Oblig 1

a) $u'' + \omega^2 u = f(t)$, $u(0) = I$, $u'(0) = V$, $t \in (0, T]$

$$[D_t D_t u + \omega^2 u = f]^n$$

$$\rightarrow \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + \omega^2 u^n = f^n$$

To find u^1, first solve for $n=0$

$$\frac{u^1 - 2u^0 + u^{-1}}{\Delta t^2} + \omega^2 u^0 = f^0$$

$$u^1 = \Delta t^2 (f^0 - \omega^2 u^0) + 2u^0 - u^{-1}$$

$$u^0 = u(0) = I$$

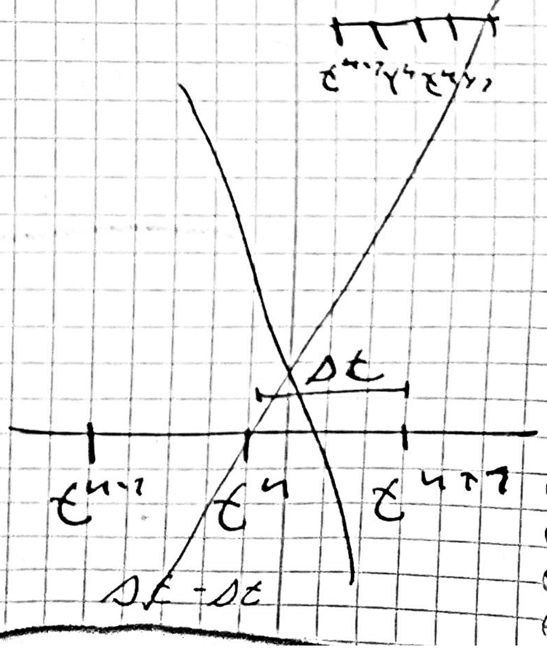
To find u^{-1}, discretize $u'(0) = V$

$$\cancel{u'(t) \approx [D_{2t} u]^n}$$

$$u'(t_n) = [D_{2t} u]^n = \frac{u^{n+1} - u^{n-1}}{2\Delta t}$$

$$n = 0 \Rightarrow \frac{u^1 - u^{-1}}{2\Delta t} = u'(0) = V$$

$$u^{-1} = u^1 - 2V\Delta t$$



$$u^1 = \Delta t^2 (f^0 - \omega^2 I) + 2I^0 - u^1 + 2V\Delta t$$

$$u^1 = \frac{\Delta t^2}{2}(f^0 - \omega^2 I) + I + V\Delta t$$

b) $u_e(t) = ct + d$

$u_e(0) = \cancel{c}\,0 + d = I$

$u_e'(0) = c + 0 = V$

$\underline{u_e(t) = Vt + I}$

Plug into ODE

$$\underbrace{u_e''}_{=0} + \omega^2 u_e = f(t)$$

$\underline{f(t) = \omega^2(Vt + I)}$

$$[D_t D_t t]^n = \frac{t^{n+1} - 2t^n + t^{n-1}}{\Delta t^2} = \frac{(t^{n+1} - t^n) - (t^n - t^{n-1})}{\Delta t^2}$$

$$= \frac{\Delta t - \Delta t}{\Delta t^2} = \underline{0}$$

$$[D_t D_t (ct + d)]^n = c\overbrace{[D_t D_t t]^n}^{=0} + [D_t D_t d]^n = 0$$

$\underline{[D_t D_t d]^n = 0}$

The discretized eq. is:

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + \omega^2 u^n = f^n$$

$$\Rightarrow u^{n+1} = \Delta t^2 (f^n + \omega^2 u^n) + 2u^n - u^{n-1}$$

Plugging u_e into the eq.

$$u_e^{n+1} = \Delta t^2 (f^n - \omega^2 u_e^n) + 2u_e^n - u_e^{n-1}$$

$$V(t+\Delta t) + I = \Delta t^2 (\omega^2 (Vt+I) - \omega^2 (Vt+I)) + 2(Vt+I) - (V(t-\Delta t) + I)$$

$$= 0 + 2Vt + 2I - Vt + V\Delta t - I$$

$$= Vt + I + V\Delta t = \underline{\underline{V(t+\Delta t) + I = u_e^{n+1}}}$$

c) See code

d) — " —

e) ~~the~~ From the code, we see that the cubic polynomial gives rise to a non-zero residual for u^1, meaning the discrete equations are <u>not</u> fulfilled.