AM213B Assignment #3

Problem 1 (Theoretical)

Part 1: Derive the stability function $\phi(z)$ for each of the two RK methods below

- Predictor-corrector method (Heun's method)
- Classic 4-th order Runge-Kutta method (RK4)

Hint: Check your expression of $\phi(z)$ with the theorem we studied.

Theorem: If an RK method is *p*-th order accurate, then it must satisfies

$\phi(z)=e^{z}+O(|z|^{p+1})$ for any complex z with small $|z|$

Part 2: Study the zero-stability for each of the two LMMs below

- $u_{n+2}-2u_{n+1}+u_{n}=hf(u_{n+1},t_{n+1})-hf(u_{n},t_{n})$
- $u_{n+2}-u_{n}=h\left[\frac{1}{3}f(u_{n+2},t_{n+2})+\frac{4}{3}f(u_{n+1},t_{n+1})+\frac{1}{3}f(u_{n},t_{n})\right]$

Problem 2 (Theoretical)

Consider the Runge-Kutta method described by Butcher tableau

Butcher tableau:
$$\begin{array}{c|cc} \alpha & \alpha & 0 \\ 1 & 1-\alpha & \alpha \\ \hline & 1-\alpha & \alpha \end{array}$$

where $\alpha > 0$. This is called a 2s-DIRK (2-Stage Diagonally Implicit Runge-Kutta) method.

The matrix A of a DIRK method is lower triangular so $\{k_1, k_2, k_3, \ldots\}$ can be solved sequentially. The first row of A gives an equation on k_1 without involving $\{k_2, k_3, \ldots\}$. The second row of A gives an equation on k_2 without involving $\{k_3, \ldots\}$ where k_1 is already known. This is in contrast to a fully implicit Runge-Kutta where $\{k_1, k_2, k_3, \ldots\}$ has to be solved simultaneously in a joint system.

Part 1: Show that method is second order for $\alpha = 1-1/\sqrt{2}$.

Hint: check the internal consistency condition, the condition for first order and the additional condition for second order.

Part 2: Apply the 2s-DIRK to solving $u' = \gamma u$.

Derive the expressions for k_1, k_2 and the stability function $\phi(z)$.

$$k_1 = \frac{z}{1-\alpha z} u_n \,, \qquad k_2 = \frac{(1-\alpha z)z + (1-\alpha)z^2}{(1-\alpha z)^2} u_n$$

$$\phi(z) = \frac{1+(1-2\alpha)z}{(1-\alpha z)^2}$$

Part 3: Suppose the 2s-DIRK is A-stable for $\alpha = 1 - 1/\sqrt{2}$ (see Problem 4 below).

Show that it satisfies the second condition of L-stability.

Problem 3 (Theoretical)

Consider the implicit 2-step method below

$$u_{n+2} - u_n = h\left[\frac{1}{3} f(u_{n+2}, t_{n+2}) + \frac{4}{3} f(u_{n+1}, t_{n+1}) + \frac{1}{3} f(u_n, t_n)\right]$$

Part 1: Use Taylor expansion to show $e_n(h) = O(h^5)$.

Hint: Expand everything around t_{n+1}.

Part 2: The stability polynomial is

$$\pi(\xi, z) = \left(\xi^2 - 1\right) - z\left(\frac{1}{3}\xi^2 + \frac{4}{3}\xi + \frac{1}{3}\right)$$

Consider $z = -\varepsilon$ with small $\varepsilon > 0$. We examine the two roots of $\pi(\xi, -\varepsilon)$.

Show that the two roots $\xi_1(\varepsilon)$ and $\xi_2(\varepsilon)$ have the expansions

$$\xi_1(\varepsilon) = 1 - \varepsilon + O(\varepsilon^2)\,, \qquad \xi_2(\varepsilon) = -\left(1 + \frac{\varepsilon}{3}\right) + O(\varepsilon^2)$$

Therefore, $z = -\varepsilon$ is NOT in the region of absolute stability.

Remarks:

- This method demonstrates that the Dahlquist barrier on accuracy of implicit LMMs: $p \le r + 2$ is actually attainable.
- Although this method has the 4th order, it is not practically useful. When applied to solving $u' = -u$, the numerical solution contains a decaying mode (corresponding to the exact solution), and an oscillating and exponentially growing mode (which will eventually ruin the numerical solution). This is similar to the situation with the 2-step midpoint method.

Problem 4 (Computational)

Plot the region of absolute stability (S) for each of the methods below.

- Predictor-corrector method (Heun's method)
- Classic 4-th order Runge-Kutta method (RK4)
- 2s-DIRK method with $\alpha = 1 - 1/\sqrt{2}$
- 2s-DIRK method with $\alpha = 0.5$

Hint: Look at the sample code on how to plot contours of $f(x, y)$.

Problem 5 (Computational)

Read the sample code implementing a 3-stage DIRK method. Understand the code and modify the code to implement the 2s-DIRK method with $\alpha = 1 - 1/\sqrt{2}$.

Solve the IVP below to T = 30.

$$\begin{cases} u' = -\Big(0.5 + \exp\big(20\cos(1.3t)\big)\Big)\sinh\big(u - \cos(t)\big) \\ u(0) = 0 \end{cases}$$

Part 1:

Plot the numerical solution $u(t)$ vs t of the 2s-DIRK for $h = 2^{-5}$.

Plot $\cos(t)$ vs t in the same figure for comparison.

Does the solution $u(t)$ always follow the function $\cos(t)$ very closely?

Part 2:

Use loglog to plot $\left|u(t) - \cos(t)\right|$ vs $\Big(0.5 + \exp\big(20\cos(1.3t)\big)\Big)$ for $t \in (0, 30]$.

For what value of $\cos(1.3t)$, does $u(t)$ follow $\cos(t)$ closely?

Problem 6 (continue with the IVP in Problem 5)

Implement the backward Euler and the 2s-DIRK method with $\alpha = 1 - 1/\sqrt{2}$. Use each of these two methods to solve the IVP to T = 30. Try time steps $h = \frac{1}{2^3}, \frac{1}{2^4}, \ldots, \frac{1}{2^8}$.

For each numerical method, carry out numerical error estimation.

Part 1:

For each method, plot the estimated error vs t for $h = 2^{-5}$. Plot the two curves in ONE figure to compare the two. Use the logarithmic scale for the errors.

Part 2:

In a separate figure, plot the two curves of estimated error vs t for $h = 2^{-7}$.