HW1 Report

Jensen Davies

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Problem 1

Submitted as handwritten PDF.

Problem 2

In this problem we utilize the trapezoidal rule and Simpsons rule to numerically calculate the integral

$$I \equiv \int_1^3 \sqrt{2 + cos^3(x)}\, e^{sin(x)}\, \mathrm{d}x.$$

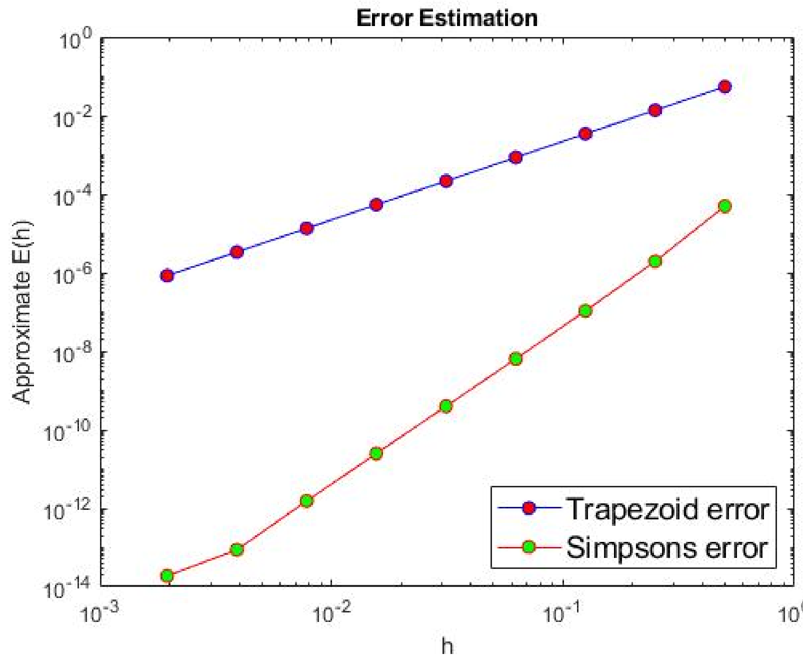
These methods are carried out for varying values of N, where $N = 2^2, 2^3, \ldots, 2^{10}$ are our numerical resolutions. The numerical solutions for the methods for $N = 2^{10}$ are

$$I_{trap} = 5.948926538426832, \qquad I_{simp} = 5.948926749149473.$$

We can see the two values are quite similar (numerically the same in single precision) since

$$|I_{trap} - I_{simp}| = 2.1072\,\text{e-07}.$$

Below is a plot of the estimated error, where each method is labeled clearly.

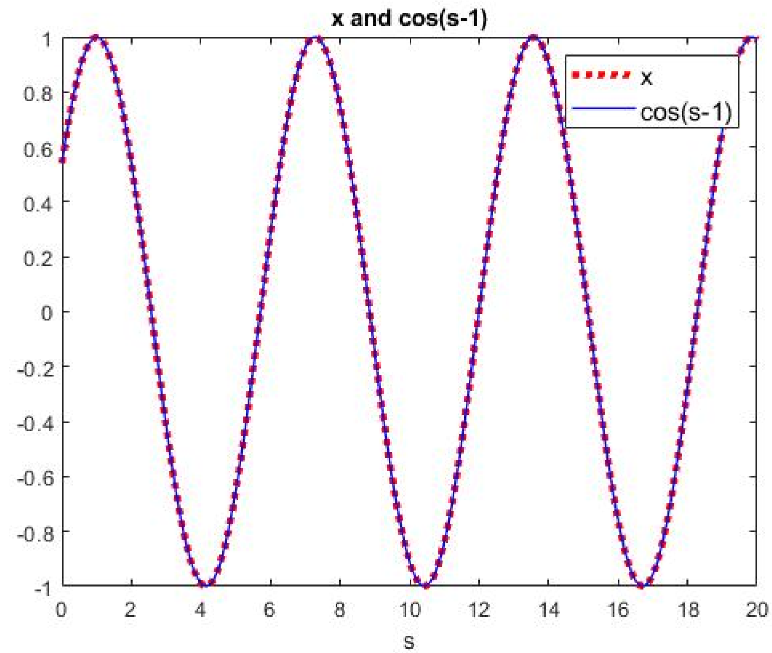


Problem 3

This problem utilizes Newton's method to find the roots of the function

$$f(x) = x - \alpha + \beta \sinh(x - cos(s-1)) = 0,$$

where $s = [0 : 0.1 : 20]$, $\alpha = 0.9$, $\beta = 50000$. As requested, below is a figure of the solution x versus s and $cos(s-1)$ versus s in the same figure. As we can see, the two plots coincide.

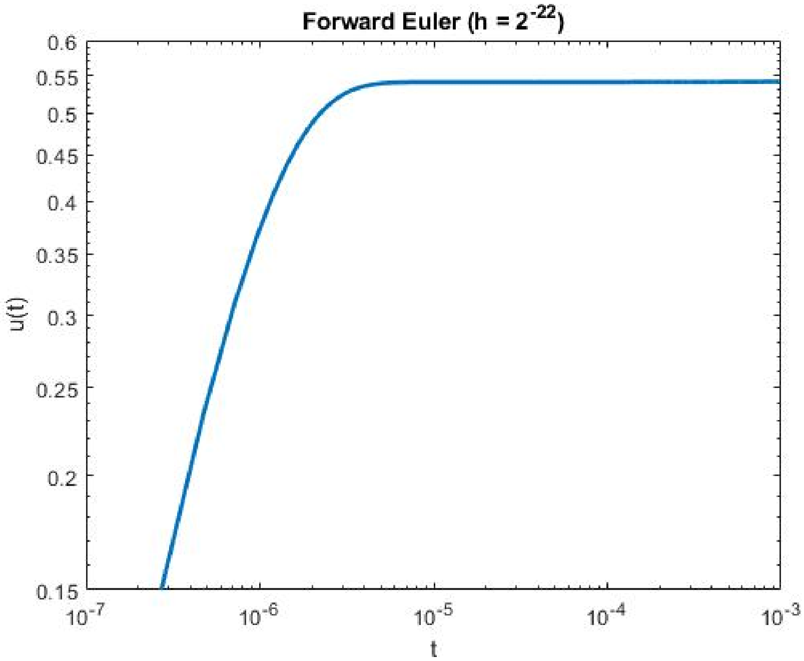


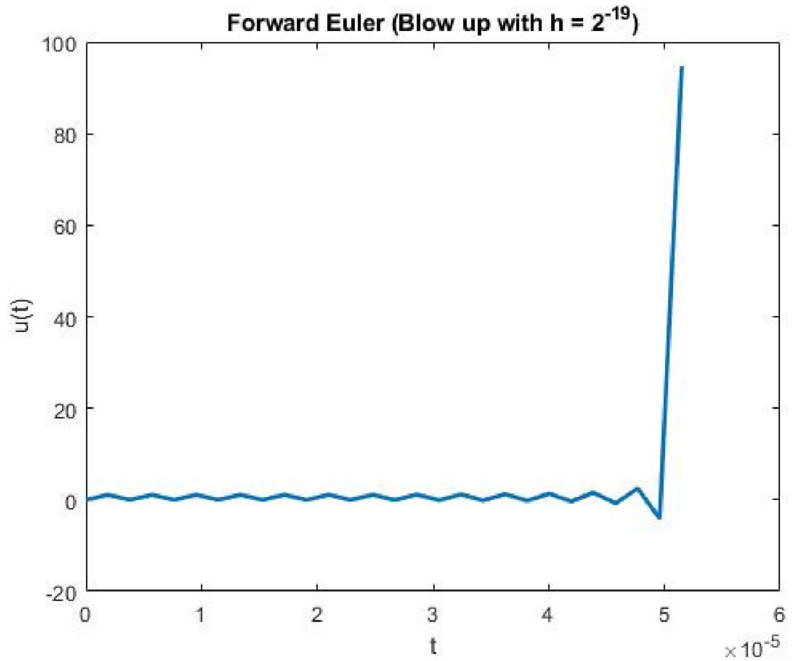
Problem 4

In this problem, we implemented the forward Euler method and backward Euler method to solve the initial value problem

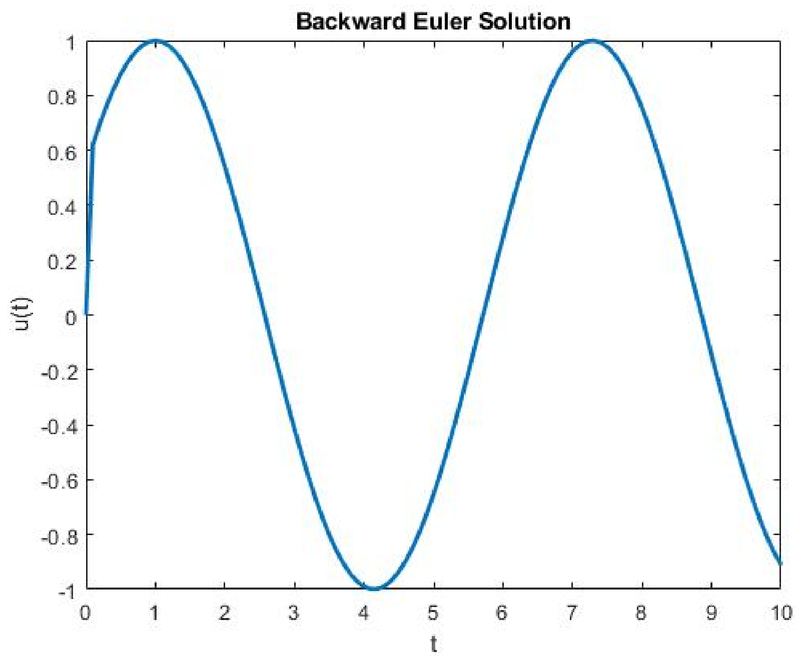
$$\begin{cases} u' = -\lambda \sinh\big(u - \cos(t-1)\big), & \lambda = 10^6 \\ u(0) = 0 \end{cases} .$$

We numerically solve the IVP to $T = 2^{-10}$ with forward Euler, and to $T = 10$ for the backward Euler method. For the forward Euler method, we found that the numerical solution remains bounded when $h \leq 2^{-20}$. If h is any larger, the numerical solution diverges. Provided below are figures for a small enough step size, and a step size too large.



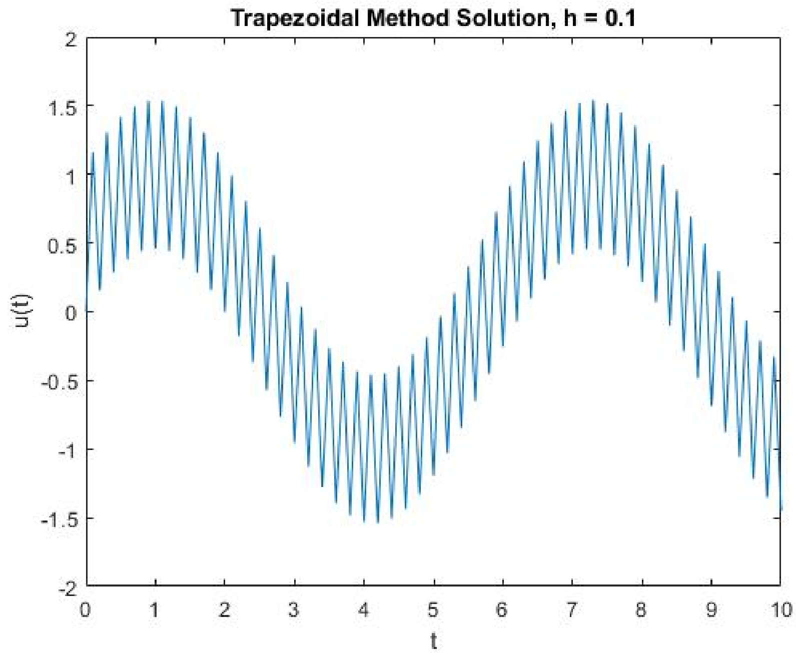


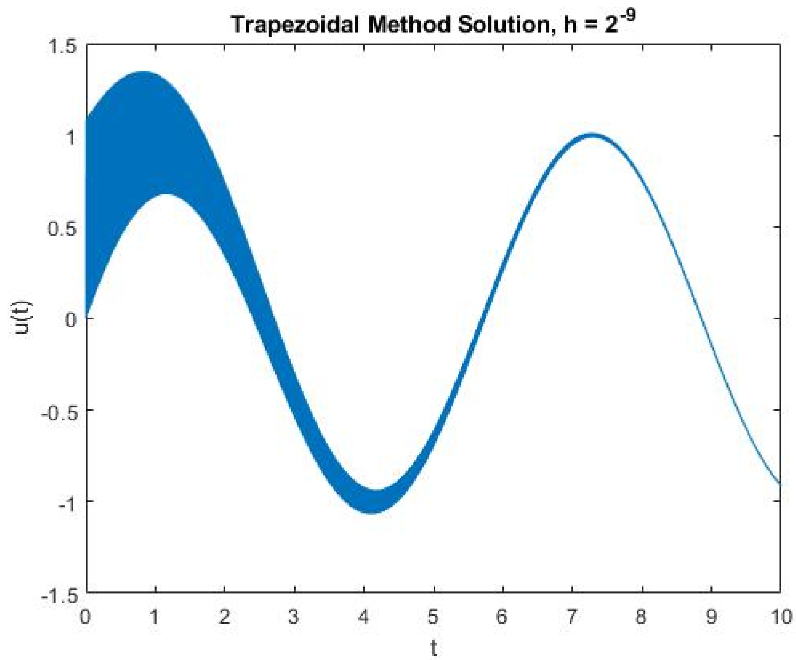
As for for the backward Euler method, we utilize a much larger step size of $h = 0.1$. Below is a figure of the numerical solution plotted versus t, solved up to $T = 10$.



Problem 5

In this problem we implement the trapezoidal method to solve the same IVP in problem 4. We numerically solve up to $T = 10$, with a step size of $h = 0.1$. Below are two figures for varying time step sizes; when $h = 0.1$ we can see clearly that there is oscillation in the numerical solution. As h becomes much smaller, visually the oscillation is much denser looking, and is constricted towards $t = 0$.


Trapezoidal Method Solution, h = 0.1
u(t)
t
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
8
9
10

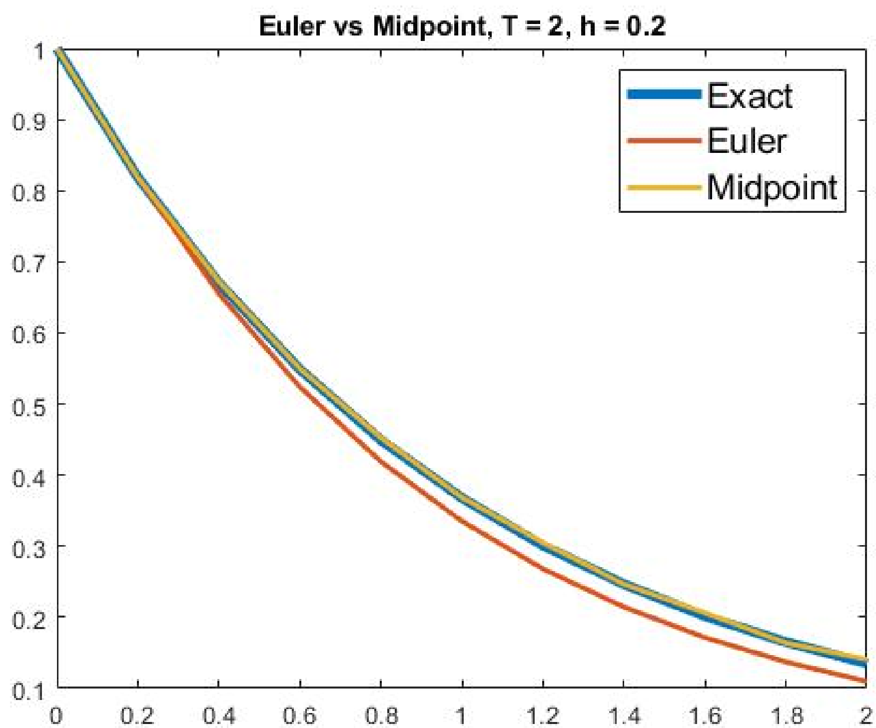

Trapezoidal Method Solution, h = 2^{-9}
u(t)
t
1.5
1
0.5
0
-0.5
-1
-1.5
0
1
2
3
4
5
6
7
8
9
10

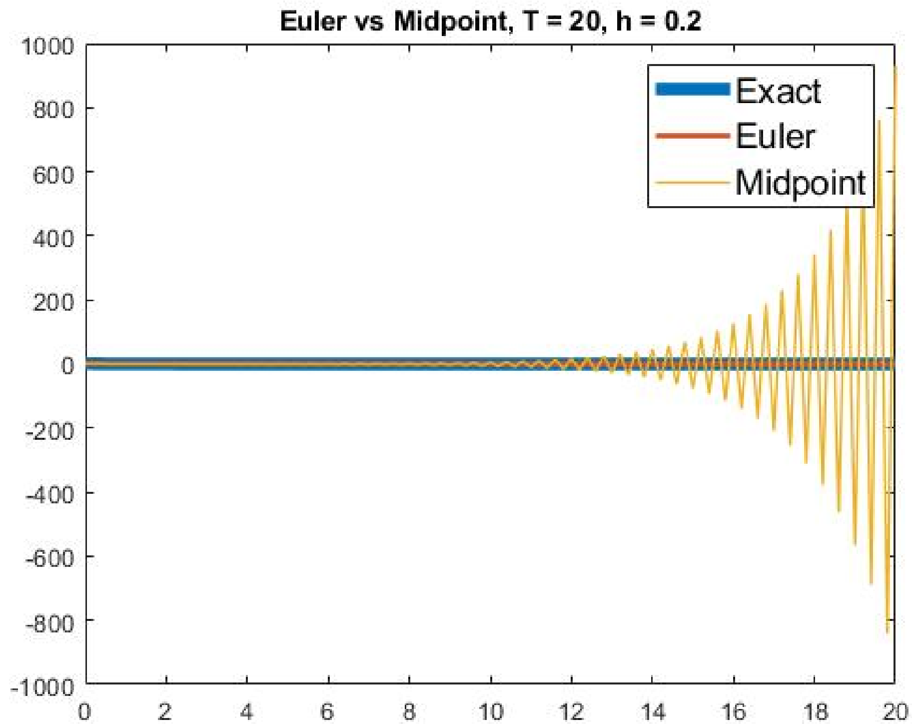
Problem 6

This problem illustrates the inaccuracy of the 2-step midpoint method. We solve the IVP

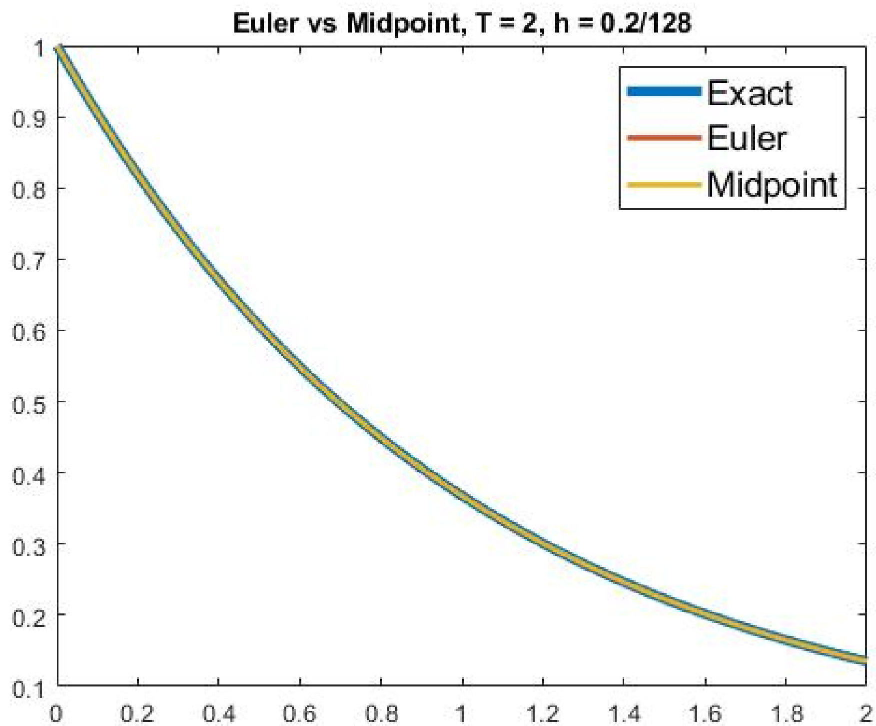
$$\begin{cases} u' = -u \\ u(0) = 1 \end{cases}$$

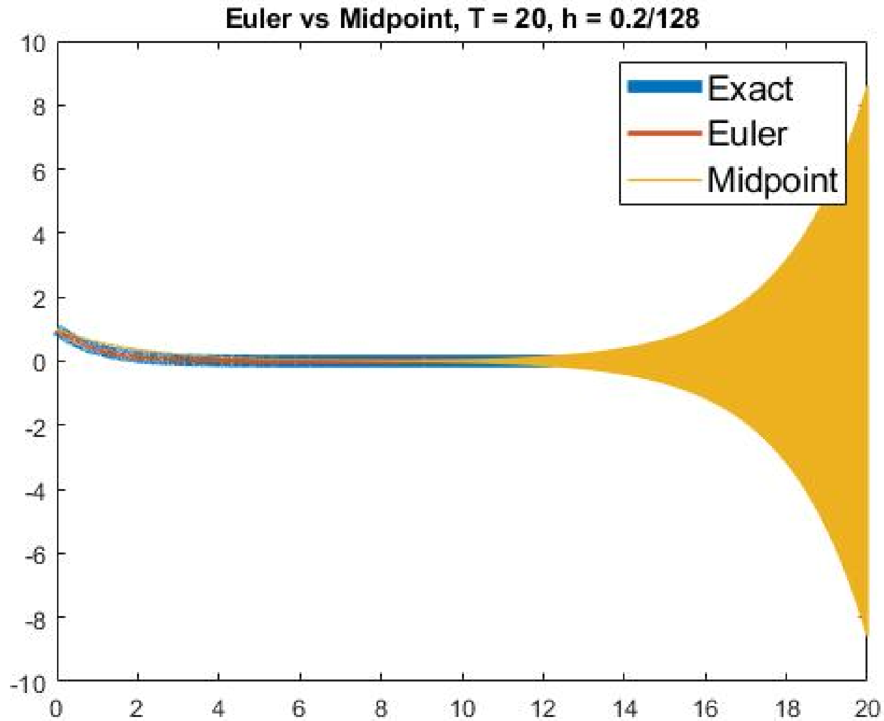
using both the Euler method and the 2-step midpoint method, both with a step size of $h = 0.2$. The exact solution to this IVP is $u_{exact}(t) = e^{-t}$, which we utilize to start the midpoint method. Below is a figure of the numerical results of both methods, strictly for $h = 0.2$. It's clear that the midpoint method is more accurate in this time period.





Evidently, for the longer time period of $T = 20$, the numerical solution isn't well behaved, growing in error as t increases. After correcting my midpoint method, for the larger time step size of $h = 0.2$, we see that the midpoint method is more accurate than the Euler method. When the time step size is reduced significantly, (i.e. to $h = 0.2/128$) there is a large reduction in the growth of error for the midpoint method. For $T = 2$, both the midpoint and Euler method visually appear quite accurate. However, the midpoint method ultimately fails to be well behaved for larger values of T, i.e. $T = 20$. This can be seen below.


Euler vs Midpoint, T = 2, h = 0.2/128
Exact
Euler
Midpoint
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



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(Note to prof: My LaTeX environment compresses these figures a lot, so the coloring is odd. I tried to find good colors, sorry if it isn't clear for the denser figures).

Theory

Wednesday, March 31, 2021 5:21 PM

Problem 1 (Theoretical)

Suppose E_n satisfies the recursive inequality

$$E_{n+1} \le (1+Ch)E_n + h^2 \quad \text{for } n \ge 0$$
$$E_0 = 0$$

where $C > 0$ is a constant independent of h and n.

Derive that $E_N \le \frac{e^{CT}-1}{C} h$ for $Nh \le T$

P1)

Starting with

$E_{n+1} \le (1+Ch)E_n + h^2$, multiply both sides by $(1+Ch)^{-(n+1)}$

$$\Rightarrow (1+Ch)^{-(n+1)} E_{n+1} \le (1+Ch)^{-n} E_n + h^2 (1+Ch)^{-(n+1)}$$

$$\Rightarrow (1+Ch)^{-(n+1)} E_{n+1} - (1+Ch)^{-n} E_n \le h^2 (1+Ch)^{-(n+1)}$$

Then, summing from $n=0$ to $n=N-1$ and utilizing that $E_0=0$,

$$(1+Ch)^{-N} E_N \le h^2 \sum_{n=0}^{N-1} (1+Ch)^{-(n+1)}$$

$$\le h^2 \cdot (1+Ch)^{-1} \cdot \frac{1-(1+Ch)^{-N}}{1-(1+Ch)^{-1}}$$

$1-(1+Ch)$

$$= h^2 \cdot \frac{1 - (1+Ch)^{-N}}{\cancel{1} + Ch - \cancel{1}}$$

$$= h^{\cancel{2}} \frac{1-(1+Ch)^{-N}}{C\cancel{h}}$$

$$= h \cdot \frac{1-(1+Ch)^{-N}}{C}.$$

Since $\displaystyle\sum_{n=0}^{N-1} r^{n+1} = r\left(\frac{1-r^N}{1-r}\right)$.

If we multiply ~~both~~ sides by $(1+Ch)^N$,

$$\Rightarrow \quad E_N \le h \cdot \frac{(1+Ch)^N - 1}{C}$$

Because $1+\alpha \le e^{\alpha} \ \forall \alpha$, we know

$1+Ch \le e^{Ch}$.

Hence,

$$E_N \le \frac{e^{ChN} - 1}{C} h.$$

Since $C>0$, we can then write

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$$E_N \le \frac{e^{CT}-1}{C} \cdot h, \qquad Nh \le T.$$