AM213B Assignment #2

Problem 1 (Theoretical)

Suppose k satisfies the equation

$$k = h\exp(1+k) \qquad \text{where } h \text{ is a small quantity}$$

Recall the approach of iterative expansion we used in lecture.

Start with $k = O(h)$. Expand k iteratively into

$$k = a_1 h + a_2 h^2 + \cdots$$

Find the coefficients a_1 and a_2.

Problem 2 (Theoretical)

Consider the Runge-Kutta (RK) method specified by the Butcher tableau below

$$\begin{array}{c|c} c^T & A \\ \hline & b \end{array} = \begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Write out the method in the form of $k_1 = \cdots$, $k_2 = \cdots$, ..., $u_{n+1} = u_n + \cdots$

Support $f(u, t)$ satisfies $\left| f(u,t) - f(v,t) \right| \le C \left| u - v \right|$ for all u, v, and t.

Part 1: Show that the method is stable.

Part 2: Use Taylor expansion to show $e_n(h) = O(h^2)$.

Part 3: What is the order of its global error $E_N(h)$?

Problem 3 (Theoretical)

Recall that in lecture, we carried out polynomial interpolation based on 3 points and used the polynomial interpolation to derive

- 3-step Adams-Bashforth method and
- 2-step Adams-Moulton method

Carry out polynomial interpolation based on 2 points and use it to derive

- 2-step Adams-Bashforth method and
- 1-step Adams-Moulton method

Problem 4

Use the classic 4th-order Runge-Kutta method (RK4) to solve the IVP below

$$y'' - \mu\left(2 - \exp(y'^2)\right)y' + y = 0$$
$$y(0) = y_0, \qquad y'(0) = v_0$$

Before applying RK4, you need to convert it into a first order ODE system.

$$\frac{d\vec{w}(t)}{dt} = F(\vec{w}, t), \qquad \vec{w}(0) = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$$

Use $y_0 = 3$, $v_0 = 0.5$, and $h = 0.025$. Solve the IVP to $T = 30$ respectively for

$\mu = 0.5$, $\quad \mu = 2 \quad$ and $\quad \mu = 4$

Plot y(t) vs t and y′(t) vs t in 3 figures (one for each μ)

Plot y′(t) vs y(t) in 3 figures (one for each μ)

Hint: Look at the sample code on how to implement RK methods.

Problem 5 (continue with the IVP in Problem 4)

Use $y_0 = 3$, $v_0 = 0.5$, and $\mu = 4$. Use RK4 to solve the IVP to $T = 30$.

Run simulations, respectively, with time step sizes $h = \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5},, \frac{1}{2^6} \cdots$

Use these numerical solutions to estimate errors in numerical solutions

$$E_n(h) = \frac{1}{1-(0.5)^4}\left(w_n(h) - w_{2n}\left(\frac{h}{2}\right)\right)$$

Here $E_n(h)$ is the error associated with numerical solution $w_n(h)$.

Part 1: For each time step size h, consider the maximum error over $t \in [0, 30]$.

$$E_{\max}(h) = \max_{nh \in [0,30]} \left\| E_n(h) \right\|$$

Plot $E_{\max}(h)$ vs h in a log-log plot.

Part 2: From the sequence $h = \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5},, \frac{1}{2^6} \cdots$, find a time step h_c such that

$$E_{\max}(h) < 5\times 10^{-8}$$

Report the value of h_c.

Plot $\left\| E_n(h) \right\|$ vs t_n respectively for time step sizes h_c and $h_c/2$. Use the logarithmic scale for $\left\| E_n(h) \right\|$. Plot the two curves in ONE figure for comparison.

Hint:

In the estimation of error, $w_n(h)$ is compared with $w_{2n}(h/2)$, NOT $w_n(h/2)$. Look at the sample code on how to estimate error in numerical solution of ODE systems.

Problem 6 (continue with the IVP in Problem 4)

The Fehlberg method is an embedded Runge-Kutta method with orders 5 and 4. Implement the Fehlberg method to solve the IVP to $T = 30$.

Use $y_0 = 3$, $v_0 = 0.5$, $\mu = 4$ and $h = 0.025$.

In each time step, the error is estimated as:

$$E_n^{(\text{Fehlberg})}(h) \approx \frac{e_n(h)}{h} \approx \frac{\left\| w_{n+1} - \tilde{w}_{n+1} \right\|}{h}$$

where w_{n+1} and $\tilde{w}_{n+1}$ are respectively the results of the 5th-order method and the 4th-order method in the Fehlberg method. In each time step, both w_{n+1} and $\tilde{w}_{n+1}$ are calculated from w_n in the Fehlberg method:

$$w_{n+1} = w_n + \sum_{i=1}^{p} b_i k_i$$

$$\tilde{w}_{n+1} = w_n + \sum_{i=1}^{p} \tilde{b}_i k_i$$

After calculating $E_n^{(\text{Fehlberg})}(h)$, at the end of each time step, $\tilde{w}_{n+1}$ is discarded. In the next time step, both methods are started with w_{n+1}.

Part 1: Plot $E_n^{(\text{Fehlberg})}(h)$ vs t_n. Use the logarithmic scale for the error.

Part 2: Estimate the error using $E_n(h) = \dfrac{1}{1-(0.5)^5} \left\| w_n(h) - w_{2n}\left(\dfrac{h}{2}\right) \right\|$.

Plot $E_n^{(\text{Fehlberg})}(h)$ vs t_n and $E_n(h)$ vs t_n in ONE figure for comparison. Use the logarithmic scale for the errors.

Hint:

The specifications (p, matrix A, vectors c, b5 and b4) of the Fehlberg method are given in the folder of sample code on how to implement RK methods