Series Representations of L??vy Processes from the Perspective of Point Processes

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Series representations of Lévy processes from the perspective of point processes

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ABSTRACT Several methods of generating series representations of a Lévy process are presented under a unified approach and a new rejection method is introduced in this context. The connection of such representations with the Lévy–Itô integral representation is precisely established. Four series representations of a gamma process are given as illustrations of these methods.

1 From Lévy–Itô to series representations. Introduction.

Let $\{X(t) : t \in [0,1]\}$ be a Lévy process in $\mathbb{R}^d$ with the characteristic function given by

$$E\exp(iuX(t)) = \exp t[iua + \int_{\mathbb{R}_0^d} (e^{iux} - 1 - iuxI(|x| \le 1))\, Q(dx)] \tag{1.1}$$

where $a \in \mathbb{R}^d$ and Q is a Lévy measure on $\mathbb{R}_0^d$ ($\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$). Assume that the paths of X are right–continuous and have left–hand limits (abbreviated as rcll). By the Lévy–Itô integral representation, a.s. for each $t \ge 0$,

$$X(t) = ta + \int_{|x|\le 1} x\,[(N([0,t],dx) - tQ(dx)] + \int_{|x|>1} x\, N([0,t],dx) \tag{1.2}$$

where N is the point process of jumps of X: $N = \sum_{\{t:\ \Delta X(t)\neq 0\}} \delta_{(t,\Delta X(t))}$, (see, e.g., [Ka2], Th. 13.4). N is a Poisson point process with the mean measure $Leb \times Q$ on $[0,1] \times \mathbb{R}_0^d$.

There are many ways to represent the process of jumps N in the form

$$N = \sum_{i=1}^{\infty} \delta_{(U_i, J_i)} \tag{1.3}$$

where $\{J_i\}$ is a sequence of random variables in $\mathbb{R}^d$ independent of the sequence $\{U_i\}$ of i.i.d. uniform on $[0,1]$ random variables. We will discuss several types of such representations in Section 3. Put

$$X_n(t) = ta + \int_{n^{-1}\le|x|\le 1} x\,[N([0,t],dx) - tQ(dx)] + \int_{|x|>1} x\, N([0,t],dx). \tag{1.4}$$

Since by (1.3)

$$X_n(t) = \sum_{i\in\Lambda_n} J_i 1(U_i \le t) - tb_n,$$

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where $\Lambda_n = \Lambda_n(\omega) = \{i \geq 1 : |J_i(\omega)| \geq n^{-1}\}$ and $b_n = \int_{n^{-1} \leq |x| \leq 1} x\, Q(dx) - a$, we get

$$\sum_{i \in \Lambda_n} J_i 1(U_i \leq t) - t b_n \to X(t) \quad a.s. \tag{1.5}$$

as $n \to \infty$. Shortly speaking, a series representation of X is obtained by replacing the random set Λ_n in (1.5) by a nonrandom one, $\{1, \ldots, n\}$, a procedure that yields

$$X(t) = \sum_{i=1}^{\infty} (J_i 1(U_i \leq t) - t c_i) \quad a.s. \tag{1.6}$$

for a suitable sequence of centers $\{c_i\}$ (see Section 5 where this theme is concluded). It turns out that various properties of the series (1.6), such as its speed of convergence or the need to center with certain c_i's, depend on the specific representation of the J_i's. Such series representations can be used for a study or simulation of linear and nonlinear functionals of Lévy processes. Therefore, one may want to look for a representation of the J_i's that has a probabilistically simple structure or/and is easy to simulate on a computer. In Section 6 we give examples of different representations of a gamma process that illustrate these points.

2 Almost sure representations of point processes.

The method that we are going to describe is much more general but we will limit our discussion to Poisson point processes.

Proposition 2.1 *Let M and N be Poisson point processes on Borel spaces S and T_0, with control measures μ and ν, respectively, and defined on possibly different probability spaces. Suppose that T_0 is a Borel subset of some Borel space T and that for some measurable mapping $h : S \to T$,*

$$\nu = \mu \circ h^{-1} \qquad \text{on} \quad \mathcal{B}(T_0). \tag{2.1}$$

Then

$$N \stackrel{d}{=} M \circ h^{-1}. \tag{2.2}$$

Assume in addition that N is defined on a probability space which is rich enough, that is, there exists on this probability space a uniform $\mathcal{U}[0,1]$ random variable that is independent of N, and also that

$$M = \sum_{i=1}^{\infty} \delta_{s_i} \tag{2.3}$$

for some S–valued random elements s_i, $i \geq 1$. Then there exists a sequence $\{\tilde{s}_i\}_{i \geq 1}$ of S–valued random elements defined on the same probability space as N such that

$$\{\tilde{s}_i\}_{i \geq 1} \stackrel{d}{=} \{s_i\}_{i \geq 1} \tag{2.4}$$

and

$$N = \sum_{i=1}^{\infty} \delta_{h(\tilde{s}_i)} \quad a.s. \tag{2.5}$$

The proof of the second part of Proposition 2.1 is an application of the following result from [Ka2], Corollary 5.11 (see also Theorem 5.10).

Proposition 2.2 *Let ξ and η be random elements, defined on possibly different probability spaces, and taking values in Borel spaces S and T, respectively. Suppose that $\eta \stackrel{d}{=} f(\xi)$ for some measurable mapping $f : S \to T$, and that η is defined on a probability space rich enough to support an independent of η uniform $\mathcal{U}[0,1]$ random variable. Then there exists a random element $\tilde{\xi}$ in S, defined on the same probability space as η, such that $\tilde{\xi} \stackrel{d}{=} \xi$ and $\eta = f(\tilde{\xi})$ a.s.*

Proof of Proposition 2.1: The first part of this proposition is obvious; we will only prove (2.4)–(2.5). Let $\mathcal{A} = \{A_1, A_2, \dots\}$ be a countable field generating $\mathcal{B}(T_0)$. Define a random element η in $(\mathbb{Z}_+ \cup \{\infty\})^\infty$ by

$$\eta = (N(A_1), N(A_2), \dots).$$

Define also a random element ξ in S^∞ by

$$\xi = (s_1, s_2, \dots).$$

In view of (2.2), $M \circ h^{-1} = \sum_{i=1}^\infty \delta_{h(s_i)}$ has the same distribution as N. Therefore,

$$\eta \stackrel{d}{=} f(\xi) := (M \circ h^{-1}(A_1), M \circ h^{-1}(A_2), \dots).$$

By Proposition 2.2, there exists a random element $\tilde{\xi} = (\tilde{s}_1, \tilde{s}_2, \dots)$, defined on the same probability space as N and taking values in S^∞, such that $\tilde{\xi} \stackrel{d}{=} \xi$ and

$$N(A_k) = \sum_{i=1}^\infty \delta_{h(\tilde{s}_i)}(A_k) \quad a.s., \ k = 1, 2, \dots$$

Since the measures $N(\cdot, \omega)$ and $\sum_{i=1}^\infty \delta_{h(\tilde{s}_i(\omega))}(\cdot)$ are equal on the field $\mathcal{A}$ for each ω from a set of probability 1, they are equal a.s. on $\mathcal{B}(T_0)$.

In essence, Proposition 2.1 allows one to replace the equality $N \stackrel{d}{=} M \circ h^{-1}$ by the equality $N = M \circ h^{-1}$ a.s., provided the probability base $(\Omega, \mathcal{F}, P)$ of N is rich enough. The latter condition does not represent any serious restriction from a probabilistic point of view. Indeed, one can always enlarge the original probability space to $(\Omega \times [0,1], \mathcal{F} \times \mathcal{B}([0,1]), P \times Leb)$ to accommodate an independent of N uniformly distributed random variable. Therefore, from now on, we will assume that N is defined on a rich enough probability space.

3 Several methods of representing the Poisson process of jumps.

Throughout this paper $\{\Gamma_i\}_{i \geq 1}$ will denote a sequence of partial sums of i.i.d. standard exponential random variables and $\{U_i\}_{i \geq 1}$ will stand for an i.i.d. sequence of $\mathcal{U}[0,1]$ random variables that is independent of $\{\Gamma_i\}_{i \geq 1}$. Let $\{X(t) : t \in [0,1]\}$ be a Lévy process given by (1.1) and let N be its Poisson process of jumps on $[0,1] \times \mathbb{R}_0^d$. We will give several representations of N in the form (1.3) with J_i's defined by different methods.

(A). The Inverse Lévy Measure Method (Khintchine [Khi], Ferguson and Klass [F-K]).

Let $d = 1$ and suppose that Q is concentrated on $(0, \infty)$. Define the inverse of the tail of Q by

$$Q^{\leftarrow}(u) = \inf\{x > 0 : Q([x, \infty)) < u\}, \qquad u > 0. \tag{3.1}$$

Then

$$N \stackrel{d}{=} \sum_{i=1}^{\infty} \delta_{(U_i, Q^{\leftarrow}(\Gamma_i))}. \tag{3.2}$$

Indeed, $M := \sum \delta_{(U_i,\Gamma_i)}$ is a marked Poisson point process on $S = [0,1] \times (0,\infty)$ with mean measure $\mu = Leb \times Leb$. Let $h : S \to T$ be given by $h(u, \gamma) := (u, Q^{\leftarrow}(\gamma))$. Here $T = [0,1] \times [0,\infty)$ and $T_0 = [0,1] \times (0,\infty)$. Since $\mu \circ h^{-1} = Leb \times Q$ on $\mathcal{B}(T_0)$, $M \circ h^{-1} \stackrel{d}{=} N$ by (2.2). From (2.4)–(2.5) we infer that the sequences $\{U_i\}$, $\{\Gamma_i\}$ can be defined on the same probability space as N and

$$N = \sum_{i=1}^{\infty} \delta_{(U_i, Q^{\leftarrow}(\Gamma_i))} \quad a.s. \tag{3.3}$$

(B) LePage's Method (LePage [LP2]).

The representation is based on a radial decomposition of Q of the form

$$Q(A) = \int_{S^{d-1}} \int_0^{\infty} I_A(xv)\, \rho(dx, v)\sigma(dv), \quad A \in \mathcal{B}(\mathbb{R}_0^d) \tag{3.4}$$

where σ is a probability measure on the unit sphere S^{d-1} of $\mathbb{R}^d$ and $\{\rho(\cdot, v)\}_{v \in S^{d-1}}$ is a measurable family of Lévy measures on $(0,\infty)$. Put

$$\rho^{\leftarrow}(u, v) := \inf\{x > 0 : \rho([x,\infty), v) < u\}. \tag{3.5}$$

Let $\{V_i\}$ be an i.i.d. sequence in S^{d-1} with the common distribution σ such that $\{V_i\}$ is independent of $\{\Gamma_i, U_i\}$. Then

$$N \stackrel{d}{=} \sum_{i=1}^{\infty} \delta_{(U_i, \rho^{\leftarrow}(\Gamma_i, V_i)V_i)}. \tag{3.6}$$

Sometimes a simplification in (3.5) can be achieved by taking a radial decomposition (3.4) with respect to a Borel subset D of $\mathbb{R}^d$ in place of S^{d-1}. For example, to represent semistable random vectors we take $D = \{r < |x| \le 1\}$, where $r \in (0,1)$ is a parameter (see [Ro3], Example 4.11).

To justify (3.6), consider a marked Poisson point process $M := \sum \delta_{(U_i,\Gamma_i,V_i)}$ on $S = [0,1] \times (0,\infty) \times D$ ($D = S^{d-1}$, or arbitrary) with mean measure $\mu = Leb \times Leb \times \sigma$. Let $h : S \to T$ be given by $h(u, \gamma, v) = (u, \rho^{\leftarrow}(\gamma, v)v)$. Here $T = [0,1] \times \mathbb{R}^d$ and $T_0 = [0,1] \times \mathbb{R}_0^d$. Since $\mu \circ h^{-1} = Leb \times Q$ on $\mathcal{B}(T_0)$, $M \circ h^{-1} \stackrel{d}{=} N$ by (2.2). From (2.4)–(2.5) we infer that the sequences $\{U_i\}$, $\{\Gamma_i\}$, $\{V_i\}$ can be defined on the same probability space as N and

$$N = \sum_{i=1}^{\infty} \delta_{(U_i, \rho^{\leftarrow}(\Gamma_i, V_i)V_i)} \quad a.s. \tag{3.7}$$

(C) Bondesson's Method (Bondesson [Bon]).

Suppose that Q can be represented as

$$Q(A) = \int_0^\infty F(A/g(t))\,dt, \qquad A \in \mathcal{B}(\mathbb{R}_0^d) \tag{3.8}$$

where F is a probability distribution on $\mathbb{R}^d$ and $g : [0,\infty) \to [0,\infty)$ is a nonincreasing function. Let $\{V_i\}$ be an i.i.d. sequence with common distribution F independent of $\{\Gamma_i, U_i\}$. Then

$$N \stackrel{d}{=} \sum_{i=1}^\infty \delta_{(U_i, g(\Gamma_i)V_i)}. \tag{3.9}$$

Indeed, $M := \sum \delta_{(U_i,\Gamma_i,V_i)}$ is a marked Poisson point process on $S = [0,1] \times (0,\infty) \times \mathbb{R}^d$ with mean measure $\mu = Leb \times Leb \times F$. Let $h : S \to T$ be given by $h(u,\gamma,v) := (u, g(\gamma)v)$. Here $T = [0,1] \times \mathbb{R}^d$ and $T_0 = [0,1] \times \mathbb{R}_0^d$. Since $\mu \circ h^{-1} = Leb \times Q$ on $\mathcal{B}(T_0)$, $M \circ h^{-1} \stackrel{d}{=} N$ by (2.2). From (2.4)–(2.5) we infer that the sequences $\{U_i\}$, $\{\Gamma_i\}$, $\{V_i\}$ can be defined on the same probability space as N and

$$N = \sum_{i=1}^\infty \delta_{(U_i, g(\Gamma_i)V_i)} \quad a.s. \tag{3.10}$$

(D) Thinning Method (Rosiński [Ro3]).

Choose any probability distribution F on $\mathbb{R}^d$ such that Q is absolutely continuous with respect to F, otherwise F is arbitrary. Let $\{V_i\}_{i\geq 1}$ be an i.i.d. sequence with common distribution F. Assume that $\{U_i\}_{i\geq 1}$, $\{V_i\}_{i\geq 1}$ and $\{\Gamma_i\}_{i\geq 1}$ are independent of each other. Set

$$J_i = \begin{cases} V_i & \text{if } \frac{dQ}{dF}(V_i) \geq \Gamma_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

Then

$$N \stackrel{d}{=} \sum_{i=1}^\infty \delta_{(U_i, J_i)}. \tag{3.12}$$

Thus the jumps J_i of X are obtained by a random thinning of an almost arbitrary i.i.d. sequence. To justify (3.12) we check (2.1) for $h(u,\gamma,v) = (u, H(\gamma,v))$, where $H(\gamma,v) = v$ if $\frac{dQ}{dF}(v) \geq \gamma$ and $= 0$ otherwise. ($S = [0,1] \times (0,\infty) \times \mathbb{R}^d$, $\mu = Leb \times Leb \times F$, $T = [0,1] \times \mathbb{R}^d$ and $T_0 = [0,1] \times \mathbb{R}_0^d$.) By Proposition 2.1, the sequences $\{U_i\}$, $\{\Gamma_i\}$, $\{V_i\}$ can be defined on the same probability space as N such that

$$N = \sum_{i=1}^\infty \delta_{(U_i, J_i)} \quad a.s. \tag{3.13}$$

In practical applications of this method, e.g., for computer simulation of N, one only needs to consider nonzero jumps $\{J_k^*\}$ selected from $\{J_i\}$. We have $J_k^* = J_{\tau_k}$, where $\tau_i < \tau_2 < \ldots$ is a finite or infinite (with probability one) random sequence of indices (depending on whether $Q(\mathbb{R}_0^d)$ is finite or infinite). If Card$(\{\tau_k\}) = \infty$ a.s., then set $U_k^* = U_{\tau_k}$, $k \geq 1$. If $K :=$

Card$(\{\tau_k\}) < \infty$ a.s., then put $U_k^* = U_{\tau_k}$, $k \le K$ and $U_k^* = U_{\tau_K+k-K}$, for $k > K$. $\{U_k^*\}_{k\ge1}$ is an i.i.d. sequence of $\mathcal{U}[0,1]$ random variables independent of $\{\Gamma_i, V_i\}$ and by (3.13),

$$N = \sum_{k=1}^{\infty} \delta_{(U_k^*, J_k^*)} \quad a.s. \tag{3.14}$$

This form is much more efficient for simulation of N than (3.13) because here one only needs to generate a uniform random variable U_k^* (the location of a jump) when the corresponding jump size is nonzero.

(E) Rejection Method.

This is a new method in the context of Lévy processes that is being introduced in this paper. Let $\{X_0(t) : t \in [0,1]\}$ be a Lévy process in $\mathbb{R}^d$ with Lévy measure Q_0 such that

$$\frac{dQ}{dQ_0} \le 1. \tag{3.15}$$

Let N_0 be the process of jumps of X_0 admitting a representation

$$N_0 \stackrel{d}{=} \sum_{i=1}^{\infty} \delta_{(U_i, J_i^0)} \tag{3.16}$$

where $\{U_i\}_{i\ge1}$ is as above. Let $\{W_i\}$ be an i.i.d. sequence of uniform $\mathcal{U}[0,1]$ random variables that is also independent of $\{U_i, J_i^0\}$. Define

$$J_i = \begin{cases} J_i^0 & \text{if } \frac{dQ}{dQ_0}(J_i^0) \ge W_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

Then

$$N \stackrel{d}{=} \sum_{i=1}^{\infty} \delta_{(U_i, J_i)}. \tag{3.18}$$

Indeed, $M := \sum \delta_{(U_i, W_i, J_i^0)}$ is a marked Poisson point process on $S = [0,1] \times [0,1] \times \mathbb{R}_0^d$ with control measure $\mu = Leb \times Leb \times Q_0$. (3.18) follows by (2.1)–(2.2) verified for $h(u,w,j) = (u, j1(\frac{dQ}{dQ_0}(j) \ge w))$. From the second part of Proposition 2.1 we infer that the sequences $\{U_i, W_i, J_i^0\}$ can be defined on the same probability space as N such that

$$N = \sum_{i=1}^{\infty} \delta_{(U_i, J_i)} \quad a.s. \tag{3.19}$$

The key to this method is to find an easy to generate Lévy process X_0 from which only a small finite number of jumps must be removed to get the jumps of X. A similar reasoning as in the conclusion of the thinning method can be made to justify that one needs to generate a uniform random variable only when the jump size is nonzero (J_i^0 is accepted).

(F) Shot Noise Method.

The idea of Bondesson [Bon] was to obtain a nonnegative infinitely divisible random variable as a shot noise. Independently, Rosiński [Ro1] took the same approach in the study of series

representations of infinitely divisible random vectors in Banach spaces (see also [Ro3]). This method can be described as follows. Consider a disintegration of Q in the form

$$Q(A) = \int_0^\infty \sigma(r; A)\, dr, \quad A \in \mathcal{B}(\mathbb{R}_0^d) \tag{3.20}$$

where σ is a probability kernel from $(0,\infty)$ to $\mathbb{R}^d$. Let $\{J_i\}_{i\geq 1}$ be a sequence of $\mathbb{R}^d$-valued conditionally independent random variables given $\{\Gamma_i\}_{i\geq 1}$ such that

$$P(J_i \in A | \{\Gamma_j\}_{j\geq 1}, \{J_j\}_{j\neq i}) = \sigma(\Gamma_i, A), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{3.21}$$

By Proposition 3.8 in [Re2], $\sum_{i=1}^\infty \delta_{(\Gamma_i, J_i)}$ is a Poisson point process on $(0,\infty)\times\mathbb{R}^d$ with mean measure λ, $\lambda(dr, dv) = dr\sigma(r; dv)$. Then $\sum_{i=1}^\infty \delta_{J_i}$, restricted to $\mathcal{B}(\mathbb{R}_0^d)$, is a Poisson point process with mean measure Q.

The probabilistic structure of the J_i's can be better understood when we notice that

$$J_i = H(\Gamma_i, V_i) \tag{3.22}$$

for some i.i.d. sequence $\{V_i\}$, that is independent of $\{U_i, \Gamma_i\}$, and a jointly measurable $\mathbb{R}^d$-valued function H such that

$$P(H(r, V_i) \in A) = \sigma(r; A), \qquad A \in \mathcal{B}(\mathbb{R}^d),\ r > 0. \tag{3.23}$$

Indeed, the existence of a function H satisfying (3.23) with V_i uniformly distributed on $[0,1]$ is a standard fact (see, e.g., Lemma 2.22 in [Ka2]). Clearly (3.22)-(3.23) imply (3.21) (the distribution and the range of V_i's are irrelevant under (3.23)).

As before, let $\{U_i\}$ be independent of $\{\Gamma_i, V_i\}$. By Proposition 2.1, $\{U_i, \Gamma_i, V_i\}_{i\geq 1}$ can be based on the same probability space as N such that

$$N = \sum_{i=1}^\infty \delta_{(U_i, H(\Gamma_i, V_i))} \quad a.s. \tag{3.24}$$

Remark 3.1 *The probability kernel σ in (3.20) is not unique. Moreover, every sequence $\{J_i\}$ of jumps of X considered in this paper can be viewed as a special case of (3.22). In all these cases the function $r \to |H(r,v)|$ is nonincreasing.*

4 Convergence of a generalized shot noise series.

Let $\{V_i\}_{i\geq 1}$ be an i.i.d. sequence of random elements in a measurable space S with common distribution F. Assume that $\{V_i\}_{i\geq 1}$ is independent of the sequence $\{\Gamma_i\}_{i\geq 1}$ of partial sums of standard exponential random variables. Let

$$H : (0,\infty)\times S \to \mathbb{R}^d \tag{4.1}$$

be a measurable function such that for each $v \in S$

$$r \to |H(r,v)| \quad \text{is nonincreasing.} \tag{4.2}$$

The goal of this section is to characterize the almost sure convergence of $\sum H(\Gamma_i, V_i)$.

$H(\Gamma_i, V_i)$ can be viewed as an effect at time zero of a shot V_i that happened Γ_i time units ago and (4.2) says that the magnitude of this effect decreases as the time elapsed from the moment of the shot increases. $\sum H(\Gamma_i, V_i)$ represents the total shot noise at time zero.

Define measures on $\mathbb{R}^d$ by

$$\sigma(r;\cdot) = P\big(H(r, V_i) \in \cdot\big), \quad r > 0 \tag{4.3}$$

and

$$Q(\cdot) = \int_0^\infty \sigma(r;\cdot)\, dr. \tag{4.4}$$

Put

$$A(s) = \int_0^s \int_{|x|\le 1} x\, \sigma(r; dx) dr, \quad s \ge 0. \tag{4.5}$$

The following theorem characterizes the almost sure convergence of a generalized shot noise. Its proof combines and simplifies certain arguments from [Ro3] (given there for a Banach space valued function H).

Theorem 4.1 (A) *$\sum_{i=1}^\infty H(\Gamma_i, V_i)$ converges a.s. if and only if*
(i) Q is a Lévy measure on $\mathbb{R}_0^d$, i.e.,

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\, Q(dx) < \infty,$$

and
(ii) $a := \lim_{s\to\infty} A(s)$ exists in $\mathbb{R}^d$.
If (i) and (ii) are satisfied, then $\mathcal{L}\big(\sum_{i=1}^\infty H(\Gamma_i, V_i)\big)$ is infinitely divisible with characteristic function $\phi(u)$ given by

$$\phi(u) = \exp[iua + \int_{\mathbb{R}_0^d} (e^{iux} - 1 - iuxI(|x| \le 1))\, Q(dx)]. \tag{4.6}$$

(B) *If only (i) holds, then $\sum_{i=1}^\infty [H(\Gamma_i, V_i) - c_i]$ converges a.s. for $c_i = A(i) - A(i-1)$. In this case the characteristic function of $\mathcal{L}\big(\sum_{i=1}^\infty [H(\Gamma_i, V_i) - c_i]\big)$ is given by (4.6) with $a = 0$.*

Proof: Define a stochastic process

$$Y(s) = \sum_{\{i:\ \Gamma_i \le s\}} H(\Gamma_i, V_i) - A(s), \quad s \ge 0. \tag{4.7}$$

Y has rcll paths and can be written as

$$Y(s) = \int_{[0,s]\times S} H(r, v)\, [M(dr, dv) - drF(dv)], \tag{4.8}$$

where $M = \sum_{i=1}^\infty \delta_{(\Gamma_i, V_i)}$ is a marked Poisson process with mean measure $Leb \times F$. Hence Y is a process with independent (but not stationary) increments. Furthermore, $Y(s)$ is a centered compound Poisson random variable with Lévy measure Q_s given by

$$Q_s(A) = \int_0^s \sigma(r; A) dr \nearrow Q(A), \quad \text{as } s \nearrow \infty. \tag{4.9}$$

We first claim that

$$\lim_{s\to\infty} Y(s) = Y(\infty) \quad \text{exists a.s.} \tag{4.10}$$

if and only if (i) holds. Indeed, (i) and (4.9) imply that $\lim_{s\to\infty}\mathcal{L}(Y(s))$ exists. Since Y has independent increments, $Y(s_k)=\sum_{i=1}^{k}(Y(s_i)-Y(s_{i-1}))$ converges also a.s., as $k\to\infty$, for each increasing sequence $s_k\nearrow\infty$ with $s_0=0$. The latter property implies (4.10) because Y has rcll paths. We also note that

$$E\exp(iuY(\infty))=\phi(y)e^{-iua}. \tag{4.11}$$

Conversely, implies that $\lim_{s\to\infty}\mathcal{L}(Y(s))$ exists. Hence the Lévy measures Q_s converge vaguely to some Lévy measure on each continuity set bounded away from the origin. By (4.9), that Lévy measure must coincide with the measure Q.

Now we will prove (A). If (i) and (ii) are satisfied, then by (4.10)

$$\sum_{i=1}^{n}H(\Gamma_i,V_i)=Y(\Gamma_n)+A(\Gamma_n)\to Y(\infty)+a$$

a.s. as $n\to\infty$; (4.6) follows from (4). Conversely, if $\sum_{i=1}^{\infty}H(\Gamma_i,V_i)$ converges a.s., then $Y(s)+A(s)=\sum_{\{i:\,\Gamma_i\le s\}}H(\Gamma_i,V_i)$ converges a.s. to the same limit as $s\to\infty$. Since the Lévy measure of $Y(s)+A(s)$ is Q_s, we get that Q is a Lévy measure by the same argument as above. Then by (4.10), $A(s)=(Y(s)+A(s))-Y(s)$ converges as $n\to\infty$. This concludes the proof of (A).

To prove part (B) we observe as above that

$$\sum_{i=1}^{n}H(\Gamma_i,V_i)-A(\Gamma_n)=Y(\Gamma_n)\to Y(\infty) \tag{4.12}$$

a.s. as $n\to\infty$. Therefore, it is enough to show that

$$A(\Gamma_n)-A(n)\to 0\quad a.s.\text{ as }\ n\to\infty. \tag{4.13}$$

We have

$$|A(\Gamma_n)-A(n)|\le\int_{\Gamma_n\wedge n}^{\Gamma_n\vee n}\int_{\mathbb{R}^d}(|x|\wedge 1)\,\sigma(r;dx)dr. \tag{4.14}$$

Put

$$g(r)=\int_{\mathbb{R}^d}(|x|\wedge 1)\,\sigma(r;dx)=E(|H(r,V_1)|\wedge 1).$$

By (4.2), g is nonincreasing and square integrable. Indeed, by Jensen's inequality,

$$\begin{aligned}\int_0^\infty[g(r)]^2\,dr &\le \int_0^\infty\int_{\mathbb{R}^d}(|x|^2\wedge 1)\,\sigma(r;dx)dr\\ &= \int_{\mathbb{R}^d}(|x|^2\wedge 1)\,Q(dx)<\infty.\end{aligned} \tag{4.15}$$

From (4.14),

$$\begin{aligned}|A(\Gamma_n)-A(n)| &\le |\Gamma_n-n|g(\Gamma_n\wedge n)\\ &= g(n/2)|\Gamma_n-n|\big[g(\Gamma_n\wedge n)/g(n/2)\big]\end{aligned}$$

Since $n^{-1}\Gamma_n\to 1$ a.s. and g is nonincreasing,

$$\limsup_{n\to\infty}g(\Gamma_n\wedge n)/g(n/2)\le 1.$$

Using the Hájek-Rényi- Chow inequality ([C-T], page 243) and (4.15) we get, for every $\epsilon > 0$,

$$P(\sup_{n\geq k} g(n/2)|\Gamma_n - n| \geq \epsilon) \leq \epsilon^{-2}\big(kg^2(k/2) + \sum_{n>k} g^2(n/2)\big) \to 0$$

as $k \to \infty$. Hence $\lim_{n\to\infty} g(n/2)|\Gamma_n - n| = 0$ a.s., which completes the proof of (4.13) and concludes the proof of this theorem.

Remark 4.1 (a) *The monotonicity condition (4.2) was crucial in the proof of part (B), of the existence of nonrandom centers. However, it was not used in the proof of part (A).*
(b) *Condition (ii) is automatically satisfied when $\sigma(r;\cdot)$ is symmetric or $\int_{|x|\leq 1} |x|\, Q(dx) < \infty$. If the latter integral is infinite and Q is not symmetric, then it is still possible to get (ii) under a special choice of H for Q. Such representations without centering can be obtained by the thinning method which will be studied in detail in a forthcoming paper.*

5 From Lévy–Itô to series representations. Conclusion.

Now we return to the notation of Section 1. Suppose that

$$N = \sum_{i=1}^{\infty} \delta_{(U_i, H(\Gamma_i, V_i))} \quad a.s. \tag{5.1}$$

on $\mathcal{B}([0,1]\times\mathbb{R}_0^d)$, where $\{U_i\}_{i\geq 1}$ is independent of $\{\Gamma_i.V_i\}_{i\geq 1}$ and $H(\Gamma_i, V_i)$ is described at the beginning of Section 4. All representations of N given in Section 3 are of this form. Let

$$M := \sum_{i=1}^{\infty} \delta_{(U_i, \Gamma_i, V_i)}. \tag{5.2}$$

M is a marked Poisson point process with mean measure $Leb \times Leb \times F$. By a change of variable in (1.2),

$$\begin{aligned} X(t) &= at + \int_{|H(r,v)|\leq 1} H(r,v)\,(M([0,t],dr,dv) - tdrF(dv)) \\ &\quad + \int_{|H(r,v)|>1} H(r,v)\, M([0,t],dr,dv). \end{aligned} \tag{5.3}$$

Define

$$\begin{aligned} \tilde{X}_s(t) &= at + \int_{r\in[0,s],\ |H(r,v)|\leq 1} H(r,v)\,(M([0,t],dr,dv) - tdrF(dv)) \\ &\quad + \int_{r\in[0,s],\ |H(r,v)|>1} H(r,v)\, M([0,t],dr,dv) \end{aligned} \tag{5.4}$$

for $s \geq 0$, $t \in [0,1]$, and observe that

$$\tilde{X}_s(t) = at + \sum_{\{i:\ \Gamma_i \leq s\}} H(\Gamma_i, V_i)1(U_i \leq t) - tA(s)$$

where $A(s)$ is given by (4.5). The process $\tilde{X}_s(t)$ has independent increments both with respect to $s \in [0,\infty)$ and $t \in [0,1]$. Fix $t \in [0,1]$. Since $\tilde{X}_s(t)$ has rcll paths in s and $\tilde{X}_{s_n}(t) \to X(t)$

a.e. by (5.3), for every sequence $s_n \nearrow \infty$, we infer that $\lim_{s\to\infty} \tilde{X}_s(t) = X(t)$ a.s. Therefore, as $n \to \infty$,

$$\tilde{X}_{\Gamma_n}(t) = at + \sum_{i=1}^{n} H(\Gamma_i, V_i)1(U_i \le t) - tA(\Gamma_n) \to X(t) \quad a.s. \tag{5.5}$$

Since $A(\Gamma_n) - A(n) \to 0$ by (4.13),

$$at + \sum_{i=1}^{\infty} \big(H(\Gamma_i, V_i)1(U_i \le t) - tc_i\big) = X(t) \quad a.s. \tag{5.6}$$

where $c_i = A(i) - A(i-1)$. This is a series representation of a Lévy process. Conditionally on $\{\Gamma_i\}$, (5.6) is a series of independent random variables. Unconditionally, the terms $H(\Gamma_i, V_i)$ are not independent but are Markovian. Again, the usefulness of such representations depends on the choice of H, that can be different for different problems. An important fact is the following.

Theorem 5.1 *Series (5.6) converges a.s. uniformly on* $[0,1]$.

Proof: Let $\|\cdot\|$ denote the supremum norm of a function in $D[0,1]$. Fix $\epsilon > 0$ and define a sequence $w_n \nearrow \infty$ inductively. Put $w_0 = 0$ and for $n \ge 1$ let $w_n > w_{n-1} + 1$ satisfy

$$\begin{aligned} q_n &:= P\big(\sup_{w_{n-1}\le r<s} \|\tilde{X}_s - \tilde{X}_r\| > \epsilon\big) \\ &\le P\big(\sup_{w_{n-1}\le r<s\le w_n} \|\tilde{X}_s - \tilde{X}_r\| > \epsilon\big) + n^{-1}. \end{aligned}$$

Since paths of $\tilde{X}$ are rcll, one can find a partition $w_{n-1} = w_1^n < w_2^n < \cdots < w_{k_n}^n = w_n$ such that

$$q_n \le P\big(\max_{i<j\le k_n} \|\tilde{X}_{w_j^n} - \tilde{X}_{w_i^n}\| > \epsilon\big) + 2n^{-1}.$$

Form a sequence $s_j \nearrow \infty$ from $w_1^1 < \cdots < w_{k_1}^1 < w_1^2 < \cdots < w_{k_2}^2 < \cdots$ and define $Y_j(t) := \tilde{X}_{s_j}(t) - \tilde{X}_{s_{j-1}}(t)$. $\{Y_j(t) : t \in [0,1]\}$ are mutually independent processes with paths in $D[0,1]$ such that $\sum_{j=1}^{\infty} Y_j(t)$ converges a.s. to $X(t) - ta$, for each t. Applying a result of Kallenberg [Ka1], we infer that $\sum_{j=1}^{\infty} Y_j(t)$ converges a.s uniformly on $[0,1]$. Hence $\lim_{n\to\infty} q_n = 0$. Since $\epsilon > 0$ is arbitrary, we get that

$$\lim_{w\to\infty} \sup_{w\le r<s} \|\tilde{X}_s - \tilde{X}_r\| = 0 \quad a.s. \tag{5.7}$$

Consequently,

$$\lim_{n\to\infty} \sup_{n\le k<m} \|\tilde{X}_{\Gamma_m} - \tilde{X}_{\Gamma_k}\| = 0 \quad a.s.$$

In view of (5.5) and (4.13)

$$\lim_{n\to\infty} \sup_{n\le k<m} \sup_{0\le t\le 1} \Big|\sum_{i=k+1}^{m} H(\Gamma_i, V_i)1(U_i \le t) - t(A(m) - A(k))\Big| = 0 \quad a.s.$$

which proves the theorem.

Remark 5.1 *If H is given by the inverse Lévy measure method then $X_n = \tilde{X}_{s_n}$ for $s_n = Q([n^{-1}, \infty))$ (see (1.4)). In that case Theorem 5.1 can be derived directly from the Lévy-Itô theory and (4.13). However, in general, there is no direct relationship between X_n and $\tilde{X}_s$. In particular, there is no uniform bound on the size of jumps of $X - \tilde{X}_s$ as opposed to $X - X_n$.*

Remark 5.2 *Both (5.6) and $\tilde{X}_s$ can be used for simulating of Lévy processes. For $t = 1$ we get*

$$X(1) = a + \sum_{i=1}^{\infty} \big(H(\Gamma_i, V_i) - c_i\big) \quad a.s. \tag{5.8}$$

This series can be used for the simulation of an infinitely divisible random variable with distribution $G = \mathcal{L}(X(1))$. However, more direct and much faster methods can be designed specifically for sampling from G, as it is in the case of stable random variables (see, e.g., Janicki and Weron [J-W]). This fact is not surprising because $\{H(\Gamma_i, V_i)\}_{i\geq 1}$ gives us more information than we need for G, we know the sequence of jumps of $\{X(t,\omega)\}_{0\leq t\leq 1}$ under a certain enumeration. Despite of this, if the series (5.8) converges fast, then (5.8) can still give a good method for simulating i.i.d. samples from G.

6 Four series representations of a gamma process.

Let $\{X(t) : t \in [0,1]\}$ be a Lévy gamma process with parameters a, b; a and b are shape and scale parameters, respectively, of a gamma random variable $X(1)$. The Lévy measure Q of $X(1)$ is concentrated on $(0,\infty)$ and its tail is given by

$$Q([x,\infty)) = \int_x^{\infty} u^{-1} e^{-u/b} a \, du = a E_1(x/b),$$

where $E_p(x) = \int_x^{\infty} u^{-p} e^{-u} \, du$ is the exponential integral function.

The inverse Lévy measure method yields the following representation of X:

$$X(t) = \sum_{i=1}^{\infty} b E_1^{-1}(\Gamma_i/a) 1(U_i \leq t).$$

The obvious difficulty here is that E_1^{-1} can not be given explicitly (neither E_1). However, Wolpert and Ickstadt [W-I] developed a procedure for an approximate evaluation of E_1^{-1} in S-PLUS (MathSoft, 1997) and used it for the simulation of X from the above formula.

Bondesson [Bon] obtained a gamma random variable as a shot noise variable with exponential shot-distribution $F(x) = 1 - e^{-x/b}$ and exponential response function $g(t) = e^{-t/a}$. Thus Bondesson's method gives the following representation of X:

$$X(t) = \sum_{i=1}^{\infty} b e^{-\Gamma_i/a} V_i 1(U_i \leq t),$$

where $\{V_i\}$ is an i.i.d. sequence of standard exponential random variables that is independent of $\{\Gamma_i, U_i\}$.

Now we will give the third representation of a gamma process based on the thinning method. Let $\{V_i\}$ be an i.i.d. sequence of standard exponential random variables that is independent of $\{\Gamma_i, U_i\}$. Let

$$T = \{i \geq 1 : \ V_i\Gamma_i \leq a\} = \{\tau_1 < \tau_2 < \dots\}.$$

Then

$$X(t) = \sum_{i=1}^{\infty} bV_{\tau_i} 1(U_i \leq t).$$

($J_i^* = V_{\tau_i}$ in the notation of Section 3 D.)

The fourth representation is based on the rejection method. As a Lévy measure Q_0 we choose $Q_0(dx) = ax^{-1}(1+x/b)^{-1}$. Jumps of the corresponding process X_0 are found from the inverse Lévy measure method:

$$J_i^0 = Q_0^{\leftarrow}(\Gamma_i) = \frac{b}{e^{\Gamma_i/a} - 1}.$$

Then the jumps $\{J_i\}$ of the gamma process are selected from $\{J_i^0\}$ by the following procedure

$$J_i = \begin{cases} J_i^0 & \text{if } \ (1 + J_i^0/b)\exp(-J_i^0/b) \geq W_i, \\ 0 & \text{otherwise} \end{cases}$$

where $\{W_i\}$ is an i.i.d. sequence of uniform on $[0,1]$ random variables that is independent of $\{\Gamma_i, U_i\}$ (see Section 3 E). Then

$$X(t) = \sum_{i=1}^{\infty} J_i 1(U_i \leq t)$$

represents a gamma process with shape a and scale b. The average number of discarded J_i^0's in this method equals $a\gamma$, where $\gamma = .5772\dots$ is the Euler constant. This method is very fast in computer implementation. These results will be part of a more comprehensive study of the approximation of gamma processes and random fields in a forthcoming paper. Finally we note that the results of Section 3 and Theorem 5.1 imply that the above series representations of the process X hold a.s. uniformly in $t \in [0,1]$.

7 Historical remarks and generalizations.

Khintchine [Khi] established series representations of real infinitely divisible random variables using the inverse Lévy measure method in the following form. If X is an infinitely divisible random variable with Lévy measure Q concentrated on the positive half-line, then $X \stackrel{d}{=} G + \sum_{i=1}^{\infty}[Q^{\leftarrow}(\Gamma_i) - c_i]$, where G is a Gaussian random variable independent of $\{\Gamma_i\}$ and $c_i = \int_{i-1}^{i} Q^{\leftarrow}(x)/(Q^{\leftarrow}(x)^2 + 1)\, dx$. In general, if Q is concentrated on the whole real line, then X is represented in distribution as the difference of two independent series of the above form (see Hauptsatz III, pages 100 and 112 in [Khi]). This pioneering work of Khintchine on series representations is mentioned in Ken–iti Sato recent monograph [Sat], page 144; we simply adapt the pertinent part of Hauptsatz III, [Khi] to the present notation. The inverse Lévy measure method was rediscovered by Ferguson and Klass [F-K] in their study of series expansions of independent increment processes. Kallenberg [Ka1] showed the uniform convergence of the series expansions from [F-K] and Resnick [Re1] explained their connection to the Lévy-Itô integral representation (see also Remark (5.1)).

The work of LePage, Woodroofe, and Zinn [LP-W-Z] and of LePage [LP1] on series representations of stable random vectors led to a rapid progress of the theory of stable processes and their applications (see the monograph [S-T] of Samorodnitsky and Taqqu). LePage [LP1] gave a representation of Lévy stable processes and observed that general symmetric stable processes can be viewed as conditionally Gaussian. Infinitely divisible processes having this important property, that is strictly related to their series representations, were called of type G by Marcus [Mar] and studied in the context of path continuity. A systematic investigation of type G random variables and processes appears in Rosiński [Ro5]. LePage presented his method for general infinitely divisible random vectors in [LP2].

Bondesson [Bon] considered the simulation of nonnegative infinitely divisible random variables and developed their series representations as a special case of a generalized shot noise. Rosiński [Ro1], [Ro3] used the same approach as a general pattern for series representations of Banach space valued infinitely divisible random vectors. This approach was also used for representing general infinitely divisible processes, see Rosiński [Ro2].

The thinning method was introduced in Rosiński [Ro3], Example 4.6 (see also [Ro4]). There are many papers using and extending series representations based on the inverse Lévy measure or LePage's methods that are not listed here. A major application of series representations by the thinning method appears in Talagrand [Tal] in his study of the boundedness of general infinitely divisible processes. As far as we know, the rejection method given here is new (in the context of Lévy processes).

There is an immediate way to extend the series representations of this paper to Lévy processes indexed by an arbitrary finite interval $[0,T]$. One just needs to replace the uniform on $[0,1]$ sequence $\{U_i\}$ by a uniform on $[0,T]$ sequence and then replace Γ_i by Γ_i/T in (5.6). Theorem 4.1 gives a simple general method for the verification of the validity of this and other series representations of nonhomogeneous processes, infinitely divisible random measures, and random fields.

8 REFERENCES

[Bon] L. Bondesson, *On simulation from infinitely divisible distributions*, Adv. Appl. Probab. **14** (1982) 855-869.

[C-T] Y.S. Chow and H. Teicher, Probability Theory: Independence, Interchangeability, Martingales *Springer* 1978.

[F-K] T.S. Ferguson and M. J. Klass, *A representation of independent increment processes without Gaussian components*, Ann. Math. Statist. **43** (1972) 1634-1643.

[J-W] A. Janicki and A. Weron, Simulation and Chaotic Behavior of α-Stable Stochastic Processes *Marcel Dekker* 1994

[Ka1] O. Kallenberg, *Series of random processes without discontinuities of the second kind*, Ann. Probab. **2** (1974) 729-737.

[Ka2] O. Kallenberg, Foundations of Modern Probability *Springer* 1997.

[Khi] A. Ya. Khintchine, Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze. Mat. Sbornik **2** (1937) 79–119.

[Mar] M.B. Marcus, ξ-radial processes and random Fourier series. *Mem. Amer. Math. Soc.* **368** (1978).

[LP1] R. LePage, *Multidimensional infinitely divisible variables and processes. I*, Technical Rept. **292** Dept. Statistics, Stanford University. (1980)

[LP2] R. LePage, *Multidimensional infinitely divisible variables and processes. II*, Lecture Notes Math. **860** Springer-Verlag (1980), pp. 279–284.

[LP-W-Z] R. LePage, M. Woodroofe, and J. Zinn, *Convergence to a stable distribution via order statistics*, Ann. Probab. **9** (1981) 624-632.

[R-R] B.S. Rajput and J. Rosiński, *Spectral representations of infinitely divisible processes*, Probab. Th. Rel. Fields **82** (1989) 451–487.

[Re1] S.I. Resnick, *An extremal decomposition of a process with stationary, independent increments*, Tech. Rept. 79, Dept. Statistics, Stanford Univ. (1976).

[Re2] S.I. Resnick, Extreme Values, Regular Variation, and Point Processes *Springer* 1987.

[Ro1] J. Rosiński, *Series representations of infinitely divisible random vectors and a generalized shot noise in Banach spaces*, University of North Carolina Center for Stochastic Processes, Technical Report No. 195 (1987).

[Ro2] J. Rosiński, *On path properties of certain infinitely divisible processes*, Stoch. Proc. Appl. **33** (1989) 73-87.

[Ro3] J. Rosiński, *On series representations of infinitely divisible random vectors*, Ann. Probab. **18** (1990) 405–430.

[Ro4] J. Rosiński, *An application of series representations for zero–one laws for infinitely divisible random vectors*. In: Probability in Banach Spaces 7, Progress in Probability **21** Birkhäuser, (1990) pp. 189–199.

[Ro5] J. Rosiński, *On a class of infinitely divisible processes represented as mixtures of Gaussian processes*. In: Stable Processes and Related Topics. S. Cambanis, G. Samorodnitsky, and M.S. Taqqu, eds. Birkhäuser (1991) pp. 27–41.

[S-T] G. Samorodnitsky and M.S. Taqqu, Non-Gaussian Stable Processes; *Chapman & Hall* 1994.

[Sat] K. Sato, Lévy Processes and Infinitely Divisible Distributions; *Cambridge University Press* 1999.

[Tal] M. Talagrand, *Regularity of infinitely divisible processes*, Ann. Probab. **21** (1993) 362–432.

[W-I] R.L. Wolpert and K. Ickstadt, *Simulation of Lévy random fields*. In: Practical Nonparametric and Semiparametric Bayesian Statistics. Müller, Dey & Sinha, eds. Springer–Verlag (1998) pp. 227-242.

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