HW6 Report

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Problems 1 and 2:

Both problem 1 and problem 2 are theoretical and are appended in a handwritten format to the end of this report.

Problem 3:

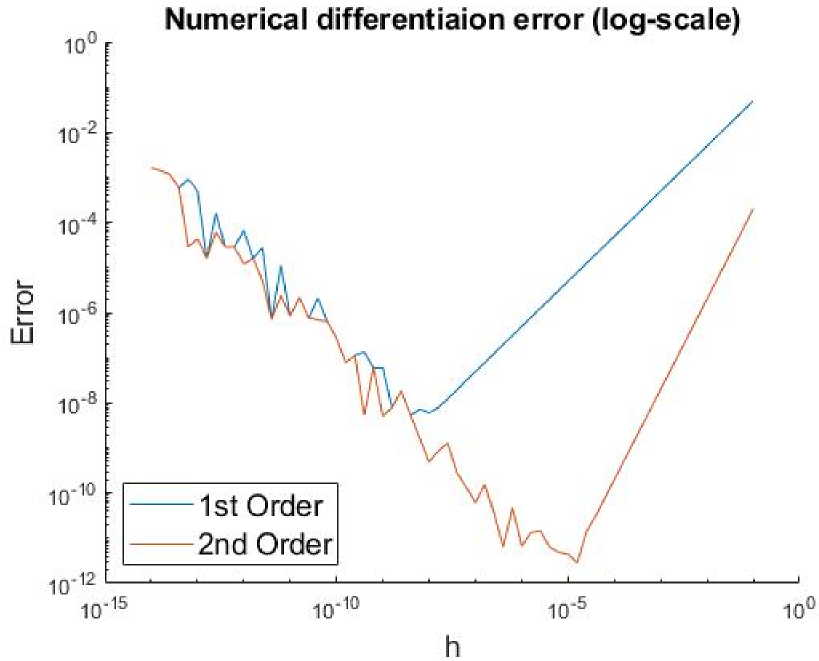
In this problem, we use numerical differentiation (first and second order) to approximate

$$\frac{d}{dx}\sin(x), \quad x = 1.45.$$

The purpose of this is to illustrate that there is an optimal value of h to achieve minimum error, and that a higher order method results in less error. We compare our numerical derivatives against the values of the actual derivative, $\cos(x)$. Below, we provide a loglog plot of both

$$E_{T,1}(h) = |q_1(h) - \cos(x)|, \quad E_{T,2}(h) = |q_2(h) - \cos(x)|,$$

versus h, where $q_1(h)$ and $q_2(h)$ are the first and second order approximations of the derivative, respectively.



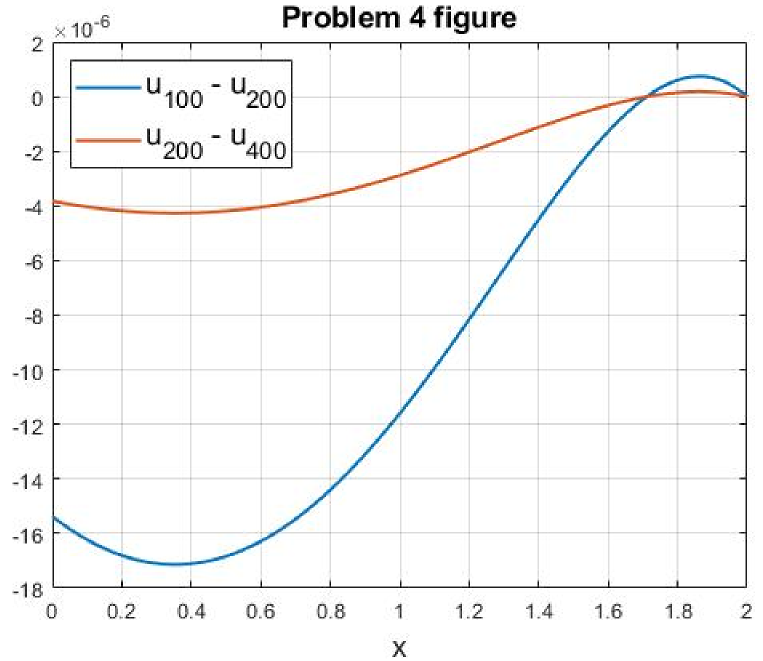
Problem 4:

In this problem, we continue with the same IBVP from problem 6 in assignment 5,

$$\begin{cases} u_t = u_{xx}, \quad x \in (0, L), \ \ t > 0 \\ u(x,0) = p(x), \quad x \in (0, L) \\ u_x(0,t) - \alpha u(0,t) = 0., \quad u(L,t) = q(t) \end{cases}$$

with $L = 2,\ \alpha = 0.4,\ p(x) = (1 - 0.5x)^2,\ q(t) = 2\sin^2(t)$.

In this problem, we solve the IBVP with differing spatial grids, namely when $N = 100,\ N = 200,$ and $N = 400$. The method we use is the FTCS method, and we solve to $T = 3$ with $\Delta t = 10^{-5}$. We then use the spline function in MATLAB to map the solution for each value of N to the grid $x = [0 : 0.002 : 1]^* L$. Below we plot $(u_{\{N=100\}} - u_{\{N=200\}})$ vs x and then $(u_{\{N=200\}} - u_{\{N=400\}})$ vs x.



Problem 5:

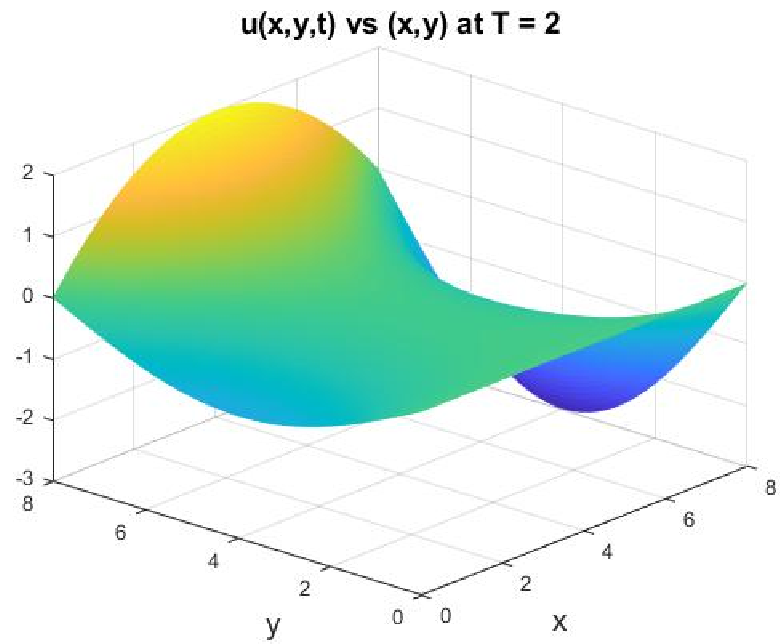
In this problem, we solve the 2D IBVP problem

$$\begin{cases} u_t = u_{xx} + u_{yy}, \quad (x, y) \in (0, 8) \times (0, 8), \;\; t > 0 \\ u(x, y, 0) = f(x, y) \\ u(0, y, t) = g_L(y, t), \quad u(8, y, t) = g_R(y, t) \\ u(x, 0, t) = g_B(x, t), \quad u(x, 8, t) = g_T(x, t) \end{cases}$$

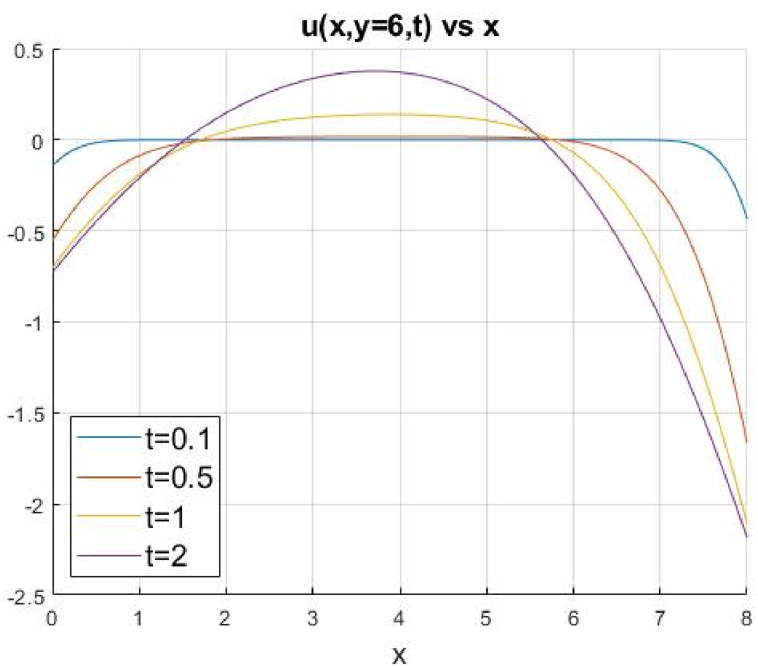
where

$$f(x, y) = 0$$
$$g_L(y, t) = -\sin(\pi y/8)\tanh(2t), \quad g_R(y, t) = -3\sin(\pi y/8)\tanh(2t)$$
$$g_B(x, t) = 0, g_T(x, t) = x(1 - x/8)\tanh(2t).$$

We solve to $T = 2$ using the 2D FTCS method, with $\Delta x = \Delta y = 0.08$ and $\Delta t = 1.25 \times 10^{-3}$. Below we plot $u(x, 6)$ vs x at $t = 0.1,\ 0.5\, 0.1\, 2$ in one figure, and $u(x, y, t)$ vs (x, y) at $T = 2$ as a surface in another figure.



u(x,y,t) vs (x,y) at T = 2
2
1
0
-1
-2
-3
8
6
4
2
0
y
0
2
4
6
8
x



u(x,y=6,t) vs x
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
1
2
3
4
5
6
7
8
x
t=0.1
t=0.5
t=1
t=2

Problem 6:

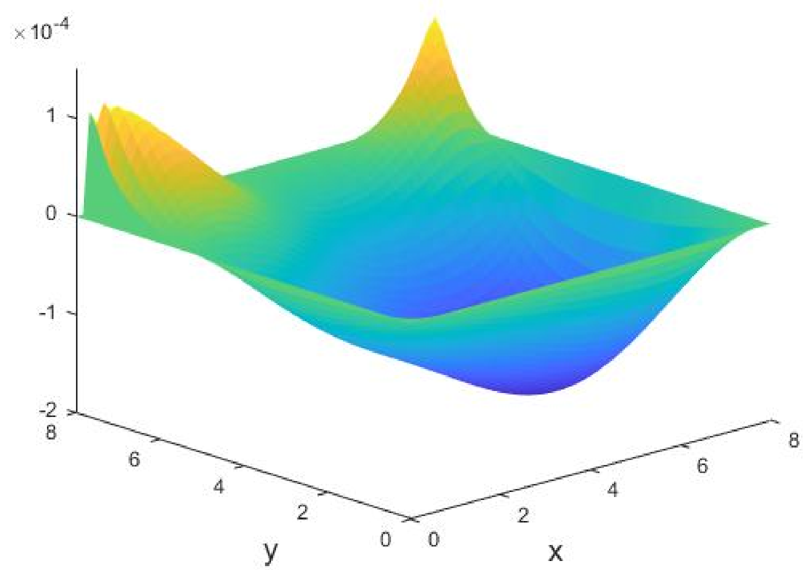
This problem is a continuation of the problem above. Using $\Delta x = \Delta y = 0.08$ and $\Delta t = 1.25 \times 10^{-3}$, we first solve the IBVP to $T = 20$. For each time level, we calculate the max value of $|u_t|$ at all of the internal points. We store the maximum at each t_n in the vector

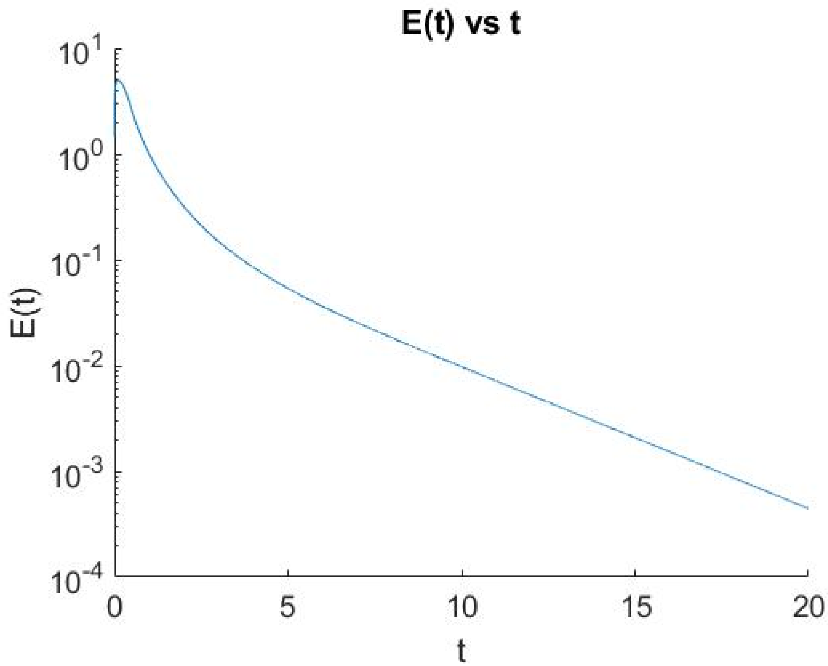
$$E(t_n) = \max \frac{|u_{i,j}^{n+1} - u_{i,j}^{n}|}{\Delta t}, \quad 1 \le i \le N_x - 1, \quad 1 \le j \le N_y - 1.$$

We then plot $E(t)$ against t, where $E(t)$ is on a logarithmic scale (t on a linear scale).

We also re-solve the IBVP using $\Delta x = \Delta y = 0.16$ and $\Delta t = 1.25 \times 10^{-3}$, using the new solution to calculate $u_{\{\Delta=0.16\}} - u_{\{\Delta=0.08\}}$ for $T = 20$, and then plot the result vs (x, y) as a surface.

$u_{\{\Delta=0.16\}} - u_{\{\Delta=0.08\}}$ at T = 20.




E(t) vs t
E(t)
t
10^1
10^0
10^-1
10^-2
10^-3
10^-4
0
5
10
15
20

Theory

Thursday, May 6, 2021 10:51 AM

Problem 1 (Theoretical)

Consider the Lax-Friedrichs method for the general case ($a > 0$ or $a < 0$)

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

P1

Part 1: Carry out von Neumann stability analysis.

Part 2: Use Taylor expansion to find the local truncation error e_i^n.

Convert r back to $\Delta t/(\Delta x)$. Find coefficients of $(\Delta t)^2$, $(\Delta t)(\Delta x)$ and $(\Delta x)^2$ in e_i^n.

Part 3: Answer the two questions below.

For fixed $\frac{\Delta t}{\Delta x} = r$, write $\frac{e_i^n}{\Delta t}$ in terms of Δx only. Do we have $\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = 0$?

For fixed $\frac{\Delta t}{(\Delta x)^2} = c$, write $\frac{e_i^n}{\Delta t}$ in terms of Δx only. Do we have $\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = 0$?

Part 1: We have

$$U_i^{n+1} = \frac{U_{i+1}^n + U_{i-1}^n}{2} - \frac{ar}{2}\left(U_{i+1}^n - U_{i-1}^n\right), \quad r = \frac{\Delta t}{\Delta x}$$

$(a < 0$ or $a > 0)$.

First, write

$$L_{num}(U^{n+1}) = L_{num}(U^n)$$

$\Rightarrow$ Already in this form.

We try $U_i^n = \rho^n \cdot e^{\sqrt{-1}\xi i \Delta x}$.

$$\Rightarrow \rho^{n+1} \cdot e^{\sqrt{-1}\xi i \Delta x} = \frac{1}{2}\left[\rho^n \cdot e^{\sqrt{-1}\xi(i+1)\Delta x} + \rho^n e^{\sqrt{-1}\xi(i-1)\Delta x}\right] - \frac{ar}{2}\left[\rho^n e^{\sqrt{-1}\xi(i+1)\Delta x} - \rho^n e^{\sqrt{-1}\xi(i-1)\Delta x}\right]$$

Divive by $\rho^n e^{\sqrt{-1}\xi i \Delta x}$:

$$\rho = \frac{1}{2}\left[e^{\sqrt{-1}\xi\Delta x} + e^{-\sqrt{-1}\xi\Delta x}\right] - \frac{ar}{2}\left[e^{\sqrt{-1}\xi\Delta x} - e^{-\sqrt{-1}\xi\Delta x}\right]$$

$$= \cosh(\sqrt{-1}\,\xi\Delta x) - ar\sinh(\sqrt{-1}\,\xi\Delta x)$$

$$= \cos(\xi\Delta x) - ar\cdot i\sin(\xi\Delta x),$$

since $\cosh(i\cdot\alpha) = \cos(\alpha)$ and $\sinh(i\cdot\alpha) = i\sin(\alpha)$.

Considering $|\rho|^2$, we have

$$|\rho|^2 = [\cos(\xi\Delta x) - i\,ar\sin(\xi\Delta x)][\cos(\xi\Delta x) + i\,ar\sin(\xi\Delta x)]$$

$$= \cos^2(\xi\Delta x) - iar\cancel{\cos(\xi\Delta x)\sin(\xi\Delta x)} + iar\cancel{\cos(\xi\Delta x)\sin(\xi\Delta x)} + a^2r^2\sin^2(\xi\Delta x)$$

$$= \cos^2(\xi\Delta x) + a^2r^2\sin^2(\xi\Delta x)$$

$\Rightarrow$ We need $|\rho| \leq 1 + C\Delta t$ for small Δt and all ξ.

$$|\rho|^2 = \cos^2(\xi\Delta x) + a^2r^2\sin^2(\xi\Delta x)$$

$$\Rightarrow |\rho| = \sqrt{\cos^2(\xi\Delta x) + (ar)^2\sin^2(\xi\Delta x)}$$

$$|\rho| = \sqrt{1 - \sin^2(\xi\Delta x) + (ar)^2\sin^2(\xi\Delta x)}$$

$$= \sqrt{1 + (a^2r^2 - 1)\sin^2(\xi\Delta x)}$$

Case 1:

Let $ar > 1$. Then, $a^2r^2 > 1$.

Set $k = a^2r^2 - 1 > 0$, giving us

$$|\rho| = \sqrt{1 + k\sin^2(\xi\Delta x)} > 1.$$

Case 2:

Now, let $ar < -1$, $\Rightarrow$ $a^2r^2 > 1$ again,

giving $|\rho| > 1$.

Case 3:

Hence, let $|ar| \leq 1$, $\Rightarrow$ $a^2r^2 \leq 1$

$$\Rightarrow \quad k := a^2r^2 - 1 \leq 0$$

$$\Rightarrow \quad |\rho| = \sqrt{1 + k\sin^2(\xi\Delta x)} \leq 1.$$

Thus, $|\rho| \leq 1$ requires $|ar| \leq 1$.

Therefore, the Lax-Friedrichs method is stable for $|ar| \leq 1$.

Part 2:

Ask:

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

We WTS $e_i^n(\Delta x, \Delta t) = \Delta t \cdot O(\Delta t + \Delta x)$.

We know

$$e_i^n(\Delta x, \Delta t) = u(x_i, t_{n+1}) - L_{num}(u(x_i, t_n))$$

$$\Rightarrow \quad e_i^n(\Delta x, \Delta t) = u(x_i, t_{n+1}) - \frac{1}{2}\left[u(x_{i+1}, t_n) + u(x_{i-1}, t_n)\right] + \frac{ar}{2}\left[u(x_{i+1}, t_n) - u(x_{i-1}, t_n)\right]$$

$$+\frac{ar}{2}\left[u(x_{i+1},t_n) - u(x_{i-1},t_n)\right]$$

$$\Rightarrow \quad e_i^n(\Delta x, \Delta t) = \overbrace{u(x_i, t_n+\Delta t)}^{①} - \frac{1}{2}\left[\overbrace{u(x_i+\Delta x, t_n) + u(x_i-\Delta x, t_n)}^{②}\right]$$

$$+\frac{ar}{2}\left[\underbrace{u(x_i+\Delta x, t_n) - u(x_i-\Delta x, t_n)}_{③}\right]$$

Expanding ①, ②, ③ around (x_i, t_n): Let u denote $u\big|_{(x_i,t_n)}$.

①: $u(x_i, t_n+\Delta t) = u + u_t\Delta t + \frac{1}{2}u_{tt}\Delta t^2 + \mathcal{O}(\Delta t^3)$

②: $u(x_i+\Delta x, t_n) + u(x_i-\Delta x, t_n)$

$$= u + \cancel{u_x\Delta x} + \frac{1}{2}u_{xx}\Delta x^2 + \mathcal{O}(\Delta x^3) + u - \cancel{u_x\Delta x} + \frac{1}{2}u_{xx}\Delta x + \mathcal{O}(\Delta x^4)$$

$$= 2u + u_{xx}\Delta x^2 + \mathcal{O}(\Delta x^4)$$

odd powers of Δx cancel.

③: $u(x_i+\Delta x, t_n) - u(x_i-\Delta x, t_n)$

$$= \cancel{u} + u_x\Delta x + \cancel{\frac{1}{2}u_{xx}\Delta x^2} + \mathcal{O}(\Delta x^3) - \left[\cancel{u} - u_x\Delta x + \cancel{\frac{1}{2}u_{xx}\Delta x^2} + \mathcal{O}(\Delta x^3)\right]$$

$$= 2u_x\Delta x + \mathcal{O}(\Delta x^3)$$

Plugging in these expansions...

$$e_i^n(\Delta x, \Delta t) = \overbrace{u(x_i, t_n+\Delta t)}^{①} - \frac{1}{2}\left[\overbrace{u(x_i+\Delta x, t_n) + u(x_i-\Delta x, t_n)}^{②}\right] + \frac{ar}{2}\left[\underbrace{u(x_i+\Delta x, t_n) - u(x_i-\Delta x, t_n)}_{③}\right]$$

$$\Rightarrow \quad e_i^n(\Delta x, \Delta t) = \cancel{u} + u_t\Delta t + \frac{1}{2}u_{tt}\Delta t^2 + \mathcal{O}(\Delta t^3)$$

$$\cancel{-\frac{1}{2}}\left[\cancel{2u} + u_{xx}\Delta x^2 + \mathcal{O}(\Delta x^4)\right] + \frac{a}{\cancel{2}}\cdot\frac{\Delta t}{\Delta x}\left[\cancel{2}u_x\Delta x + \mathcal{O}(\Delta x^3)\right]$$

$$= (u_t + au_x)\Delta t + \frac{1}{2}u_{tt}\Delta t^2 - \frac{1}{2}u_{xx}\Delta x^2$$

$$= (u_t + a u_x)\Delta t + \frac{1}{2} u_{tt}\Delta t^2 - \frac{1}{2} u_{xx}\Delta x^2$$
$$+ \mathcal{O}(\Delta t^3) + \mathcal{O}(\Delta x^4) + \mathcal{O}(\Delta x^2)\Delta t$$

From the model equation for hyperbolic PDEs:

$$u_t + a u_x = 0$$

$$\Rightarrow \quad e_i^n(\Delta x, \Delta t) = \frac{1}{2}\left[u_{tt}\Delta t^2 - u_{xx}\Delta x^2\right] + \Delta t\,\mathcal{O}(\Delta t^2 + \Delta x^2) + \mathcal{O}(\Delta x^4).$$

Part 3: We have that

$$e_i^n = \frac{1}{2} u_{tt}\Delta t^2 - \frac{1}{2} u_{xx}\Delta x^2 + \Delta t\,\mathcal{O}(\Delta t^2 + \Delta x^2) + \mathcal{O}(\Delta x^4)$$

$$\Rightarrow \frac{e_i^n}{\Delta t} = \frac{1}{2} u_{tt}\Delta t - \frac{1}{2} u_{xx}\frac{\Delta x^2}{\Delta t} + \mathcal{O}(\Delta t^2 + \Delta x^2) + \frac{1}{\Delta t}\mathcal{O}(\Delta x^4)$$

For fixed $r = \frac{\Delta t}{\Delta x}$:

$r = \frac{\Delta t}{\Delta x}$: $\Rightarrow \Delta t = \Delta x \cdot r \Rightarrow \Delta t^2 = \Delta x^2 r^2$. Thus, using $u_{tt} = a^2 u_{xx}$

$$\frac{e_i^n}{\Delta t} = \frac{a^2 r}{2} u_{xx}\Delta x - \frac{1}{2} u_{xx}\Delta x r^{-1} + \mathcal{O}(r^2\Delta x^2 + \Delta x^2) + \frac{1}{r}\mathcal{O}(\Delta x^3)$$

Hence,

$$\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = 0.$$

For fixed $c = \frac{\Delta t}{\Delta x^2}$, we have:

$$c = \frac{\Delta t}{\Delta x^2} \Rightarrow \Delta t = \Delta x^2 c \Rightarrow \Delta t^2 = c^2 \Delta x^4.$$

Thus,

$$\frac{e_i^n}{\Delta t} = \frac{1}{2} u_{tt} \Delta t - \frac{1}{2} u_{xx} \frac{\Delta x^2}{\Delta t} + O(\Delta t^2 + \Delta x^2) + \frac{1}{\Delta t} O(\Delta x^4)$$

$$\Rightarrow \frac{e_i^n}{\Delta t} = \frac{a^2 c}{2} u_{xx} \Delta x^2 - \frac{1}{2c} u_{xx} + O(c^2 \Delta x^4) + \frac{1}{c} O(\Delta x^2)$$

We can see that

$$\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = -\frac{1}{2c} \cdot u_{xx} \neq 0.$$

P2

Problem 2 (Theoretical)

Carry out von Neumann stability analysis on each of the methods below

i) the BTCS method for the general case ($a > 0$ or $a < 0$)

$$u_i^{n+1} = u_i^n - \frac{ar}{2}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

ii) the implicit upwind method for the case of $a > 0$

$$u_i^{n+1} = u_i^n - ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

Hint: Examine $|1/\rho|^2$.

i) We need

$$L_{num}(u^{n+1}) = L_{num}(u^n)$$

[illegible]

$$\Rightarrow \quad u_i^{n+1} + \frac{ar}{2}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) = u_i^{n}$$

We try $u_i^n = \rho^n e^{\sqrt{-1}\,\xi i \Delta x}$:

$$\rho^{n+1} e^{-\sqrt{-1}\,\xi i \Delta x} + \frac{ar}{2}\left[\rho^{n+1} e^{\sqrt{-1}\,\xi (i+1)\Delta x} - \rho^{n+1} e^{\sqrt{-1}\,\xi (i-1)\Delta x}\right] = \rho^{n} e^{\sqrt{-1}\,\xi i \Delta x}$$

Dividing by $\rho^n e^{\sqrt{-1}\,\xi i \Delta x}$:

$$\rho + \frac{ar}{2}\left[\rho e^{\sqrt{-1}\,\xi \Delta x} - \rho e^{-\sqrt{-1}\,\xi \Delta x}\right] = 1$$

$$\Rightarrow \quad \rho\left[1 + ar\sinh(\sqrt{-1}\,\xi\Delta x)\right] = 1$$

Using $\sinh(i\alpha) = i\sin(\alpha)$:

$$\rho\left[1 + i\, ar\sin(\xi\Delta x)\right] = 1$$

Hence,

$$\rho = \frac{1}{1 + i\, ar\sin(\xi\Delta x)} \qquad |z| = \sqrt{a^2 + b^2}$$

$$\Rightarrow \quad |\rho| = \frac{1}{\sqrt{1 + a^2 r^2 \sin^2(\xi\Delta x)}}$$

Notice that $\sqrt{1 + a^2 r^2 \sin^2(\xi\Delta x)} > 0$.

Hence $|\rho| \leq 1$ regardless of ar.

Hence, $|\rho| \leq 1$ regardless of ar.

$\therefore$ BTCS is unconditionally stable.

ii) We need

$$L_{num}(u^{n+1}) = L_{num}(u^n),$$

$$\Rightarrow \quad u_i^{n+1} + ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right) = u_i^n.$$

We try $u_i^n = \rho^n e^{\sqrt{-1}\xi i \Delta x}$:

$$\rho^{n+1} e^{\sqrt{-1}\xi i\Delta x} + ar\left[\rho^{n+1} e^{\sqrt{-1}\xi i \Delta x} - \rho^{n+1} e^{\sqrt{-1}\xi(i-1)\Delta x}\right] = \rho^n e^{\sqrt{-1}\xi i \Delta x}$$

Dividing by $\rho^n e^{\sqrt{-1}\xi i \Delta x}$:

$$\rho + ar\left[\rho - \rho e^{-\sqrt{-1}\xi\Delta x}\right] = 1$$

$$\Rightarrow \quad \rho\left[1 + ar\left(1 - e^{-\sqrt{-1}\xi\Delta x}\right)\right] = 1$$

We have $e^{-\sqrt{-1}\xi\Delta x} = \cos(\xi\Delta x) - i\sin(\xi\Delta x)$

$$\Rightarrow \quad \rho = \frac{1}{1 + ar[1 - \cos(\xi\Delta x) + i\sin(\xi\Delta x)]}$$

$$\Rightarrow \quad \rho = \frac{1}{1 + ar - ar\cos(\xi\Delta x) + iar\sin(\xi\Delta x)}$$

Hence,

$$|\rho| = \frac{1}{\sqrt{[1 + ar - ar\cos(\xi\Delta x)]^2 + a^2r^2\sin^2(\xi\Delta x)}}$$

Working in the denominator:

$$(1 + ar - ar\cos(\xi\Delta x))^2 = 1 + ar - ar\cos(\xi\Delta x) + ar + a^2r^2 - a^2r^2\cos(\xi\Delta x) - ar\cos(\xi\Delta x) - a^2r^2\cos(\xi\Delta x) + a^2r^2\cos^2(\xi\Delta x)$$

$$= \overbrace{1 + 2ar + a^2r^2}^{(1+ar)^2} - 2ar\cos(\xi\Delta x) - 2a^2r^2\cos(\xi\Delta x) + a^2r^2\cos^2(\xi\Delta x)$$

Hence, considering the hint, we have

$$\frac{1}{|\rho|^2} = 1 + 2ar + a^2r^2 - 2ar\cos(\xi\Delta x) - 2a^2r^2\cos(\xi\Delta x) + a^2r^2\cancel{\cos^2(\xi\Delta x)} + a^2r^2\cancel{\sin^2(\xi\Delta x)}$$

$$= 1 + 2ar + 2a^2r^2 - 2ar\cos(\xi\Delta x) - 2a^2r^2\cos(\xi\Delta x)$$

$$= 1 + 2ar(1 + ar - \cos(\xi\Delta x) - ar\cos(\xi\Delta x))$$

$$= 1 + 2ar\left[1 + ar - (1+ar)\cos(\xi\Delta x)\right]$$

$$= 1 + 2ar\left[(1+ar)(1-\cos(\xi\Delta x))\right]$$

$$= 1 + \left[2(ar + a^2r^2)(1-\cos(\xi\Delta x))\right]$$

In Upwind method, $a > 0$.

$$\Rightarrow \frac{1}{|\rho|} \geq 1, \quad \text{hence } |\rho| \leq 1.$$

$\therefore$ the implicit upwind method is unconditionally stable.