + k_d$	2.42e-02	2.41e-02	2.69e-02	2.68e-02
135	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) + k_d$	1.59e-02	1.40e-02	1.77e-02	1.55e-02
136	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) + k_d$	1.29e-02	1.29e-02	1.42e-02	1.42e-02
137	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) + k_d$	1.30e-02	1.30e-02	1.44e-02	1.43e-02
138	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) + k_d$	1.29e-02	1.29e-02	1.42e-02	1.42e-02
139	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) + k_d$	1.29e-02	1.28e-02	1.42e-02	1.42e-02
140	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) + k_d$	1.28e-02	1.28e-02	1.42e-02	1.42e-02
141	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) + k_d$	1.29e-02	1.28e-02	1.42e-02	1.42e-02
142	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) + k_d$	1.28e-02	1.28e-02	1.42e-02	1.42e-02
143	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) + k_d$	1.29e-02	1.28e-02	1.42e-02	1.42e-02
144	$\log(p) = c + \text{poly}\,(\text{bc}(t), 3) * k_d$	3.55e-02	1.61e-02	3.96e-02	1.78e-02
145	$\log(p) = c + \text{poly}\,(\text{bc}(t), 4) * k_d$	1.87e-02	1.86e-02	2.09e-02	2.08e-02
146	$\log(p) = c + \text{poly}\,(\text{bc}(t), 5) * k_d$	7.62e-03	4.58e-03	8.52e-03	5.12e-03
147	$\log(p) = c + \text{poly}\,(\text{bc}(t), 6) * k_d$	7.63e-04	6.91e-04	8.48e-04	7.67e-04
148	$\log(p) = c + \text{poly}\,(\text{bc}(t), 7) * k_d$	5.90e-04	5.51e-04	6.56e-04	6.13e-04
149	$\log(p) = c + \text{poly}\,(\text{bc}(t), 8) * k_d$	3.17e-04	2.91e-04	3.49e-04	3.20e-04
150	$\log(p) = c + \text{poly}\,(\text{bc}(t), 9) * k_d$	3.64e-04	3.17e-04	4.02e-04	3.49e-04
151	$\log(p) = c + \text{poly}\,(\text{bc}(t), 10) * k_d$	3.64e-04	3.35e-04	4.03e-04	3.70e-04
152	$\log(p) = c + \text{poly}\,(\text{bc}(t), 11) * k_d$	2.85e-04	2.53e-04	3.13e-04	2.77e-04
153	$\log(p) = c + \text{poly}\,(\text{bc}(t), 12) * k_d$	2.23e-04	2.10e-04	2.43e-04	2.27e-04
154	$\log(p) = c + \text{poly}\,(\text{bc}(t), 13) * k_d$	2.33e-04	2.17e-04	2.54e-04	2.36e-04

Table A13: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
155	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3)$	5.70e-02	2.09e-02	2.31e-02	2.22e-02
156	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4)$	2.55e-02	2.42e-02	2.70e-02	2.63e-02
157	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5)$	1.48e-02	1.48e-02	1.54e-02	1.54e-02
158	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 6)$	1.35e-02	1.28e-02	1.47e-02	1.40e-02
159	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 7)$	1.24e-02	1.19e-02	1.31e-02	1.30e-02
160	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 8)$	1.16e-02	1.16e-02	1.27e-02	1.27e-02
161	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 9)$	1.17e-02	1.16e-02	1.28e-02	1.28e-02
162	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 10)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
163	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 11)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
164	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
165	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13)$	1.15e-02	1.15e-02	1.26e-02	1.26e-02
166	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3) + k$	5.66e-02	1.93e-02	2.10e-02	1.99e-02
167	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4) + k$	2.42e-02	2.28e-02	2.52e-02	2.45e-02
168	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5) + k$	1.23e-02	1.23e-02	1.20e-02	1.19e-02
169	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 6) + k$	1.07e-02	9.82e-03	1.11e-02	1.00e-02
170	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 7) + k$	9.26e-03	8.61e-03	8.78e-03	8.63e-03
171	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 8) + k$	8.24e-03	8.21e-03	8.24e-03	8.20e-03
172	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 9) + k$	8.31e-03	8.26e-03	8.33e-03	8.26e-03
173	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 10) + k$	8.08e-03	8.06e-03	8.06e-03	8.04e-03
174	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 11) + k$	8.08e-03	8.06e-03	8.07e-03	8.04e-03
175	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 12) + k$	8.07e-03	8.05e-03	8.05e-03	8.03e-03
176	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 13) + k$	8.07e-03	8.05e-03	8.05e-03	8.03e-03
177	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 3) * k$	5.56e-02	1.77e-02	1.97e-02	1.85e-02
178	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 4) * k$	2.28e-02	2.13e-02	2.40e-02	2.31e-02
179	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 5) * k$	1.00e-02	1.00e-02	9.87e-03	9.87e-03
180	$\text{bc}(p) = c + \text{poly}\,(\text{bc}(t), 6) * k$	8.01e-03	6.86e-03	8.75e-03	7.43e-03

Table A13: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
181	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * k$	5.72e-03	5.06e-03	5.62e-03	5.47e-03
182	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * k$	4.61e-03	4.57e-03	5.04e-03	5.00e-03
183	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * k$	4.68e-03	4.63e-03	5.14e-03	5.08e-03
184	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * k$	4.33e-03	4.33e-03	4.76e-03	4.76e-03
185	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k$	4.34e-03	4.33e-03	4.77e-03	4.76e-03
186	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k$	4.31e-03	4.31e-03	4.74e-03	4.74e-03
187	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k$	4.31e-03	4.31e-03	4.74e-03	4.74e-03
188	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + \log(k)$	5.65e-02	1.90e-02	2.07e-02	1.96e-02
189	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + \log(k)$	2.40e-02	2.26e-02	2.49e-02	2.42e-02
190	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + \log(k)$	1.19e-02	1.19e-02	1.14e-02	1.14e-02
191	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + \log(k)$	1.03e-02	9.33e-03	1.05e-02	9.34e-03
192	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + \log(k)$	8.74e-03	8.05e-03	7.99e-03	7.83e-03
193	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + \log(k)$	7.65e-03	7.62e-03	7.40e-03	7.36e-03
194	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + \log(k)$	7.72e-03	7.66e-03	7.50e-03	7.42e-03
195	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + \log(k)$	7.47e-03	7.45e-03	7.20e-03	7.18e-03
196	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + \log(k)$	7.48e-03	7.46e-03	7.21e-03	7.18e-03
197	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + \log(k)$	7.46e-03	7.45e-03	7.19e-03	7.17e-03
198	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + \log(k)$	7.46e-03	7.45e-03	7.19e-03	7.17e-03
199	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * \log(k)$	5.55e-02	1.72e-02	1.91e-02	1.79e-02
200	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * \log(k)$	2.23e-02	2.09e-02	2.35e-02	2.26e-02
201	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * \log(k)$	9.10e-03	9.08e-03	8.72e-03	8.72e-03
202	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * \log(k)$	6.83e-03	5.43e-03	7.43e-03	5.82e-03
203	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * \log(k)$	3.91e-03	2.83e-03	3.21e-03	2.94e-03
204	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * \log(k)$	1.95e-03	1.87e-03	2.07e-03	1.98e-03
205	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * \log(k)$	2.13e-03	2.00e-03	2.31e-03	2.16e-03
206	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * \log(k)$	1.16e-03	1.15e-03	1.27e-03	1.25e-03

Table A13: Performance of the models for the second case with Engle-Granger and Johansen as underlying tests. The RMSE and cRMSE were calculated over the whole distribution and over the lower tail of the distribution. The cRMSE reflects the RMSE after correcting for values ranging between 0 and 1. *(continued)*

	Model	RMSE	cRMSE	RMSE	cRMSE
207	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * \log(k)$	1.18e-03	1.15e-03	1.29e-03	1.27e-03
208	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * \log(k)$	1.08e-03	1.08e-03	1.19e-03	1.19e-03
209	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * \log(k)$	1.08e-03	1.08e-03	1.19e-03	1.19e-03
210	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) + k_d$	5.65e-02	1.90e-02	2.07e-02	1.96e-02
211	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) + k_d$	2.40e-02	2.26e-02	2.49e-02	2.42e-02
212	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) + k_d$	1.19e-02	1.19e-02	1.14e-02	1.14e-02
213	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) + k_d$	1.03e-02	9.32e-03	1.05e-02	9.33e-03
214	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) + k_d$	8.73e-03	8.04e-03	7.98e-03	7.82e-03
215	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) + k_d$	7.64e-03	7.61e-03	7.39e-03	7.35e-03
216	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) + k_d$	7.71e-03	7.65e-03	7.49e-03	7.41e-03
217	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) + k_d$	7.46e-03	7.45e-03	7.19e-03	7.17e-03
218	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) + k_d$	7.47e-03	7.45e-03	7.20e-03	7.17e-03
219	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) + k_d$	7.45e-03	7.44e-03	7.18e-03	7.16e-03
220	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) + k_d$	7.45e-03	7.44e-03	7.18e-03	7.17e-03
221	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 3) * k_d$	5.54e-02	1.71e-02	1.91e-02	1.78e-02
222	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 4) * k_d$	2.23e-02	2.09e-02	2.34e-02	2.26e-02
223	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 5) * k_d$	9.03e-03	9.02e-03	8.65e-03	8.65e-03
224	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 6) * k_d$	6.75e-03	5.33e-03	7.35e-03	5.70e-03
225	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 7) * k_d$	2.23e-03	2.09e-03	2.31e-03	2.16e-03
226	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 8) * k_d$	1.85e-03	1.68e-03	2.01e-03	1.82e-03
227	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 9) * k_d$	5.97e-04	5.78e-04	5.90e-04	5.66e-04
228	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 10) * k_d$	5.27e-04	4.60e-04	5.69e-04	4.91e-04
229	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 11) * k_d$	1.85e-04	1.83e-04	1.90e-04	1.88e-04
230	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 12) * k_d$	1.84e-04	1.78e-04	1.93e-04	1.86e-04
231	$\mathrm{bc}(p) = c + \mathrm{poly}\,(\mathrm{bc}(t), 13) * k_d$	1.45e-04	1.45e-04	1.53e-04	1.52e-04

Figure A1: Simulated against approximated p-values over the whole distribution for all cases and all underlying tests.

Figure A2: Simulated vs. approximated p-values for the lower tail of the distribution for all cases and all underlying test.

Figure A3: Simulated against approximated p-values over the whole distribution for all cases and EJ test type.

Figure A4: Simulated vs. approximated p-values for the lower tail of the distribution for all cases and EJ test type.

Figure A5: Corrected (blue) and uncorrected (red) p-value predictions for all cases and all underlying tests.

Figure A6: Corrected (blue) and uncorrected (red) p-value predictions for all cases using Engle-Granger and Johansen as underlying tests.

Eidesstattliche Versicherung

Ich versichere an Eides statt durch meine Unterschrift, dass ich die vorstehende Arbeit selbständig und ohne fremde Hilfe angefertigt und alle Stellen, die ich wörtlich oder annähernd wörtlich aus Veröffentlichungen entnommen habe, als solche kenntlich gemacht habe, mich auch keiner anderen als der angegebenen Literatur oder sonstiger Hilfsmittel bedient habe. Die Arbeit hat in dieser oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen.

Essen, den _____ _____

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