HW9 Report

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Problem 1

In this problem, we work with the following system of conservation laws,

$$\begin{cases} \vec{w}_t + \vec{F}(\vec{w})_x = 0, \quad \vec{w}(x,t) = \begin{bmatrix} w_1(x,t) \\ w_2(x,t) \end{bmatrix}, \quad \vec{F}(\vec{w}) = \begin{bmatrix} w_1 w_2/2 \\ (w_1^2 + w_2^2)/4 \end{bmatrix} \\ \vec{w}(x,0) = \begin{cases} (2.5,\, 0.25)^T, & x \leq 0 \\ (0.75,\, 0.25)^T, & x > 0 \end{cases} \end{cases}$$

and solve it by implementing the Richtmyer 2-step Lax-Wendroff method. Our computational domain is $[L_1,\, L_2] = [-2,\, 2]$. Our grid is defined as

$$x_i = L_1 + (i - 0.5)\Delta x, \quad i = 0, 1, \ldots, N+1, \quad \Delta x = \frac{L2 - L1}{N}, \quad N = 400.$$

We solve to time $t = 1.5$, using artificial boundary conditions, $\vec{w}_0^n = \vec{w}_1^n,\ \ \vec{w}_{N+1}^n = \vec{w}_N^n$, and $r = 0.5$.

Below we produce Figure 1, which is a plot of $w_1(x,t)$ vs x and $w_2(x,t)$ vs x.

Problem 2

We consider a linear conservation law with variable coefficients,

$$\begin{cases} u_t + (a(x)u)_x = b(x)u, \quad a(x) = \sin(x) + \cos(x), \quad b(x) = -\sin(x) \\ u(x,0) = \cos^2(x). \end{cases}$$

We solve the IVP via the Richtmyer 2-step Lax-Wendroff method, and draw comparisons against another numerical solution via method of characteristics. Our computational domain is $[L_1,\, L_2] = [0,\, 4\pi]$, with $\Delta x = \frac{L_2 - L_1}{N}, \quad x_i = L_1 + (i - 0.5)\Delta x, \quad i = 0, 1, \ldots, N+1$. We solve to $t = 0.8$, with $N = 400$ and $r = 1/\pi$. Below we plot Figure 2, which contains both numerical solutions vs x.

Problem 3

This problem is a continuation of the IVP in problem 2. We treat the method of characteristics solution as the exact solution, and then apply the Richtmyer 2s-LW method with differing resolutions ($N = 400, N = 800$). We use the numerical solutions at differing resolutions to estimate the error in $u(x,t)$. Below we provide Figure 3, which is a plot of the estimated error vs x and the exact error vs x at time $t = 0.8$.

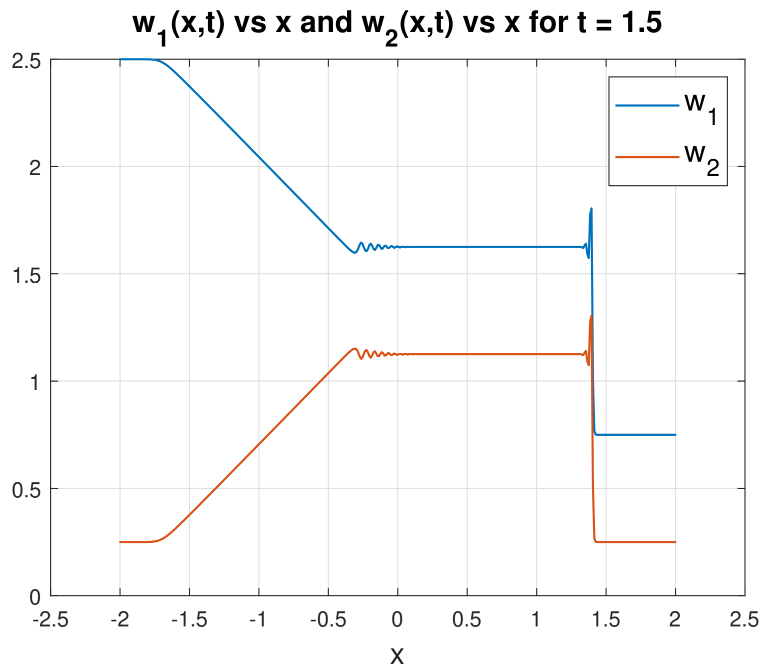

Figure 1: $w_1(x,t)$ vs x, $w_2(x,t)$ vs x, $t = 1.5$

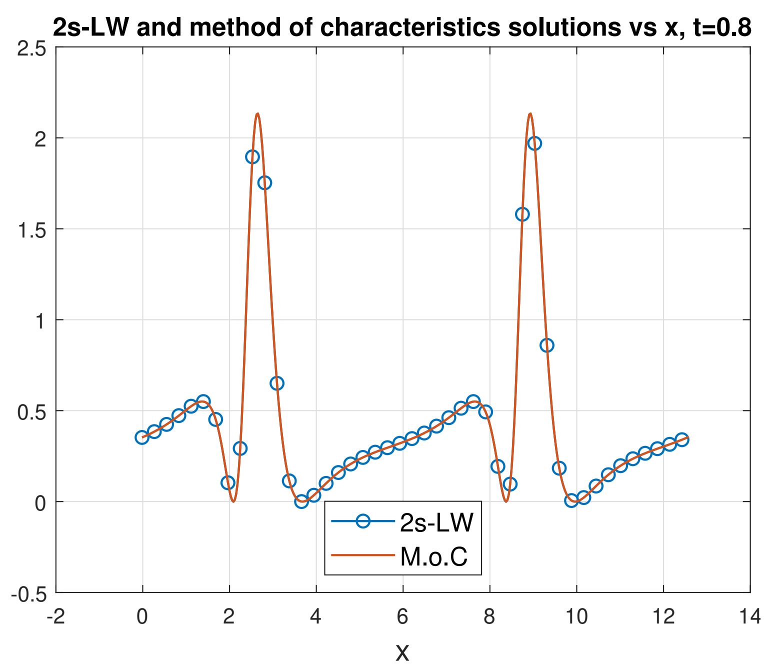

Figure 2: Richtmyer 2s-LW vs x and MoC vs x

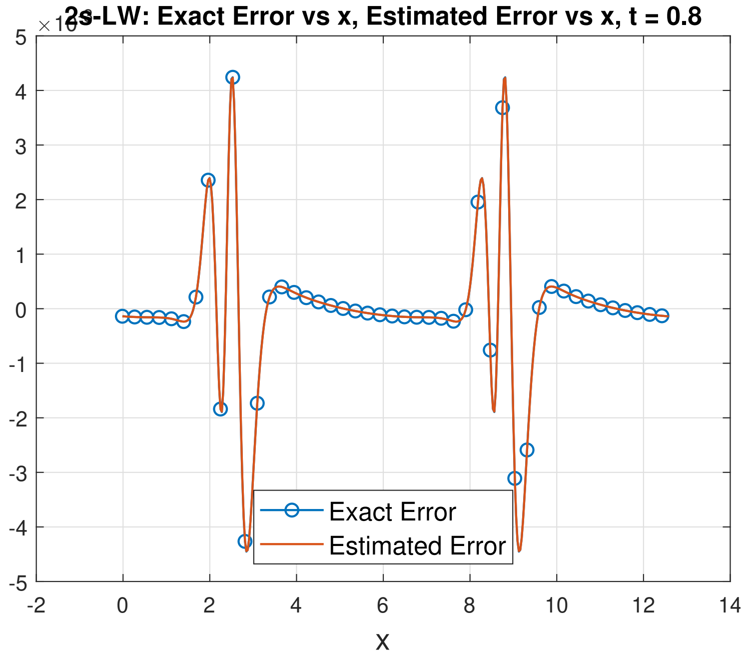

Figure 3: Estimated error vs x and Exact error vs x

Problem 4

In this problem, we consider the "virtually" one-dimensional IVP,

$$\begin{cases} u_t + (a^{(x)}(x,y)u)_x = 0, \quad a^{(x)}(x,y) = \sin(x)\sin(y) \\ u(x,y,0) = \sin^2(x+y) \end{cases}$$

with the discretization

$$x_i = L_1 + (i-0.5)\Delta x, \quad \Delta x = \frac{L_2 - L_1}{N}, \quad i = 0, 1, \ldots, N+1$$
$$y_j = L_1 + (j-0.5)\Delta y, \quad \Delta y = \frac{L_2 - L_1}{N}, \quad j = 0, 1, \ldots, N+1.$$

This problem is virtually 1D, since we're fixing the values of x on which we perform our method. We utilize periodic boundary conditions, and use the $2s-LW$ method with $N = 200$ and $r = 1/\pi$, as in previous problems. We solve the IVP to $t = 1$. Below we provide Figure 4 and Figure 5.

Problem 5

Problem 5 is extremely similar to problem 4, however we instead fix our values of y we perform our method on (hence this problem is also virtually 1D) to solve our IVP for the remaining dimension. The IVP is

$$\begin{cases} u_t + (a^{(y)}(x,y)u)_y = 0, \quad a^{(y)}(x,y) = 1 - \exp(\sin(x+y)) \\ u(x,y,0) = \sin^2(x+y). \end{cases}$$

Below we provide Figure 6, which is a contour plot showing $u(x,y)$ at time $t = 1.0$.

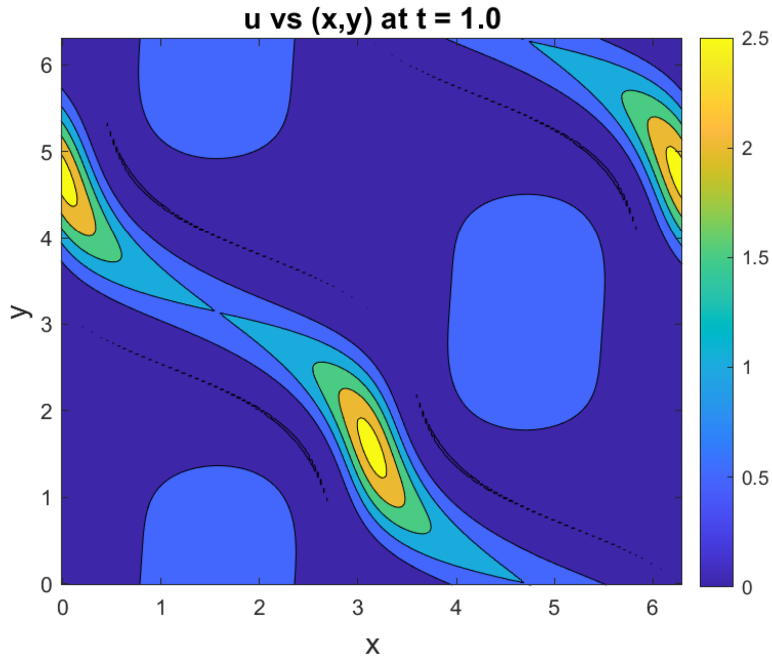

Figure 4: Contour of u vs (x, y) at time $t = 1.0$.

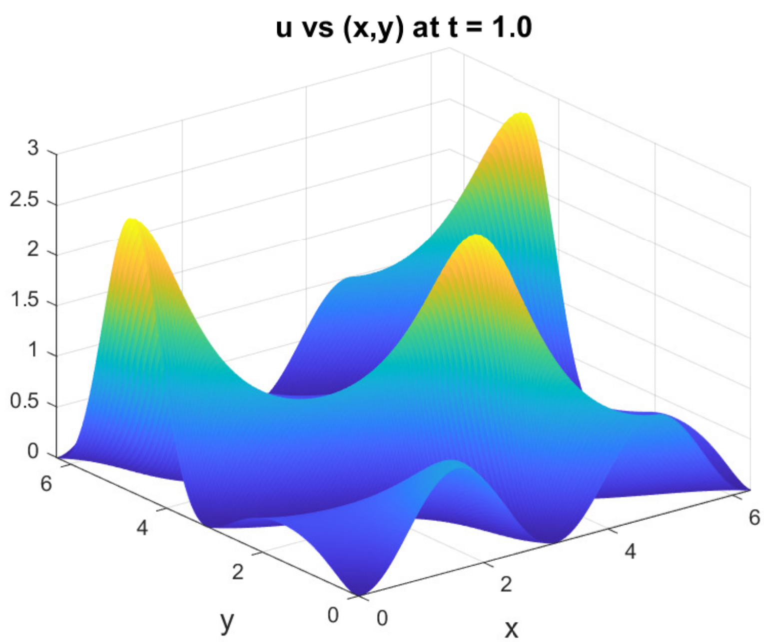

Figure 5: Surface plot for $u(x, y, t)$ at time $t = 1.0$.

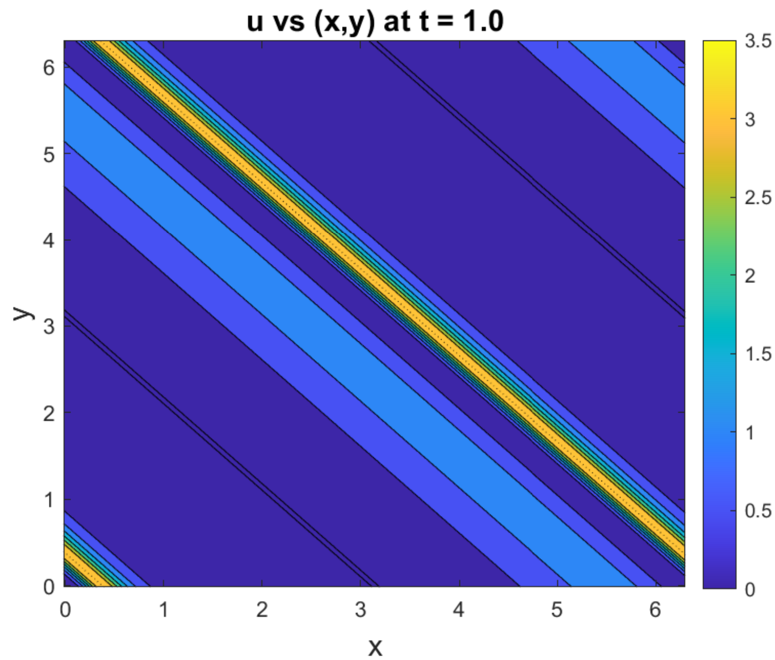

Figure 6: Contour of u vs (x, y) at time $t = 1.0$.

Problem 6

This problem utilizes the implementation of problems 4 and 5 to apply the first order split-operator method with the 2s-LW method in each direction. The IVP is a combination of the previous two IVPs,

$$\begin{cases} u_t + (a^{(x)}(x,y)u)_x + (a^{(y)}(x,y)u)_y = 0, \\ \quad a^{(x)}(x,y) = \sin(x)\sin(y), \quad a^{(y)}(x,y) = 1 - \exp(\sin(x+y)) \\ u(x,y,0) = \sin^2(x+y). \end{cases}$$

We provide two plots, Figure 7 and Figure 8, of u vs (x, y) at time $t = 1.0$.

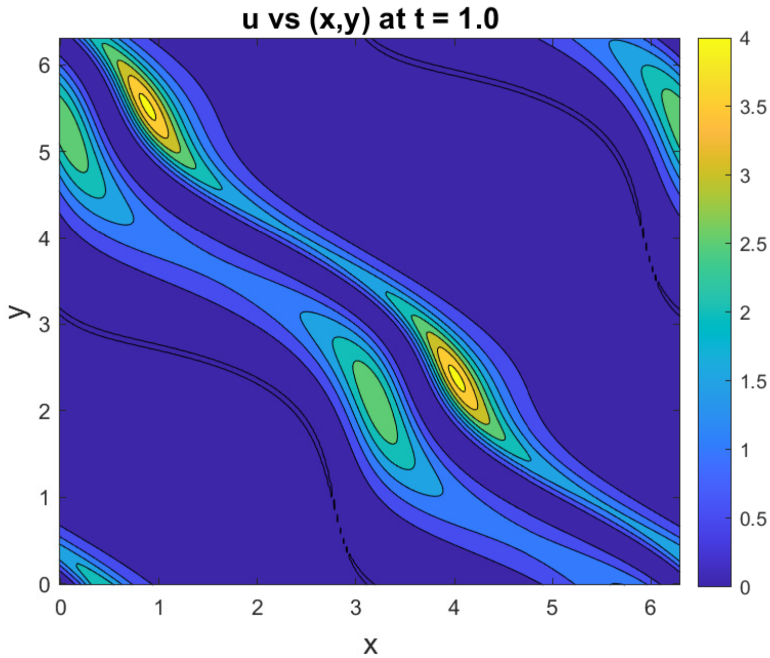

Figure 7: Contour of u vs (x, y) at time $t = 1.0$.

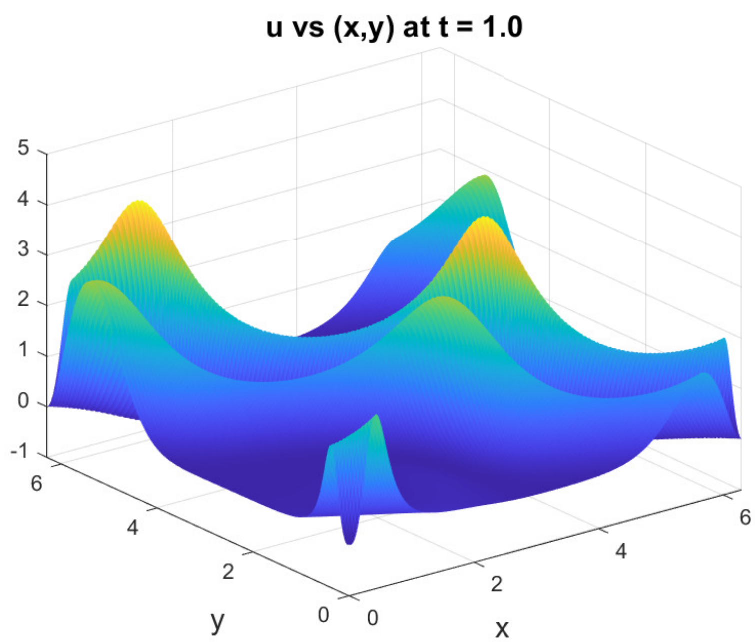

Figure 8: Surface of u vs (x, y) at time $t = 1.0$.