IN5270 Project 1

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1 Method

Our goal is to solve the two-dimensional, standard, linear wave equation, with damping.

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(q(x,y)\frac{\partial u}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(q(x,y)\frac{\partial u}{\partial y}\Big) + f(x,y,t) \tag{1}$$

As solving this analytically can prove quite difficult, we will use a numerical approach. The first step is to discretize the equation by finite difference. This is done by the following formula.

$$\Big[D_t D_t u + bD_{2t}u = D_x q^{-x} D_x u + D_y q^{-y} D_y u + f\Big]_{i,j}^n \tag{2}$$

where

$$[D_t D_t u]_{i,j}^n = \frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2},$$

$$[bD_{2t}u]_{i,j}^n = b\frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t}, [f(x,y,t)]_{i,j}^n = f(x_i, y_i, t_n) = f_{i,j}^n$$

To discretize $\frac{\partial}{\partial x}\Big(q(x,y)\frac{\partial u}{\partial x}\Big)$, we define a temporary variable $\phi = q(x)\frac{\partial u}{\partial x}$ and discretize this first:

$$\Big[\frac{\partial \phi}{\partial x}\Big]_i^n = \frac{\phi_{i+1/2} - \phi_{i-1/2}}{\Delta x}$$

$$\phi_{i+1/2} = q_{i+1/2}\Big[\frac{\partial u}{\partial x}\Big]_{i+1/2}^n \approx q_{i+1/2}\frac{u_{i+1}^n - u_i^n}{\Delta x}$$

$$\phi_{i-1/2} = q_{i-1/2}\Big[\frac{\partial u}{\partial x}\Big]_{i-1/2}^n \approx q_{i-1/2}\frac{u_i^n - u_{-1}^n}{\Delta x}$$

leading to

$$\Big[\frac{\partial}{\partial x}\Big(q(x)\frac{\partial u}{\partial x}\Big)\Big]_i^n \approx \frac{1}{\Delta x^2}\Big(q_{i+1/2}(u_{i+1}^n - u_i^n) - q_{i-1/2}(u_i^n - u_{i-1}^n)\Big) \tag{3}$$

and similar for y:

$$
\begin{align}
\Big[\frac{\partial}{\partial y}\Big(q(y)\frac{\partial u}{\partial y}\Big)\Big]_j^n \approx \frac{1}{\Delta y^2}\Big(q_{j+1/2}(u^n_{j+1} - u^n_j) - q_{j-1/2}(u^n_j - u^n_{j-1})\Big) \tag{4} \\
\tag{5}
\end{align}
$$

The entire discretized equation then becomes

$$
\begin{aligned}
\frac{u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j}}{\Delta t^2} + b\frac{u^{n+1}_{i,j} - u^{n-1}_{i,j}}{2\Delta t} &= \frac{1}{\Delta x^2}\Big(q_{i+1/2}(u^n_{i+1} - u^n_i) - q_{i-1/2}(u^n_i - u^n_{i-1})\Big) \\
&+ \frac{1}{\Delta y^2}\Big(q_{j+1/2}(u^n_{j+1} - u^n_j) - q_{j-1/2}(u^n_j - u^n_{j-1})\Big) + f^n_{i,j}
\end{aligned}
$$

Now we rearrange and use the arithmetic mean $q_{i+1/2} = \frac{1}{2}(q_i + q_{i+1})$:

$$
\begin{aligned}
&u^{n+1}_{i,j} + b\frac{\Delta t}{2}(u^{n+1}_{i,j} - u^{n-1}_{i,j}) \\
&= -u^{n-1}_{i,j} + 2u^n_{i,j} + \frac{\Delta t^2}{\Delta x^2}\Big[\frac{1}{2}(q_{i,j} + q_{i+1,j})(u^n_{i+1,j} - u^n_{i,j}) - \frac{1}{2}(q_{i-1,j} + q_{i,j})(u^n_{i,j} - u^n_{i-1,j}) \\
&+ \frac{\Delta t^2}{\Delta y^2}\Big[\frac{1}{2}(q_{i,j} + q_{i,j+1})(u^n_{i,j} - u^n_{i,j-1}) - \frac{1}{2}(q_{i,j-1} + q_{i,j})(u^n_{i,j} - u^n_{i,j-1})\Big] + \Delta t^2 f^n_{i,j}
\end{aligned}
$$

Isolating $u^{n+1}_{i,j}$ on the left side, we obtain the general scheme for the interior points:

$$
\begin{aligned}
u^{n+1}_{i,j} = \Big(b\frac{\Delta t}{2} + 1\Big)^{-1}\Big[&u^{n-1}_{i,j}(b\frac{\Delta t}{2} - 1) + 2u^n_{i,j} \\
&+ \frac{\Delta t^2}{\Delta x^2}\Big[\frac{1}{2}(q_{i,j} + q_{i+1,j})(u^n_{i+1,j} - u^n_{i,j}) - \frac{1}{2}(q_{i-1,j} + q_{i,j})(u^n_{i,j} - u^n_{i-1,j})\Big] \\
&+ \frac{\Delta t^2}{\Delta y^2}\Big[\frac{1}{2}(q_{i,j} + q_{i,j+1})(u^n_{i,j+1} - u^n_{i,j}) - \frac{1}{2}(q_{i,j-1} + q_{i,j})(u^n_{i,j} - u^n_{i,j-1})\Big] \\
&+ \Delta t^2 f^n_{i,j}\Big]
\end{aligned}
$$

The first time step is found by inserting $n = 0$.

$$
\begin{aligned}
u^1_{i,j} = \Big(b\frac{\Delta t}{2} + 1\Big)^{-1}\Big[&u^{-1}_{i,j}(b\frac{\Delta t}{2} - 1) + 2u^0_{i,j} \\
&+ \frac{\Delta t^2}{\Delta x^2}\Big[\frac{1}{2}(q_{i,j} + q_{i+1,j})(u^0_{i+1,j} - u^0_{i,j}) - \frac{1}{2}(q_{i-1,j} + q_{i,j})(u^0_{i,j} - u^0_{i-1,j})\Big] \\
&+ \frac{\Delta t^2}{\Delta y^2}\Big[\frac{1}{2}(q_{i,j} + q_{i,j+1})(u^0_{i,j+1} - u^0_{i,j}) - \frac{1}{2}(q_{i,j-1} + q_{i,j})(u^0_{i,j} - u^0_{i,j-1})\Big] \\
&+ \Delta t^2 f^0_{i,j}\Big]
\end{aligned}
$$

Since we already know the initial conditions

$$
\begin{aligned}
u(x, y, 0) &= I(x, y) \\
u_t(x, y, 0) &= V(x, y),
\end{aligned}
$$

it will only be necessary to find the unknown term $u_{i,j}^{-1}$. Fortunately, this can be found by the initial conditions

$$\Big[D_{2t}u = V(x,y)\Big]_{i,j}^{0}$$
$$u_{i,j}^{-1} = u_{i,j}^{1} - 2\Delta t V_{i,j}$$

Putting this into our equation.

$$\begin{aligned}
u_{i,j}^{1} = &\Big(b\frac{\Delta t}{2}+1\Big)^{-1}\Big[(u_{i,j}^{1} - 2\Delta t V_{i,j})(b\frac{\Delta t}{2}-1) + 2u_{i,j}^{0}\\
&+ \frac{\Delta t^2}{\Delta x^2}\Big[\frac{1}{2}(q_{i,j}+q_{i+1,j})(u_{i+1,j}^{0}-u_{i,j}^{0}) - \frac{1}{2}(q_{i-1,j}+q_{i,j})(u_{i,j}^{0}-u_{i-1,j}^{0})\Big]\\
&+ \frac{\Delta t^2}{\Delta y^2}\Big[\frac{1}{2}(q_{i,j}+q_{i,j+1})(u_{i,j+1}^{0}-u_{i,j}^{0}) - \frac{1}{2}(q_{i,j-1}+q_{i,j})(u_{i,j}^{0}-u_{i,j-1}^{0})\Big]\\
&+ \Delta t^2 f_{i,j}^{0}\Big]
\end{aligned}$$

Thus, we end up with the following scheme for the first time step.

$$\begin{aligned}
u_{i,j}^{1} = &-\Delta t V_{i,j}(b\frac{\Delta t}{2}-1) + u_{i,j}^{0}\\
&+ \frac{\Delta t^2}{4\Delta x^2}\Big[(q_{i,j}+q_{i+1,j})(u_{i+1,j}^{0}-u_{i,j}^{0}) - (q_{i-1,j}+q_{i,j})(u_{i,j}^{0}-u_{i-1,j}^{0})\Big]\\
&+ \frac{\Delta t^2}{4\Delta y^2}\Big[(q_{i,j}+q_{i,j+1})(u_{i,j+1}^{0}-u_{i,j}^{0}) - (q_{i,j-1}+q_{i,j})(u_{i,j}^{0}-u_{i,j-1}^{0})\Big]\\
&+ \frac{1}{2}\Delta t^2 f_{i,j}^{0}
\end{aligned}$$

Since $u_{i,j}^{0} = I(x_i, y_i)$ and similarly for other spacial steps, we then have

$$\begin{aligned}
u_{i,j}^{1} = &-\Delta t V_{i,j}(b\frac{\Delta t}{2}-1) + I_{i,j}\\
&+ \frac{\Delta t^2}{4\Delta x^2}\Big[(q_{i,j}+q_{i+1,j})(I_{i+1,j}-I_{i,j}) - (q_{i-1,j}+q_{i,j})(I_{i,j}-I_{i-1,j})\Big]\\
&+ \frac{\Delta t^2}{4\Delta y^2}\Big[(q_{i,j}+q_{i,j+1})(I_{i,j+1}-I_{i,j}) - (q_{i,j-1}+q_{i,j})(I_{i,j}-I_{i,j-1})\Big]\\
&+ \frac{1}{2}\Delta t^2 f_{i,j}^{0}
\end{aligned}$$

To discretize the Neumann boundary conditions $\frac{\partial u}{\partial n} = 0$, we will be using ghost cells with our general scheme. Therefore, no additional scheme is needed.

1.1 Constant solution

We will firstly assume the solution is constant:

$$u_e(x,y,t) = c$$

In this case, all derivatives are 0 and the PDE reduces to

$$f(x, y, t) = 0$$

This means b and q can be set to anything and still give the correct solution. It also means $I = c$ and $V = 0$. By plugging this into the discretized equations, and setting all $u^n_{i,j} = 0$, we get

$$\begin{aligned}
u^1_{i,j} &= 0 + c \\
&+ \frac{\Delta t^2}{4\Delta x^2}\Big[(q_{i,j} + q_{i+1,j})(c - c) - (q_{i-1,j} + q_{i,j})(c - c)\Big] \\
&+ \frac{\Delta t^2}{4\Delta y^2}\Big[(q_{i,j} + q_{i,j+1})(c - c) - (q_{i,j-1} + q_{i,j})(c - c)\Big] \\
&+ 0 = c
\end{aligned}$$

1.2 Implementation

For implementation and testing of the schemes, see the program.