HW2 Report

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April 14, 2021

Theory Problems (1-3)

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Problem 4

In this problem, we implement the numerical method for Runge-Kutta (RK4) to solve the IVP

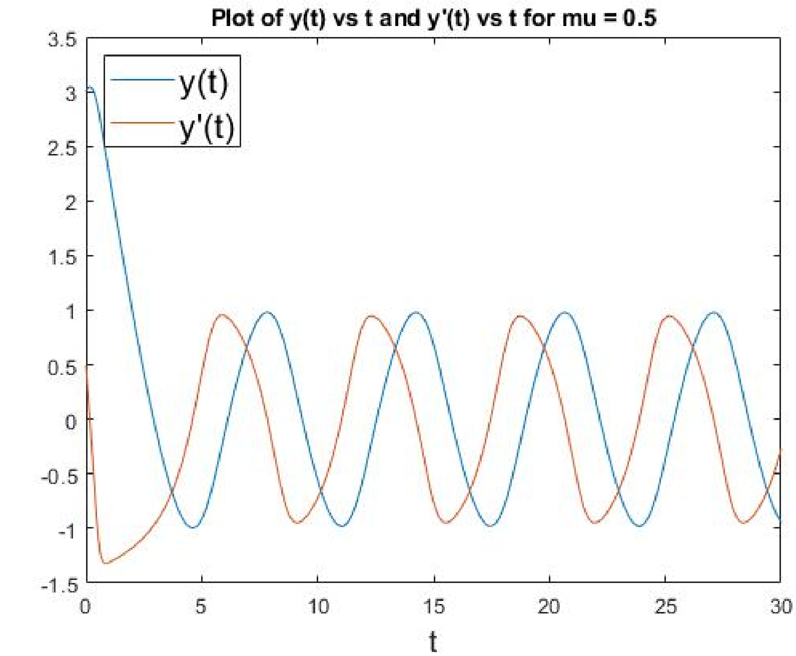
$$\begin{cases} y'' - \mu\big(2 - \exp(y'^2)\big)y' + y = 0 \\ y(0) = y_0, \quad y'(0) = v_0 \end{cases}$$

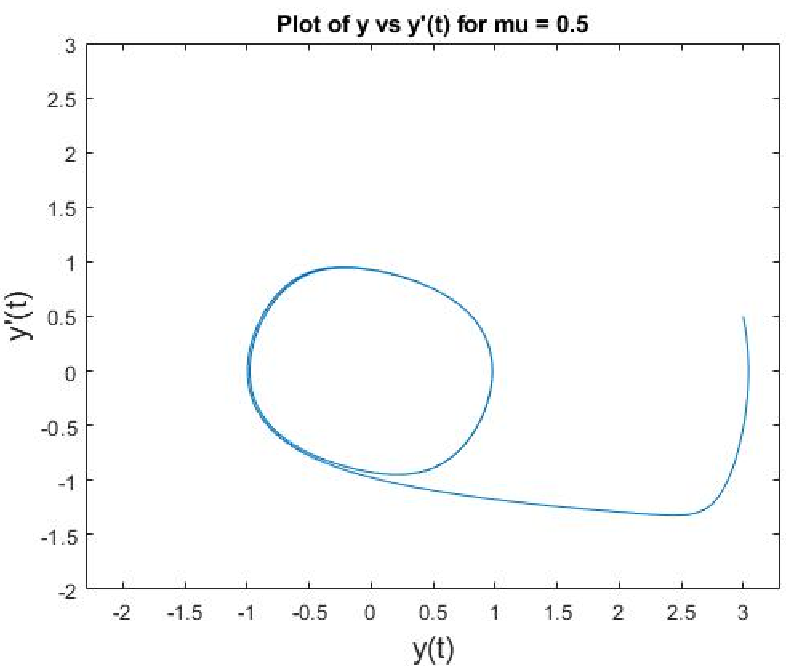
up to $T = 30$, where $y_o = 3,\ v_0 = 0.5,\ h = 0.025,$ and $\mu \in \{0.5,\,2,\,4\}$. We numerically solve this IVP for each value of μ and create six figures below, where the first three are plots of $y(t)$ and $y'(t)$ both versus t, and the remaining three are simply $y'(t)$ vs $y(t)$. In the following problems we're using methods for first order ODEs, hence we must convert the given second order ODE into a system of first order ODEs by letting

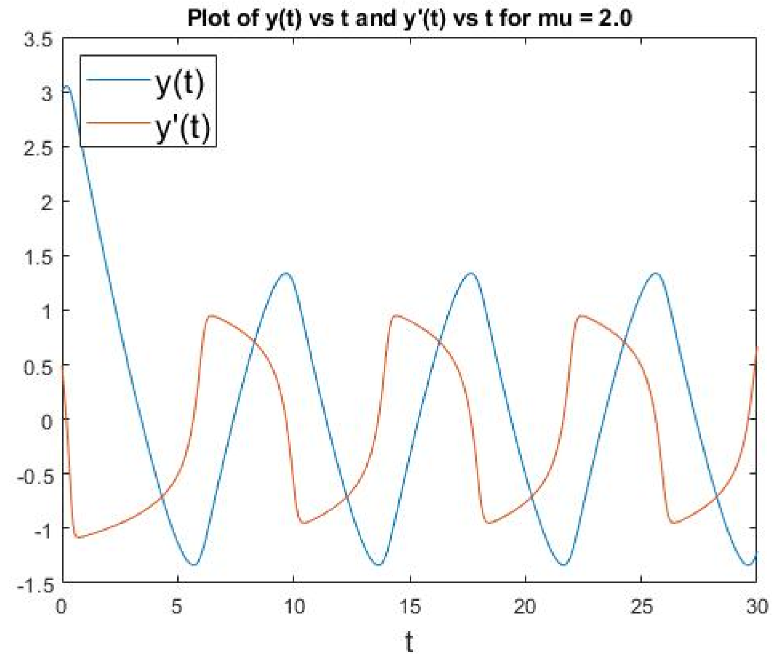
$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$

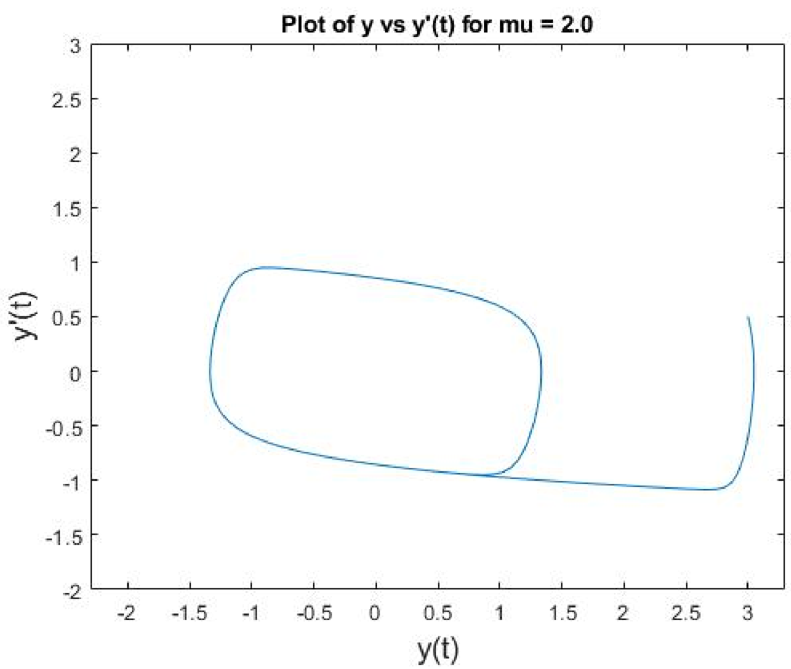
This implies that

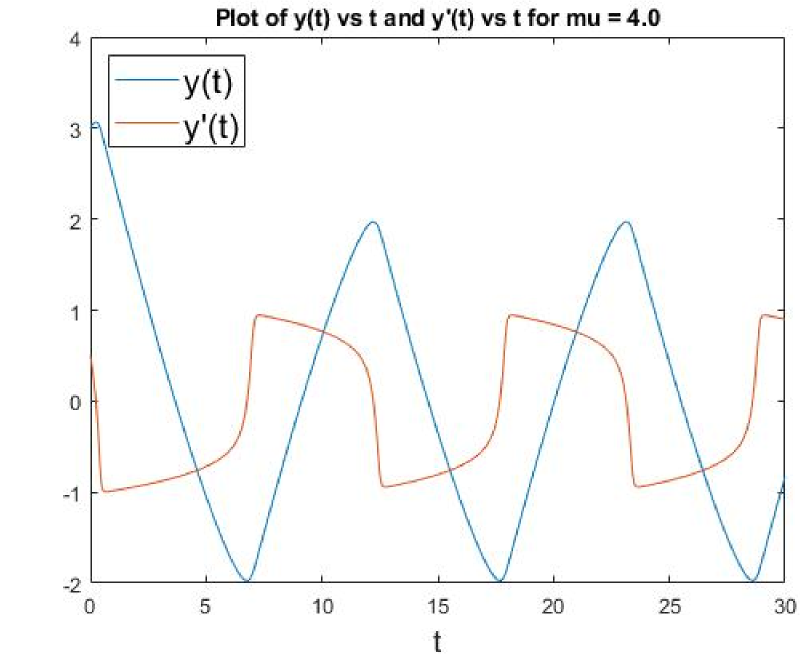
$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = \begin{bmatrix} w_1'(t) \\ w_2'(t) \end{bmatrix} = \begin{bmatrix} w_2(t) \\ \mu(2 - e^{w_2^2})w_2 - w_1 \end{bmatrix}.$$


Plot of y(t) vs t and y'(t) vs t for mu = 0.5
y(t)
y'(t)
t

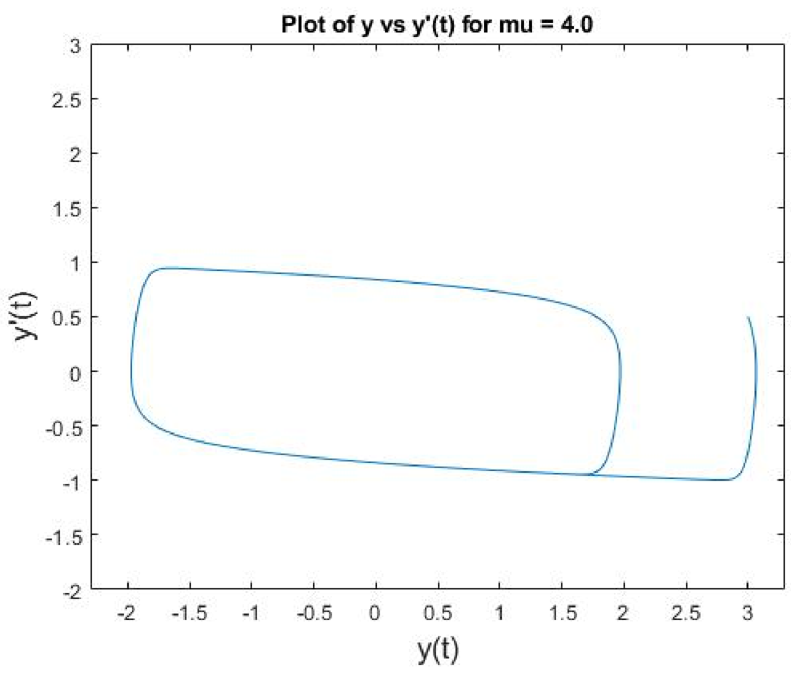

Plot of y vs y'(t) for mu = 0.5
y'(t)
y(t)


Plot of y(t) vs t and y'(t) vs t for mu = 2.0
y(t)
y'(t)
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30
t


Plot of y vs y'(t) for mu = 2.0
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y'(t)
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
y(t)



Plot of y(t) vs t and y'(t) vs t for mu = 4.0
y(t)
y'(t)
4
3
2
1
0
-1
-2
0
5
10
15
20
25
30
t



Plot of y vs y'(t) for mu = 4.0
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y'(t)
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
y(t)

Problem 5

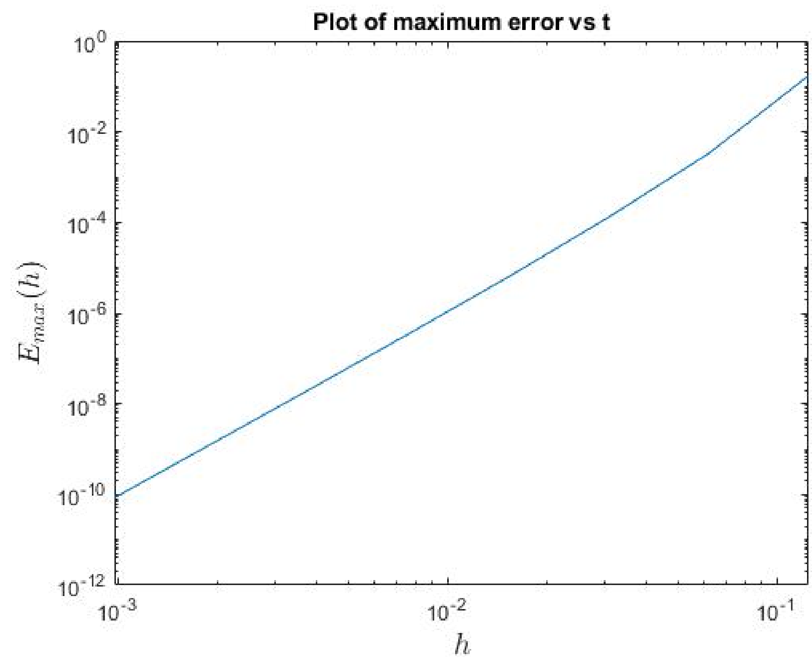
In this problem, we perform numerical error analysis for the same IVP in problem 4. Using $y_0 = 3$, $v_0 = 0.5$, $\mu = 4$, we again solve the IVP to $T = 30$. However, we now vary our step sizes so that $h = \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6} \dots$. We estimate errors in the numerical solution using

$$E_n(h) = \frac{1}{1-(0.5)^4}\Big(w_n(h) - w_{2n}(\frac{h}{2})\Big),$$

where $E_n(h)$ is the error associated with the numerical solution $w_n(h)$. Then, for each time step size, we calculate

$$E_{\max}(h) = \max_{nh\in[0,\,30]} \|E_n(h)\|$$

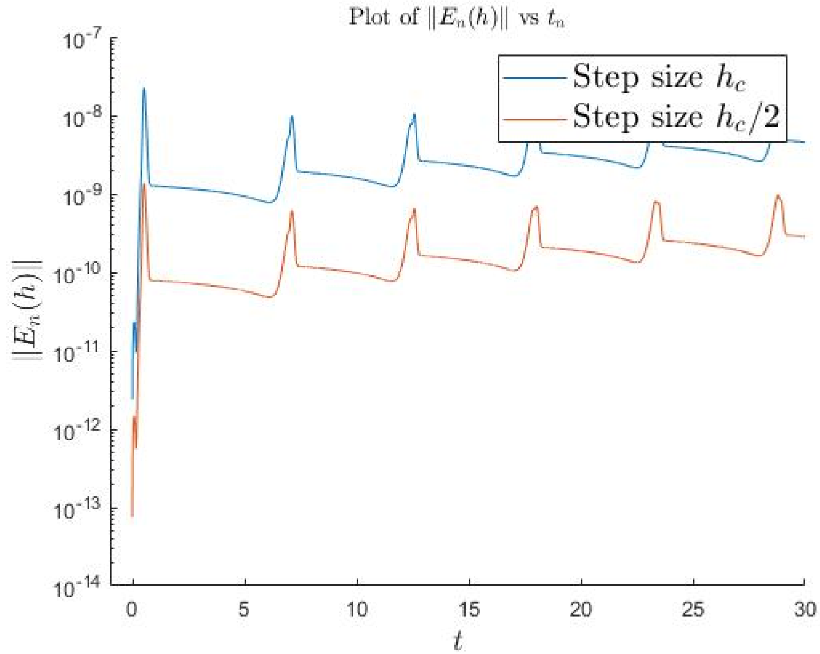
and plot the values for $E_{\max}(h)$ versus h in a log-log scale plot, pictured below.



We can see that as the length of the step size increases, the maximum error increases. When $h = \frac{1}{2^8} := h_c$, we have that

$$E_{\max}(h) < 5 \times 10^{-8}.$$

Pictured below is one figure containing a plot of $\|E_n(h)\|$ versus t_n respectively, for time step lengths h_c and $h_c/2$.

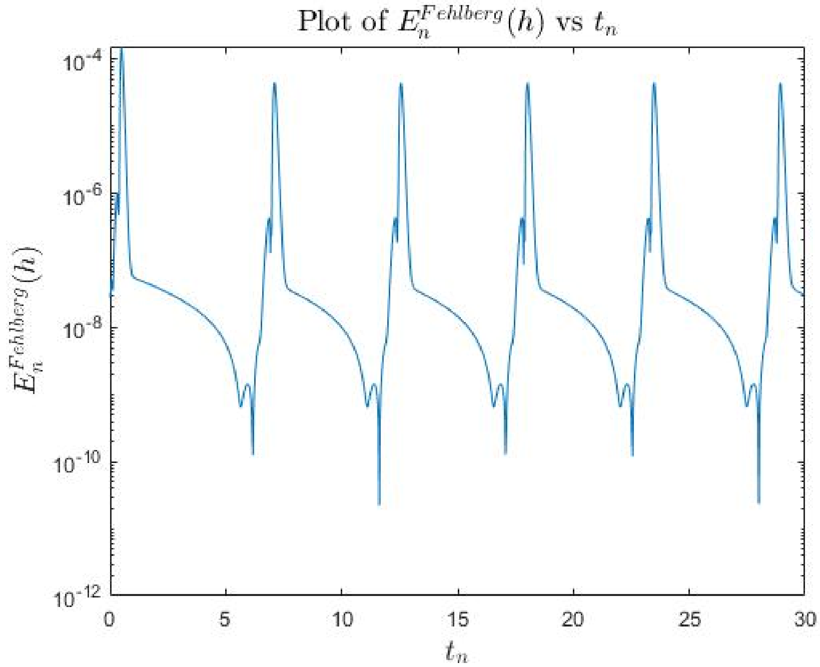


Problem 6

In this problem, we implement the embedded Runge-Kutta method with orders 5 and 4 (Fehlberg 45) and solve the IVP in problem 4 to $T = 30$, using $y_0 = 3,\ v_0 = 0.5,\ \mu = 4,\ h = 0.025$. For each time step, we compute

$$E_n^{(\text{Fehlberg})}(h) \approx \frac{e_n(h)}{h} = \frac{\|w_{n+1} - \tilde{w}_{n+1}\|}{h},$$

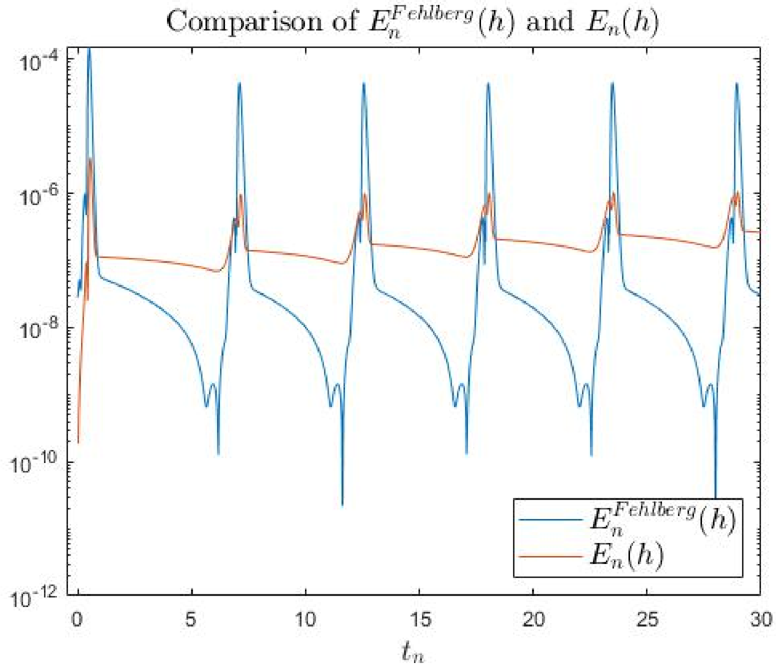
where, respectively, w_{n+1} and $\tilde{w}_{n+1}$ are the vector results of the 5th order and 4th order RK methods in the Fehlberg method. Below is a logarthmic plot of $E_n^{(\text{Fehlberg})}(h)$ versus t_n.



We also estimate the error $E_n(h)$ by

$$E_n(h) = \frac{1}{1-(0.5)^4}\Big(w_n(h) - w_{2n}(\frac{h}{2})\Big),$$

and then (in the figure below) plot $E_n^{(\mathrm{Fehlberg})}(h)$ versus t_n as well as $E_n(h)$ versus t_n,


Comparison of $E_n^{Fehlberg}(h)$ and $E_n(h)$
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
0
5
10
15
20
25
30
t_n
$E_n^{Fehlberg}(h)$
$E_n(h)$

.

Theory

Wednesday, April 7, 2021 10:31 AM

Problem 1 (Theoretical)

P1

Suppose k satisfies the equation

$$k = h\exp(1+k) \quad \text{where } h \text{ is a small quantity}$$

Recall the approach of iterative expansion we used in lecture.

Start with $k = O(h)$. Expand k iteratively into

$$k = a_1 h + a_2 h^2 + \cdots$$

Find the coefficients a_1 and a_2.

We have $k = h \cdot \exp(1+k)$.

Assuming k is bounded, we start with $k = O(h)$.

$$\Rightarrow k = h \cdot \exp(1 + O(h)) = h \cdot e \cdot \exp(O(h))$$

Applying a Taylor expansion to $\exp(O(h))$,

$$k = he(1 + O(h)) = eh + O(h^2).$$

Assuming that k can be written as

$k = a_1 h + O(h^2)$, we have

$$k = eh + O(h^2) = a_1 h + O(h^2) \Rightarrow eh = a_1 h,$$

hence $\boxed{a_1 = e.}$

To find a_2, we write

$$k = h \cdot \exp(1 + eh + O(h^2)) = he \cdot \exp(eh) \cdot \exp(O(h^2)).$$

Expanding $\exp(O(h^2))$ as

$$\exp(O(h^2)) = 1 + O(h^2), \text{ then}$$

$$\ldots \exp(O(h^2)) = he \cdot \exp(eh)(1 + O(h^2))$$

$$\exp(O(h^2)) = 1 + O(h^2)$$

$$he \cdot \exp(eh) \cdot \exp(O(h^2)) = he \cdot \exp(eh)(1 + O(h^2))$$
$$= he \cdot \exp(eh) + O(h^3).$$

Expanding $\exp(eh)$ as $\exp(eh) = 1 + eh + O(h^2)$,

We have

$$K = he \cdot (1 + eh + O(h^2)) + O(h^3)$$

$$\Rightarrow K = eh + e^2h^2 + O(h^3) = eh + a_2h^2 + O(h^3)$$

Hence,

$$eh + e^2h^2 = eh + a_2h^2$$

$$\Rightarrow \quad e^2h^2 = a_2h^2$$

So, $$\boxed{a_2 = e^2.}$$

P2

Problem 2 (Theoretical)

Consider the Runge-Kutta (RK) method specified by the Butcher tableau below

$$\begin{array}{c|c} c^T & A \\ \hline & b \end{array} = \begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Write out the method in the form of $k_1 = \cdots$, $k_2 = \cdots$, ..., $u_{n+1} = u_n + \cdots$

Support $f(u, t)$ satisfies $|f(u,t) - f(v,t)| \le C|u - v|$ for all u, v, and t.

Part 1: Show that the method is stable.

Part 2: Use Taylor expansion to show $e_n(h) = O(h^2)$.

Part 3: What is the order of its global error $E_N(h)$?

We're given the Butcher tableau

$$\begin{array}{c|c} c^T & A \\ \hline & b \end{array} = \begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

$$\Rightarrow P = 2, \quad A = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}, \quad b = (\tfrac{1}{2}, \tfrac{1}{2}), \quad c = (0, \tfrac{1}{2})$$

$$\Rightarrow \begin{cases} k_1 = h \cdot f(u_n, t_n) \\ k_2 = h \cdot f(u_n + \frac{1}{2}k_1, t_n + \frac{1}{2}h) \\ u_{n+1} = u_n + \frac{1}{2}k_1 + \frac{1}{2}k_2 \end{cases}$$

<u>Part 1</u>: Show the method is stable.

<u>Proof</u>:

We're given f is LC.

$$u_{n+1} = u_n + \tfrac{1}{2}h \cdot f(u_n, t_n) + \tfrac{1}{2}hf(u_n + \tfrac{1}{2}hf(u_n, t_n), t_n + \tfrac{1}{2}h)$$

$$v_{n+1} = v_n + \tfrac{1}{2}h \cdot f(v_n, t_n) + \tfrac{1}{2}hf(v_n + \tfrac{1}{2}hf(v_n, t_n), t_n + \tfrac{1}{2}h)$$

$$\Rightarrow$$

$$|u_{n+1} - v_{n+1}| = \Big|(u_n - v_n) + \tfrac{1}{2}h\Big[f(u_n, t_n) + f(u_n + \tfrac{1}{2}hf(u_n, t_n), t_n + \tfrac{1}{2}h) - f(v_n, t_n) - f(v_n + \tfrac{1}{2}hf(v_n, t_n), t_n + \tfrac{1}{2}h)\Big]\Big|$$

$$= \Big|(u_n - v_n) + \tfrac{1}{2}h\Big[\underbrace{f(u_n, t_n) - f(v_n, t_n)}$$

$$\Rightarrow \Big| (u_n - v_n) + \tfrac{1}{2}h \Big[\underbrace{f(u_n,t_n) - f(v_n,t_n)}_{a}$$

$$+ \underbrace{f\left(u_n + \tfrac{1}{2}hf(u_n,t_n), t_n + \tfrac{1}{2}h\right) - f\left(v_n + \tfrac{1}{2}hf(v_n,t_n), t_n + \tfrac{1}{2}h\right)}_{b} \Big] \Big|$$

$$\Rightarrow |u_{n+1} - v_{n+1}| \le |(u_n - v_n)| + \tfrac{1}{2}h|a+b|$$

Since f is LC,

$$a \le C|u_n - v_n|$$

and

$$b \le C\left|u_n + \tfrac{1}{2}hf(u_n,t_n) - \left(v_n + \tfrac{1}{2}hf(v_n,t_n)\right)\right|.$$

$$\Rightarrow \quad b \le C\left|(u_n - v_n) + \tfrac{1}{2}h(f(u_n,t_n) - f(v_n,t_n))\right|$$

$$\Rightarrow b \le C|u_n - v_n| + \tfrac{C}{2}h|f(u_n,t_n) - f(v_n,t_n)|$$

$$\Rightarrow b \le C|u_n - v_n| + \tfrac{C^2}{2}h|u_n - v_n|$$

$$b \le \left(C + \tfrac{C^2}{2}h\right)|u_n - v_n|$$

$$\Rightarrow |a+b| \le C|u_n - v_n| + \left(C + \tfrac{C^2}{2}h\right)|u_n - v_n|$$

$$\le \left(2C + \tfrac{C^2}{2}h\right)|u_n - v_n|$$

Hence,

$$|u_{n+1} - v_{n+1}| \le |u_n - v_n| + \tfrac{1}{2}h|a+b|$$

$\Rightarrow$

$$|U_{n+1} - V_{n+1}| \leq |U_n - V_n| + \tfrac{1}{2}h\left(2c + \tfrac{c^2}{2}h\right)|U_n - V_n|$$

$$\leq \left(1 + ch + \frac{c^2h^2}{4}\right)|U_n - V_n| .$$

For small enough h, $\frac{c^2h^2}{4} \to 0$.

Hence

$$|U_{n+1} - V_{n+1}| \leq (1+ch)|U_n - V_n|,$$

so the method is stable. $\blacksquare$

Part 2:

We have that

$$U_{n+1} = U_n + \tfrac{1}{2}hf(U_n, t_n) + \tfrac{1}{2}hf\left(U_n + \tfrac{1}{2}hf(U_n, t_n), t_n + \tfrac{1}{2}h\right)$$

$\Rightarrow$ $e_n(h) =$

$$U(t_{n+1}) - U(t_n) - \tfrac{1}{2}hf(U(t_n), t_n) - \tfrac{1}{2}hf\left(U(t_n) + \tfrac{1}{2}f(U(t_n), t_n), t_n + \tfrac{1}{2}h\right)$$

Write $U(t_{n+1}) = U(t_n + h)$, then

$$U(t_n+h) - U(t_n) - \frac{h}{2}\left[f(U(t_n), t_n) + f\left(U(t_n) + \tfrac{1}{2}f(U(t_n), t_n), t_n + \tfrac{1}{2}h\right)\right]$$

Applying a Taylor expansion to $U(t_n+h)$, we have

Applying a Taylor expansion to $u(t_n+h)$, we have

$$u(t_n+h) = u(t_n) + u'(t_n)h + u''(t_n)\frac{h^2}{2!} + \ldots$$

Substituting into $e_n(h)$, we get

$$u'(t_n)\cdot h - \frac{1}{2}h\left[f(u(t_n),t_n) + f\left(u(t_n)+\frac{h}{2}f(u(t_n),t_n),t_n+\frac{1}{2}h\right)\right]+O(h^2)$$

Since $u(t_n)$ is an exact solution, we know

$u'(t_n) = f(u(t_n),t_n)$.

Thus,

$$\frac{h}{2}f(u(t_n),t_n) - \frac{h}{2}\underbrace{f\left(u(t_n)+\frac{h}{2}f(u(t_n),t_n),\ t_n+\frac{h}{2}\right)}_{\text{expand this further}} + O(h^2)$$

$$\Rightarrow e_n(h) = \frac{h}{2}f(u(t_n),t_n) - \frac{h}{2}\left[f(u(t_n),t_n) + O(h)\right] + O(h^2)$$

$$= \frac{h}{2}\cancel{f(u(t_n),t_n)} - \frac{h}{2}\cancel{f(u(t_n),t_n)} + O(h^2)$$

Hence, $e_n(h) = O(h^2)$.

<u>Part 3</u>

From lecture 2, we can apply the theorem relating global error and the LTE. Since we proved in Part 1 that the method is stable, and in Part 2 we showed $e_n(h)=O(h^2)$, We know

$$E_N(h) = O(h).$$

Problem 3 (Theoretical)

P3

Recall that in lecture, we carried out polynomial interpolation based on 3 points and used the polynomial interpolation to derive

- 3-step Adams-Bashforth method and
- 2-step Adams-Moulton method

Carry out polynomial interpolation based on 2 points and use it to derive

- 2-step Adams-Bashforth method and
- 1-step Adams-Moulton method

To derive Adams-Bashforth 2, we let $s = 1 \Rightarrow r = s+1 = 2$.

$$\Rightarrow \{t_n, t_{n+1}\}. \qquad \begin{aligned} t_n &\longrightarrow X_1 = -s = -1 \\ t_{n+1} &\longrightarrow X_2 = -(s-1) = 0. \end{aligned}$$

Recall $x = \dfrac{t - t_{n+2}}{h}$.

Then, the method is given by

$$U_{n+2} = U_{n+1} + \int_{t_{n+1}}^{t_{n+2}} p\left(\frac{t - t_{n+2}}{h}\right) dt$$

$$= U_{n+1} + \int_0^1 P(x)\,dx.$$ We must calculate $P(x)$.

$$P(x) = f(U_n, t_n)\,P_1(x) + f(U_{n+1}, t_{n+1})\,P_2(x),$$

where

$$P_j(x) := \prod_{k=1, k\neq j}^{m} \left(\frac{x - x_k}{x_j - x_k}\right), \quad j = 1, 2. \qquad \begin{cases} x_1 = -1 \\ x_2 = 0 \end{cases}$$

$$\Rightarrow P_1(x) = \left(\frac{x-0}{-1-0}\right) = -x$$

$$P_2(x) = \left(\frac{x+1}{0+1}\right) = x+1.$$

$$\Rightarrow \quad u_{n+2} = u_{n+1} + hf(u_n, t_n)\int_0^1 P_1(x)\,dx + hf(u_{n+1}, t_{n+1})\int_0^1 P_2(x)\,dx$$

We have

$$\int_0^1 P_1(x)\,dx = \int_0^1 (-x)\,dx = -\frac{1}{2}x^2\Big|_0^1 = -\frac{1}{2}$$

$$\int_0^1 P_2(x)\,dx = \int_0^1 (x+1)\,dx = \left(\frac{1}{2}x^2 + x\right)\Big|_0^1 = \frac{3}{2}$$

$$\Rightarrow \quad u_{n+2} = u_{n+1} + -\frac{1}{2}hf(u_n, t_n) + \frac{3}{2}hf(u_{n+1}, t_{n+1})$$

Hence, AB-2:

$$\boxed{u_{n+2} = u_{n+1} + \frac{h}{2}\left[-f(u_n, t_n) + 3f(u_{n+1}, t_{n+1})\right]}$$

Now, we derive Adams–Moulton 1 method. This is the same as the trapezoidal method.

Here, $s = 1 \Rightarrow r = s = 1$

$$\begin{cases} t_n \longrightarrow x_1 = -s = -1 \\ t_{n+1} \rightarrow x_2 = -(s-1) = 0 \end{cases}$$

Here, $s = 1 \Rightarrow r = s = 1$

$$\begin{cases} t_n \longrightarrow x_1 = -s = -1 \\ t_{n+1} \rightarrow x_2 = -(s-1) = 0 \end{cases}$$

So,

$$U_{n+1} = U_n + \int_{t_n}^{t_{n+1}} P\left(\frac{t - t_{n+1}}{h}\right) dt$$

$$= U_n + \int_{-1}^{0} P(x)\, dx.$$

$$P(x) = hf(U_n, t_n)\, P_1(x) + hf(U_{n+1}, t_{n+1})\, P_2(x)$$

$$P_j(x) = \prod_{k=1, k\neq j}^{m} \left(\frac{x - x_k}{x_j - x_k}\right), \ j = 1, 2 \qquad x_1 = -1, \quad x_2 = 0$$

$$\Rightarrow P_1(x) = \frac{x}{-1} = -x \quad \Rightarrow \int_{-1}^{0} (-x)\, dx = -\frac{1}{2}x^2 \Big|_{-1}^{0} = \frac{1}{2}$$

$$P_2(x) = \frac{x+1}{1} = x+1. \Rightarrow \int_{-1}^{0} (x+1)\, dx = \left(\frac{1}{2}x^2 + x\right)\Big|_{-1}^{0} = \frac{1}{2}$$

Thus, AM-1 method is

$$U_{n+1} = U_n + \frac{1}{2} h f(U_n, t_n) + \frac{1}{2} h f(U_{n+1}, t_{n+1})$$

$$\Rightarrow \boxed{U_{n+1} = U_n + \frac{h}{2}\left[f(U_n, t_n) + f(U_{n+1}, t_{n+1})\right]}$$

(Also trapezoidal).