HW1 Report

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Problem 1

Proof. On one hand, since A is an upper-triangular matrix, we know that the inverse of A is also upper-triangular. But on the other hand, since A is unitary, we know

$$A^*A = I \implies A^{-1} = A^*.$$

We then know $A^* = A^{-1}$ is a lower-triangular matrix, and hence diagonal (upper and lower triangular at the same time implies diagonal). Thus, $A = (A^*)^*$ is diagonal as well. The same holds for lower-triangular matrices following the same logic. $\square$

Problem 2

a)

Proof. Since $\lambda \neq 0$ is an eigenvalue of A, we know $Ax = \lambda x$. We know A is invertible, hence

$$x = \lambda A^{-1}x \implies A^{-1}x = \frac{1}{\lambda}x,$$

i.e., $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. $\square$

b)

Proof. If λ is an eigenvalue of AB, then $ABx = \lambda x$. Let $y = Bx$, giving us $Ay = \lambda x$. If we left multiply by B, we have

$$BAy = \lambda Bx = \lambda y.$$

Thus AB and BA share the same eigenvalues. $\square$

c)

Proof. We want to show that $\det(A - \lambda I) = \det(A^* - \lambda I) = \det(A^T - \lambda I)$, since $A \in \mathbb{R}^{n\times n}$. Consider

$$\det(A^T - \lambda I) = \det(A^T - \lambda I^T) = \det[(A - \lambda I)^T].$$

Since $\det(X) = \det(X^T)$, we can conclude that

$$\det(A^T - \lambda I) = \det[(A - \lambda I)^T] = \det(A - \lambda I).$$

Hence, A and A^T share the same eigenvalues. Note, if $A \in \mathbb{C}^{n\times n}$, then the eigenvalues of A^* are the complex conjugates of the eigenvalues of A. □

Problem 3

a)

Proof. If x is an eigenvector of Hermitian A, then we know

$$\langle Ax, x\rangle = \langle \lambda x, x\rangle = \langle x, A^* x\rangle = \langle x, \lambda x\rangle.$$

Hence, we have that

$$\lambda\langle x, x\rangle = \bar{\lambda}\langle x, x\rangle \implies \lambda = \bar{\lambda},$$

implying that $\mathbf{Im}(\lambda) = 0$.

□

b)

Proof. We have that $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$, with $\lambda_1 \neq \lambda_2$. We want to show that $\langle x, y\rangle = 0$. Notice that

$$\langle \lambda_1 x, y\rangle = \langle Ax, y\rangle = x^* Ay = x^* \lambda_2 y = \langle \lambda_2 x, y\rangle$$

This implies that

$$\langle \lambda_1 x, y\rangle - \langle \lambda_2 x, y\rangle = \lambda_1 x^* y - \lambda_2 x^* y = (\lambda_1 - \lambda_2) x^* y = 0.$$

Since λ_1, λ_2 are distinct, it must be that $x^* y = \langle x, y\rangle = 0$. □

Problem 4

a)

Proof. Consider the chain of equalities

$$\langle x, x\rangle = \langle A^* Ax, Ax\rangle = \langle Ax, Ax\rangle = \langle \lambda x, \lambda x\rangle = |\lambda|^2\langle x, x\rangle \implies |\lambda|^2 = 1.$$

Hence, $|\lambda| = 1$. □

b)

Proof. For unitary $A \in \mathbb{C}^{n\times n}$, we have that

$$\|A\|_F = \sqrt{\mathbf{trace}(A^* A)} = \sqrt{\mathbf{trace}(I)} = \sqrt{n} \neq 1.$$

□

Problem 5

a)

Proof. Let x be an eigenvector for skew-Hermitian A. Then, we have that

$$\langle Ax, x\rangle = \langle \lambda x, x\rangle = \langle x, A^* x\rangle = \langle x, -Ax\rangle = \langle x, -\bar{\lambda} x\rangle.$$

Thus, we have that

$$\lambda\langle x, x\rangle = -\bar{\lambda}\langle x, x\rangle \implies \lambda = -\bar{\lambda}.$$

Hence, $\mathbf{Re}(\lambda) = 0$. □

b)

Proof. In order to show that $I - A$ is nonsingular, it suffices to show it fails to have $\lambda = 0$ as an eigenvalue. Let λ be an eigenvalue of A. Consider

$$(I - A)x = (I + A)x = Ax + Ix = Ax + x = \lambda x + x = (\lambda + 1)x.$$

This chain of equalities shows us that $(I - A)x = (\lambda + 1)x$, that is, $\lambda + 1$ is an eigenvalue of $I - A$. Since λ is an eigenvalue of A and A is skew-symmetric, from part a), we know that λ is purely imaginary. Hence, $1 + \lambda$ is nonzero. Thus $I - A$ is nonsingular.

□

c)

Proof. We have that

$$\begin{aligned} BB^* &= (I - A)^{-1}(I + A)(I - A)(I + A)^{-1} \\ &= (I - A)^{-1}(I - A)(I + A)(I + A)^{-1} \\ &= I \end{aligned}$$

since $(I + A)(I - A) = (I - A)(I + A)$.

□

Problem 6

Proof. We want to show that

$$\rho(A) = \lambda_{max}(A) \leq \|A\| = \sup_{\|x\|=1} \|Ax\|.$$

Let v be the unit-eigenvector associated with the maximal eigenvalue $\lambda_{max}(A)$. Then, by definition of supremum, we have that

$$\begin{aligned}
\|Av\| = \|\lambda_{max}(A)v\| &\leq \sup_{\|x\|=1} \|Ax\| \\
= |\lambda_{max}(A)|\|v\| &\leq \sup_{\|x\|=1} \|Ax\| \\
= |\lambda_{max}(A)| = \rho(A) &\leq \sup_{\|x\|=1} \|Ax\| = \|A\|.
\end{aligned}$$

□

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a)

Proof. We have that

$$\begin{aligned}
\|A\|_2 &= \sqrt{\lambda_{max}(A^*A)} = \sqrt{\lambda_{max}[(uv^*)^*(uv^*)]} \\
&= \sqrt{\|u\|_2^2 \lambda_{max}(vv^*)} \\
&= \|u\|_2 \sqrt{\lambda_{max}(vv^*)}
\end{aligned}$$

Now, consider the other outer product matrix $B = vv^*$. Notice that

$$Bv = (vv^*)v = \|v\|_2^2 v,$$

which implies that v is an eigenvector of vv^* with eigenvalue $\|v\|_2^2$. Also notice that

$$B = [\bar{v_1}v \mid \bar{v_2}v \mid \ \ldots \ \mid \bar{v_n}v],$$

showing us that every column of B is dependent on v. Hence, $\mathbf{rank}(B) = \mathbf{rank}(vv^*) = 1$, implying that $\lambda = \|v\|_2^2$ is the only, and maximal, eigenvalue of vv^*. Therefore,

$$\|A\|_2 = \|u\|_2 \sqrt{\lambda_{max}(vv^*)} = \|u\|_2 \|v\|_2.$$

□

b)

Proof. By definition, we have that

$$\begin{aligned}
\|A\|_F &= \sqrt{\mathbf{trace}(A^*A)} = \sqrt{\mathbf{trace}[(uv^*)^*(uv^*)]} \\
&= \sqrt{\mathbf{trace}(\|u\|_2^2 vv^*)} \\
&= \|u\|_2 \sqrt{\mathbf{trace}(vv^*)}
\end{aligned}$$

As shown in part a), the only eigenvalue of $vv*$ is $\lambda = \|v\|_2^2$. Since the trace of a matrix is equal to the sum of its eigenvalues, $\textbf{trace}(vv^*) = \|v\|_2^2$.

Hence,

$$\|A\|_F = \|u\|_2 \sqrt{\textbf{trace}(vv^*)} = \|u\|_2 \|v\|_2.$$

$\square$

Problem 8

a)

Proof. We know that A and B are unitarily equivalent, hence $A = QBQ^*$ for unitary $Q \in \mathbb{C}^{m \times m}$. The SVD of B is $B = U\Sigma V^*$, where Σ contains the singular values of B along the diagonal and U, V are unitary. Thus,

$$A = (QU)\Sigma(V^*Q).$$

Since the product of two unitary matrices is also unitary, the SVD of A is the above, notably with the same Σ. $\square$

b)

Proof. Assume that A is unitarily equivalent to B. Let B be the identity matrix and choose A to be any other unitary matrix such that $A \neq B$. Then, we know that $A^*A = I$, meaning A shares the same singular values as B. But $A = QBQ^* = I$. Hence we have $A = I$, and $A \neq B = I$. This is a contradiction.

$\square$

Problem 9

a)

We have $f(x_1, x_2) = x_1 + x_2$. Since $f \in C^{(1)}$,

$$K(x) = \frac{\|J(x)\|_\infty \|x\|_\infty}{\|f(x)\|_\infty} = \frac{2\max(|x_1|, |x_2|)}{|x_1 + x_2|}.$$

Thus, as $x_1 \to -x_2$ or $x_2 \to -x_1$, our relative condition number $K(x) \to \infty$.

b)

Let $x_1 = \alpha x_2$. Then, we can write f as a univariate function,

$$g(x) = \alpha x^2.$$

Calculating the condition number for g gives us:

$$K(x) = \frac{2|\alpha||x|^2}{|\alpha||x|^2} = 2.$$

c)

We have $f(x) = (x-2)^9$. Hence,

$$K(x) = \frac{|9(x-2)^8||x|}{|(x-2)^9|} = \frac{9(x-2)^8|x|}{(x-2)^8|x-2|} = \frac{9|x|}{|x-2|}.$$

As $x \to 2$, we see that $K(x) \to \infty$.

Problem 10

a)

Submitted separately as Matlab code.

b)

Submitted separately as Matlab code.

c)

We see that around $x = 2$, the two curves do not coincide by (what I found to be) a surprisingly large variance. This makes sense, since our relative condition number for f is extremely large around $x = 2$.