AM213B Assignment #9

Problem 1 (Computational)

Consider the IVP of a system of conservation laws

$$\begin{cases} \vec{w}_t + \vec{F}(\vec{w})_x = 0, \quad \vec{w}(x,t) = \begin{pmatrix} w_1(x,t) \\ w_2(x,t) \end{pmatrix}, \quad \vec{F}(\vec{w}) = \begin{pmatrix} w_1 w_2 / 2 \\ (w_1^2 + w_2^2)/4 \end{pmatrix} \\ \vec{w}(x,0) = \begin{cases} (2.5, 0.25)^T, & x \le 0 \\ (0.75, 0.25)^T, & x > 0 \end{cases} \end{cases} \qquad \text{(IVP-1)}$$

Implement the Richtmyer 2-step Lax-Wendroff method (2s-LW)

$$\vec{w}^*_{i+1/2} = \frac{\vec{w}^n_{i+1} + \vec{w}^n_i}{2} - \frac{r}{2}\left(\vec{F}(\vec{w}^n_{i+1}) - \vec{F}(\vec{w}^n_i)\right), \qquad r = \frac{\Delta t}{\Delta x}$$

$$\vec{F}^{(\mathrm{LW})}_{i+1/2} = \vec{F}(\vec{w}^*_{i+1/2})$$

$$\vec{w}^{n+1}_i = \vec{w}^n_i - r\left(\vec{F}^{(\mathrm{LW})}_{i+1/2} - \vec{F}^{(\mathrm{LW})}_{i-1/2}\right)$$

We select $[L_1, L_2]$ with $L_1 = -2$ and $L_2 = 2$ as the computational domain.

We use the finite volume discretization: viewing x_i as the center of cell i.

$$\Delta x = \frac{L_2 - L_1}{N}, \qquad x_i = L_1 + (i - 0.5)\Delta x, \quad i = 0, 1, \ldots, N+1$$

$$x_0 = L_1 - 0.5\Delta x, \quad x_1 = L_1 + 0.5\Delta x, \ldots, \quad x_N = L_2 - 0.5\Delta x, \quad x_{N+1} = L_2 + 0.5\Delta x$$

We use artificial boundary conditions: $\vec{w}^n_0 = \vec{w}^n_1$, $\vec{w}^n_{N+1} = \vec{w}^n_N$.

Use $N = 400$ and $r = \Delta t/\Delta x = 0.5$ to solve (IVP-1) to $t = 1.5$.

Plot in one figure, $w_1(x, t)$ vs x and $w_2(x, t)$ vs x at $t = 1.5$.

Remark: The 2s-LW is versatile and is relatively easy to implement. In contrast, the upwind method requires eigenvalues and eigenvectors of Jacobian $\partial\vec{F}/\partial\vec{w}$.

Problem 2 (Computational)

Consider the IVP of a linear conservation law with variable coefficients.

$$\begin{cases} u_t + (a(x)u)_x = b(x)u, \quad a(x) = \sin x + \cos x, \quad b(x) = -\sin x \\ u(x,0) = \cos^2 x \end{cases} \qquad \text{(IVP-2)}$$

(IVP-2) is equivalent to the IVP in Problem 6 of Assignment 7, which you solved using the method of characteristics. You will need the code from Assignment 7.

Implement the Richtmyer 2-step Lax-Wendroff method (2s-LW)

$$u_{i+1/2}^{*} = \frac{u_{i+1}^{n} + u_{i}^{n}}{2} - \frac{r}{2}\left(a_{i+1}u_{i+1}^{n} - a_{i}u_{i}^{n}\right) + \frac{\Delta t}{2}\cdot\frac{b_{i+1}+b_{i}}{2}\cdot\frac{u_{i+1}^{n}+u_{i}^{n}}{2}, \qquad r = \frac{\Delta t}{\Delta x}$$

$$u_{i}^{n+1} = u_{i}^{n} - r\left(\frac{a_{i+1}+a_{i}}{2}u_{i+1/2}^{*} - \frac{a_{i}+a_{i-1}}{2}u_{i-1/2}^{*}\right) + \Delta t\cdot b_{i}\cdot\frac{u_{i+1/2}^{*}+u_{i-1/2}^{*}}{2}$$

We select $[L_1, L_2]$ with $L_1 = 0$ and $L_2 = 4\pi$ as the computational domain.

We use the finite volume discretization: viewing x_i as the center of cell i.

$$\Delta x = \frac{L_2 - L_1}{N}, \qquad x_i = L_1 + (i - 0.5)\Delta x, \quad i = 0, 1, \ldots, N+1$$

We use periodic boundary conditions: $u_0^n = u_N^n, \quad u_{N+1}^n = u_1^n$.

Use $N = 400$ and $r = \Delta t/\Delta x = 1/\pi$ to solve (IVP-2) to $t = 0.8$ (the value of r is selected to make sure we arrive at $t = 0.8$ in integer number of steps).

Plot in one figure, the solution of 2s-LW vs x and the solution of method of characteristic vs x (using your code from Assignment 7) at $t = 0.8$.

Important suggestion: In 2s-LW, each time step has

Input: $\{u_i^n, 0 \le i \le N+1\}$, $\{a_i\}$, $\{b_i\}$, r, Δt

Output: $\{u_i^{n+1}, 0 \le i \le N+1\}$

Write "LW_1dt.m" to advance one time step (with periodic BCs incorporated)

```
function [u2]=LW_1dt(u, a, b, r, dt)
```

This function will be very useful in Problems 2-6.

Problem 3 (Computational)

Continue with (IVP-2) in Problem 2. Treat the solution of method of characteristics as the "exact" solution. Let $u_{\{N=400\}}$ denote the numerical solution of 2s-LW for $N = 400$.

Use the "exact" solution to calculate the error in $u_{\{N=400\}}$.

Use $u_{\{N=400\}}$ and $u_{\{N=800\}}$ to estimate the error in $u_{\{N=400\}}$.

Plot in one figure, the estimated error vs x and the exact error vs x at $t = 0.8$. Use linear scales for both x and the errors.

Hint: Use "spline to" map $u_{\{N=800\}}$ from $x_{\{N=800\}}$ to $x_{\{N=400\}}$.

Problem 4 (Computational)

Consider the IVP for $u(x, y, t)$ that is virtually one-dimensional.

$$\begin{cases} u_t + (a^{(x)}(x,y)u)_x = 0, \qquad a^{(x)}(x,y) = \sin(x)\sin(y) \\ u(x,y,0) = \sin^2(x+y) \end{cases} \qquad \text{(IVP-3X)}$$

We select $[L_1, L_2] \times [L_1, L_2]$ with $L_1 = 0$ and $L_2 = 2\pi$ as the computational domain.

We use the finite volume discretization:

$$\Delta x = \frac{L_2 - L_1}{N}, \qquad x_i = L_1 + (i - 0.5)\Delta x, \quad i = 0, 1, \ldots, N+1$$

$$\Delta y = \frac{L_2 - L_1}{N}, \qquad y_j = L_1 + (j - 0.5)\Delta x, \quad j = 0, 1, \ldots, N+1$$

$$u_{i,j}^n \approx u(x_i, y_j, t_n)$$

We use periodic boundary conditions: $u_{0,j}^n = u_{N,j}^n, \quad u_{N+1,j}^n = u_{1,j}^n$.

Use 2s-LW with N = 200 and $r = \Delta t/\Delta x = 1/\pi$ to solve (IVP-3X) to t = 1.0.

Part 1: Plot u vs (x, y) at t = 1.0, using contourf with colorbar.

Part 2: Plot u vs (x, y) at t = 1.0, using surf.

Hint: In each time step, at every y_j, treat $\{u_{i,j}^n\}$ as a vector and apply "LW_1dt.m".

```
for j=1:N+1
    u1=u(j, :); a=ax(j, :); b=0*a;
    u2=LW_1dt(u, a, b, r, dt);
    u(j, :)= u2;
end
```

Problem 5 (Computational)

Consider the IVP for $u(x, y, t)$ that is virtually one-dimensional.

$$\begin{cases} u_t + (a^{(y)}(x,y)u)_y = 0, \qquad a^{(y)}(x,y) = 1 - \exp(\sin(x+y)) \\ u(x,y,0) = \sin^2(x+y) \end{cases} \qquad \text{(IVP-3Y)}$$

Use the same domain, grid and discretization as in Problem 4.

We use periodic boundary conditions: $u_{i,0}^n = u_{i,N}^n, \quad u_{i,N+1}^n = u_{i,1}^n$.

Use 2s-LW with N = 200 and $r = \Delta t/\Delta y = 1/\pi$ to solve (IVP-3Y) to t = 1.0.

Plot u vs (x, y) at $t = 1.0$, using contourf with colorbar.

Hint: In each time step, at every x_i, treat $\{u_{i,j}^n\}$ as a vector and apply "LW_1dt.m".

```
for i=1:N+1
    u1=u(:, i)'; a=ay(:, i)'; b=0*a;      % assuming "LW_1dt" works with row vectors
    u2=LW_1dt(u, a, b, r, dt);
    u(:, i)= u2';
end
```

Problem 6 (Computational)

Consider the 2D IVP for $u(x, y, t)$

$$\begin{cases} u_t + (a^{(x)}(x,y)u)_x + (a^{(y)}(x,y)u)_y = 0 \\ \qquad a^{(x)}(x,y) = \sin(x)\sin(y), \quad a^{(y)}(x,y) = 1 - \exp(\sin(x+y)) \\ u(x,y,0) = \sin^2(x+y) \end{cases} \qquad \text{(IVP-3)}$$

Implement the first order split-operator method with 2s-LW in each direction.

Use $N = 200$ and $r = \Delta t/\Delta x = 1/\pi$ to solve (IVP-3) to $t = 1.0$.

Part 1: Plot u vs (x, y) at $t = 1.0$, using contourf with colorbar.

Part 2: Plot u vs (x, y) at $t = 1.0$, using surf.

Hint: In each time step, first advance (IVP-3X) for Δt; then advance (IVP-3Y) for Δt.