# HW2 Part 2 Theory

Friday, February 5, 2021 12:02 PM

1) 1. The Schur decomposition theorem states that if $A \in \mathbb{C}^{m\times m}$, then there exist a unitary matrix $Q$ and an upper triangular matrix $U$ such that $A = QUQ^{-1}$. Use the Schur decomposition theorem to show that a real symmetric matrix $A$ is diagonalizable by an orthogonal matrix, i.e., $\exists$ an orthogonal matrix $Q$ such that $Q^T AQ = D$, where $D$ is a diagonal matrix with its eigenvalues in the diagonal.

From the Schur-decomposition, we can decompose $A$ as

$$A = QUQ^T, \quad Q \text{ orthogonal}, \quad U \text{ upper-triang.}$$

This implies $Q^TAQ = U$. Since $A$ is symmetric,

$U^T = (Q^TAQ)^T = Q^TAQ = U$, showing us $U$ is symmetric. Hence $U$ is diagonal.

Also, $U = Q^TAQ$ implies $A$ is similar to $U$.

Therefore they share the same eigenvalues (and must lie along $\mathrm{diag}(U)$).

Lec. Notes.

2) 2. Show that if $A \in \mathbb{R}^{m\times m}$ is symmetric positive definite if and only if its eigenvalues $\lambda_i$ are all positive. (Hint: Use Problem 1.)

**Proof:**

($\Rightarrow$) Let $A$ be positive definite and let $x$ be an eigenvector of $A$. Then,

$$\lambda x^T x = \lambda \|x\|_2^2 = x^T A x > 0.$$

Since $\|x\|_2^2 > 0$, we can write

$$\lambda = \frac{x^T A x}{\|x\|_2^2} > 0, \quad \text{hence} \quad \lambda > 0.$$

($\Leftarrow$) Assume $A$ is symmetric and has strictly positive eigenvalues. Then, by ... theorem for symmetric matrices, strictly positive eigenvalues.

spectral theorem for symmetric matrices;

$$A = PDP^T,$$ where $P$ is orthogonal and has the eigenvectors of $A$ as columns, $D$ diagonal w/ eig. values as entries.

Let $\{p_1, p_2, \ldots, p_m\}$ form an orthonormal basis for $\mathbb{R}^m$. Then, any $x \in \mathbb{R}^m$ can be expressed as

$$x = a_1 p_1 + a_2 p_2 + \ldots + a_m p_m, \quad a_i \in \mathbb{R}.$$

Since we want to show $x^T A x > 0$,

consider

$$x^T P D P^T x = (P^T x) D P^T x.$$

We have that

$$P^T x = [p_1^T \mid p_2^T \mid \ldots \mid p_m^T] x.$$

For column $p_i$, we have that

$$p_i^T x = p_i^T(a_1 p_1 + \ldots + a_i p_i + \ldots + a_m p_m) = a_i \|p_i\|^2$$

since $p_i^T p_j = 0$ when $i \neq j$.

Thus, $$(P^T x)^T D (P^T x) = \left[a_1\|P_1\|^2 \ \dots \ a_m\|P_m\|^2\right] D \begin{bmatrix} a_1\|P_1\|^2 \\ \vdots \\ a_m\|P_m\|^2 \end{bmatrix}$$

$$\Rightarrow \quad (P^T x)^T D (P^T x) = \lambda_1 a_1^2 \|P_1\|^4 + \lambda_2 a_2^2 \|P_2\|^4 + \dots$$
$$\dots + \lambda_m a_m^2 \|P_m\|^4 > 0,$$

since $\lambda_i > 0$ by assumption.

3) **3.** What can you say about the diagonal entries of a symmetric positive definite matrix? Justify your assertion.

The diagonal entries of a positive definite matrix must be positive.

<u>Proof</u>: Towards a contradiction, assume the $i^{th}$ diagonal entry of $A$ is negative, i.e. $a_{ii} < 0$. Since $A$ is positive definite,

$$x^T A x > 0 \quad \text{for all nonzero } x.$$

Hence, let $x = e_i$, the $i^{th}$ standard basis vector.

Then $e_i^T A e_i = a_{ii} > 0$. ↯

**4.** Suppose $A \in \mathbb{C}^{m\times m}$ is written in the block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad (1)$$

where $A_{11} \in \mathbb{C}^{n\times n}$ and $A_{22} \in \mathbb{C}^{(m-n)\times(m-n)}$. Assume that $A$ satisfies the condition: $A$ has an $LU$ decomposition if and only if the upper-left $k \times k$ block matrix $A_{1:k,1:k}$ is nonsingular for each $k$ with $1 \le k \le m$.

(a) Verify the formula

$$\begin{pmatrix} I & \mathbf{0} \\ -A_{21}A_{11}^{-1} & I \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}, \qquad (2)$$

which "eliminate" the block $A_{21}$ from $A$. The matrix $A_{22}-A_{21}A_{11}^{-1}A_{12}$ is known as the Schur complement of $A_{11}$ in $A$, denoted as $A/A_{11}$.

(b) Suppose that after applying $n$ steps of Gaussian elimination on the matrix $A$ in (1), $A_{21}$ is eliminated row by row, resulting in a matrix

$$\begin{pmatrix} A_{11} & C \\ \mathbf{0} & D \end{pmatrix}. \qquad (3)$$

Show that the bottom-right $(m-n)\times(m-n)$ block matrix $D$ is again $A_{22} - A_{21}A_{11}^{-1}A_{12}$. (Note: Part (b) is separate from Part (a)).

4)

(a) In order for the matrix multiplication in (2) to be defined,

$A_{12}$ is $n\times(m-n)$

$A_{21}$ is $(m-n)\times n$.

Now, computing, we get:

$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ -A_{21}\overset{I}{\cancel{A_{11}^{-1}A_{11}}} + A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix} \checkmark$$

(b) As stated, A has an LU decomposition, which is formed by applying n steps of Gaussian elimination. Thus,

$=LU$

Thus,

$$\begin{pmatrix} L^{-1} & 0 \\ ? & I \end{pmatrix}\begin{pmatrix} \overset{=LU}{A_{11}} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$ is equivalent to applying

n steps of Gaussian elimination. Let $? = -A_{21}U^{-1}L^{-1}$

$$= \begin{pmatrix} U & L^{-1}A_{12} \\ -A_{21}(LU)^{-1}A_{11} + A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}.$$

$U^{-1}L^{-1} = A_{11}^{-1}$

Hence, $D = A_{22} - A_{21}A_{11}^{-1}A_{12}$, as desired. $\blacksquare$

5) **5.** Suppose $A \in \mathbb{C}^{m\times m}$ is *strictly column diagonally dominant*, i.e.,

$$|a_{kk}| > \sum_{j\neq k} |a_{jk}|. \qquad (3)$$

Show that if Gaussian elimination with partial pivoting is applied to $A$, no row swapping takes place.

Clearly, no swapping occurs when the 1st column is eliminated, since $\underbrace{|a_{11}| > |a_{21}| + |a_{31}| + \dots + |a_{m1}|}_{(*)}$, meaning $|a_{11}| > |a_{i1}|$, $i = 1, \dots, m$. The question becomes: Did the elementary row opperations preserve the next matrices property of being diagonally column dominant, or destroy it?

From the algorithm, after one iteration of GE, We can describe the entries of our matrix as

$$c_{jk} = a_{jk} - \frac{a_{1k}}{a_{11}} a_{j1}, \qquad j > 1.$$

We want to show

$$|c_{kk}| \geq \sum_{j\neq k}^{m} |c_{jk}| \Rightarrow \text{WTS} \sum_{j\neq k}^{m} \left| a_{jk} - \frac{a_{1k}}{a_{11}} a_{j1} \right| \leq |c_{kk}|$$

$$|c_{kk}| \geq \sum_{\substack{j \neq k \\ j=2}} |c_{jk}| \Rightarrow \text{WTS} \sum_{\substack{j \neq k \\ j=2}} \left| a_{jk} - \frac{a_{1k}}{a_{11}} a_{j1} \right| \leq |c_{kk}|$$

$$= \left| a_{kk} - \frac{a_{1k}}{a_{11}} a_{k1} \right|$$

Since $|a_{kk}| > \sum_{j \neq k}^{m} |a_{jk}|$ (1), we know that:

$$\sum_{j \neq k}^{m} \frac{|a_{jk}|}{|a_{kk}|} < 1, \ \forall k \quad (2) \quad \Rightarrow \sum_{j=2}^{m} \frac{|a_{j1}|}{|a_{11}|} < 1 \text{ for } k=1.$$

$$\Rightarrow \sum_{\substack{j \neq k \\ j=2}}^{m} \frac{|a_{j1}|}{|a_{11}|} + \frac{|a_{k1}|}{|a_{11}|} < 1 \quad (3)$$

Hence, we have that

$$\sum_{\substack{j \neq k \\ j=2}}^{m} \left| a_{jk} - \frac{a_{1k}}{a_{11}} a_{j1} \right| \leq \sum_{\substack{j \neq k \\ j=2}}^{m} |a_{jk}| + \sum_{\substack{j \neq k \\ j=2}}^{m} |a_{j1}| \cdot \frac{|a_{1k}|}{|a_{11}|}, \quad \text{Triangle Ineq.}$$

Because $\sum_{\substack{j \neq k \\ j=2}}^{m} |a_{jk}| < |a_{kk}| - |a_{1k}|$ (from (1)), we have

$$\sum_{\substack{j \neq k \\ j=2}}^{m} \left| a_{jk} - \frac{a_{1k}}{a_{11}} a_{j1} \right| < |a_{kk}| - |a_{1k}| + |a_{1k}| \sum_{\substack{j \neq k \\ j=2}} \frac{|a_{j1}|}{|a_{11}|}$$

$$\leq |a_{kk}| - |a_{1k}| + |a_{1k}| \left( 1 - \frac{|a_{k1}|}{|a_{11}|} \right), \text{ by (3)}$$

$$\leq |a_{kk}| - |a_{1k}| \frac{|a_{k1}|}{|a_{11}|}, \quad \text{distributed and cancel}$$

$$\leq \underbrace{\left| a_{kk} - a_{1k} \frac{a_{k1}}{a_{11}} \right|}$$

$$:= |c_{kk}|.$$

Thus, the submatrix of $A$, say $C$

$$C_{ij} = \text{GE modified entry of } A, \text{ for } i,j = 2, \dots, m,$$

remains diagonally column dominant.

Therefore, the next iteration requires no row swaps.

$\blacksquare$