HW7 Report

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Problem 1

Problems 1 is appended to the bottom of this report in a handwritten format.

Problem 2:

In this problem, we consider the 2D IBVP of the heat equation,

$$\begin{cases} u_t = u_{xx} + u_{yy}, \quad (x,y) \in (0,8)\times(0,8), \;\; t>0 \\ u(x,y,0) = f(x,y) \\ u_x(0,y,t) = g_L(y,t), \quad u(8,y,t) = g_R(y,t) \\ u(x,0,t) = g_B(x,t), \quad u(x,8,t) = g_T(x,t) \end{cases}$$

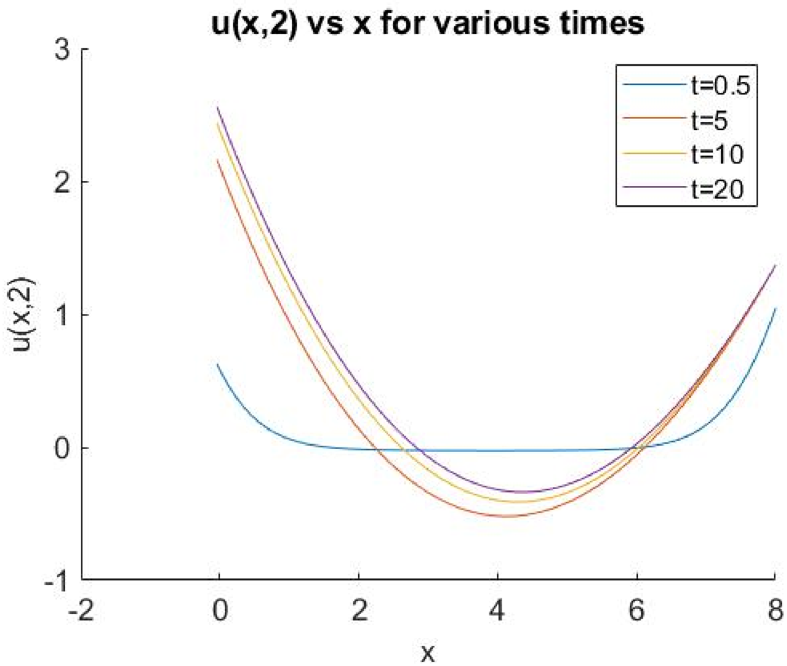
where

$$\begin{aligned} f(x,y) &= 0 \\ g_L(y,t) &= -2\sin(\pi y/8)\tanh(2t), \quad g_R(y,t) = 2\sin(\pi y/8)\tanh(2t) \\ g_B(x,t) &= g_T(x,t) = -x(1-x/8)\tanh(2t). \end{aligned}$$

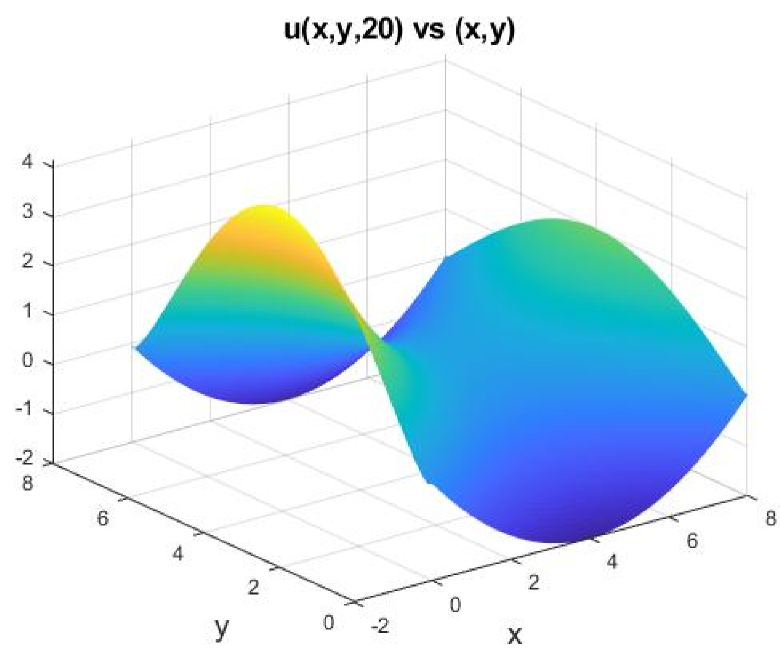
As derived in the assignment, we have that the left boundary satisfies

$$u^n_{0,j} = u^n_{1,j} - \Delta x\, g_L(y_j, t_n).$$

We slightly modify our FTCS method from the previous assignment to account for our left boundary condition, and then we solve the IBVP using $N_x = N_y = 100,\ \Delta t = 1.25\times10^{-3},\ T = 20$. We provide a plot of $u(x,2)$ vs x at $t = 0.5, 5, 10, 20$ in a single figure, as well as a surface plot of numerical solution u vs (x,y) at $T = 20$.



u(x,2) vs x for various times
t=0.5
t=5
t=10
t=20
u(x,2)
x
3
2
1
0
-1
-2
0
2
4
6
8



u(x,y,20) vs (x,y)
4
3
2
1
0
-1
-2
8
6
4
2
0
y
-2
0
2
4
6
8
x

Problem 3:

In this problem, we consider the IBVP

$$\begin{cases} u_t + au_x = 0, & x \in (-0.5, 2.5),\ t > 0 \\ u(x,0) = f(x), & x \in (-0.5, 2.5) \\ u(-0.5, t) = g(t), & t > 0, \end{cases}$$

where $a = 1.5$, $f(x) = -\cos(\pi x)$, $g(t) = \sin(a\pi t)$. We implement the Upwind, Lax-Friedrichs, and Lax-Wendroff methods to solve this IBVP. Both the Lax-Friedrichs and Lax-Wendroff methods require artificial numerical boundary conditions, which we first set to be

$$u_N^n = u_{ext}(x_N, t_n), \quad \text{at } x_N = 2.5,$$

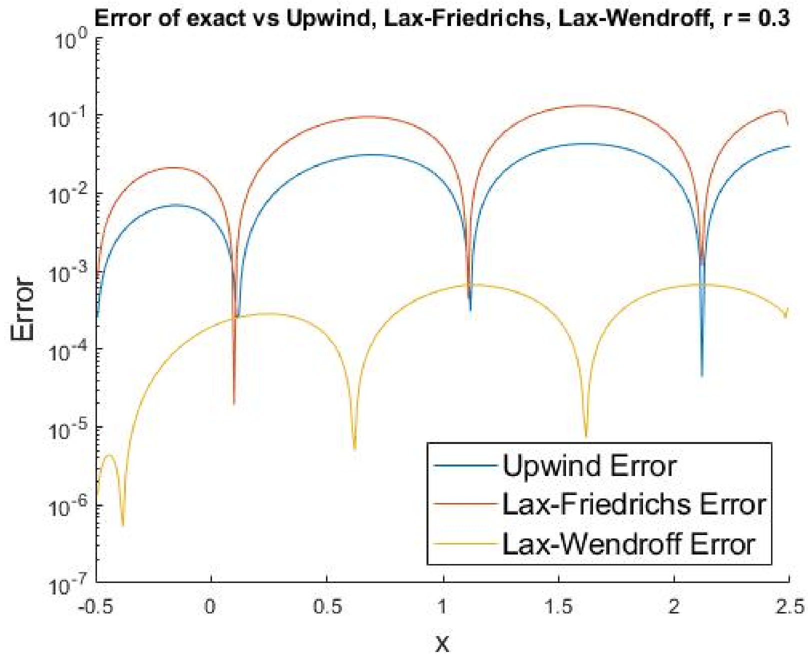
where

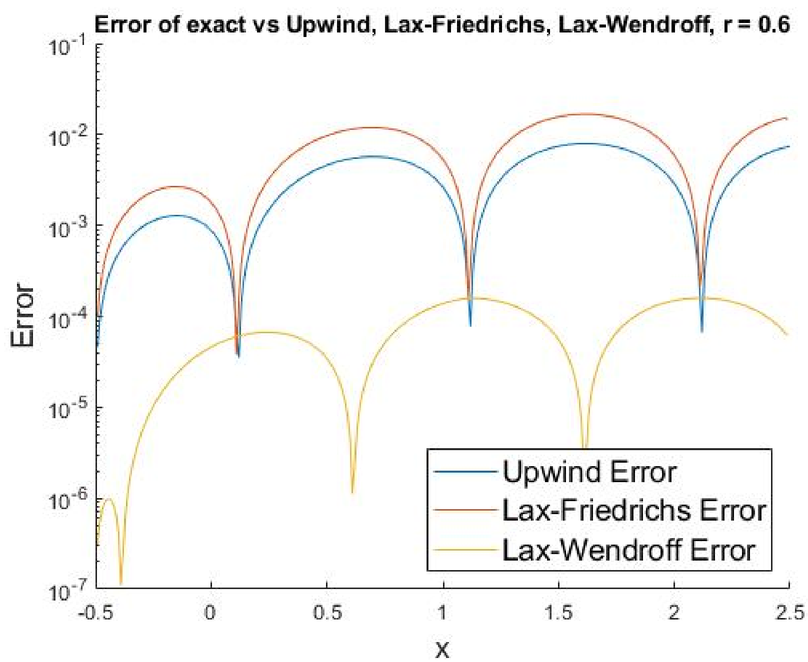
$$u_{ext}(x,t) = \begin{cases} f(x - at), & x - at > -0.5 \\ g(t - (x + 0.5)/a), & x - at \le -0.5. \end{cases}$$

In our computations, we set $N = 300$ and calculate $\Delta t = r\,\Delta x$, where $r = 0.3$, and then $r = 0.6$. For each value of r, we produce a numerical solution which we calculate error with against the exact solution,

$$E_i^n = |u_{ext}(x_i, t_n) - u_i^n|.$$

We then plot the errors vs x at the time $t = 1.08$ for each method in one figure. We produce two figures total, one for each value of r. As seen below, the Lax-Wendroff method has significantly less error than the Upwind method and Lax-Friedrichs method. Also, we observe that $r = 0.6$ value results in the smallest error.


Error of exact vs Upwind, Lax-Friedrichs, Lax-Wendroff, r = 0.3
Error
x
Upwind Error
Lax-Friedrichs Error
Lax-Wendroff Error

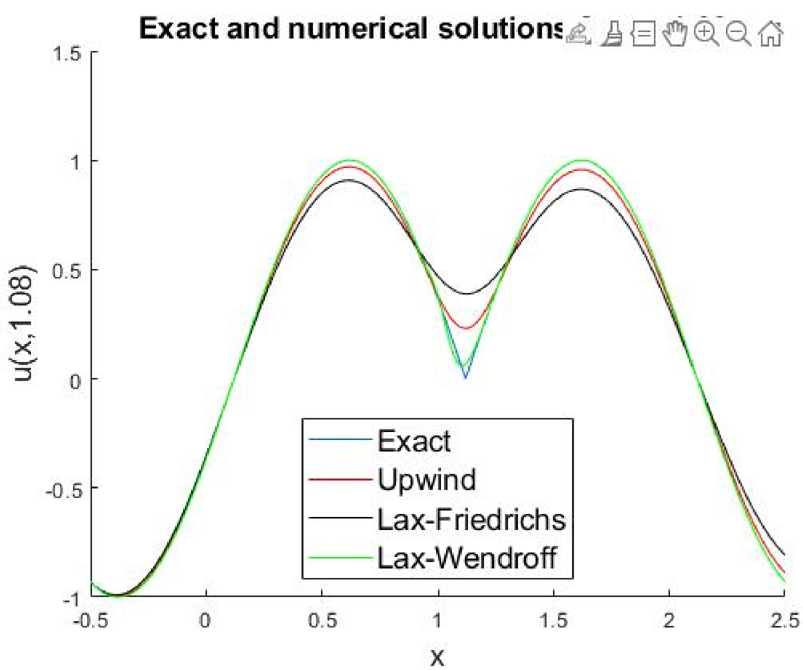

Error of exact vs Upwind, Lax-Friedrichs, Lax-Wendroff, r = 0.6
Error
x
Upwind Error
Lax-Friedrichs Error
Lax-Wendroff Error

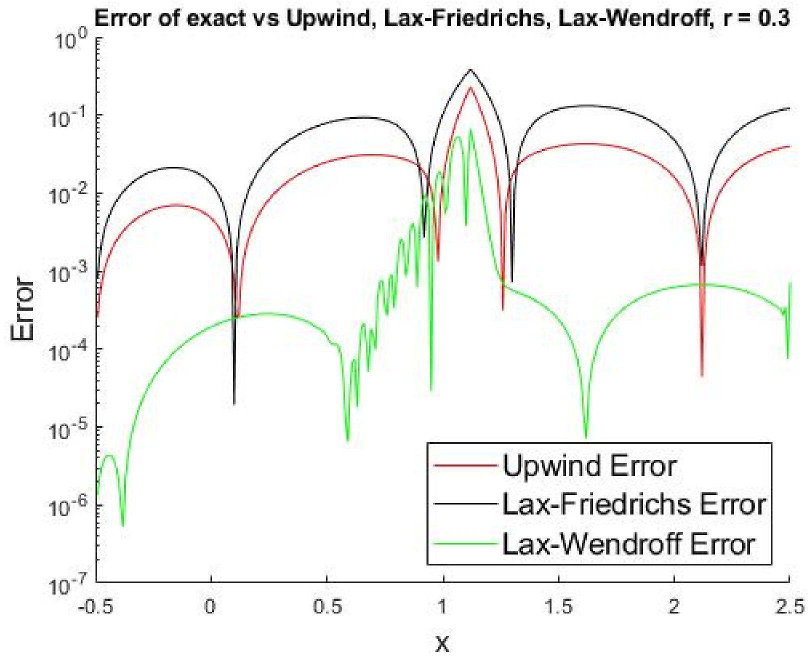
Problem 4

Problem 4 is an extension of the previous problem. Working with the same IBVP as before, we set $f(x) = \cos(\pi x), \quad x \in (-0.5, 2.5)$. We also change our artificial numerical boundary condition at $x_N = 2.5$ to

$$u_N^n = 2u_{N-1}^n - u_{N-2}^n.$$

We solve the IBVP with the same three previous methods. In our computations, we set $N = 300, \ r = 0.3$. We then plot the numerical solutions of the three methods at $t = 1.08$ in one figure, and also plot errors vs x for each method at $t = 1.08$ in another figure. Notice that the cusp of the exact solution corresponds to a spike in the errors of the numerical solutions, as expected. Also, our artificial numerical boundary condition does not encroach into our computational region.





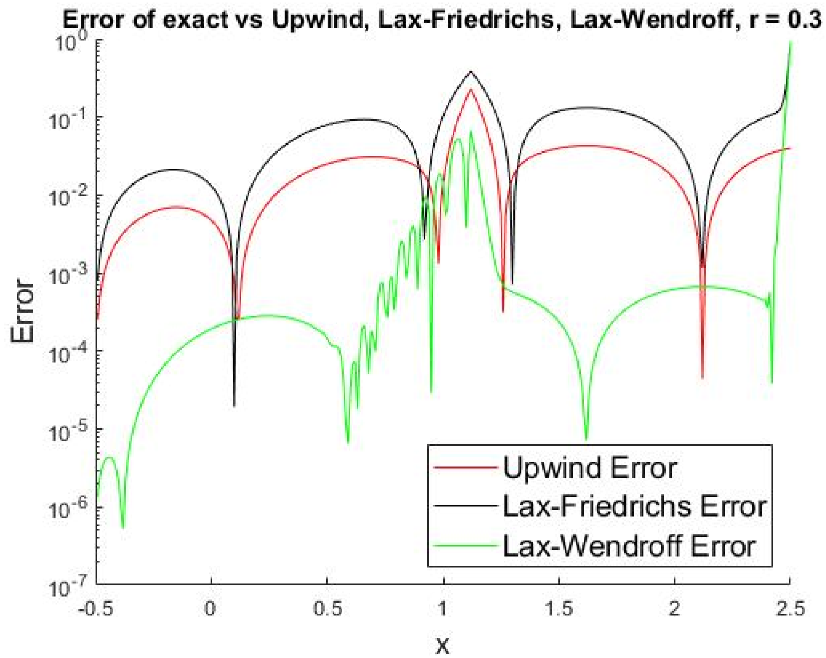
Problem 5:

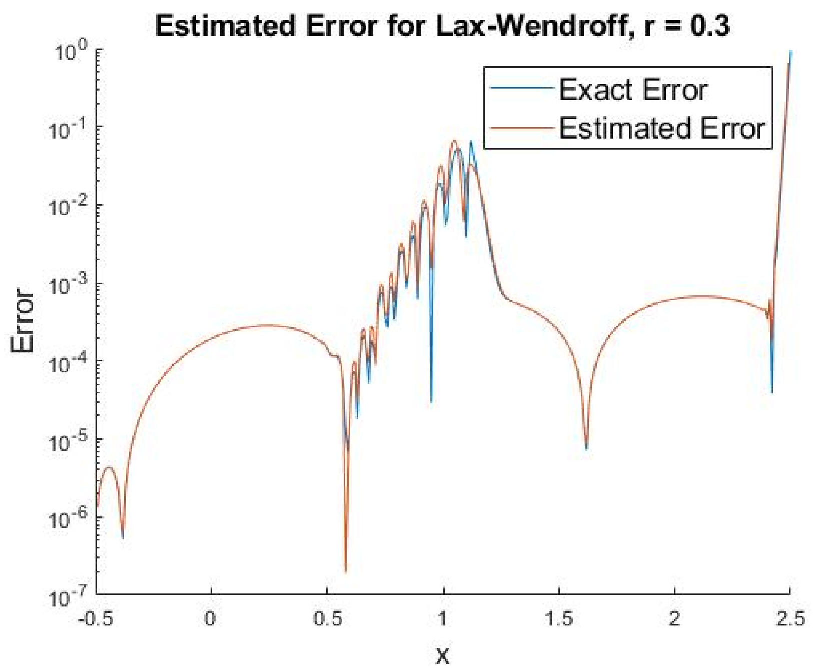
Problem 5 is a continuation of the IBVP in problem 4. We change the artificial numerical boundary condition at $x_N = 2.5$ to

$$u_N^n = 0.$$

In our computations, we set $N = 300,\ r = 0.3$ We then provide a plot of the exact errors vs x at time $t = 1.08$ in a single figure. Notice that while our ad hoc boundary condition does effect our numerical solution, it does not affect the vast majority of our computation region.

We then also study the Lax-Wendroff method slightly closer. We produce two numerical solutions, one with $N = 300$, and the second with $N = 600$. We use these two solutions to estimate the error at $t = 1.08$, as shown in one figure below.


Error of exact vs Upwind, Lax-Friedrichs, Lax-Wendroff, r = 0.3
Upwind Error
Lax-Friedrichs Error
Lax-Wendroff Error
Error
x

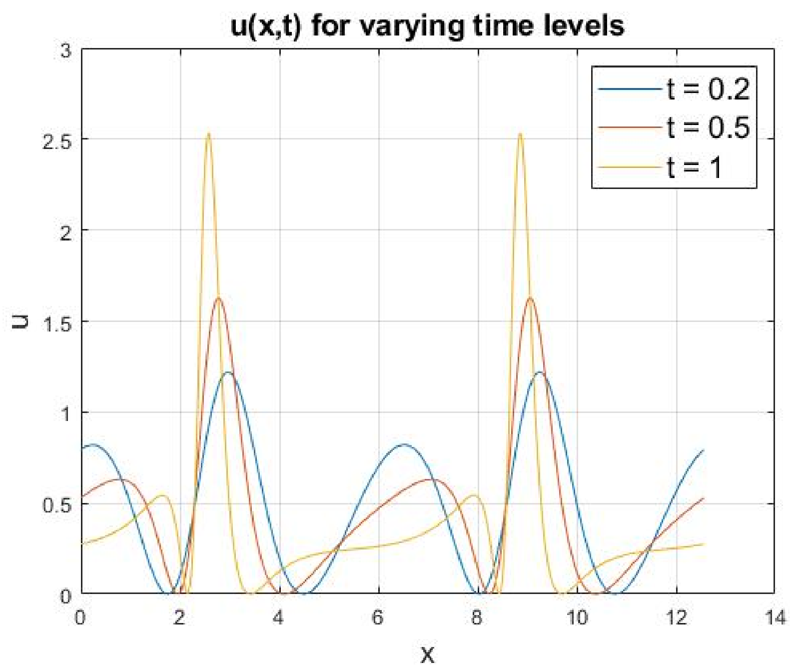

Estimated Error for Lax-Wendroff, r = 0.3
Exact Error
Estimated Error
Error
x

Problem 6:

In this problem, implement the method of characteristics to solve the IVP

$$\begin{cases} u_t + (\sin(x) + \cos(x))u_x = -\cos(x)u, & x \in (-\infty, +\infty), \ t > 0 \\ u(x,0) = \cos^2(x), & x \in (-\infty, +\infty) \end{cases}$$

We utilize a previously coded RK4 ODE solver for both the backtracing portion of the method, and the forward evolution portion. In computations, we calculate our solution at $t = 0.2, 0.5, 1$. At each time level we use 400 points to represent the numerical solution u for $x \in [0, 4\pi]$. We then plot $u(x,t)$ vs x for $x \in [0, 4\pi]$ at all three time levels in a single figure.



Theory

Thursday, May 13, 2021 11:37 AM

P1

Problem 1 (Theoretical)

Consider the Lax-Friedrichs method for solving $u_t + a\, u_x = 0$

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

On the RHS, we write $\frac{u_{i+1}^n + u_{i-1}^n}{2}$ as $u_i^n + \underbrace{\frac{1}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}_{\text{Added viscosity}}$.

We know that the Lax-Friedrichs method has too much added viscosity. So we consider a modified version of Lax-Friedrichs

$$u_i^{n+1} = u_i^n + \frac{q}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \quad r = \frac{\Delta t}{\Delta x}, \quad 0 \le q \le 1 \qquad \text{(LF-2)}$$

Part 1: Find the modified PDE of (LF-2).

Part 2: Find the modified PDE of the implicit upwind method

$$u_i^{n+1} = u_i^n - ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

Hint: Expanding around (x_i, t_{n+1}) will make it easier.

Part 1: We have the modified Lax-Friedrichs method:

$$U_i^{n+1} = U_i^n + \frac{q}{2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right) - \frac{ar}{2}\left(U_{i+1}^n - U_{i-1}^n\right),$$

with $r = \frac{\Delta t}{\Delta x}$, $0 \le q \le 1$.

Assume that $U_i^n = w(x_i, t_n)$ satisfies a perturbed PDE. Then, we have

$$\underbrace{w(x_i, t_n+\Delta t) - w(x_i, t_n)}_{①} = \frac{q}{2}\Big[\underbrace{w(x_{i+1}, t_n) - 2w(x_i, t_n) + w(x_{i-1}, t_n)}_{②}\Big]$$

$$- \frac{ar}{2}\Big[\underbrace{w(x_{i+1}, t_n) - w(x_{i-1}, t_n)}_{③}\Big]$$

We now expand ①, ②, ③ around (x_i, t_n).

Let $w \equiv w(x_i, t_n)$.

①: $w(x_i, t_n + \Delta t) - w(x_i, t_n)$

(1): $w(x_i, t_n + \Delta t) - w(x_i, t_n)$

$$= w + w_t \Delta t + \tfrac{1}{2} w_{tt} \Delta t^2 + O(\Delta t^3) - w$$

$$= w_t \Delta t + \tfrac{1}{2} w_{tt} \Delta t^2 + O(\Delta t^3).$$

(2): $w(x_i + \Delta x, t_n) - 2w(x_i, t_n) + w(x_i - \Delta x, t_n)$

$$= w + w_x \Delta x + \tfrac{1}{2} w_{xx} \Delta x^2 - 2w$$

$$+ w - w_x \Delta x + \tfrac{1}{2} w_x \Delta x^2 + O(\Delta x^4)$$ only even powers survive.

$$= w_{xx} \Delta x^2 + O(\Delta x^4)$$

(3): $w(x_i + \Delta x, t_n) - w(x_i - \Delta x, t_n)$

$$= w + w_x \Delta x + \tfrac{1}{2} w_{xx} \Delta x^2 + O(\Delta x^3) - \left[w - w_x \Delta x + \tfrac{1}{2} w_{xx} \Delta x^2 - O(\Delta x^3) \right]$$

$$= 2 w_x \Delta x + O(\Delta x^3)$$

$$\overbrace{w(x_i, t_n + \Delta t) - w(x_i, t_n)}^{①} = \frac{q}{2} \Big[\overbrace{w(x_{i+1}, t_n) - 2w(x_i, t_n) + w(x_{i-1}, t_n)}^{②} \Big] - \frac{ar}{2} \Big[\underbrace{w(x_{i+1}, t_n) - w(x_{i-1}, t_n)}_{③} \Big]$$

Substituting into the modified method, we have

$$w_t \Delta t + \tfrac{1}{2} w_{tt} \Delta t^2 + O(\Delta t^3) = \frac{q}{2} \left[w_{xx} \Delta x^2 + O(\Delta x^4) \right] - \frac{ar}{2} \left[2 w_x \Delta x + O(\Delta x^3) \right]$$

$$\Rightarrow w_t \Delta t + \tfrac{1}{2} w_{tt} \Delta t^2 + O(\Delta t^3)$$

$$= \frac{q}{2} w_{xx} \Delta x^2 - ar \cdot w_x \Delta x + O(\Delta x^3)$$

Dividing by Δt and rearranging, we have

Dividing by Δt and rearranging, we have

$$w_t = -aw_x + \frac{g}{2r} w_{xx} \Delta x - \frac{1}{2} w_{tt} \Delta t + O(\Delta t^2) + \frac{1}{\Delta t} O(\Delta x^3)$$

Following the iterative approach provided in lecture, we can convert w_{tt} to a spatial derivative.

We have

$$w_t = -aw_x + O(\Delta x + \Delta t)$$

$$\Rightarrow \quad w_{tt} = (-aw_x)_t + O(\Delta x + \Delta t)$$

$$= -a(w_t)_x + O(\Delta x + \Delta t)$$

Plugging in w_t again gives

$$w_{tt} = -a(-aw_x)_x + O(\Delta x + \Delta t)$$
$$= a^2 w_{xx} + O(\Delta x + \Delta t).$$

Thus, we have that

$$w_t = -aw_x + \frac{g}{2r} w_{xx} \Delta x - \frac{a^2}{2} w_{xx} \Delta t + O(\Delta t^2) + \frac{1}{\Delta t} O(\Delta x^3)$$

$$\Rightarrow \quad w_t = -aw_x + \frac{1}{2}(gr^{-1}\Delta x - a^2 \Delta t) w_{xx} + O(\Delta t^2) + \frac{1}{\Delta t} O(\Delta x^3)$$

Leaving off higher order terms, we have that

$$\boxed{w_t = -a w_x + \sigma w_{xx}, \quad \sigma = \frac{\Delta x}{2}\left(q r^{-1} - a^2 r\right), \quad r = \frac{\Delta t}{\Delta x}.}$$

Part 2: We have the implicit upwind method:

$$U_i^{n+1} = U_i^n - ar\left(U_i^{n+1} - U_{i-1}^{n+1}\right), \quad r = \frac{\Delta t}{\Delta x}.$$

Assume that $U_i^n = w(x_i, t_n)$. Then, we can write

$$\underbrace{w(x_i, t_n+\Delta t) - w(x_i, t_n)}_{①} = -ar\left[\underbrace{w(x_i, t_n+\Delta t) - w(x_i - \Delta x, t_n+\Delta t)}_{②}\right]$$

Expanding ①, ② around $(x_i, t_{n+1}) = (x_i, t_n+\Delta t)$

Let $w \equiv w(x_i, t_n+\Delta t)$.

Then,

① : $w(x_i, t_n+\Delta t) - w(x_i, t_{n+1} - \Delta t)$

$$= w - \left[w - w_t \Delta t + \tfrac{1}{2} w_{tt} \Delta t^2 - O(\Delta t^3)\right]$$

$$= w_t \Delta t - \tfrac{1}{2} w_{tt} \Delta t^2 + O(\Delta t^3)$$

② : $w(x_i, t_n+\Delta t) - w(x_i - \Delta x, t_n+\Delta t)$

$$= w - \left[w - w_x \Delta x + \tfrac{1}{2} w_{xx} \Delta x^2 - O(\Delta x^3)\right]$$

$$= w - \left[w - w_x \Delta x + \tfrac{1}{2} w_{xx} \Delta x^2 - O(\Delta x^3) \right]$$

$$= w_x \Delta x - \tfrac{1}{2} w_{xx} \Delta x^2 + O(\Delta x^3).$$

Substituting into the method: $\underbrace{w(x_i, t_n+\Delta t) - w(x_i, t_n)}_{①} = -ar\Big[\underbrace{w(x_i, t_n+\Delta t) - w(x_i-\Delta x, t_n+\Delta t)}_{②}\Big]$

$$\Rightarrow \quad w_t \Delta t - \tfrac{1}{2} w_{tt} \Delta t^2 + O(\Delta t^3) = -a \frac{\Delta t}{\Delta x} \left[w_x \Delta x - \tfrac{1}{2} w_{xx} \Delta x^2 + O(\Delta x^3) \right]$$

$$\Rightarrow w_t \Delta t - \frac{1}{2} w_{tt} \Delta t^2 + O(\Delta t^3)$$

$$= -a \Delta t\, w_x + \frac{a}{2} w_{xx} \Delta t \Delta x + \Delta t\, O(\Delta x^2)$$

Hence, dividing by Δt and rearranging some terms,

$$w_t = -a w_x + \frac{1}{2} w_{tt} \Delta t + \frac{a}{2} w_{xx} \Delta x + O(\Delta t^2) + O(\Delta x^2)$$

Using $w_{tt} = a^2 w_{xx}$, we have

$$w_t = -a w_x + \frac{a^2}{2} w_{xx} \Delta t + \frac{a}{2} w_{xx} \Delta x + O(\Delta t^2) + O(\Delta x^2)$$

$$\Rightarrow \quad w_t = -a w_x + \frac{a \Delta x}{2} (ar + 1) w_{xx} + O(\Delta t^2 + \Delta x^2)$$

$\Rightarrow$ Modified PDE is:

$$\boxed{w_t = -a w_x + \sigma w_{xx}, \quad \sigma = \frac{a \Delta x}{2} (ar+1), \; r = \frac{\Delta t}{\Delta x}}$$