Theory

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Problem 1 (Theoretical)

Consider the Lax-Friedrichs method for the general case ($a > 0$ or $a < 0$)

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

P1

Part 1: Carry out von Neumann stability analysis.

Part 2: Use Taylor expansion to find the local truncation error e_i^n.

Convert r back to $\Delta t/(\Delta x)$. Find coefficients of $(\Delta t)^2$, $(\Delta t)(\Delta x)$ and $(\Delta x)^2$ in e_i^n.

Part 3: Answer the two questions below.

For fixed $\frac{\Delta t}{\Delta x} = r$, write $\frac{e_i^n}{\Delta t}$ in terms of Δx only. Do we have $\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = 0$?

For fixed $\frac{\Delta t}{(\Delta x)^2} = c$, write $\frac{e_i^n}{\Delta t}$ in terms of Δx only. Do we have $\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = 0$?

Part 1: We have

$$U_i^{n+1} = \frac{U_{i+1}^n + U_{i-1}^n}{2} - \frac{ar}{2}\left(U_{i+1}^n - U_{i-1}^n\right), \quad r = \frac{\Delta t}{\Delta x}$$

$(a < 0$ or $a > 0)$.

First, write

$$L_{num}(U^{n+1}) = L_{num}(U^n)$$

$\Rightarrow$ Already in this form.

We try $U_i^n = \rho^n \cdot e^{\sqrt{-1}\xi i \Delta x}$.

$$\Rightarrow \rho^{n+1} \cdot e^{\sqrt{-1}\xi i \Delta x} = \frac{1}{2}\left[\rho^n \cdot e^{\sqrt{-1}\xi(i+1)\Delta x} + \rho^n e^{\sqrt{-1}\xi(i-1)\Delta x}\right]$$
$$- \frac{ar}{2}\left[\rho^n e^{\sqrt{-1}\xi(i+1)\Delta x} - \rho^n e^{\sqrt{-1}\xi(i-1)\Delta x}\right]$$

Divive by $\rho^n e^{\sqrt{-1}\xi i \Delta x}$:

$$\rho = \frac{1}{2}\left[e^{\sqrt{-1}\xi\Delta x} + e^{-\sqrt{-1}\xi\Delta x}\right] - \frac{ar}{2}\left[e^{\sqrt{-1}\xi\Delta x} - e^{-\sqrt{-1}\xi\Delta x}\right]$$

$$= \cosh(\sqrt{-1}\,\xi\Delta x) - ar\sinh(\sqrt{-1}\,\xi\Delta x)$$

$$= \cos(\xi\Delta x) - ar \cdot i\sin(\xi\Delta x),$$

since $\cosh(i\cdot\alpha) = \cos(\alpha)$ and $\sinh(i\cdot\alpha) = i\sin(\alpha)$.

Considering $|\rho|^2$, we have

$$|\rho|^2 = \left[\cos(\xi\Delta x) - i\,ar\sin(\xi\Delta x)\right]\left[\cos(\xi\Delta x) + i\,ar\sin(\xi\Delta x)\right]$$

$$= \cos^2(\xi\Delta x) - i\,ar\cancel{\cos(\xi\Delta x)\sin(\xi\Delta x)} + i\,ar\cancel{\cos(\xi\Delta x)\sin(\xi\Delta x)}$$
$$+ a^2r^2\sin^2(\xi\Delta x)$$

$$= \cos^2(\xi\Delta x) + a^2r^2\sin^2(\xi\Delta x)$$

$\Rightarrow$ We need $|\rho| \leq 1 + C\Delta t$ for small Δt and all ξ.

$$|\rho|^2 = \cos^2(\xi\Delta x) + a^2r^2\sin^2(\xi\Delta x)$$

$$\Rightarrow |\rho| = \sqrt{\cos^2(\xi\Delta x) + (ar)^2\sin^2(\xi\Delta x)}$$

$$|\rho| = \sqrt{1 - \sin^2(\xi\Delta x) + (ar)^2\sin^2(\xi\Delta x)}$$

$$= \sqrt{1 + (a^2r^2 - 1)\sin^2(\xi\Delta x)}$$

Case 1:

Let $ar > 1$. Then, $a^2r^2 > 1$.

Set $k = a^2r^2 - 1 > 0$, giving us

$$|\rho| = \sqrt{1 + k\sin^2(\xi\Delta x)} > 1.$$

Case 2:

Now, let $ar < -1$, $\Rightarrow$ $a^2r^2 > 1$ again,

giving $|\rho| > 1$.

Case 3:

Hence, let $|ar| \leq 1$, $\Rightarrow$ $a^2r^2 \leq 1$

$$\Rightarrow \quad k := a^2r^2 - 1 \leq 0$$

$$\Rightarrow \quad |\rho| = \sqrt{1 + k\sin^2(\xi\Delta x)} \leq 1.$$

Thus, $|\rho| \leq 1$ requires $|ar| \leq 1$.

Therefore, the Lax-Friedrichs method is stable for $|ar| \leq 1$.

Part 2: Ask:

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

We WTS $e_i^n(\Delta x, \Delta t) = \Delta t \cdot O(\Delta t + \Delta x)$.

We know

$$e_i^n(\Delta x, \Delta t) = u(x_i, t_{n+1}) - L_{num}(u(x_i, t_n))$$

$$\Rightarrow \quad e_i^n(\Delta x, \Delta t) = u(x_i, t_{n+1}) - \frac{1}{2}\left[u(x_{i+1}, t_n) + u(x_{i-1}, t_n)\right] + \frac{ar}{2}\left[u(x_{i+1}, t_n) - u(x_{i-1}, t_n)\right]$$

$$+ \frac{ar}{2}\left[u(x_{i+1}, t_n) - u(x_{i-1}, t_n)\right]$$

$$\Rightarrow \quad e_i^n(\Delta x, \Delta t) = \overbrace{u(x_i, t_n + \Delta t)}^{①} - \frac{1}{2}\left[\overbrace{u(x_i + \Delta x, t_n) + u(x_i - \Delta x, t_n)}^{②}\right]$$

$$+ \frac{ar}{2}\left[\underbrace{u(x_i + \Delta x, t_n) - u(x_i - \Delta x, t_n)}_{③}\right]$$

Expanding ①, ②, ③ around (x_i, t_n): Let u denote $u\big|_{(x_i, t_n)}$.

①: $u(x_i, t_n + \Delta t) = u + u_t \Delta t + \frac{1}{2} u_{tt} \Delta t^2 + \mathcal{O}(\Delta t^3)$

②: $u(x_i + \Delta x, t_n) + u(x_i - \Delta x, t_n)$

$$= u + \cancel{u_x \Delta x} + \frac{1}{2} u_{xx} \Delta x^2 + \mathcal{O}(\Delta x^3) + u - \cancel{u_x \Delta x} + \frac{1}{2} u_{xx} \Delta x + \mathcal{O}(\Delta x^4)$$

$$= 2u + u_{xx} \Delta x^2 + \mathcal{O}(\Delta x^4)$$

odd powers of Δx cancel.

③: $u(x_i + \Delta x, t_n) - u(x_i - \Delta x, t_n)$

$$= \cancel{u} + u_x \Delta x + \cancel{\frac{1}{2} u_{xx} \Delta x^2} + \mathcal{O}(\Delta x^3) - \left[\cancel{u} - u_x \Delta x + \cancel{\frac{1}{2} u_{xx} \Delta x^2} + \mathcal{O}(\Delta x^3)\right]$$

$$= 2u_x \Delta x + \mathcal{O}(\Delta x^3)$$

Plugging in these expansions...

$$e_i^n(\Delta x, \Delta t) = \overbrace{u(x_i, t_n + \Delta t)}^{①} - \frac{1}{2}\left[\overbrace{u(x_i + \Delta x, t_n) + u(x_i - \Delta x, t_n)}^{②}\right] + \frac{ar}{2}\left[\underbrace{u(x_i + \Delta x, t_n) - u(x_i - \Delta x, t_n)}_{③}\right]$$

$$\Rightarrow \quad e_i^n(\Delta x, \Delta t) = \cancel{u} + u_t \Delta t + \frac{1}{2} u_{tt} \Delta t^2 + \mathcal{O}(\Delta t^3)$$

$$- \cancel{\frac{1}{2}}\left[\cancel{2}u + u_{xx} \Delta x^2 + \mathcal{O}(\Delta x^4)\right] + \frac{a}{\cancel{2}} \cdot \frac{\Delta t}{\Delta x}\left[\cancel{2} u_x \Delta x + \mathcal{O}(\Delta x^3)\right]$$

$$= (u_t + a u_x) \Delta t + \frac{1}{2} u_{tt} \Delta t^2 - \frac{1}{2} u_{xx} \Delta x^2$$

$$= (u_t + a u_x)\Delta t + \frac{1}{2}u_{tt}\Delta t^2 - \frac{1}{2}u_{xx}\Delta x^2$$

$$+ \mathcal{O}(\Delta t^3) + \mathcal{O}(\Delta x^4) + \mathcal{O}(\Delta x^2)\Delta t$$

From the model equation for hyperbolic PDEs:

$$u_t + a u_x = 0$$

$$\Rightarrow \quad e_i^n(\Delta x, \Delta t) = \frac{1}{2}\left[u_{tt}\Delta t^2 - u_{xx}\Delta x^2\right] + \Delta t\,\mathcal{O}(\Delta t^2 + \Delta x^2) + \mathcal{O}(\Delta x^4).$$

<u>Part 3</u>: We have that

$$e_i^n = \frac{1}{2}u_{tt}\Delta t^2 - \frac{1}{2}u_{xx}\Delta x^2 + \Delta t\,\mathcal{O}(\Delta t^2 + \Delta x^2) + \mathcal{O}(\Delta x^4)$$

$$\Rightarrow \frac{e_i^n}{\Delta t} = \frac{1}{2}u_{tt}\Delta t - \frac{1}{2}u_{xx}\frac{\Delta x^2}{\Delta t} + \mathcal{O}(\Delta t^2 + \Delta x^2) + \frac{1}{\Delta t}\mathcal{O}(\Delta x^4)$$

<u>For fixed $r = \frac{\Delta t}{\Delta x}$</u>:

$r = \frac{\Delta t}{\Delta x}$: $\Rightarrow \Delta t = \Delta x \cdot r \Rightarrow \Delta t^2 = \Delta x^2 r^2$. Thus, using $u_{tt} = a^2 u_{xx}$

$$\frac{e_i^n}{\Delta t} = \frac{a^2 r}{2}u_{xx}\Delta x - \frac{1}{2}u_{xx}\Delta x r^{-1} + \mathcal{O}(r^2\Delta x^2 + \Delta x^2) + \frac{1}{r}\mathcal{O}(\Delta x^3)$$

Hence,

$$\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = 0.$$

For fixed $c = \dfrac{\Delta t}{\Delta x^2}$, we have:

$$c = \frac{\Delta t}{\Delta x^2} \Rightarrow \Delta t = \Delta x^2 c \Rightarrow \Delta t^2 = c^2 \Delta x^4. \quad \text{Thus,}$$

$$\frac{e_i^n}{\Delta t} = \frac{1}{2} u_{tt} \Delta t - \frac{1}{2} u_{xx} \frac{\Delta x^2}{\Delta t} + O(\Delta t^2 + \Delta x^2) + \frac{1}{\Delta t} O(\Delta x^4)$$

$$\Rightarrow \quad \frac{e_i^n}{\Delta t} = \frac{a^2 c}{2} u_{xx} \Delta x^2 - \frac{1}{2c} u_{xx} + O(c^2 \Delta x^4) + \frac{1}{c} O(\Delta x^2)$$

We can see that

$$\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = -\frac{1}{2c} \cdot u_{xx} \neq 0.$$

Problem 2 (Theoretical)

P2

Carry out von Neumann stability analysis on each of the methods below

i) the BTCS method for the general case ($a > 0$ or $a < 0$)

$$u_i^{n+1} = u_i^n - \frac{ar}{2}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

ii) the implicit upwind method for the case of $a > 0$

$$u_i^{n+1} = u_i^n - ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

Hint: Examine $|1/\rho|^2$.

i) We need

$$L_{num}(u^{n+1}) = L_{num}(u^n)$$

$$u^{n+1} \quad ar \left(u^{n+1} \quad u^{n+1} \right) \quad u^n$$

$$\Rightarrow \quad u_i^{n+1} + \frac{ar}{2}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) = u_i^n$$

We try $u_i^n = \rho^n e^{\sqrt{-1}\,\xi i \Delta x}$:

$$\rho^{n+1} e^{-\sqrt{-1}\,\xi i\Delta x} + \frac{ar}{2}\left[\rho^{n+1} e^{\sqrt{-1}\,\xi(i+1)\Delta x} - \rho^{n+1} e^{\sqrt{-1}\,\xi(i-1)\Delta x}\right] = \rho^n e^{\sqrt{-1}\,\xi i\Delta x}$$

Dividing by $\rho^n e^{\sqrt{-1}\,\xi i \Delta x}$:

$$\rho + \frac{ar}{2}\left[\rho e^{\sqrt{-1}\,\xi\Delta x} - \rho e^{-\sqrt{-1}\,\xi\Delta x}\right] = 1$$

$$\Rightarrow \quad \rho\left[1 + ar\sinh(\sqrt{-1}\,\xi\Delta x)\right] = 1$$

Using $\sinh(i\alpha) = i\sin(\alpha)$:

$$\rho\left[1 + i\,ar\sin(\xi\Delta x)\right] = 1$$

Hence,

$$\rho = \frac{1}{1 + i\,ar\sin(\xi\Delta x)} \qquad |z| = \sqrt{a^2+b^2}$$

$$\Rightarrow \quad |\rho| = \frac{1}{\sqrt{1 + a^2r^2\sin^2(\xi\Delta x)}}$$

Notice that $\sqrt{1+a^2r^2\sin^2(\xi\Delta x)} > 0$.

Hence $|\rho| \leq 1$ regardless of ar.

Hence, $|\rho| \leq 1$ regardless of ar.

$\therefore$ BTCS is unconditionally stable.

ii) We need

$$L_{num}(u^{n+1}) = L_{num}(u^{n}),$$

$$\Rightarrow \quad u_i^{n+1} + ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right) = u_i^{n}.$$

We try $u_i^n = \rho^n e^{\sqrt{-1}\,\xi i \Delta x}$:

$$\rho^{n+1} e^{\sqrt{-1}\,\xi i \Delta x} + ar\left[\rho^{n+1} e^{\sqrt{-1}\,\xi i \Delta x} - \rho^{n+1} e^{\sqrt{-1}\,\xi (i-1)\Delta x}\right] = \rho^{n} e^{\sqrt{-1}\,\xi i \Delta x}$$

Dividing by $\rho^n e^{\sqrt{-1}\,\xi i \Delta x}$:

$$\rho + ar\left[\rho - \rho e^{-\sqrt{-1}\,\xi \Delta x}\right] = 1$$

$$\Rightarrow \quad \rho\left[1 + ar\left(1 - e^{-\sqrt{-1}\,\xi \Delta x}\right)\right] = 1$$

We have $e^{-\sqrt{-1}\,\xi \Delta x} = \cos(\xi \Delta x) - i\sin(\xi \Delta x)$

$$\Rightarrow \quad \rho = \frac{1}{1 + ar[1 - \cos(\xi\Delta x) + i\sin(\xi\Delta x)]}$$

$$\Rightarrow \quad \rho = \frac{1}{1 + ar - ar\cos(\xi\Delta x) + iar\sin(\xi\Delta x)}$$

Hence,

$$|\rho| = \frac{1}{\sqrt{[1 + ar - ar\cos(\xi\Delta x)]^2 + a^2r^2\sin^2(\xi\Delta x)}}$$

Working in the denominator:

$$(1 + ar - ar\cos(\xi\Delta x))^2 = 1 + ar - ar\cos(\xi\Delta x) + ar + a^2r^2$$
$$- a^2r^2\cos(\xi\Delta x) - ar\cos(\xi\Delta x)$$
$$- a^2r^2\cos(\xi\Delta x) + a^2r^2\cos^2(\xi\Delta x)$$

$$= \overbrace{1 + 2ar + a^2r^2}^{(1+ar)^2} - 2ar\cos(\xi\Delta x) - 2a^2r^2\cos(\xi\Delta x) + a^2r^2\cos^2(\xi\Delta x)$$

Hence, considering the hint, we have

$$\frac{1}{|\rho|^2} = 1 + 2ar + a^2r^2 - 2ar\cos(\xi\Delta x) - 2a^2r^2\cos(\xi\Delta x) + a^2r^2\cancel{\cos^2(\xi\Delta x)} + a^2r^2\cancel{\sin^2(\xi\Delta x)}$$

$$= 1 + 2ar + 2a^2r^2 - 2ar\cos(\xi\Delta x) - 2a^2r^2\cos(\xi\Delta x)$$

$$= 1 + 2ar(1 + ar - \cos(\xi\Delta x) - ar\cos(\xi\Delta x))$$

$$= 1 + 2ar\left[1 + ar - (1+ar)\cos(\xi\Delta x)\right]$$

$$= 1 + 2ar\left[(1+ar)(1-\cos(\xi\Delta x))\right]$$

$$= 1 + \left[2(ar + a^2r^2)(1-\cos(\xi\Delta x))\right]$$

In Upwind method, $a > 0$.

$\Rightarrow \frac{1}{|\rho|} \geq 1$, hence $|\rho| \leq 1$.

$\therefore$ the implicit upwind method is unconditionally stable.