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On simulation of tempered stable random variates

Reiichiro Kawai [a,*], Hiroki Masuda [b]

[a] Department of Mathematics, University of Leicester, Leicester LE1 7RH, UK

[b] Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, Japan

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ABSTRACT

Various simulation methods for tempered stable random variates with stability index greater than one are investigated with a view towards practical implementation, in particular cases of very small scale parameter, which correspond to increments of a tempered stable Lévy process with a very short stepsize. Methods under consideration are based on acceptance–rejection sampling, a Gaussian approximation of a small jump component, and infinite shot noise series representations. Numerical results are presented to discuss advantages, limitations and trade-off issues between approximation error and required computing effort. With a given computing budget, an approximative acceptance–rejection sampling technique Baeumer and Meerschaert (2009) [11] is both most efficient and handiest in the case of very small scale parameter and moreover, any desired level of accuracy may be attained with a small amount of additional computing effort.

1. Introduction

The class of tempered stable distributions was first proposed in [1]. Several featuring properties of tempered stable laws and processes were revealed in [2], such as a stable-like behavior over short intervals, the absolute continuity with respect to its short-range limiting stable process, an aggregational Gaussianity and an infinite shot noise series representation in closed form. Tempered stable distributions and processes have been used in a variety of applications, such as statistical physics [3,4], mathematical finance [5], financial econometrics [6] and mathematical biology [7], to mention just a few. Simulation of the tempered stable distribution have thus been of great practical interest, in particular, for validation and estimation purposes.

On the one hand, it is well known that their increments with stability index smaller than one can be simulated exactly through, either single or double, acceptance–rejection sampling. (See Section 2.2 for more details.) To the best of our knowledge, on the other hand, there exist no *practically* exact simulation methods for tempered stable random variates with stability index greater than one. The purpose of this paper is to investigate and weigh various possible simulation techniques and discuss their advantages, limitations and trade-off between approximation error and computing effort, with a full view towards practical implementation. We pay particular attention to the case of very small scale parameters, which corresponds to increments of tempered stable Lévy processes with a very short stepsize. This was motivated by application to approximation of Lévy driven stochastic differential equations such as the Euler scheme, for which we have to have many small time independent increments of the Lévy process. Moreover, the generation is extremely important in simulation experiments concerning statistics, such as parameter estimation, for high frequently observed Lévy driven processes.

* Corresponding author.
E-mail addresses: reiichiro.kawai@gmail.com (R. Kawai), hiroki@math.kyushu-u.ac.jp (H. Masuda).

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The importance of our comparative study is clearly reinforced by ever-increasing practical demand of statistical inference for processes with jumps and wide applicability of the tempered stable process in modeling. (See, for example, [5,8–10], and the references therein.) This work was initiated when the approximative simulation method of Section 3.2 due to [11] came to our attention, as we have long been aware of the fact that simulation methods with the Gaussian approximation of [12] are not as efficient as often claimed in the literature, due to the unaddressed practical trade-off between the accuracy of the normal approximation of small jumps and the required computing time to generate the compound Poisson component for large jumps. In this paper, we illustrate through numerous numerical results that simulation methods based on the Gaussian approximation are not necessarily the best and that different techniques are worth testing whenever available.

The rest of this paper is organized as follows. After summarizing background material in brief on stable and tempered stable distributions in Section 2, we discuss acceptance–rejection sampling methods of [13,11]. The method of [13] yields an exact simulation in principle, but requires very time-consuming numerical integration for each fundamental quantity, while the approach of [11] provides an approximative, yet very handy and efficient, simulation method. In Section 4, we investigate simulation methods based on a suitable decomposition of the Lévy measure into small and large jump components. We apply the well known Gaussian approximation of [12] to the small jump component. We also propose further compound Poisson extraction schemes on the small jump component. In Section 5, we discuss yet another simulation method based on infinite shot noise series representations. In principle, only the infinite shot noise series provides an exact simulation method for tempered stable Lévy *processes* since it simulates complete information of sample paths, that is, size, direction and timing of every single jump. A closed form of such a series representation is given in [2] (first introduced in his discussion section of the article [6]). From a computational point of view, however, the form of infinite sum has raised important issues of finite truncation to be addressed. (See [14].)

2. Preliminaries

Let us begin this preliminary section with the notations which will be used throughout the paper. We denote by $\mathbb{R}$ the one dimensional Euclidean space with the norm $|\cdot|$, $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$. Let $\mathbb{N}$ be the collection of positive integers with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote by $\overset{\mathscr{L}}{=}$ and $\overset{\mathscr{L}}{\to}$, respectively, identity and convergence in law. We write $f_L(z)$ for a smooth probability density function of a distribution L. We fix $(\Omega, \mathscr{F}, \mathbb{P})$ as our underlying probability space. We denote by $C_b^\infty(\mathbb{R};\ \mathbb{R})$ by the class of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ which, together with all their derivatives, are bounded. Finally, the gamma function $\Gamma(s) := \int_0^{+\infty} x^{s-1}\mathrm{e}^{-x}\mathrm{d}x$, $s > 0$, can be extended to negative s thanks to analytic continuation and $\Gamma(s+1) = s\Gamma(s)$. In particular, $\Gamma(-s) < 0$ for $s \in (0, 1)$, while $\Gamma(-s) > 0$ for $s \in (1, 2)$.

2.1. Spectrally positive stable processes

Let $\{L_t^{(\mathrm{s})} :\ t \geq 0\}$ be a *totally positively skewed* stable (Lévy) process satisfying

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}yL_t^{(\mathrm{s})}}\right] &= \exp\left[ta\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)|y|^\alpha\left(1 - \mathrm{i}\tan\frac{\pi\alpha}{2}\mathrm{sgn}(y)\right)\right]\\ &= \begin{cases}\exp\left[t\displaystyle\int_{\mathbb{R}_+}\left(\mathrm{e}^{\mathrm{i}yz} - 1\right)\frac{a}{z^{\alpha+1}}\mathrm{d}z\right], & \text{if } \alpha \in (0, 1),\\ \exp\left[t\displaystyle\int_{\mathbb{R}_+}\left(\mathrm{e}^{\mathrm{i}yz} - 1 - \mathrm{i}yz\right)\frac{a}{z^{\alpha+1}}\mathrm{d}z\right], & \text{if } \alpha \in (1, 2),\end{cases}\end{aligned} \tag{2.1}$$

with some $a > 0$. Throughout this paper, we exclude the case $\alpha = 1$. We write $S(\alpha, a) := \mathscr{L}(L_1^{(\mathrm{s})})$. Note that the random variable $L_1^{(\mathrm{s})}$ takes values only in $\mathbb{R}_+$ if $\alpha \in (0, 1)$, while in $\mathbb{R}$ if $\alpha \in (1, 2)$. It holds that for each $t > 0$, $\mathscr{L}(L_t^{(\mathrm{s})}) = S(\alpha, ta)$, and by the scaling property, $\mathscr{L}(t^{-1/\alpha}L_t^{(\mathrm{s})}) = S(\alpha, a)$. Note that the distribution $S(\alpha, ta)$ has density $t^{-1/\alpha}f_{S(\alpha,a)}(t^{-1/\alpha}x)$. The distribution $S(\alpha, a)$ can be simulated in the exact sense through the well known representation, due to Chambers et al. [15],

$$S(\alpha, a) \overset{\mathscr{L}}{=} (-a\Gamma(-\alpha)\cos(\pi\alpha/2))^{1/\alpha}\,\frac{\sin(\alpha U + \theta)}{(\cos U\cos\theta)^{1/\alpha}}\left(\frac{\cos((1-\alpha)U - \theta)}{E}\right)^{\frac{1-\alpha}{\alpha}}, \tag{2.2}$$

where $\theta := \arctan(\tan(\pi\alpha/2))$, U is a uniform random variable on $(-\pi/2, \pi/2)$ and E is a standard exponential random variable independent of U. See [16] for complete details on the stable distribution.

2.2. Spectrally positive tempered stable processes

Let $\{L_t^{(\mathrm{ts})} :\ t \geq 0\}$ be a *centered* and *totally positively skewed* tempered stable (Lévy) process satisfying

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}yL_t^{(\mathrm{ts})}}\right] = \exp\left[t\int_{\mathbb{R}_+}\left(\mathrm{e}^{\mathrm{i}yz} - 1 - \mathrm{i}yz\right)a\frac{\mathrm{e}^{-bz}}{z^{\alpha+1}}\mathrm{d}z\right] = \exp\left[ta\Gamma(-\alpha)\left((b - \mathrm{i}y)^\alpha - b^\alpha + \mathrm{i}y\alpha b^{\alpha-1}\right)\right].$$

When $\alpha \in (0,1)$, by adding back the centering term as $L_t^{(\mathrm{ts})} + t\Gamma(1-\alpha)ab^{\alpha-1}$, we can recover the tempered stable subordinator. Throughout the paper, we will use the notations

$$TS(\alpha,a,b) := \mathscr{L}\left(L_1^{(\mathrm{ts})}\right), \tag{2.3}$$

and

$$TS'(\alpha,a,b) := \mathscr{L}\left(L_1^{(\mathrm{ts})} + \Gamma(1-\alpha)ab^{\alpha-1}\right). \tag{2.4}$$

It is known that

$$\frac{\mathrm{e}^{-bz}}{\mathbb{E}\left[\mathrm{e}^{-bL_1^{(\mathrm{s})}}\right]} f_{S(\alpha,a)}(z) = \mathrm{e}^{-bz-a\Gamma(-\alpha)b^{\alpha}} f_{S(\alpha,a)}(z) = f_{TS'(\alpha,a,b)}(z), \tag{2.5}$$

and clearly

$$\begin{aligned} f_{TS(\alpha,a,b)}(z) &= f_{TS'(\alpha,a,b)}(z - \Gamma(1-\alpha)ab^{\alpha-1}) \\ &= \mathrm{e}^{-bz-a(\alpha+1)\Gamma(-\alpha)b^{\alpha}} f_{S(\alpha,a)}(z - \Gamma(1-\alpha)ab^{\alpha-1}). \end{aligned} \tag{2.6}$$

Those hold for every $\alpha \in (0,1)\cup(1,2)$.

Let us first focus on the case $\alpha \in (0,1)$ with

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}y(L_t^{(\mathrm{ts})}+t\Gamma(1-\alpha)ab^{\alpha-1})}\right] = \exp\left[t\int_{\mathbb{R}_+}\left(\mathrm{e}^{\mathrm{i}yz}-1\right)a\frac{\mathrm{e}^{-bz}}{z^{\alpha+1}}\mathrm{d}z\right] = \exp\left[ta\Gamma(-\alpha)\left((b-\mathrm{i}y)^{\alpha}-b^{\alpha}\right)\right].$$

(Note that this never holds for $\alpha \in (1,2)$.) Based upon this fact and the density function (2.5), it is well known (for example, [11,17]) that when $\alpha \in (0,1)$, the tempered stable distribution $TS'(\alpha,a,b)$ can be simulated *exactly* through acceptance–rejection sampling as follows.

Algorithm 0.

Step 1. Generate U as uniform $(0,1)$ and V as $S(\alpha,a)$.

Step 2. If $U \le \mathrm{e}^{-bV}$, exit with V. Otherwise, return to Step 1.

For example, this exact acceptance–rejection sampling paves the way for efficient simulation of tempered stable Ornstein–Uhlenbeck processes. In particular, the acceptance rate increases to 1 as $\Delta \downarrow 0$ and decreases to $\mathrm{e}^{a\Gamma(-\alpha)b^{\alpha}}$ as Δ tends to infinity. (For more details, see [18,19].) Moreover, another exact double rejection method is developed in [17] based on the Zolotarev integral representation [16] of the density function $f_{S(\alpha,a)}(z)$. This double rejection is not handy anymore but has an interesting feature that it is more efficient than Algorithm 0 when Δ is large. Although Algorithm 0 is not necessarily the most efficient, it is still an exact and very handy method. For this reason, in this paper, we do not consider the range $\alpha \in (0,1)$.

In what follows, we concentrate on the case $\alpha \in (1,2)$. In particular, Algorithm 0 cannot be simply extended to the range $\alpha \in (1,2)$, due to the support of the whole real line $\mathbb{R}$ where the exponential tilting explodes at either positive or negative infinity. (See Section 3.2 for details.) Without loss of generality, we focus on a *centered* and *totally positively skewed* tempered stable random variate $X(\Delta)$ satisfying

$$\begin{aligned} \varphi_{\Delta}(y) := \mathbb{E}\left[\mathrm{e}^{\mathrm{i}yX(\Delta)}\right] &= \exp\left[\Delta\int_{\mathbb{R}_+}\left(\mathrm{e}^{\mathrm{i}yz}-1-\mathrm{i}yz\right)a\frac{\mathrm{e}^{-bz}}{z^{\alpha+1}}\mathrm{d}z\right] \\ &= \exp\left[\Delta a\Gamma(-\alpha)\left((b-\mathrm{i}y)^{\alpha}-b^{\alpha}+\mathrm{i}y\alpha b^{\alpha-1}\right)\right]. \end{aligned} \tag{2.7}$$

Let us note that simulation of increments of general infinite-variation tempered stable Lévy processes (with bilateral jumps) is within our scope, as this can be done simply through a convolution of two independent tempered stable random variables totally skewed in the opposite directions. (See Remark 3.1 for a related discussion.)

Remark 2.1. Let $L_t^{(\mathrm{s})}$ and $L_t^{(\mathrm{ts})} + t\Gamma(1-\alpha)ab^{\alpha-1}$ be random variables respectively with distributions $S(\alpha,ta)$ under the probability measure $\mathbb{Q}$ and $TS'(\alpha,ta,b)$ under $\mathbb{P}$. It is a straightforward application of Theorem 33.3 of Sato [20] to evaluate an expected value related to tempered stable random variables by the density transform

$$\mathbb{E}_{\mathbb{P}}\left[\Phi\left(L_t^{(\mathrm{ts})}\right)\right] = \mathbb{E}_{\mathbb{Q}}\left[\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right|_{\mathscr{G}}\Phi\left(L_t^{(\mathrm{s})}\right)\right], \tag{2.8}$$

with $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}_{\mathbb{P}}[|\Phi(L_t^{(\mathrm{ts})})|] < +\infty$. Here, the Radon–Nykodym derivative is given in closed form $(\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q})|_{\mathscr{G}} = \mathrm{e}^{-bL_t^{(\mathrm{s})}}/\mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{-bL_t^{(\mathrm{s})}}]$, $\mathbb{Q}$-*a.s.*, where $\mathscr{G}$ is the minimal σ-field generated by the random variable $L_t^{(\mathrm{s})}$. The equality (2.8) is valid for every $\alpha \in (0,1)\cup(1,2)$. Evaluation of expectations based on (2.8) does not require simulation of $L_t^{(\mathrm{ts})}$, but only requires simulation of $L_t^{(\mathrm{s})}$, which is simple through the representation (2.2). This density transform formulation is found useful in the computation of Greeks under an asset price model driven by tempered stable processes. (See [21] for details.) However, this formulation is not valid for simulation of replications but only valid for evaluation of expectations. □

3. Acceptance–rejection sampling

In this section, we discuss two (one exact and the other approximative) acceptance–rejection sampling techniques for simulation of tempered stable random variables.

3.1. Exact sampling using density function

It holds by the well known result of [13] that for $z \in \mathbb{R}$,

$$f_{TS(\alpha,\Delta a,b)}(z) \le \min\left[C_1(\Delta), \frac{C_2(\Delta)}{z^2}\right] := q_\Delta(z), \tag{3.1}$$

where

$$C_1(\Delta) := \frac{1}{2\pi}\int_{\mathbb{R}} |\varphi_\Delta(y)|\,\mathrm{d}y, \qquad C_2(\Delta) := \frac{1}{2\pi}\int_{\mathbb{R}} \left|\varphi''_\Delta(y)\right|\mathrm{d}y, \tag{3.2}$$

and that

$$C_3(\Delta) := \int_{\mathbb{R}} q_\Delta(z) = 4\sqrt{C_1(\Delta)C_2(\Delta)}.$$

Let U_1 and U_2 be iid uniform random variables on $(-1, +1)$. It is also shown that the random variable V defined by

$$V := \sqrt{\frac{C_2(\Delta)}{C_1(\Delta)}}\frac{U_1}{U_2} \tag{3.3}$$

has density $(C_3(\Delta))^{-1}q_\Delta(z)$. Based on the above facts, we can employ a simulation method based on acceptance–rejection sampling as follows.

Algorithm 1.

Step 1. Generate V as (3.3) and U as $U(0, 1)$ independent of V. If $|V| < \sqrt{C_2(\Delta)/C_1(\Delta)}$, then go to Step 3.

Step 2. If

$$C_2(\Delta)U < f_{TS(\alpha,\Delta a,b)}(V)V^2,$$

then exit with $Y_1(\Delta) \leftarrow V$. Otherwise, go to Step 1.

Step 3. If

$$C_1(\Delta)U < f_{TS(\alpha,\Delta a,b)}(V),$$

then exit with $Y_1(\Delta) \leftarrow V$. Otherwise, go to Step 1.

This is an exact simulation algorithm, that is, $Y_1(\Delta) \overset{\mathscr{L}}{=} X(\Delta)$. The expected number of times Step 1 is executed is $C_3(\Delta)$, while the acceptance rate at Step 1 is given by

$$p_1(\Delta) := \frac{1}{C_3(\Delta)}.$$

As discussed in Remark 4 of [17], Algorithm 1 has already been enhanced in terms of constant shift in the sense that the tempered stable distribution is centered in our setting (2.7).

By recalling (2.7) and observing that

$$\varphi''_\Delta(y) = -\varphi_\Delta(y)\Delta a\left[\Delta a\Gamma(1-\alpha)^2\left(b^{\alpha-1} - (b - \mathrm{i}y)^{\alpha-1}\right)^2 + \Gamma(2-\alpha)(b-\mathrm{i}y)^{\alpha-2}\right],$$

it seems difficult to obtain $C_1(\Delta)$ and $C_2(\Delta)$ in closed form. We thus need to compute $C_1(\Delta)$ and $C_2(\Delta)$ based on (3.2) through some numerical integration techniques. Note that numerical integration here does not have to be extremely accurate, as long as the inequality (3.1) holds true.

The important point in question is how to prepare $f_{TS(\alpha,\Delta a,b)}(V)$ in Step 2 and 3 of Algorithm 1, where as mentioned, the density $f_{TS(\alpha,\Delta a,b)}(z)$ is unavailable in closed form for any $\alpha \in (1, 2)$. One straightforward approach is to compute the density by the Fourier inverse of the characteristic function (2.7), while the other is to compute the density $f_{S(\alpha,\Delta a)}(z)$ of the associated stable distribution in order to use the relation (2.6). It would be more sensible to take the latter route since some math tools provide a function that returns density values of the stable distribution, such as `dstable` in R language. With the help of such existing functions, we may either (i) compute $f_{S(\alpha,\Delta a)}(V - \Delta\Gamma(1-\alpha)ab^{\alpha-1})$ whenever required for each V, (ii) compute $f_{S(\alpha,\Delta a)}(v_k)$ at several pre-selected points $\{v_k\}_{k\in\mathbb{N}}$ and use interpolation for each replication, or could be (iii) a combination of them. Note that they are, strictly speaking, both approximative since numerical integration is used. In principle, the choice is up to how many replications to be generated.

Table 1
Numerical results of the acceptance rate $p_1(\Delta)$ of Algorithm 1 under various parameter settings. We fix $a = 1$ here.

b	α	$\Delta = 0.001$	$\Delta = 0.010$	$\Delta = 0.100$	$\Delta = 1.000$
0.1	1.2	0.280	0.317	0.382	0.483
	1.5	0.483	0.499	0.529	0.573
	1.8	0.615	0.618	0.624	0.631
1.0	1.2	0.328	0.400	0.505	0.597
	1.5	0.512	0.550	0.596	0.626
	1.8	0.623	0.629	0.634	0.636
2.0	1.2	0.350	0.435	0.544	0.615
	1.5	0.527	0.571	0.612	0.632
	1.8	0.627	0.632	0.635	0.637

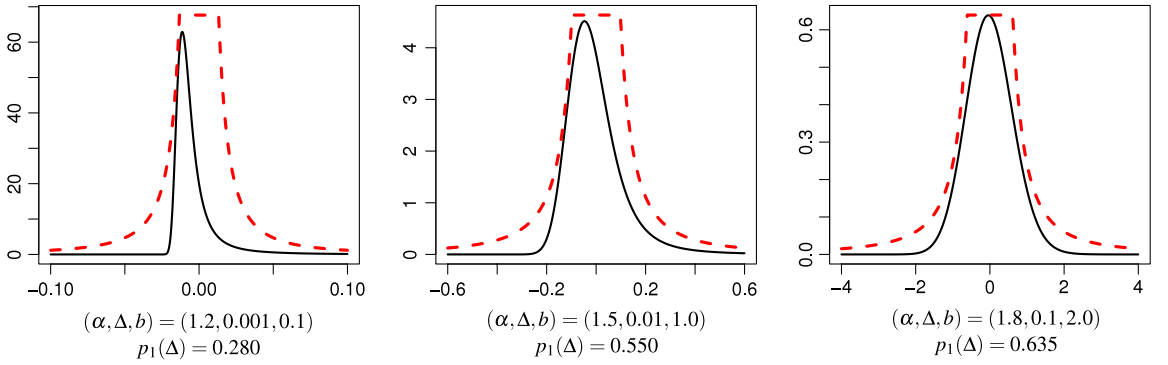

Fig. 1. Comparison of $f_{TS(\alpha,\Delta a,b)}(z)$ (solid line) and $q_\Delta(z)$ (dotted line) in the inequality (3.1). We fix $a = 1$ here.

To discuss the efficiency of this numerical approach, the key quantities are the constants $C_1(\Delta)$ and $C_2(\Delta)$ and the acceptance rate $p_1(\Delta)$. As $\Delta \downarrow 0$, it tends to be more expensive to compute $C_1(\Delta)$ due to $\lim_{\Delta\downarrow 0} |\varphi_\Delta(y)| = 1$ for each $y \in \mathbb{R}$, while less expensive to compute $C_2(\Delta)$ due to $\lim_{\Delta\downarrow 0} |\varphi''_\Delta(y)| = 0$. It however seems difficult to discuss the computing time required for both in total. We report in Table 1 the acceptance rate $p_1(\Delta)$ for various parameter settings. It seems safe to conclude that Algorithm 1 tends to be more efficient (i) with a larger Δ, (ii) with a larger α, and (iii) with a larger b. In other words, Algorithm 1 is more efficient when the tempered stable distribution is closer to a Gaussian distribution. (It is known that the tempered stable distribution approaches to a Gaussian distribution with larger Δ, b and α, while it is closer to a stable distribution with smaller Δ and b. See, for example, Section 3 of [2].) We also provide in Fig. 1 comparisons of the density $f_{TS(\alpha,\Delta a,b)}(z)$ and its bounding function $q_\Delta(z)$ in the inequality (3.1). In conclusion, it seems sensible to employ this approach for simulation of increments over a longer time stepsize, but not for simulation of small increments, for example in approximation of stochastic differential equations.

Remark 3.1. For simulation of the bilateral tempered stable distribution, that is, with a characteristic function

$$y \mapsto \exp\left[\Delta \int_{\mathbb{R}_0} \left(e^{iyz} - 1 - iyz\right)\left(a_+ \frac{e^{-b_+ z}}{z^{\alpha_+ + 1}} \mathbb{1}_{\mathbb{R}_+}(z) + a_- \frac{e^{-b_- |z|}}{|z|^{\alpha_- + 1}} \mathbb{1}_{\mathbb{R}_-}(z)\right) dz\right],$$

we need to implement Algorithm 1 at least twice; once for the positive component and the other for the negative. This is so because the simple relation (2.6) does not hold simultaneously for both the positive and negative components. □

3.2. Approximative sampling with stable proposal distribution

The second acceptance–rejection sampling is an approximative method of [11]. Let us first state the algorithm.

Algorithm 2.

Step 0. Fix $c > 0$.

Step 1. Generate U as uniform (0, 1) and $V(\Delta)$ as $S(\alpha, \Delta a)$.

Step 2. If $U \le e^{-b(V(\Delta)+c)}$, exit with $Y_2(\Delta, c) \leftarrow V(\Delta) - \Delta\Gamma(1-\alpha)ab^{\alpha-1}$. Otherwise, return to Step 1.

This is not an exact simulation algorithm, that is, $\mathscr{L}(Y_2(\Delta, c)) \neq TS(\alpha, \Delta a, b)$ for any $c \in \mathbb{R}_+$, due to the support of the whole real line $\mathbb{R}$, rather than the half line $\mathbb{R}_+$ for $\alpha \in (0, 1)$. The constant shift $-\Delta\Gamma(1-\alpha)ab^{\alpha-1}$ in Step 2 accounts for the difference between (2.3) and (2.4). Basic properties of Algorithm 2 are discussed in [11]. The acceptance rate at Step 2

Table 2
Numerical results of distribution error and acceptance rate for different levels c.

$\Delta = 0.01$				$\Delta = 0.10$			
c	$D_{\mathrm{KS}}(\Delta, c)$	$p_2(\Delta, c)$	$\frac{D_{\mathrm{KS}}(\Delta,c)}{p_2(\Delta,c)}$	c	$D_{\mathrm{KS}}(\Delta, c)$	$p_2(\Delta, c)$	$\frac{D_{\mathrm{KS}}(\Delta,c)}{p_2(\Delta,c)}$
0.00	2.48E−2	0.954	2.60E−2	0.0	1.26E−1	0.878	1.43E−1
0.06	1.58E−2	0.931	1.69E−2	0.6	3.68E−2	0.661	5.57E−2
0.12	6.28E−3	0.896	7.02E−3	0.8	1.29E−2	0.560	2.31E−2
0.13	5.03E−3	0.889	5.66E−3	1.0	2.65E−3	0.464	5.71E−3
0.14	4.73E−3	0.881	5.37E−3	1.1	9.48E−4	0.421	2.25E−3
0.15	4.73E−3	0.874	5.42E−3	1.2	9.10E−4	0.381	2.39E−3
0.18	4.73E−3	0.850	5.57E−3	1.3	9.10E−4	0.345	2.64E−3

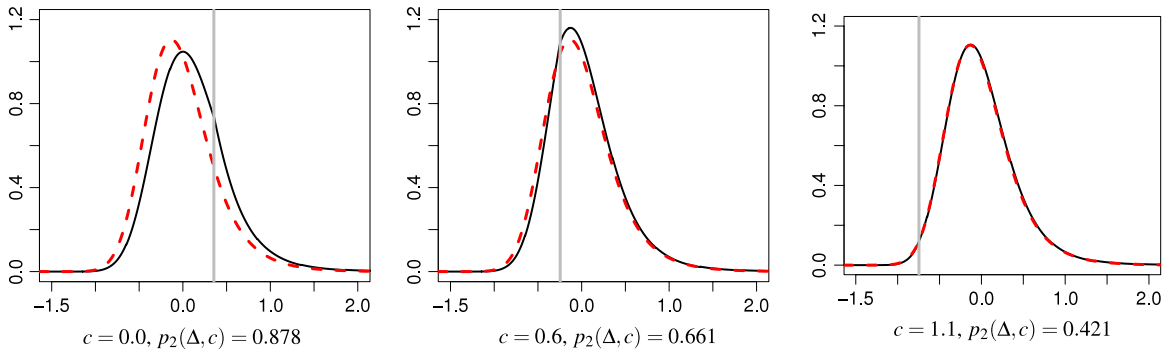

Fig. 2. Comparison of two density functions $f_{\mathscr{L}(Y_2(\Delta,c))}(z)$ (solid line) and $f_{TS(\alpha,\Delta a,b)}(z)$ (dotted line) under $(\alpha, a, b) = (1.5,\ 1.0,\ 1.0)$ and $\Delta = 0.10$. The horizontal line indicates $x = -c - \Delta\Gamma(1-\alpha)ab^{\alpha-1}$.

of Algorithm 2 is

$$p_2(\Delta, c) := \mathbb{E}\left[\mathrm{e}^{-b(V(\Delta)+c)};\ V(\Delta) > -c\right] + \mathbb{P}\left(V(\Delta) \le -c\right).$$

The distribution function $\mathbb{P}(Y_2(\Delta, c) \le z)$ and a density function $f_{\mathscr{L}(Y_2(\Delta,c))}(z)$ are given by

$$\mathbb{P}(Y_2(\Delta, c) \le z) = \frac{1}{p_2(\Delta, c)}\left(\mathbb{P}\left(V(\Delta) \le \min(x, -c)\right) + \int_{\min(x,-c)}^{z} \mathrm{e}^{-b(y+c)} f_{S(\alpha,\Delta a)}(y)\mathrm{d}y\right),$$

$$f_{\mathscr{L}(Y_2(\Delta,c))}(z) = \begin{cases} p_2(\Delta, c)^{-1} f_{S(\alpha,\Delta a)}(z), & \text{if } z \in (-\infty,\ -c], \\ p_2(\Delta, c)^{-1}\mathrm{e}^{-b(z+c)} f_{S(\alpha,\Delta a)}(z), & \text{if } z \in (-c,\ +\infty). \end{cases}$$

The parameter c in Algorithm 2 acts as a truncation of the entire real line $\mathbb{R}$ to the domain on which the exponential tempering e^{-bz} is performed. It is also proved in Theorem 8 [11] that the density $f_{\mathscr{L}(Y_2(\Delta,c))}(z)$ converges in $L^1(\mathbb{R})$ to its target density $f_{TS(\alpha,\Delta a,b)}(z)$ as $c \uparrow +\infty$, and as a consequence, the Kolmogorov–Smirnov distance $D_{\mathrm{KS}}(\Delta, c) := D_{\mathrm{KS}}(\mathscr{L}(Y_2(\Delta, c)), TS(\alpha, \Delta a, b))$ converges to zero as well. Nevertheless, it is not sensible to simply aim at a smaller distribution error by taking $c \uparrow +\infty$, since then Algorithm 3 becomes extremely inefficient due to the low acceptance rate, that is, for each $\Delta > 0$, $\lim_{c\uparrow+\infty} p_2(\Delta, c) = 0$. (Note also that for each $c > 0$, $\lim_{\Delta\downarrow 0} p_2(\Delta, c) = 1$.)

Concerning the computing effort, as before, we wish to find c maximizing $p_2(\Delta, c)$. Asymptotic behaviors of $p_2(\Delta, c)$ with respect to c are difficult to obtain in closed form. Next, it is not clear how to choose an appropriate criterion to measure the distribution error. Natural candidates include $L^1(\mathbb{R})$- and $L^2(\mathbb{R})$-distances between $f_{\mathscr{L}(Y_2(\Delta,c))}(z)$ and $f_{TS(\alpha,\Delta a,b)}(z)$, while the Kolmogorov–Smirnov distance $D_{\mathrm{KS}}(\Delta, c) := D_{\mathrm{KS}}(\mathscr{L}(Y_2(\Delta, c)), TS(\alpha, \Delta a, b))$ is certainly valid as well. None of them are tractable in an explicit manner. Let us present in Table 2 numerical results of the quantity $D_{\mathrm{KS}}(\Delta, c)/p_2(\Delta, c)$ for different values of c. We only provide results for a single parameter set $(\alpha, a, b) = (1.5, 1.0, 1.0)$ and $\Delta = 0.1$ and $\Delta = 0.01$.

It can be observed that the Kolmogorov–Smirnov distance $D_{\mathrm{KS}}(\Delta, c)$ decreases in c, while the acceptance rate $p_2(\Delta, c)$ has to decrease as well. The quantities $D_{\mathrm{KS}}(\Delta, c)/p_2(\Delta, c)$ indicate that choosing c greater than around 0.14 for $\Delta = 0.01$ and 1.1 for $\Delta = 0.1$ would not help in total, just as can be seen from the fact that $D_{\mathrm{KS}}(\Delta, c)$ does not improve anymore, while the acceptance rate $p_2(\Delta, c)$ still gets worse. Let us however remind again that the Kolmogorov–Smirnov distance is simply one of various candidates as a measure of distribution error and the best choice of the parameter c may be different for different criteria. Finally, we draw in Fig. 2 some resulting density functions $f_{\mathscr{L}(Y_2(\Delta,c))}(z)$ with different choices of c when $\Delta = 0.1$, together with the target tempered stable density function $f_{TS(\alpha,\Delta a,b)}(z)$. We do not provide figures for $\Delta = 0.01$, while two densities are almost indistinguishable even with a very small $c > 0$.

The implementation of Algorithm 2 is very simple and requires no computation of a density function unlike in Algorithm 1. In particular, when Δ is small, the acceptance rate remains remarkably high while the distribution error is negligible. We may find an optimal parameter value of c instantaneously through a standard numerical approach, such as

the Nelder–Mead direct search method. In total, this algorithm would be a better choice than Algorithm 1 for simulation of the tempered stable distribution from a practical point of view.

Remark 3.2. The Zolotarev integral representation is known [16] even for $\alpha \in (1, 2)$, but has to be expressed separately on the positive and negative domains. It thus seems difficult to develop a double rejection method [17] of practical use. □

4. Decomposition into small and large jump components

In this section, we consider approximative simulation methods based on decomposition into a small jump component and the remaining large jump component. To be more precise, write

$$g_\Delta(z) := \Delta a \frac{\mathrm{e}^{-bz}}{z^{\alpha+1}}, \quad z \in \mathbb{R}_+,$$

and decompose the characteristic exponent $\int_{\mathbb{R}_+} (\mathrm{e}^{\mathrm{i}yz} - 1 - \mathrm{i}yz)\Delta a z^{-\alpha-1}\mathrm{e}^{-bz}\mathrm{d}z$ into three independent components as

$$\begin{aligned}\ln \mathbb{E}\left[\mathrm{e}^{\mathrm{i}yX(\Delta)}\right] &= \int_{\mathbb{R}_+} (\mathrm{e}^{\mathrm{i}yz} - 1 - \mathrm{i}yz) g_\Delta(z)\mathrm{d}z \\ &= \int_0^\varepsilon (\mathrm{e}^{\mathrm{i}yz} - 1 - \mathrm{i}yz) g_\Delta(z)\mathrm{d}z + \int_\varepsilon^{+\infty} (\mathrm{e}^{\mathrm{i}yz} - 1) g_\Delta(z)\mathrm{d}z - \mathrm{i}y \int_\varepsilon^{+\infty} z g_\Delta(z)\mathrm{d}z, \\ &=: h^{(1)}_{\varepsilon,\Delta}(y) + h^{(2)}_{\varepsilon,\Delta}(y) - h^{(3)}_{\varepsilon,\Delta}(y),\end{aligned}$$

for some $\varepsilon > 0$. First, the component $h^{(3)}_{\varepsilon,\Delta}(y)$ clearly corresponds to a constant as

$$h^{(3)}_{\varepsilon,\Delta}(y) = \mathrm{i}y \frac{\Delta a}{\alpha - 1}\left(\varepsilon^{1-\alpha}\mathrm{e}^{-b\varepsilon} - b^{\alpha-1}\Gamma(2-\alpha, b\varepsilon)\right) =: \mathrm{i}y\theta_{\varepsilon,\Delta}.$$

In what follows, we use the notations $H^{(1)}_{\varepsilon,\Delta}$ and $H^{(2)}_{\varepsilon,\Delta}$ for random variables satisfying

$$\ln \mathbb{E}\left[\mathrm{e}^{\mathrm{i}yH^{(k)}_{\varepsilon,\Delta}}\right] = h^{(k)}_{\varepsilon,\Delta}(y), \quad k = 1, 2,$$

and call $H^{(1)}_{\varepsilon,\Delta}$ the small jump component and $H^{(2)}_{\varepsilon,\Delta}$ the large jump component. It holds that for each $\varepsilon > 0$ and $\Delta > 0$,

$$X(\Delta) \stackrel{\mathscr{L}}{=} H^{(1)}_{\varepsilon,\Delta} + H^{(2)}_{\varepsilon,\Delta} - \theta_{\varepsilon,\Delta}.$$

4.1. Simulation of large jump component

We first discuss simulation of the large jump component $H^{(2)}_{\varepsilon,\Delta}$. This component is compound Poisson with intensity

$$\xi_{\varepsilon,\Delta} := \int_\varepsilon^{+\infty} \Delta a \frac{\mathrm{e}^{-bz}}{z^{\alpha+1}}\mathrm{d}z = \frac{\Delta a}{\alpha(\alpha-1)}\left(\frac{(\alpha-1)\mathrm{e}^{-b\varepsilon}}{\varepsilon^\alpha} - \frac{b\mathrm{e}^{-b\varepsilon}}{\varepsilon^{\alpha-1}} + b^\alpha\Gamma(2-\alpha, b\varepsilon)\right). \tag{4.1}$$

4.1.1. Straightforward compound Poisson simulation

The straightforward method is based on the summation of iid suitable random variables through

$$H^{(2)}_{\varepsilon,\Delta} \stackrel{\mathscr{L}}{=} \sum_{k=1}^{N_{\varepsilon,\Delta}} Y_k,$$

where $N_{\varepsilon,\Delta}$ is a Poisson random variate with intensity $\xi_{\varepsilon,\Delta}$ and $\{Y_k\}_{k\in\mathbb{N}}$ is a sequence of iid random variables with common probability density

$$\frac{1}{\xi_{\varepsilon,\Delta}}\Delta a \frac{\mathrm{e}^{-bz}}{z^{\alpha+1}}, \quad z \in (\varepsilon, +\infty).$$

This concept is indeed straightforward, while never as handy as often claimed in the literature, for mainly two reasons. First, when the truncation ε is chosen too small (which is in principle desirable), the compound Poisson intensity $\xi_{\varepsilon,\Delta}$ explodes of order $\varepsilon^{-\alpha}$. Also, we need to deal with numerical integration and inversion of the incomplete gamma function for the common distribution of the random sequence $\{Y_k\}_{k\in\mathbb{N}}$.

4.1.2. Acceptance–rejection sampling with Pareto proposal distribution

We here present an exact simulation technique for the large jump component $H^{(2)}_{\varepsilon,\Delta}$, which gets around the aforementioned numerical integration and inversion. Notice that the Lévy density of the component is bounded from

above as

$$\Delta a \frac{\mathrm{e}^{-bz}}{z^{\alpha+1}} \le \Delta a \frac{\mathrm{e}^{-b\varepsilon}}{\alpha\varepsilon^{\alpha}} \left(\frac{\alpha\varepsilon^{\alpha}}{z^{\alpha+1}} \right) =: \xi^{(0)}_{\varepsilon,\Delta} \frac{\alpha\varepsilon^{\alpha}}{z^{\alpha+1}}, \quad z \in (\varepsilon, +\infty), \tag{4.2}$$

where $\alpha\varepsilon^{\alpha} z^{-\alpha-1}$ serves as a Pareto probability density function on $(\varepsilon, +\infty)$. This Pareto random variable can easily be simulated as $\varepsilon U^{-1/\alpha}$, where $U \sim U(0, 1)$. Suppose that ε and Δ are set such that $\xi^{(0)}_{\varepsilon,\Delta} \ge 1$. Then, we can employ acceptance–rejection sampling for simulating the compound Poisson component as follows.

Algorithm 3.

Step 1. Generate U_1 and U_2 as independent uniform (0, 1) and let $V \leftarrow \varepsilon U_2^{-1/\alpha}$.

Step 2. If $U_1 \le \mathrm{e}^{-b(V-\varepsilon)}$, exit with $Y_3(\Delta) \leftarrow V$. Otherwise, return to Step 1.

The acceptance rate at Step 2 of Algorithm 3 is $1/\xi^{(0)}_{\varepsilon,\Delta}$. The expected number of times Step 1 is executed is thus $\xi^{(0)}_{\varepsilon,\Delta}$, while the expected number of times Algorithm 3 is executed for simulation of $H^{(2)}_{\varepsilon,\Delta}$ is $\xi_{\varepsilon,\Delta}$. Therefore, Step 1 will be executed $\xi_{\varepsilon,\Delta}\xi^{(0)}_{\varepsilon,\Delta}$ times on average. Due to the explosion of this expected execution time for ideally small ε, it is difficult at this stage to claim that the discussed simulation of large jump component is useful, although it is exact and straightforward to implement. It also seems difficult to find a simulation method for the jump component dominantly better than Algorithm 3.

In what follows, we assume that the compound Poisson component $H^{(2)}_{\varepsilon,\Delta}$ is always simulated through Algorithm 3 and will say that the expected *time* required for simulation of $H^{(2)}_{\varepsilon,\Delta}$ is $\xi_{\varepsilon,\Delta}\xi^{(0)}_{\varepsilon,\Delta}$.

4.2. Simulation of small jump component

In this section, we turn to simulation of the small jump component $H^{(1)}_{\varepsilon,\Delta}$. As we have just observed, simulation of large jump component is exact but computationally very demanding. Hence, in order to convince the user of the significance of the decomposition into small and large jump components for simulation purposes, it is necessary to have a very efficient and nearly exact method for small jump component.

Throughout this subsection, we will use the notation

$$\kappa_k(\varepsilon, \Delta) := \int_0^{\varepsilon} z^k g_{\Delta}(z)\mathrm{d}z = \frac{\Delta a}{b^{k-\alpha}} \gamma(k-\alpha, b\varepsilon).$$

Clearly, since the random variable $H^{(1)}_{\varepsilon,\Delta}$ consists of infinitely many jumps, the compound Poisson simulation never applies. To investigate approximative simulation techniques for $H^{(1)}_{\varepsilon,\Delta}$, let us derive its first three moments

$$\begin{aligned}
&\mathbb{E}\left[H^{(1)}_{\varepsilon,\Delta}\right] = 0,\\
&\mathrm{Var}\left(H^{(1)}_{\varepsilon,\Delta}\right) = \int_0^{\varepsilon} z^2 g_{\Delta}(z)\mathrm{d}z = \kappa_2(\varepsilon, \Delta) =: \sigma^2_{\varepsilon,\Delta},\\
&\mathbb{E}\left[\left(H^{(1)}_{\varepsilon,\Delta} - \mathbb{E}\left[H^{(1)}_{\varepsilon,\Delta}\right]\right)^3\right] = \int_0^{\varepsilon} z^3 g_{\Delta}(z)\mathrm{d}z = \kappa_3(\varepsilon, \Delta).
\end{aligned}$$

Note also that for each $\Delta > 0$, $\kappa_k(\varepsilon, \Delta) \sim \varepsilon^{k-\alpha}\Delta a/(k-\alpha)$, as $\varepsilon \downarrow 0$.

First, based on the zero mean, it would be a valid approximation to simply replace $H^{(1)}_{\varepsilon,\Delta}$ by the mean value 0. The expected time required for this approximative simulation of $X(\Delta)$ remains same as the one required for simulation of $H^{(2)}_{\varepsilon,\Delta}$ and is thus $\xi_{\varepsilon,\Delta}\xi^{(0)}_{\varepsilon,\Delta}$. For evaluation of weak approximation error, let us adopt the framework of Signahl [22] It holds by the Taylor theorem that for each $f \in C_b^{\infty}(\mathbb{R}; \mathbb{R})$,

$$\mathbb{E}[f(X(\Delta))] - \mathbb{E}\left[f\left(H^{(2)}_{\varepsilon,\Delta} - \theta_{\varepsilon,\Delta}\right)\right] = \frac{\sigma^2_{\varepsilon,\Delta}}{2}\mathbb{E}\left[f''\left(H^{(2)}_{\varepsilon,\Delta} - \theta_{\varepsilon,\Delta}\right)\right] + \frac{\kappa_3(\varepsilon, \Delta)}{6}\mathbb{E}\left[f'''\left(H^{(2)}_{\varepsilon,\Delta} - \theta_{\varepsilon,\Delta}\right)\right] + \cdots.$$

Hence, we get for each $\Delta > 0$,

$$\left|\mathbb{E}[f(X(\Delta))] - \mathbb{E}\left[f\left(H^{(2)}_{\varepsilon,\Delta} - \theta_{\varepsilon,\Delta}\right)\right]\right| = O\left(\varepsilon^{2-\alpha}\right),$$

as $\varepsilon \downarrow 0$.

Next, it would be better to replace $H^{(1)}_{\varepsilon,\Delta}$ by with a normal random variable $Z_{\varepsilon,\Delta}$, where $Z_{\varepsilon,\Delta} \sim \mathcal{N}(0, \sigma^2_{\varepsilon,\Delta})$. Let us write $X^{(0)}(\varepsilon, \Delta) := Z_{\varepsilon,\Delta} + H^{(2)}_{\varepsilon,\Delta} - \theta_{\varepsilon,\Delta}$. This Gaussian approximation was justified in [12,23] in a rigorous manner and holds true in this case since $\sigma^2_{\varepsilon,\Delta}/\varepsilon^2 \sim \Delta a\varepsilon^{-\alpha}/(2-\alpha) \uparrow +\infty$, as $\varepsilon \downarrow 0$. By the addition of this Gaussian component, the expected

time required for simulation of $X^{(0)}(\varepsilon, \Delta)$ is increased by 1 and is thus

$$\tau^{(0)}_{\varepsilon,\Delta} := 1 + \xi_{\varepsilon,\Delta}\xi^{(0)}_{\varepsilon,\Delta}. \tag{4.3}$$

Taking into account the undesirable limit $\lim_{\varepsilon\downarrow 0}\xi_{\varepsilon,\Delta}\xi^{(0)}_{\varepsilon,\Delta} = +\infty$, the addition of the Gaussian component is negligible in terms of computing effort. Now, it holds by the Taylor theorem that for each $f \in C^\infty_b(\mathbb{R};\mathbb{R})$,

$$\mathbb{E}\left[f\left(X^{(0)}(\varepsilon,\Delta)\right)\right] - \mathbb{E}\left[f\left(X(\Delta)\right)\right] = \mathbb{E}\left[\left(Z_{\varepsilon,\Delta} - H^{(1)}_{\varepsilon,\Delta}\right)f'\left(X(\Delta)\right)\right] + \frac{1}{2}\mathbb{E}\left[\left(Z_{\varepsilon,\Delta} - H^{(1)}_{\varepsilon,\Delta}\right)^2 f''\left(X(\Delta)\right)\right] + \cdots.$$

By further Taylor expansions and using the knowledge of $\mathscr{L}(Z_{\varepsilon,\Delta})$, we get

$$\left|\mathbb{E}\left[f\left(X^{(0)}(\varepsilon,\Delta)\right)\right] - \mathbb{E}\left[f\left(X(\Delta)\right)\right]\right| \sim \frac{\kappa_3(\varepsilon,\Delta)}{6}\left|\mathbb{E}\left[f'''\left(H^{(2)}_{\varepsilon,\Delta} - \theta_{\varepsilon,\Delta}\right)\right]\right|,$$

as $\varepsilon \downarrow 0$. Recall that the true distribution has characteristic function $\varphi_\Delta(y)$. Meanwhile, it is straightforward that the approximation has characteristic function

$$\begin{aligned}\varphi^{(0)}_{\varepsilon,\Delta}(y) := \mathbb{E}\left[\mathrm{e}^{\mathrm{i}yX^{(0)}(\varepsilon,\Delta)}\right] &= \varphi_\Delta(y)\exp\left[-\int_0^\varepsilon\left(\mathrm{e}^{\mathrm{i}yz} - 1 - \mathrm{i}yz + \frac{1}{2}y^2z^2\right)g_\Delta(z)\mathrm{d}z\right]\\ &= \varphi_\Delta(y)\exp\left[-\frac{|y|^3}{3!}\int_0^\varepsilon \eta(yz)z^3 g_\Delta(z)\mathrm{d}z\right],\end{aligned}$$

where η is a function from $\mathbb{R}$ to $\mathbb{C}$ satisfying $|\eta(x)| \le 1$ for $x \in \mathbb{R}$. Therefore, it holds by the Parseval theorem that

$$\begin{aligned}\rho^{(0)}_{\varepsilon,\Delta} &:= \int_{\mathbb{R}}\left|f_{TS(\alpha,\Delta a,b)}(z) - f_{\mathscr{L}(X^{(0)}(\varepsilon,\Delta))}(z)\right|^2\mathrm{d}z\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\left|\varphi_\Delta(y) - \varphi^{(0)}_{\varepsilon,\Delta}(y)\right|^2\mathrm{d}y\\ &\sim \frac{1}{2\pi}\int_{\mathbb{R}}|\varphi_\Delta(y)|^2\left|\frac{|y|^3}{3!}\int_0^\varepsilon \eta(yz)z^3 g_\Delta(z)\mathrm{d}z\right|^2\mathrm{d}y\\ &\le \frac{\kappa_3(\varepsilon,\Delta)^2}{72\pi}\int_{\mathbb{R}}\left|y^3\varphi_\Delta(y)\right|^2\mathrm{d}y = O\left(\varepsilon^{6-2\alpha}\right),\end{aligned} \tag{4.4}$$

where all the asymptotics hold when $\varepsilon \downarrow 0$.

4.2.1. Further compound Poisson of constant density

The discussed Gaussian approximation is clearly very handy. It is then natural to ask whether the approximation error can be reduced without a significant increase in computing time. A straightforward approach is to decompose the Lévy density $g_\Delta(z)$ over $(0,\varepsilon)$ into two independent components $g^{(\mathfrak{a})}_{\varepsilon,\Delta}(z) := g_\Delta(\varepsilon)$ and $g_\Delta(z) - g^{(\mathfrak{a})}_{\varepsilon,\Delta}(z)$. Accordingly, we write

$$H^{(1)}_{\varepsilon,\Delta} = J^{(\mathfrak{a})}_{\varepsilon,\Delta} + K^{(\mathfrak{a})}_{\varepsilon,\Delta}$$

for this decomposition. The first component $J^{(\mathfrak{a})}_{\varepsilon,\Delta}$ corresponds to the constant Lévy density $g^{(\mathfrak{a})}_{\varepsilon,\Delta}(\varepsilon)$ and is thus compound Poisson with intensity

$$\xi^{(\mathfrak{a})}_{\varepsilon,\Delta} := \frac{\Delta a\mathrm{e}^{-b\varepsilon}}{\varepsilon^\alpha}, \tag{4.5}$$

with iid density $U(0,\varepsilon)$. This compound Poisson component should be centered and can be simulated in the exact sense as

$$J^{(\mathfrak{a})}_{\varepsilon,\Delta} \leftarrow \sum_{k=1}^{N^{(\mathfrak{a})}_{\varepsilon,\Delta}}\varepsilon U_k - \frac{\Delta a\mathrm{e}^{-b\varepsilon}}{2\varepsilon^{\alpha-1}}.$$

The remaining component $K^{(\mathfrak{a})}_{\varepsilon,\Delta}$, corresponding to the Lévy density $g_\Delta(z) - g^{(\mathfrak{a})}_{\varepsilon,\Delta}(z)$, is still centered, is of infinite activity and is thus approximated by a normal random variable. Here, we define for each $k \in \mathbb{N}$,

$$\begin{aligned}\kappa^{(\mathfrak{a})}_k(\varepsilon,\Delta) &:= \int_0^\varepsilon z^k\left(g_\Delta(z) - g^{(\mathfrak{a})}_{\varepsilon,\Delta}(z)\right)\mathrm{d}z\\ &= \frac{\Delta a}{b^{k-\alpha}}\gamma(k-\alpha, b\varepsilon) - \frac{\Delta a\mathrm{e}^{-b\varepsilon}}{(k+1)\varepsilon^{\alpha-k}}\\ &\sim \Delta a\varepsilon^{k-\alpha}\frac{1+\alpha}{(k-\alpha)(k+1)},\end{aligned}$$

as $\varepsilon \downarrow 0$. The Gaussian approximation can easily be justified with a variance $\kappa_2^{(\mathrm{a})}(\varepsilon, \Delta)$. Let $Z_{\varepsilon,\Delta}^{(\mathrm{a})}$ be a normal random variable with mean zero and a variance $\kappa_2^{(\mathrm{a})}(\varepsilon, \Delta)$. In a similar manner to the previous case, it holds by the Taylor theorem that for each $f \in C_b^\infty(\mathbb{R}; \mathbb{R})$,

$$\mathbb{E}\left[f\left(X^{(\mathrm{a})}(\varepsilon, \Delta)\right)\right] - \mathbb{E}\left[f\left(X(\Delta)\right)\right] = \mathbb{E}\left[\left(Z_{\varepsilon,\Delta}^{(\mathrm{a})} - K_{\varepsilon,\Delta}^{(\mathrm{a})}\right) f'\left(X(\Delta)\right)\right] + \frac{1}{2}\mathbb{E}\left[\left(Z_{\varepsilon,\Delta}^{(\mathrm{a})} - K_{\varepsilon,\Delta}^{(\mathrm{a})}\right)^2 f''\left(X(\Delta)\right)\right] + \cdots,$$

where

$$X^{(\mathrm{a})}(\varepsilon, \Delta) := Z_{\varepsilon,\Delta}^{(\mathrm{a})} + J_{\varepsilon,\Delta}^{(\mathrm{a})} + H_{\varepsilon,\Delta}^{(2)} - \theta_{\varepsilon,\Delta}.$$

By further Taylor expansions of f' and f'' and using the fact that $Z_{\varepsilon,\Delta}^{(\mathrm{a})}$ is Gaussian, we get

$$\left|\mathbb{E}\left[f\left(X^{(\mathrm{a})}(\varepsilon, \Delta)\right)\right] - \mathbb{E}\left[f\left(X(\Delta)\right)\right]\right| \sim \frac{\kappa_3^{(\mathrm{a})}(\varepsilon, \Delta)}{6}\left|\mathbb{E}\left[f'''\left(H_{\varepsilon,\Delta}^{(2)} - \theta_{\varepsilon,\Delta}\right)\right]\right|,$$

as $\varepsilon \downarrow 0$. Therefore, by further introducing this compound Poisson $J_{\varepsilon,\Delta}^{(\mathrm{a})}$, we can reduce the weak error by a factor of

$$\lim_{\varepsilon\downarrow 0} \frac{\kappa_3^{(\mathrm{a})}(\varepsilon, \Delta)}{\kappa_3(\varepsilon, \Delta)} = \frac{1+\alpha}{4} \in \left(\frac{1}{2}, \frac{3}{4}\right).$$

As before, it holds by the Parseval theorem that

$$\begin{aligned}
\rho_{\varepsilon,\Delta}^{(\mathrm{a})} &:= \frac{1}{2\pi}\int_{\mathbb{R}} \left|\varphi_\Delta(y) - \varphi_{\varepsilon,\Delta}^{(\mathrm{a})}(y)\right|^2 \mathrm{d}y \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |\varphi_\Delta(y)|^2 \left|\exp\left[-\int_0^\varepsilon \left(\mathrm{e}^{\mathrm{i}yz} - 1 - \mathrm{i}yz\right)\left(g_\Delta(z) - g_{\varepsilon,\Delta}^{(\mathrm{a})}(z)\right)\mathrm{d}z\right] - 1\right|^2 \mathrm{d}y \\
&\sim \frac{1}{2\pi}\int_{\mathbb{R}} |\varphi_\Delta(y)|^2 \left|\frac{|y|^3}{3!}\int_0^\varepsilon \eta(yz) z^3 \left(g_\Delta(z) - g_{\varepsilon,\Delta}^{(\mathrm{a})}(z)\right)\mathrm{d}z\right|^2 \mathrm{d}y \\
&\le \frac{\left(\kappa_3^{(\mathrm{a})}(\varepsilon, \Delta)\right)^2}{72\pi}\int_{\mathbb{R}} \left|y^3\varphi_\Delta(y)\right|^2 \mathrm{d}y,
\end{aligned} \tag{4.6}$$

where all the asymptotics hold when $\varepsilon \downarrow 0$. Clearly, the expected time required for simulation of $X^{(\mathrm{a})}(\varepsilon, \Delta)$ is given by

$$\tau_{\varepsilon,\Delta}^{(\mathrm{a})} := 1 + \xi_{\varepsilon,\Delta}^{(\mathrm{a})} + \xi_{\varepsilon,\Delta}\xi_{\varepsilon,\Delta}^{(0)}. \tag{4.7}$$

Remark 4.1. It is straightforward that the compound Poisson component $J_{\varepsilon,\Delta}^{(\mathrm{a})}$ and the large jump component $H_{\varepsilon,\Delta}^{(2)}$ of Section 4.1 can be simulated exactly as a single compound Poisson random variable. Define

$$l_{\varepsilon,\Delta}^{(\mathrm{a})} := \frac{\xi_{\varepsilon,\Delta}^{(\mathrm{a})}}{\xi_{\varepsilon,\Delta}^{(\mathrm{a})} + \xi_{\varepsilon,\Delta}}, \qquad r_{\varepsilon,\Delta}^{(\mathrm{a})} := \frac{\xi_{\varepsilon,\Delta}}{\xi_{\varepsilon,\Delta}^{(\mathrm{a})} + \xi_{\varepsilon,\Delta}},$$

where $\xi_{\varepsilon,\Delta}^{(\mathrm{a})}$ and $\xi_{\varepsilon,\Delta}$ are defined, respectively, in (4.5) and (4.1). Clearly, $l_{\varepsilon,\Delta}^{(\mathrm{a})} + r_{\varepsilon,\Delta}^{(\mathrm{a})} = 1$. Then, it holds that

$$J_{\varepsilon,\Delta}^{(\mathrm{a})} + H_{\varepsilon,\Delta}^{(2)} \stackrel{\mathscr{L}}{=} \sum_{k=1}^{\widetilde{N}_{\varepsilon,\Delta}^{(\mathrm{a})}} Y_k^{(\mathrm{a})} - \Delta a \mathrm{e}^{-b\varepsilon} 2\varepsilon^{\alpha-1},$$

where $\widetilde{N}_{\varepsilon,\Delta}^{(\mathrm{a})}$ is a Poisson random variable with intensity $\xi_{\varepsilon,\Delta}^{(\mathrm{a})} + \xi_{\varepsilon,\Delta}$ and $\{Y_k^{(\mathrm{a})}\}_{k\in\mathbb{N}}$ is a sequence of iid random variables with common distribution which can be simulated exactly as follows.

Algorithm $3^{(\mathrm{a})}$**.**

Step 1. Generate $U_1 \leftarrow U(0, 1)$.

Step 2. If $U_1 \le l_{\varepsilon,\Delta}^{(\mathrm{a})}$, then exit with $\varepsilon U_1 / l_{\varepsilon,\Delta}^{(\mathrm{a})}$.

Step 3. Generate $U_2 \leftarrow U(0, 1)$ and let $V \leftarrow \varepsilon((U_1 - l_{\varepsilon,\Delta}^{(\mathrm{a})})/r_{\varepsilon,\Delta}^{(\mathrm{a})})^{-1/\alpha}$. If $U_1 \le \mathrm{e}^{-b(V-\varepsilon)}$, then exit with V. Otherwise, go to Step 1.

We can show that the expected total time (number of implementations of Step 1) for simulation of $J^{(\mathrm{a})}_{\varepsilon,\Delta}+H^{(2)}_{\varepsilon,\Delta}$ is

$$\left(\xi^{(\mathrm{a})}_{\varepsilon,\Delta}+\xi_{\varepsilon,\Delta}\xi^{(0)}_{\varepsilon,\Delta}\right)\frac{r^{(\mathrm{a})}_{\varepsilon,\Delta}\left(1-1/\xi^{(0)}_{\varepsilon,\Delta}\right)}{\left(1-r^{(\mathrm{a})}_{\varepsilon,\Delta}\left(1-1/\xi^{(0)}_{\varepsilon,\Delta}\right)\right)},$$

while the expected total time for simulation of $J^{(\mathrm{a})}_{\varepsilon,\Delta}$ and $H^{(2)}_{\varepsilon,\Delta}$ separately is $\xi^{(\mathrm{a})}_{\varepsilon,\Delta}+\xi_{\varepsilon,\Delta}\xi^{(0)}_{\varepsilon,\Delta}$. We can also show that the use of Algorithm 3$^{(\mathrm{a})}$ helps reduce computing effort if ε is sufficiently large, while increases it by a factor of 2 as $\varepsilon\downarrow 0$. We thus do not consider using this algorithm as our interest is mainly in a small ε. □

4.2.2. Further compound Poisson of exploding but integrable density

Before proceeding to numerical experiments, let us consider further taking more compound Poisson mass, which can be simulated exactly, out of the small jump component (and then approximate the reminder by Gaussian). To this end, we extract the density

$$g^{(\mathrm{b})}_{\varepsilon,\Delta}(z):=\Delta a\frac{\mathrm{e}^{-b\varepsilon}}{\varepsilon^{1+\delta}}\frac{1}{z^{\alpha-\delta}}, \tag{4.8}$$

for some $\delta\in(\alpha-1,\alpha)$, from the Lévy density $g_\Delta(z)$ over $(0,\varepsilon)$. As before, we decompose as

$$H^{(1)}_{\varepsilon,\Delta}=J^{(\mathrm{b})}_{\varepsilon,\Delta}+K^{(\mathrm{b})}_{\varepsilon,\Delta},$$

where $J^{(\mathrm{b})}_{\varepsilon,\Delta}$ indicates the centered compound Poisson random variable corresponding to the density (4.8), while the $K^{(\mathrm{b})}_{\varepsilon,\Delta}$ is the remaining infinite activity component to be approximated by Gaussian. The compound Poisson intensity is given by

$$\xi^{(\mathrm{b})}_{\varepsilon,\Delta}:=\int_0^\varepsilon g^{(\mathrm{b})}_{\varepsilon,\Delta}(z)\mathrm{d}z=\frac{\Delta a\mathrm{e}^{-b\varepsilon}}{(\delta-\alpha+1)\varepsilon^\alpha}. \tag{4.9}$$

Note here that this is independent of δ in the sense of asymptotics of $\varepsilon\downarrow 0$. We can derive that $J^{(\mathrm{b})}_{\varepsilon,\Delta}$ can be simulated exactly as

$$J^{(\mathrm{b})}_{\varepsilon,\Delta}\leftarrow\sum_{k=1}^{N^{(\mathrm{b})}_{\varepsilon,\Delta}}(\varepsilon^\alpha U_k)^{\frac{1}{\delta-\alpha}}-\varepsilon^{-\alpha\frac{\delta-\alpha-1}{\delta-\alpha}}\frac{a\mathrm{e}^{-b\varepsilon}(\delta-\alpha)}{(\delta-\alpha+1)^2},$$

where $N^{(\mathrm{b})}_{\varepsilon,\Delta}$ is a Poisson random variable with intensity $\xi^{(\mathrm{b})}_{\varepsilon,\Delta}$ and $\{U_k\}$ is a sequence of iid uniform random variables on $(0,1)$ as before. Using

$$\begin{aligned}\kappa^{(\mathrm{b})}_k(\varepsilon,\Delta)&:=\int_0^\varepsilon z^k\left(g_\Delta(z)-g^{(\mathrm{b})}_{\varepsilon,\Delta}(z)\right)\mathrm{d}z\\&=\frac{\Delta a}{b^{k-\alpha}}\gamma(k-\alpha,b\varepsilon)-\frac{\Delta a\mathrm{e}^{-b\varepsilon}}{(k+1+\delta-\alpha)\varepsilon^{\alpha-k}}\\&\sim\frac{\Delta a\varepsilon^{k-\alpha}}{(k-\alpha)(k+1+\delta-\alpha)},\end{aligned}$$

as $\varepsilon\downarrow 0$, we can derive

$$\left|\mathbb{E}\left[f\left(X^{(\mathrm{b})}(\varepsilon,\Delta)\right)\right]-\mathbb{E}\left[f\left(X(\Delta)\right)\right]\right|\sim\frac{\kappa^{(\mathrm{b})}_3(\varepsilon,\Delta)}{6}\left|\mathbb{E}\left[f'''\left(H^{(2)}_{\varepsilon,\Delta}-\theta_{\varepsilon,\Delta}\right)\right]\right|,$$

as $\varepsilon\downarrow 0$, where

$$X^{(\mathrm{b})}(\varepsilon,\Delta):=Z^{(\mathrm{b})}_{\varepsilon,\Delta}+J^{(\mathrm{b})}_{\varepsilon,\Delta}+H^{(2)}_{\varepsilon,\Delta}-\theta_{\varepsilon,\Delta}.$$

Therefore, by further introducing the compound Poisson $J^{(\mathrm{b})}_{\varepsilon,\Delta}$, we can reduce the weak error by a factor of

$$\lim_{\varepsilon\downarrow 0}\frac{\kappa^{(\mathrm{b})}_3(\varepsilon,\Delta)}{\kappa_3(\varepsilon,\Delta)}=\frac{1}{4+\delta-\alpha}.$$

Since δ can be taken arbitrarily in $(\alpha-1,\alpha)$, we can improve the approximation error down by a factor of $1/4$, compared to the simplest Gaussian approximation. Moreover, comparing with the one introduced in Section 4.2.1, we get

$$\lim_{\varepsilon\downarrow 0}\frac{\kappa^{(\mathrm{b})}_3(\varepsilon,\Delta)}{\kappa^{(\mathrm{a})}_3(\varepsilon,\Delta)}=\frac{4}{(1+\alpha)(4+\delta-\alpha)}.$$

Table 3
Numerical results of the approximation error and the required computing time.

$\Delta = 0.01$									
ε	Approximation error			Computing time			Algorithm 2		
	$\rho^{(0)}_{\varepsilon,\Delta}$	$\rho^{(a)}_{\varepsilon,\Delta}$	$\rho^{(b)}_{\varepsilon,\Delta}$	$\tau^{(0)}_{\varepsilon,\Delta}$	$\tau^{(a)}_{\varepsilon,\Delta}$	$\tau^{(b)}_{\varepsilon,\Delta}$	c	Error	Time
0.033	3.48E−2	2.21E−2	7.79E−3	2.15E+0	3.77E+0	1.83E+1	0.00	8.83E−2	1.04E+0
0.030	3.03E−2	1.92E−2	6.74E−3	2.55E+0	4.41E+0	2.21E+1	0.12	2.55E−2	1.11E+0
0.020	1.68E−2	1.06E−2	3.65E−3	6.33E+0	9.80E+0	4.10E+1	0.14	1.66E−2	1.13E+0
0.010	6.01E−3	3.76E−3	1.28E−3	4.46E+1	5.45E+1	1.44E+2	0.18	5.26E−3	1.17E+0
0.005	2.13E−3	1.33E−3	4.52E−4	3.53E+2	3.81E+2	6.34E+2	0.20	2.54E−3	1.19E+0
0.003	9.91E−4	6.20E−4	2.10E−4	1.64E+3	1.70E+3	2.24E+3	0.24	4.88E−4	1.24E+0
0.001	1.91E−4	1.19E−4	4.02E−5	4.44E+4	4.47E+4	4.75E+4	0.30	2.02E−4	1.32E+0
$\Delta = 0.1$									
ε	Approximation error			Computing time			Algorithm 2		
	$\rho^{(0)}_{\varepsilon,\Delta}$	$\rho^{(a)}_{\varepsilon,\Delta}$	$\rho^{(b)}_{\varepsilon,\Delta}$	$\tau^{(0)}_{\varepsilon,\Delta}$	$\tau^{(a)}_{\varepsilon,\Delta}$	$\tau^{(b)}_{\varepsilon,\Delta}$	c	Error	Time
0.130	1.59E−2	1.03E−2	4.05E−3	2.50E+0	4.38E+0	2.12E+1	0.00	2.02E−1	1.14E+0
0.120	1.43E−2	9.22E−3	3.57E−3	2.96E+0	5.09E+0	2.43E+1	0.60	6.42E−2	1.51E+0
0.100	1.10E−2	7.08E−3	2.68E−3	4.55E+0	7.42E+0	3.32E+1	0.80	2.35E−2	1.78E+0
0.010	3.71E−4	2.33E−4	7.94E−5	4.36E+3	4.46E+3	5.35E+3	1.10	1.93E−3	2.37E+0
0.005	1.32E−4	8.24E−4	2.79E−5	3.52E+4	3.55E+4	3.80E+4	1.30	1.63E−4	2.90E+0
0.001	1.18E−5	7.40E−6	2.50E−6	4.44E+6	4.44E+6	4.47E+6	1.50	9.72E−6	3.54E+0
0.0005	4.20E−6	2.60E−6	9.00E−7	3.55E+7	3.55E+7	3.56E+7	1.80	5.71E−6	4.78E+0

Hence, we can further improve the approximation error by up to a factor of $1/(1+\alpha)$. As before, it holds by the Parseval theorem that

$$\begin{aligned}
\rho^{(b)}_{\varepsilon,\Delta} &:= \frac{1}{2\pi}\int_{\mathbb{R}} \left|\varphi_\Delta(y) - \varphi^{(b)}_{\varepsilon,\Delta}(y)\right|^2 \mathrm{d}y \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |\varphi_\Delta(y)|^2 \left|\exp\left[-\int_0^\varepsilon \left(\mathrm{e}^{\mathrm{i}yz} - 1 - \mathrm{i}yz\right)\left(g_\Delta(z) - g^{(b)}_{\varepsilon,\Delta}(z)\right)\mathrm{d}z\right] - 1\right|^2 \mathrm{d}y \\
&\sim \frac{1}{2\pi}\int_{\mathbb{R}} |\varphi_\Delta(y)|^2 \left|\frac{|y|^3}{3!}\int_0^\varepsilon \eta(yz) z^3 \left(g_\Delta(z) - g^{(b)}_{\varepsilon,\Delta}(z)\right)\mathrm{d}z\right|^2 \mathrm{d}y \\
&\le \frac{\left(\kappa^{(b)}_3(\varepsilon,\Delta)\right)^2}{72\pi}\int_{\mathbb{R}} \left|y^3\varphi_\Delta(y)\right|^2 \mathrm{d}y,
\end{aligned} \tag{4.10}$$

where all the asymptotics hold when $\varepsilon \downarrow 0$. Clearly, the expected time required for simulation of $X^{(b)}(\varepsilon,\Delta)$ is given by

$$\tau^{(b)}_{\varepsilon,\Delta} := 1 + \xi^{(b)}_{\varepsilon,\Delta} + \xi_{\varepsilon,\Delta}\xi^{(0)}_{\varepsilon,\Delta}. \tag{4.11}$$

We are in a position to present some numerical results and discuss whether the decomposition into small and large jump components is useful for simulation purpose. From a practical point of view, we present in Table 3 the quantities $\rho^{(0)}_{\varepsilon,\Delta}$, $\rho^{(a)}_{\varepsilon,\Delta}$ and $\rho^{(b)}_{\varepsilon,\Delta}$, defined respectively by (4.4), (4.6) and (4.10), for the approximation error, and $\tau^{(0)}_{\varepsilon,\Delta}$, $\tau^{(a)}_{\varepsilon,\Delta}$ and $\tau^{(b)}_{\varepsilon,\Delta}$, defined respectively by (4.3), (4.7) and (4.11), for the required computing time. We computed $\rho^{(0)}_{\varepsilon,\Delta}$, $\rho^{(a)}_{\varepsilon,\Delta}$ and $\rho^{(b)}_{\varepsilon,\Delta}$ as precisely as possible by numerical integration of their definitions and did not use their asymptotic upper bounds. To check a relatively extreme setting for the technique of Section 4.2.2, we set $\delta = \alpha - 1 + 0.1$ in (4.8). In addition, to compare with Algorithm 2 of Section 3.2, we fix $(\alpha, a, b) = (1.5, 1.0, 1.0)$ and present the corresponding quantities $(\int_{\mathbb{R}} |f_{TS(\alpha,\Delta a,b)}(z) - f_{\mathscr{L}(Y_2(\Delta,c))}(z)|^2 \mathrm{d}z)^{1/2}$ and $1/p_2(\Delta,c)$ of Algorithm 2. In short, in the decomposition framework, a lot of additional computing effort is required for improvement in approximation error, either by taking a smaller truncation ε or by introducing the techniques of Sections 4.2.1 and 4.2.2. To achieve a similar level of approximation error to the method of Section 3.2, the decomposition framework requires an extraordinary larger computing effort.

5. Infinite shot noise series representation with finite truncation

It is known that every infinitely divisible random variable admits a shot noise series representation, and that the series is infinite if Lévy measure is infinite. Here, we discuss in brief such representations of the tempered stable distribution with a view towards simulation. Fix $(\lambda, \lambda_1, \lambda_2) \in \mathbb{R}^3_+$ and define $\gamma(\Delta) := (\Delta a/\alpha)^{-1/\alpha}\zeta(1/\alpha) - \Delta\Gamma(1-\alpha)ab^{\alpha-1}$ where ζ denotes the Riemann zeta function. We denote by $\{\Gamma_k\}_{k\in\mathbb{N}}$ arrival times of a standard Poisson process, by $\{U_k\}_{k\in\mathbb{N}}$ a sequence of iid uniform random variables on $[0, 1]$, by $\{E^{(1)}_k\}_{k\in\mathbb{N}}$ and $\{E^{(4)}_k\}_{k\in\mathbb{N}}$ sequences of iid standard exponential random variables, by $\{E^{(2)}_k\}_{k\in\mathbb{N}}$ a sequence of iid exponential random variables with rate $b\lambda_1$, and by $\{E^{(3)}_k\}_{k\in\mathbb{N}}$ a sequence of iid gamma random

variables with shape λ_1 and scale $(b\lambda_2)^{-1}$. Finally, define

$$\overleftarrow{q}(s) := \inf\left\{r > 0 : \int_r^{+\infty} a\frac{\mathrm{e}^{-bz}}{z^{\alpha+1}}\mathrm{d}z > s\right\}, \quad s > 0.$$

Note that this function cannot be written in closed form. Then, the tempered stable random variable $X(\Delta)$ can be written as

$$X(\Delta) - \gamma(\Delta) \overset{\mathscr{L}}{=} \sum_{k=1}^{+\infty}\left[\overleftarrow{q}\left(\frac{\Gamma_k}{\Delta}\right) - \left(\frac{\alpha k}{\Delta a}\right)^{-1/\alpha}\right] \tag{5.1}$$

$$\overset{\mathscr{L}}{=} \sum_{k=1}^{+\infty}\left[\left[\left(\frac{\alpha\Gamma_k}{\Delta a}\right)^{-1/\alpha} \wedge \frac{E_k^{(1)}U_k^{1/\alpha}}{b}\right] - \left(\frac{\alpha k}{\Delta a}\right)^{-1/\alpha}\right] \tag{5.2}$$

$$\overset{\mathscr{L}}{=} \sum_{k=1}^{+\infty}\left[E_k^{(2)}\mathbb{1}\left(\Gamma_k \le \frac{\Delta a}{\lambda b(E_k^{(2)})^{\alpha+1}}\mathrm{e}^{-b(1-\lambda)E_k^{(2)}}\right) - \left(\frac{\alpha k}{\Delta a}\right)^{-1/\alpha}\right] \tag{5.3}$$

$$\overset{\mathscr{L}}{=} \sum_{k=1}^{+\infty}\left[E_k^{(3)}\mathbb{1}\left(\Gamma_k \le \frac{\Delta a}{(b\lambda_2)^{\lambda_1}(E_k^{(3)})^{\alpha+\lambda_1}}\mathrm{e}^{-b(1-\lambda_2)E_k^{(3)}}\right) - \left(\frac{\alpha k}{\Delta a}\right)^{-1/\alpha}\right] \tag{5.4}$$

$$\overset{\mathscr{L}}{=} \sum_{k=1}^{+\infty}\left[\left(\frac{\alpha\Gamma_k}{\Delta a}\right)^{-1/\alpha}\mathbb{1}\left(\left(\frac{\alpha\Gamma_k}{\Delta a}\right)^{-1/\alpha} \le \frac{E_k^{(4)}}{b}\right) - \left(\frac{\alpha k}{\Delta a}\right)^{-1/\alpha}\right]. \tag{5.5}$$

The representation (5.1) is derived with the inverse Lévy measure method due to Ferguson and Klass [24] and LePage [25]. The representation (5.2) is derived with the generalized shot noise method due to [2], while the others (5.3)–(5.5) are due to Imai and Kawai [14] and are derived with the thinning method [26] for (5.3) and (5.4) and the rejection method [26] for (5.5). It is obviously insensible to generate the above infinite sum for sample simulation of only one increment $X(\Delta)$, while it may make sense to generate many iid replications of $X(\Delta)$. (See [27] for some techniques for computation of expectation involving series representations.) Suppose we wish to generate N of them, $N = 100\,000$ say. Let $T = N\Delta$ and let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of iid uniform random variables on $[0, T]$. Then, a tempered stable Lévy process admits the following series representation (based on (5.1), for example)

$$\left\{L_t^{(\mathrm{ts})} : t \in [0,T]\right\} \overset{\mathscr{L}}{=} \left\{\sum_{k=1}^{+\infty}\left[\overleftarrow{q}\left(\frac{\Gamma_k}{T}\right)\mathbb{1}_{[0,t]}(T_k) - \frac{t}{T}\left(\frac{\alpha k}{Ta}\right)^{-1/\alpha}\right] + \frac{t}{T}\gamma(T) : t \in [0,T]\right\}. \tag{5.6}$$

Its increments $\{L_{k\Delta}^{(\mathrm{ts})} - L_{(k-1)\Delta}^{(\mathrm{ts})}\}_{k=1,\dots,N}$ with equidistant stepsize Δ form a sequence of iid random variables with common law $\mathscr{L}(X(\Delta))$ for every $k = 1, \dots, N$.

Let us discuss in brief an issue of trade-off between computing time and finite trunction of the infinite sum. To this end, we take the representation (5.1) with the finite truncation $\{k \in \mathbb{N} : \Gamma_k \le n\}$, as this combination reveals a meaningful probabilistic structure to explain a duality to the decomposition employed in Section 4. Let ν_n denote the Lévy measure of the infinitely divisible random variable defined by

$$\sum_{\{k\in\mathbb{N}:\Gamma_k\le n\}}\left[\overleftarrow{q}(\Gamma_k) - \left(\frac{\alpha k}{a}\right)^{-1/\alpha}\right] + \gamma(1), \tag{5.7}$$

and let $\nu := \nu_{+\infty}$ denote the original Lévy measure. The decomposition of Section 4 is to divide the original Lévy measure ν into the small jump component $\nu|_{(0,\varepsilon]}$ and the large jump component $\nu|_{(\varepsilon,+\infty)}$, while the series representation (5.7) simulates the compound Poisson component $\nu_n = \nu|_{(\overleftarrow{q}(n),+\infty)}$ and discards the rest $\nu|_{(0,\overleftarrow{q}(n)]}$ near the origin. Hence, the discussion of Section 4 about the heavy computation load required for the compound Poisson component for large jumps applies to the series representation (5.7) as soon as the threshold ε of Section 4 is replaced with $\overleftarrow{q}(n)$. Namely, the simulation methods discussed in Section 4.1 are to be replaced by the series representation (5.7), while the small jump component should be treated through the techniques of Section 4.2. In addition, although a single sample path of the tempered stable Lévy process (5.6) can provide N iid random variables, it should not be ignored that additional computing of an indexed search is required for the uniform dispersion of jumps based on the indicator $\mathbb{1}_{((k-1)\Delta,k\Delta]}(\cdot)$ in (5.6). It is also known [28,14] that the representation (5.1) can express more variability of randomness than any others (5.2)–(5.5) under the common finite truncation $\{k \in \mathbb{N} : \Gamma_k \le n\}$. For those reasons, as far as tempered stable random variables are concerned, this framework does not improve the situation of Section 4 in any significant manner. Let us close this section with remarking that the choice of threshold is an important issue to be addressed whenever the decomposition of Section 4 and/or the series representation of this section are the only possibility.

6. Concluding remarks

In this paper, we have investigated various simulation methods of tempered stable laws with stability index greater than one, with primal interest in simulation of increments $X(\Delta)$ over a very short stepsize $\Delta > 0$: a suitable setting for approximation of stochastic differential equations through the Euler scheme. From a practical point of view, we have sought a simulation recipe of a good balance between computational load and approximation error, together with implementation ease. Results can be summarized as follows.

- The model-free acceptance–rejection sampling method of [17] provides an exact simulation method, in principle, but requires a lot of computing effort for computing density values. This method exhibits quite low acceptance rate when Δ is small and the stability index α is close to 2, that is, when the target is close to Gaussian.
- The acceptance–rejection sampling of [11] is approximative yet very handy with both very small computing time and approximation error. Finding an optimal value of the tuning parameter is relatively straightforward and is required only once in advance.
- The Gaussian approximation of [12] provides a different route to approximative simulation. We have shown that in this framework, the approximation error can be made very small by either simulating more large jump component or simulating more mass of the small jump component as compound Poisson random variables, while an extraordinary large amount of computing effort is additionally required for an improvement in approximation error, which may be a drawback in practice when thousands of tempered stable variates of small scale are needed.
- Infinite shot noise series representations of tempered stable laws can also be used for simulation. Even after a finite truncation of infinite shot noise series, a large computing effort needs to be paid to attain satisfactory approximation error. The trade-off between approximation error and computing time based upon the finite truncation has in principal a duality relation to the framework of the Gaussian approximation.

In conclusion, with a given computing budget, the approximative acceptance–rejection sampling of [11] is both most efficient and handiest based on numerical assessment of accuracy for simulation of increments in small time. The decomposition and the series representation are hardly competitive against Baeumer-Meerschaert as computation load for the compound Poisson part is too heavy. However, the choice of the threshold is certainly an important issue for simulation of infinitely divisible laws for which the decomposition and/or the series representation are the only possibility.

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References

[1] M.C.K. Tweedie, An index which distinguishes between some important exponential families, in: J. Ghosh, J. Roy (Eds.), Statistics: Applications and New Directions: Proc. Indian Statistical Institute Golden Jubilee International Conference, 1984, pp. 579–604.
[2] J. Rosiński, Tempering stable processes, Stochastic Processes and their Applications 117 (6) (2007) 677–707.
[3] R.N. Mantegna, H.E. Stanley, Stochastic process with ultraslow convergence to a Gaussian: The truncated Lévy flights, Physical Review Letters 73 (1994) 2946–2949.
[4] E.A. Novikov, Infinitely divisible distributions in turbulence, Physical Review E 50 (1994) R3303–R3305.
[5] P. Carr, H. Geman, D.B. Madan, M. Yor, The fine structure of asset returns: An empirical investigation, Journal of Business 75 (2002) 303–325.
[6] O.E. Barndorff-Nielsen, N. Shephard, Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics (with discussion), J. R. Statist. Soc. B 63 (2) (2001) 167–241.
[7] K.J. Palmer, M.S. Ridout, B.J.T. Morgan, Modelling cell generation times by using the tempered stable distribution, Journal of the Royal Statistical Society: Series C (Applied Statistics) 57 (4) (2008) 379–397.
[8] H. Masuda, Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein–Uhlenbeck processes, Electronic Journal of Statistics 4 (2010) 525–565.
[9] H. Masuda, Approximate quadratic estimating function for discretely observed Lévy driven SDEs with application to a noise normality test, MI Preprint Series, Kyushu University, 2010-21, 2010. Available at https://qir.kyushu-u.ac.jp/dspace/bitstream/2324/17028/1/MI2010-21.pdf.
[10] V. Todorov, G. Tauchen, Limit theorems for power variations of pure-jump processes with application to activity estimation, Annals of Applied Probability (2010) (in press).
[11] B. Baeumer, M.M. Meerschaert, Tempered stable Lévy motion and transit super-diffusion, Journal of Computational and Applied Mathematics 223 (10) (2009) 2438–2448.
[12] S. Asmussen, J. Rosiński, Approximation of small jumps of Lévy processes with a view towards simulation, Journal of Applied Probability 38 (2) (2001) 482–493.
[13] L. Devroye, On the computer generation of random variables with a given characteristic function, Computers and Mathematics with Applications 7 (6) (1981) 547–552.
[14] J. Imai, R. Kawai, On finite truncation of infinite shot noise series representation of tempered stable laws, 2010. Available at http://sites.google.com/site/reiichirokawai/.
[15] J.M. Chambers, C.L. Mallows, B.W. Stuck, A method for simulating stable random variables, Journal of the American Statistical Association 71 (354) (1976) 340–344.
[16] V.M. Zolotarev, One-Dimensional Stable Distributions, American Mathematical Society, Providence, RI, 1986.
[17] L. Devroye, Random variate generation for exponential and polynomially tilted stable distributions, ACM Transactions on Modeling and Computer Simulation 19 (4) (2009) Article No. 18.
[18] R. Kawai, H. Masuda, Exact simulation of finite variation tempered stable Ornstein–Uhlenbeck processes, 2009. Available at http://hdl.handle.net/2324/15635.

[19] R. Kawai, H. Masuda, Infinite variation tempered stable Ornstein–Uhlenbeck processes with discrete observations, 2010. Available at http://hdl.handle.net/2324/16248.
[20] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, 1999.
[21] R. Kawai, A. Takeuchi, Computation of Greeks for asset price dynamics driven by stable and tempered stable processes, 2009. Available at https://sites.google.com/site/reiichirokawai/.
[22] M. Signahl, On error rates in normal approximations and simulation schemes for Lévy processes, Stochastic Models 19 (3) (2003) 287–298.
[23] S. Cohen, J. Rosiński, Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes, Bernoulli 13 (1) (2007) 195–210.
[24] T.S. Ferguson, M.J. Klass, A representation of independent increment processes with Gaussian components, Annals of Mathematical Statistics 43 (5) (1972) 1634–1643.
[25] R. LePage, Multidimensional infinitely divisible variables and processes II, in: Lecture Notes in Mathematics, vol. 860, Springer-Verlag, Berlin, New York, Heidelberg, 1980, pp. 279–284.
[26] J. Rosiński, Series representations of Lévy processes from the perspective of point processes, in: O.-E. Barndorff-Nielsen, T. Mikosch, S.I. Resnick (Eds.), Lévy Processes – Theory and Applications, Birkhäuser, 2001, pp. 401–415.
[27] J. Imai, R. Kawai, Quasi-Monte Carlo method for infinitely divisible random vectors via series representations, SIAM Journal on Scientific Computing 32 (4) (2010) 1879–1897.
[28] J. Imai, R. Kawai, Numerical inverse Lévy measure method for infinite shot noise series representation, 2010. Available at http://sites.google.com/site/reiichirokawai/.