HW4 Report

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Problem 1

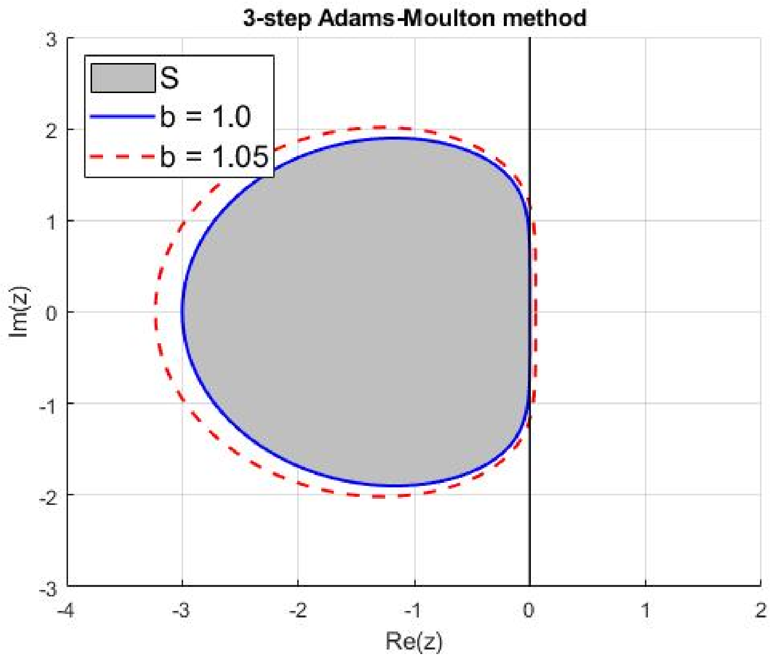
Appended in a handwritten format to the end of this report.

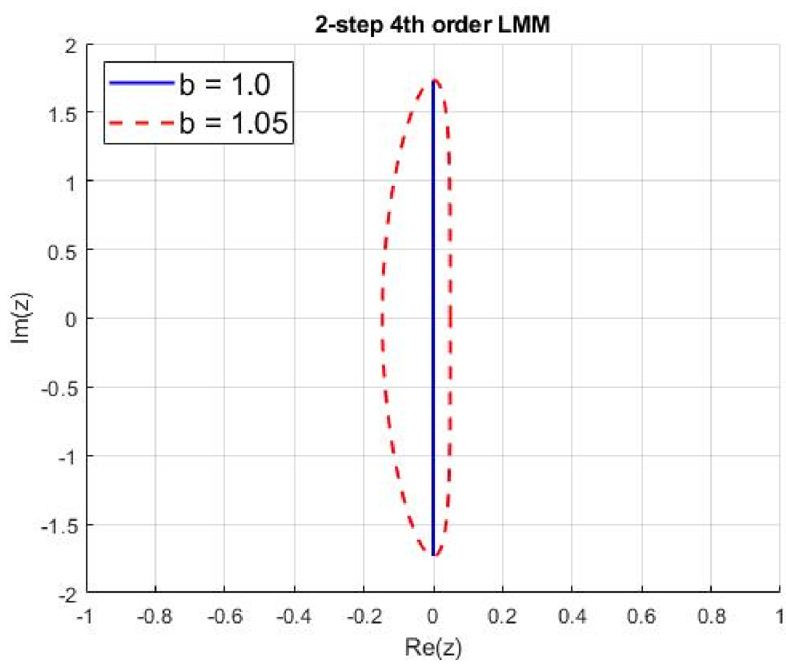
Problem 2

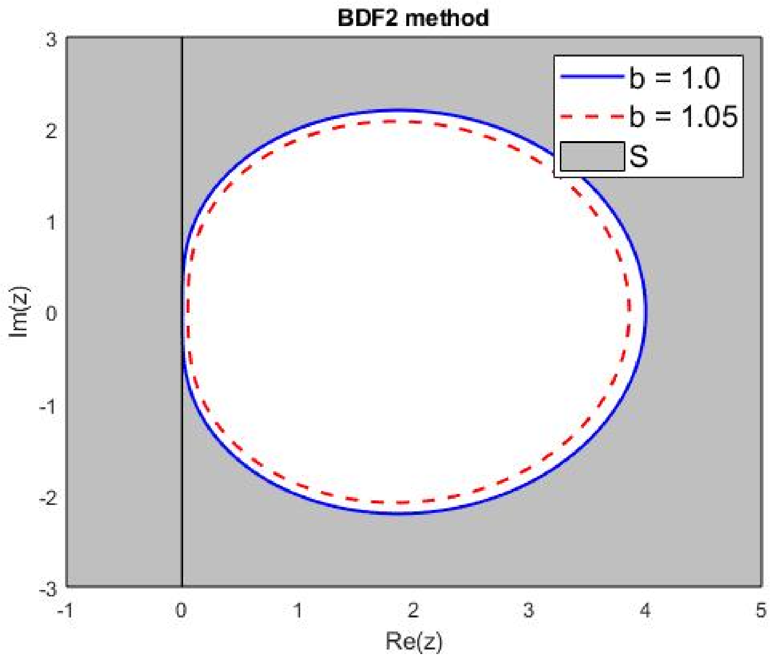
In this problem, we plot the region of absolute stability, S, for

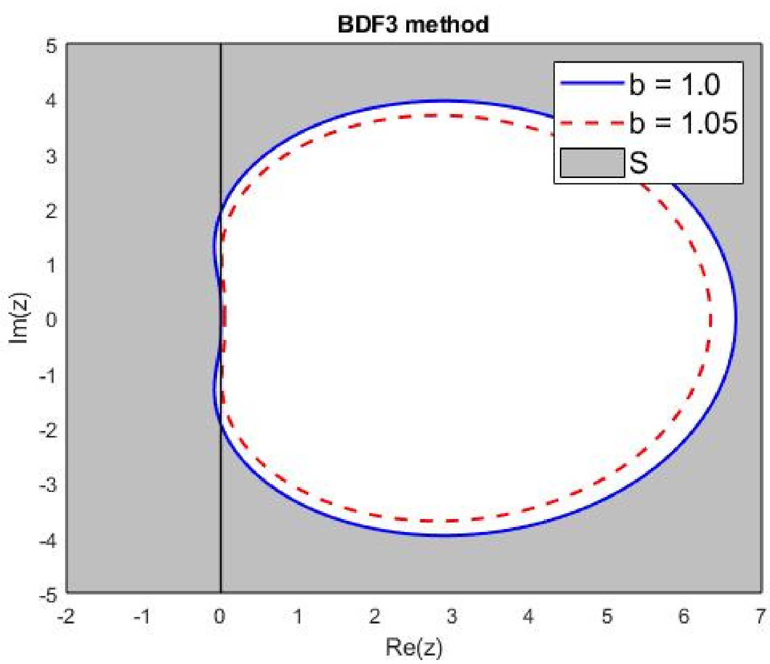
- 3-step Adams Moulton method
- the 2-step 4th-order LMM from Assignment 3,
  $u_{n+2} - u_n = h\left[\frac{1}{3}f(u_{n+2}, t_{n+2}) + \frac{4}{3}f(u_{n+1}, t_{n+1}) + \frac{1}{3}f(u_n, t_n)\right]$
- 2-step BDF
- 3-step BDF.

Below are the corresponding figures, in the order mentioned above.


3-step Adams-Moulton method
S
b = 1.0
b = 1.05
Im(z)
Re(z)
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2


2-step 4th order LMM
b = 1.0
b = 1.05
Im(z)
Re(z)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


BDF2 method
b = 1.0
b = 1.05
S
Im(z)
Re(z)

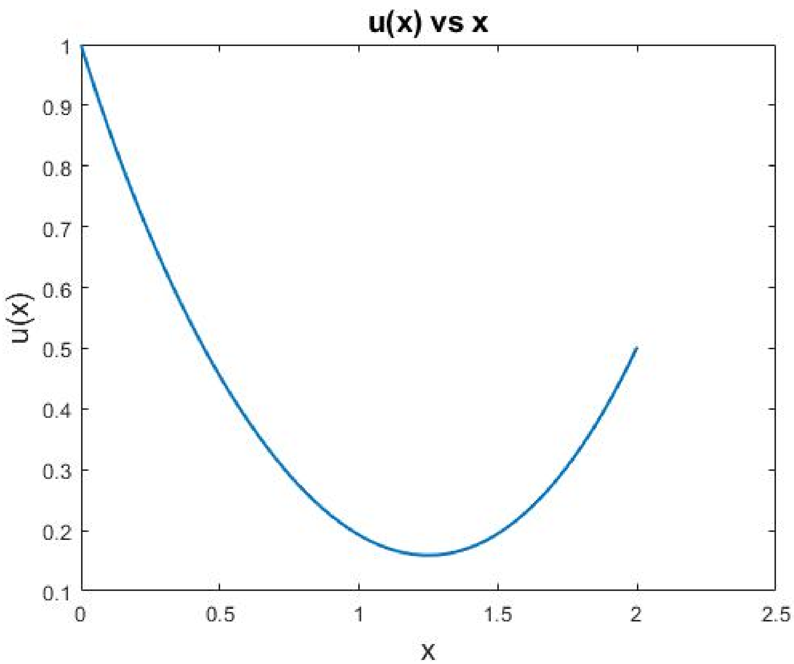

BDF3 method
b = 1.0
b = 1.05
S
Im(z)
Re(z)

Problem 3

In this problem, we solve the second order nonlinear two-point BVP

$$\begin{cases} u'' - (1 + 0.5u'^2)u = \sin x \\ u(0) = 1, \;\; u(2) = 0.5 \end{cases}$$

To do this, we first convert the system into a system of 1st order ODEs, and then apply the shooting method with RK4 and utilize Newton's method as our nonlinear equation solver to solve $G(v) = 0$. The value of v found was $v = -1.484384830636289$. Below is a plot of the solution found.



Problem 4

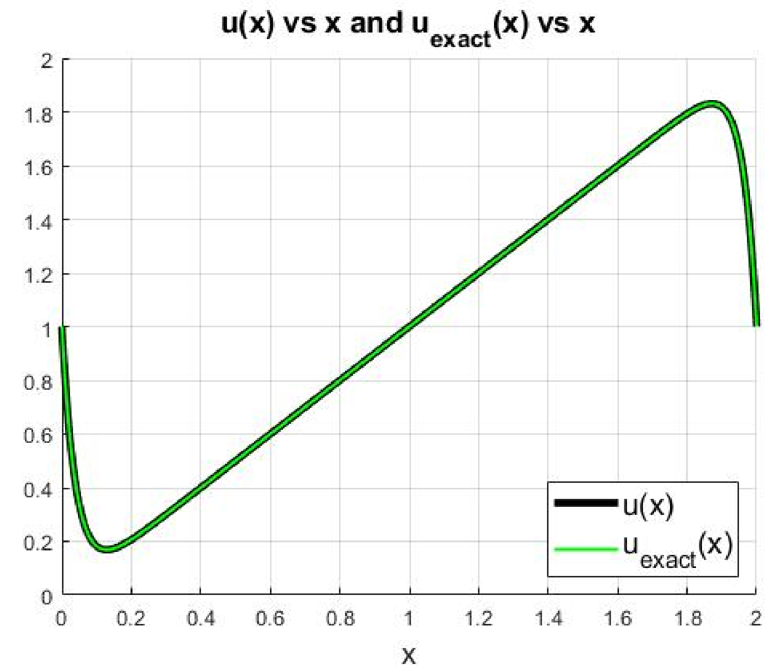
We use the finite difference method to solve the two-point BVP

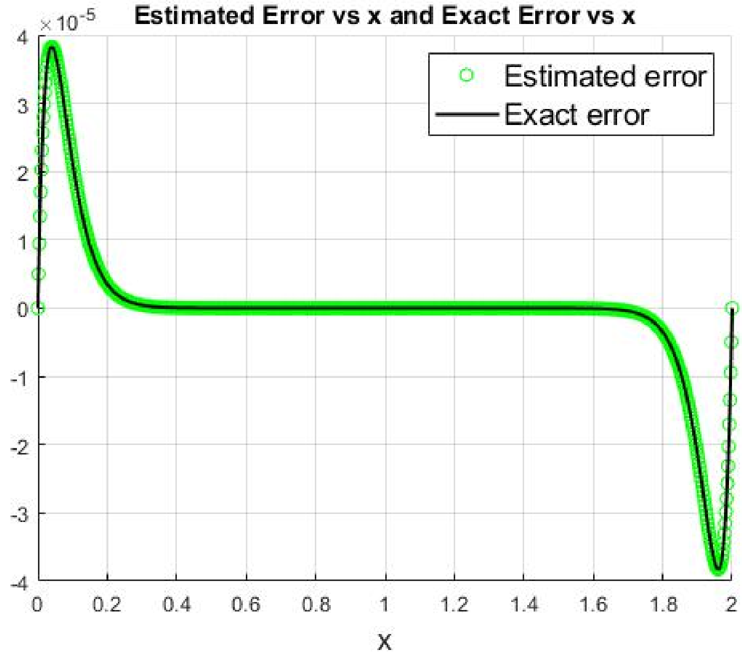
$$\begin{cases} u'' - 625u = -625x \\ u(0) = 1, \;\; u(2) = 1 \end{cases}.$$

The exact solution is

$$u_{exact}(x) = x + \frac{1 + e^{-50}}{1 - e^{-100}}\Big(e^{-25x} - e^{25(x-2)}\Big).$$

We follow the code provided to solve the system $Au = g$, and generate plots of the numerical solution vs the exact solution. We also plot both the estimated error for $u(x)$, and the exact error between $u(x)$ and $u_{exact}(x)$. As seen below, the plots of both the estimated and exact errors, as well as the numerical and exact solutions, coincide quite closely.





Problem 5

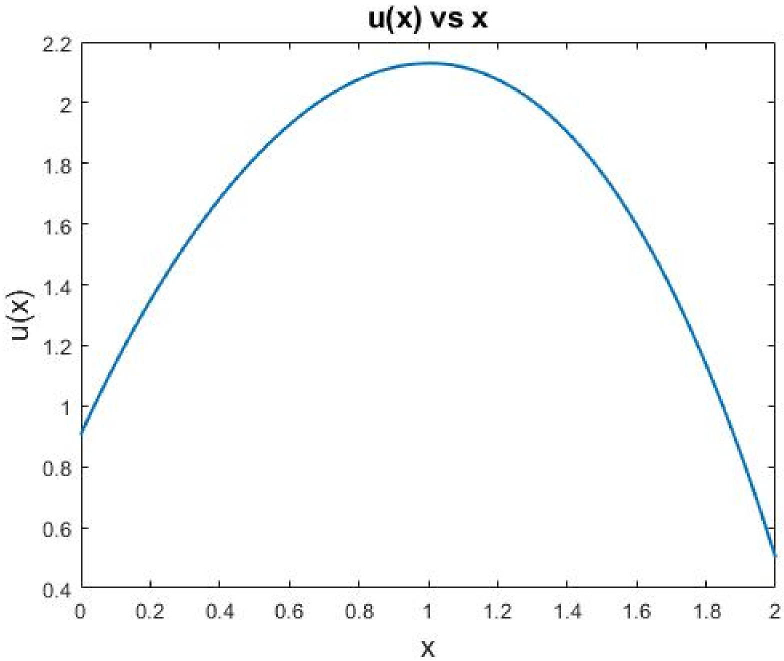
In this problem, we solve the two-point BVP below using the finite difference method.

$$\begin{cases} u'' - (1+\exp(-\sin x))u = -5 - (\sin x)^2 \\ u'(0) = 2.5, \;\; u(2) = 0.5 \end{cases}$$

We have a different type of boundary conditions than in Problem 4. This results in some of the coefficients in the matrix A and RHS vector g in the system $Au = g$ to change, as well as an adjustment to our grid in order to discretize $u'(0)$. The first entry of A and g become

$$a_{1,1} = -\frac{1}{h^2} - \frac{p_1}{2h} + q_1, \quad g_1 = r_1 + \Big(\frac{1}{h} - \frac{p_1}{2}\Big).$$

Below is a plot of the numerical solution $u(x)$ vs x.



Problem 6

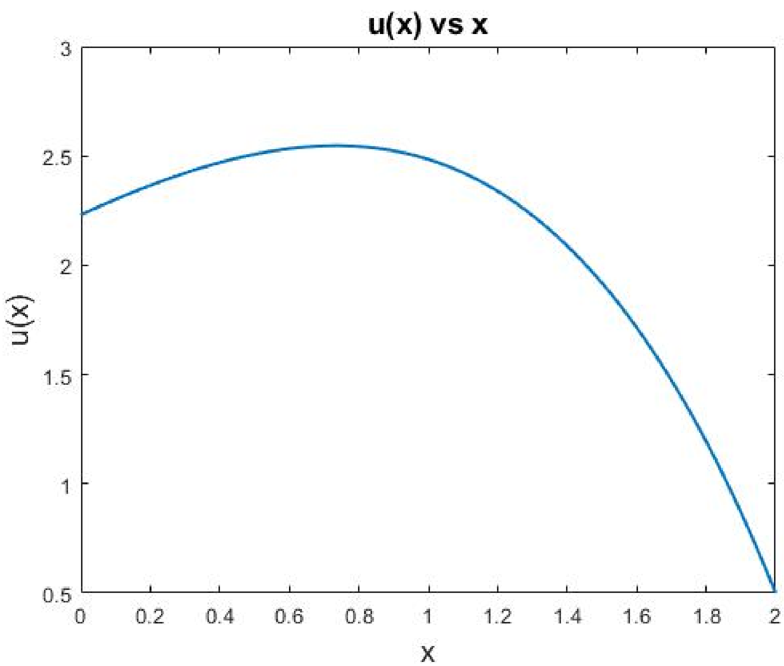
In this problem, we solve a variation of the two-point BVP from Problem 5, but with differing boundary conditions. The problem is

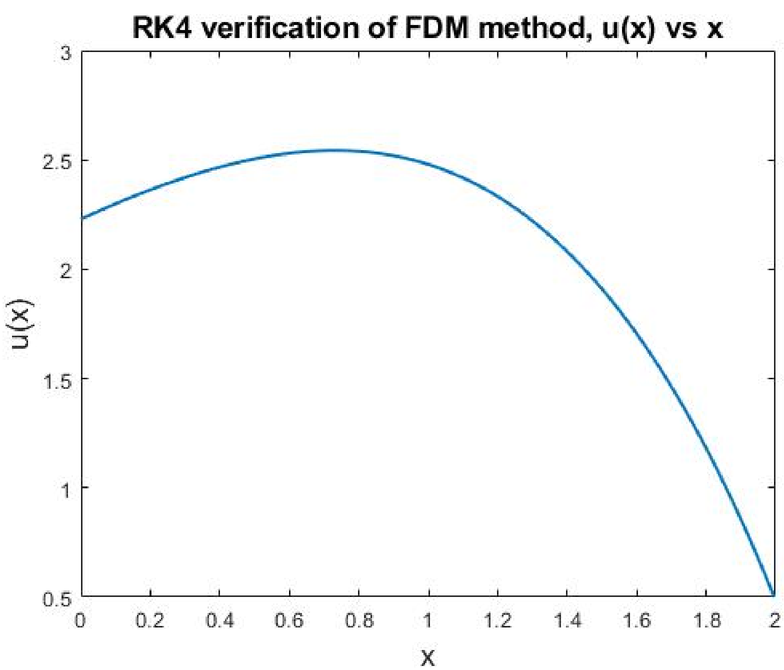
$$\begin{cases} u'' - (1 + \exp(-\sin x))u = -5 - (\sin x)^2 \\ u(0) - u'(0) = 1.5, \;\; u(2) = 0.5. \end{cases}$$

The derivation of the new first entry in matrix A and RHS vector g is included in the handwritten portion attached to this report. We have that

$$a_{1,1} = \frac{p_1 h^2 - 2h(p_1 + 3) - 4}{2h^2(2+h)} + q_1, \qquad g_1 = r_1 + \frac{\alpha(p_1 h - 2)}{h(2+h)}.$$

The numerical solution $u(x)$ vs x is plotted below. We also provide a plot that verifies the solution is correct by using the RK4 method.


u(x) vs x
u(x)
x
3
2.5
2
1.5
1
0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


RK4 verification of FDM method, u(x) vs x
u(x)
x
3
2.5
2
1.5
1
0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Theory

Wednesday, April 21, 2021 1:34 PM

Problem 1 (Theoretical)

P1

Consider the one-stage implicit RK method described by Butcher tableau

$$\begin{array}{c|c} c^T & A \\ \hline & b \end{array} = \begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

Part 1: Show that it has second order accuracy.

Hint: Check the conditions on second order accuracy.

Part 2: Derive its stability function $\phi(z)$.

Part 3: Show that it is A-stable, but not L-stable.

Part 1:

We have $\begin{array}{c|c} c^T & A \\ \hline & b \end{array} = \begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$.

To show this RK method is second order, we will show that it satisfies the first & second order consistency conditions (and internal consistency condition).

1st Order Cond:

$$\sum_{i=1}^{p} b_i = 1 \Rightarrow b_1 = b = 1 \quad \checkmark$$

2nd Order Cond:

$$\sum^{p} b_i c_i = \frac{1}{2} \Rightarrow b \cdot c = (1)\left(\frac{1}{2}\right) = \frac{1}{2} \quad \checkmark$$

$i=1$

Internal Consistency Cond:

$$C_i = \sum_{j=1}^{p} a_{ij} \Rightarrow C = a = \frac{1}{2} \quad \checkmark$$

Thus the method is second order.

Part 2:

We want to derive stability function $\phi(z)$. The method is

$$\begin{cases} k_1 = h \cdot f(u_n + \frac{1}{2}k_1, t_n + \frac{h}{2}) \\ u_{n+1} = u_n + k_1 \end{cases}$$

Apply the method to ODE $u' = \gamma u$, $z = \gamma \cdot h$

we have

$$k_1 = h \cdot \gamma (u_n + \frac{1}{2}k_1) = z u_n + \frac{z}{2} k_1$$

$$\Rightarrow \quad k_1 = \frac{z}{(1-\frac{z}{2})} u_n$$

Hence,

$$U_{n+1} = U_n + k_1 = \frac{z}{\left(1-\frac{z}{2}\right)} U_n + U_n$$

$$\Rightarrow \quad U_{n+1} = \phi(z) U_n = \left(1 + \frac{z}{1-\frac{z}{2}}\right) U_n$$

Thus,

$$\phi(z) = \left(1 + \frac{z}{1 - z/2}\right).$$

Part 3:

To show $\phi(z)$ is A-stable, we need $|\phi(z)| < 1$ to cover the LHS of the complex plane, i.e. $Re(z) < 0 \quad \forall z$

$$|\phi(z)| = \left| 1 + \frac{z}{1 - z/2} \right| < 1$$

$$\Rightarrow \quad = \left| \frac{\left(1 - \frac{z}{2}\right) + z}{1 - z/2} \right| < 1$$

$$= \left| \frac{1 + \frac{z}{2}}{1 - z/2} \right| < 1$$

Hence

$$\left|1+\frac{z}{2}\right| < \left|1-\frac{z}{2}\right|.$$

Let $z = x+iy$

$$\sqrt{\left(1+\frac{x+iy}{2}\right)\left(1+\frac{x-iy}{2}\right)} < \sqrt{\left(1-\frac{(x+iy)}{2}\right)\left(1-\frac{(x-iy)}{2}\right)}$$

$$\Rightarrow \left(1+\frac{x+iy}{2}\right)\left(1+\frac{x-iy}{2}\right) < \left(1-\frac{(x+iy)}{2}\right)\left(1-\frac{(x-iy)}{2}\right)$$

$$\Rightarrow \cancel{1} + \frac{x-iy}{2} + \frac{x+iy}{2} + \cancel{\frac{x^2+y^2}{4}} < \cancel{1} - \frac{(x-iy)}{2} - \frac{(x+iy)}{2} + \cancel{\frac{x^2+y^2}{4}}$$

$$\Rightarrow \frac{x}{2}+\frac{x}{2} < -\frac{x}{2}-\frac{x}{2}$$

$$\Rightarrow 2x < 0$$

Hence $x = \mathrm{Re}(z) < 0$.

So, method is A-stable.

However,

$$\lim \Phi(z) = \lim_{z\to\infty} 1+\frac{z}{1-\frac{z}{2}} = \lim_{z\to\infty} 1+\frac{2z}{2-z}$$

$$\lim_{z\to\infty} \Phi(z) = \lim_{z\to\infty} 1 + \frac{\quad}{1-\frac{z}{2}} - \lim_{z\to\infty} \qquad 2-z$$

$$= 1 + \lim_{z\to\infty} \frac{2z}{2-z} = 1-2 = -1 \neq 0.$$

Hence the method is not L-stable.

Problem 6 (Theoretical and computational)

Consider a slightly different version of the BVP in Problem 5.

P6

$$\begin{cases} u'' - \left(1+\exp(-\sin x)\right)u = -5 - (\sin x)^2 \\ u(0) - u'(0) = 1.5, \quad u(2) = 0.5 \end{cases} \qquad \text{(P6)}$$

Design the discretization of the finite difference method (FDM). Implement it in Matlab. Solve the BVP (P6) with $N = 1000$.

We need to discretize our boundary condition

$$u(0) - u'(0) = \alpha \qquad \left(\alpha = \frac{3}{2}\right)$$

Solution:

Write

$$u(0) = \frac{u_1 + u_0}{2}$$

$$u'(0) = \frac{u_1 - u_0}{h}$$

$u(0) - u'(0) = \alpha$ gives us

Hence, $u(0) - u'(0) = \alpha$ gives us

$$\frac{u_1 + u_0}{2} - \frac{(u_1 - u_0)}{h} = \alpha$$

$$\Rightarrow (u_1 + u_0)h - 2(u_1 - u_0) = 2\alpha h.$$

$$\Rightarrow u_0 \cdot h + u_1 \cdot h - 2u_1 + 2u_0 = 2\alpha h$$

$$\Rightarrow (2+h)u_0 + (h-2)u_1 = 2\alpha h$$

Then,

$$\boxed{u_0 = \frac{2\alpha h + (2-h)u_1}{2+h}}$$

From FDM, we have that

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + p_i \frac{(u_{i+1} - u_{i-1})}{2h} + q_i u_i = r_i, \quad 1 \leq i \leq N-1.$$

Our only difference will be for when

$i=1$:

$$\Rightarrow \frac{U_2 - 2U_1 + U_0}{h^2} + p_1 \frac{(U_2 - U_0)}{2h} + q_1 U_1 = r_1$$

Plugging in U_0 from above, we have...

$$\frac{U_2 - 2U_1}{h^2} + \frac{2\alpha h + (2-h)U_1}{h^2(2+h)} + \frac{p_1 U_2}{2h} - \frac{p_1(2\alpha h + (2-h)U_1)}{2h(2+h)} + q_1 U_1 = r_1$$

$$\frac{U_2 - 2U_1}{h^2} + \frac{2\alpha h + (2-h)U_1}{h^2(2+h)} + \frac{p_1 U_2}{2h} - \frac{(2\alpha h p_1 + p_1(2-h)U_1)}{2h(2+h)} + q_1 U_1 = r_1$$

$$\Rightarrow \frac{U_2}{h^2} + \frac{p_1 U_2}{2h} - \frac{2U_1}{h^2} + \frac{2\alpha h + (2-h)U_1}{h^2(2+h)} - \frac{2\alpha h p_1}{2h(2+h)} - \frac{p_1(2-h)U_1}{2h(2+h)}$$

$$+ q_1 U_1 = r_1$$

$$\Rightarrow \left(\frac{1}{h^2} + \frac{p_1}{2h}\right)U_2 - \frac{2U_1}{h^2} + \frac{(2-h)U_1}{h^2(2+h)} - \frac{p_1(2-h)U_1}{2h(2+h)} + q_1 U_1 = r_1 - \frac{2\alpha h}{h^2(2+h)} + \frac{2\alpha h p_1}{2h(2+h)}$$

$$\Rightarrow \left(\frac{1}{h^2} + \frac{p_1}{2h}\right)U_2 + \left[-\frac{2}{h^2} + \frac{2-h}{h^2(2+h)} - p_1\frac{(2-h)}{2h(2+h)} + q_1\right]U_1 = \underbrace{r_1 + \frac{\alpha(h \cdot p_1 - 2)}{h(2+h)}}_{g_1}$$

$$\Rightarrow \left(\frac{1}{h^2}+\frac{p_1}{2h}\right)u_2+\left[\frac{-4(2+h)+2(2-h)-p_1h(2-h)}{2h^2(2+h)}+q_1\right]u_1=g_1$$

$$\Rightarrow \left(\frac{1}{h^2}+\frac{p_1}{2h}\right)u_2+\left[\frac{-8-4h+4-2h-2p_1h+p_1h^2}{2h^2(2+h)}+q_1\right]u_1=g_1$$

$$\Rightarrow \left(\frac{1}{h^2}+\frac{p_1}{2h}\right)u_2+\left[\frac{p_1h^2-2h(p_1+3)-4}{2h^2(2+h)}+q_1\right]u_1=g_1$$

Hence, we have

$$\boxed{a_{11}=\frac{p_1h^2-2h(p_1+3)-4}{2h^2(2+h)}+q_1, \qquad g_1=r_1+\frac{\alpha(h\cdot p_1-2)}{h(2+h)}.}$$