Theory

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Problem 1 (Theoretical)

Consider the Lax-Friedrichs method for solving $u_t + a\,u_x = 0$

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

P1

On the RHS, we write $\frac{u_{i+1}^n + u_{i-1}^n}{2}$ as $u_i^n + \underbrace{\frac{1}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}_{\text{Added viscosity}}$.

We know that the Lax-Friedrichs method has too much added viscosity. So we consider a modified version of Lax-Friedrichs

$$u_i^{n+1} = u_i^n + \frac{q}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \quad r = \frac{\Delta t}{\Delta x}, \quad 0 \le q \le 1 \qquad \text{(LF-2)}$$

Part 1: Find the modified PDE of (LF-2).

Part 2: Find the modified PDE of the implicit upwind method

$$u_i^{n+1} = u_i^n - ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

Hint: Expanding around (x_i, t_{n+1}) will make it easier.

Part 1: We have the modified Lax-Friedrichs method:

$$U_i^{n+1} = U_i^n + \frac{q}{2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right) - \frac{ar}{2}\left(U_{i+1}^n - U_{i-1}^n\right),$$

with $r = \frac{\Delta t}{\Delta x}$, $0 \le q \le 1$.

satisfies a perturbed PDE

Assume that $U_i^n = w(x_i, t_n)$. Then, we have

$$\underbrace{w(x_i, t_n + \Delta t) - w(x_i, t_n)}_{①} = \frac{q}{2}\Big[\underbrace{w(x_{i+1}, t_n) - 2w(x_i, t_n) + w(x_{i-1}, t_n)}_{②}\Big]$$
$$- \frac{ar}{2}\Big[\underbrace{w(x_{i+1}, t_n) - w(x_{i-1}, t_n)}_{③}\Big]$$

We now expand ①, ②, ③ around (x_i, t_n).

Let $w \equiv w(x_i, t_n)$.

①: $w(x_i, t_n + \Delta t) - w(x_i, t_n)$

1: $w(x_i, t_n + \Delta t) - w(x_i, t_n)$

$$= W + w_t \Delta t + \frac{1}{2} w_{tt} \Delta t^2 + O(\Delta t^3) - w$$

$$= W_t \Delta t + \frac{1}{2} W_{tt} \Delta t^2 + O(\Delta t^3).$$

2: $w(x_i + \Delta x, t_n) - 2w(x_i, t_n) + w(x_i - \Delta x, t_n)$

$$= W + W_x \Delta x + \frac{1}{2} W_{xx} \Delta x^2 - 2w$$

$$+ w - W_x \Delta x + \frac{1}{2} W_x \Delta x^2 + O(\Delta x^4)$$ only even powers survive.

$$= W_{xx} \Delta x^2 + O(\Delta x^4)$$

3: $w(x_i + \Delta x, t_n) - w(x_i - \Delta x, t_n)$

$$= W + w_x \Delta x + \frac{1}{2} W_{xx} \Delta x^2 + O(\Delta x^3) - \left[W - W_x \Delta x + \frac{1}{2} W_{xx} \Delta x^2 - O(\Delta x^3)\right]$$

$$= 2W_x \Delta x + O(\Delta x^3)$$

$$\overbrace{w(x_i, t_n + \Delta t) - w(x_i, t_n)}^{1} = \frac{q}{2}\left[\overbrace{w(x_{i+1}, t_n) - 2w(x_i, t_n) + w(x_{i-1}, t_n)}^{2}\right] - \frac{ar}{2}\left[\underbrace{w(x_{i+1}, t_n) - w(x_{i-1}, t_n)}_{3}\right]$$

Substituting into the modified method, we have

$$W_t \Delta t + \frac{1}{2} W_{tt} \Delta t^2 + O(\Delta t^3) = \frac{q}{2}\left[W_{xx} \Delta x^2 + O(\Delta x^4)\right] - \frac{ar}{2}\left[2W_x \Delta x + O(\Delta x^3)\right]$$

$$\Rightarrow W_t \Delta t + \frac{1}{2} W_{tt} \Delta t^2 + O(\Delta t^3) = \frac{q}{2} W_{xx} \Delta x^2 - ar \cdot W_x \Delta x + O(\Delta x^3)$$

Dividing by Δt and rearranging, we have

Dividing by Δt and rearranging, we have

$$w_t = -aw_x + \frac{g}{2r} w_{xx}\Delta x - \frac{1}{2} w_{tt}\Delta t + O(\Delta t^2) + \frac{1}{\Delta t} O(\Delta x^3)$$

Following the iterative approach provided in lecture, we can convert w_{tt} to a spatial derivative.

We have

$$w_t = -aw_x + O(\Delta x + \Delta t)$$

$$\Rightarrow w_{tt} = (-aw_x)_t + O(\Delta x + \Delta t)$$

$$= -a(w_t)_x + O(\Delta x + \Delta t)$$

Plugging in w_t again gives

$$w_{tt} = -a(-aw_x)_x + O(\Delta x + \Delta t)$$

$$= a^2 w_{xx} + O(\Delta x + \Delta t).$$

Thus, we have that

$$w_t = -aw_x + \frac{g}{2r} w_{xx}\Delta x - \frac{a^2}{2} w_{xx}\Delta t + O(\Delta t^2) + \frac{1}{\Delta t} O(\Delta x^3)$$

$$\Rightarrow w_t = -aw_x + \frac{1}{2}(gr^{-1}\Delta x - a^2\Delta t) w_{xx} + O(\Delta t^2) + \frac{1}{\Delta t} O(\Delta x^3)$$

Leaving off higher order terms, we have that

$$\boxed{w_t = -a w_x + \sigma w_{xx}, \qquad \sigma = \frac{\Delta x}{2}\left(q r^{-1} - a^2 r\right), \quad r = \frac{\Delta t}{\Delta x}.}$$

Part 2: We have the implicit upwind method:

$$U_i^{n+1} = U_i^n - ar\left(U_i^{n+1} - U_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}.$$

Assume that $U_i^n = w(x_i, t_n)$. Then, we can write

$$\underbrace{w(x_i, t_n+\Delta t) - w(x_i, t_n)}_{①} = -ar\left[\underbrace{w(x_i, t_n+\Delta t) - w(x_i-\Delta x, t_n+\Delta t)}_{②}\right]$$

Expanding ①, ② around $(x_i, t_{n+1}) = (x_i, t_n+\Delta t)$

Let $W \equiv w(x_i, t_n+\Delta t)$.

Then,

①: $w(x_i, t_n+\Delta t) - w(x_i, t_{n+1}-\Delta t)$

$$= W - \left[W - W_t \Delta t + \tfrac{1}{2} W_{tt} \Delta t^2 - O(\Delta t^3)\right]$$

$$= W_t \Delta t - \tfrac{1}{2} W_{tt} \Delta t^2 + O(\Delta t^3)$$

②: $w(x_i, t_n+\Delta t) - w(x_i - \Delta x, t_n+\Delta t)$

$$= W - \left[W - W_x \Delta x + \tfrac{1}{2} W_{xx} \Delta x^2 - O(\Delta x^3)\right]$$

$$= w - \left[w - w_x \Delta x + \frac{1}{2} w_{xx} \Delta x^2 - \mathcal{O}(\Delta x^3) \right]$$

$$= w_x \Delta x - \frac{1}{2} w_{xx} \Delta x^2 + \mathcal{O}(\Delta x^3).$$

Substituting into the method: $\underbrace{w(x_i, t_n+\Delta t) - w(x_i, t_n)}_{①} = -ar\left[\underbrace{w(x_i, t_n+\Delta t) - w(x_i-\Delta x, t_n+\Delta t)}_{②}\right]$

$$\Rightarrow \quad w_t \Delta t - \frac{1}{2} w_{tt} \Delta t^2 + \mathcal{O}(\Delta t^3) = -a \frac{\Delta t}{\Delta x} \left[w_x \Delta x - \frac{1}{2} w_{xx} \Delta x^2 + \mathcal{O}(\Delta x^3) \right]$$

$$\Rightarrow \quad w_t \Delta t - \frac{1}{2} w_{tt} \Delta t^2 + \mathcal{O}(\Delta t^3)$$

$$= -a \Delta t\, w_x + \frac{a}{2} w_{xx} \Delta t \Delta x + \Delta t\, \mathcal{O}(\Delta x^2)$$

Hence, dividing by Δt and rearranging some terms,

$$w_t = -a w_x + \frac{1}{2} w_{tt} \Delta t + \frac{a}{2} w_{xx} \Delta x + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$$

Using $w_{tt} = a^2 w_{xx}$, we have

$$w_t = -a w_x + \frac{a^2}{2} w_{xx} \Delta t + \frac{a}{2} w_{xx} \Delta x + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$$

$$\Rightarrow \quad w_t = -a w_x + \frac{a \Delta x}{2} (ar + 1) w_{xx} + \mathcal{O}(\Delta t^2 + \Delta x^2)$$

$\Rightarrow$ Modified PDE is:

$$\boxed{w_t = -a w_x + \sigma w_{xx}, \quad \sigma = \frac{a \Delta x}{2} (ar+1), \quad r = \frac{\Delta t}{\Delta x}}$$