HW3 Report

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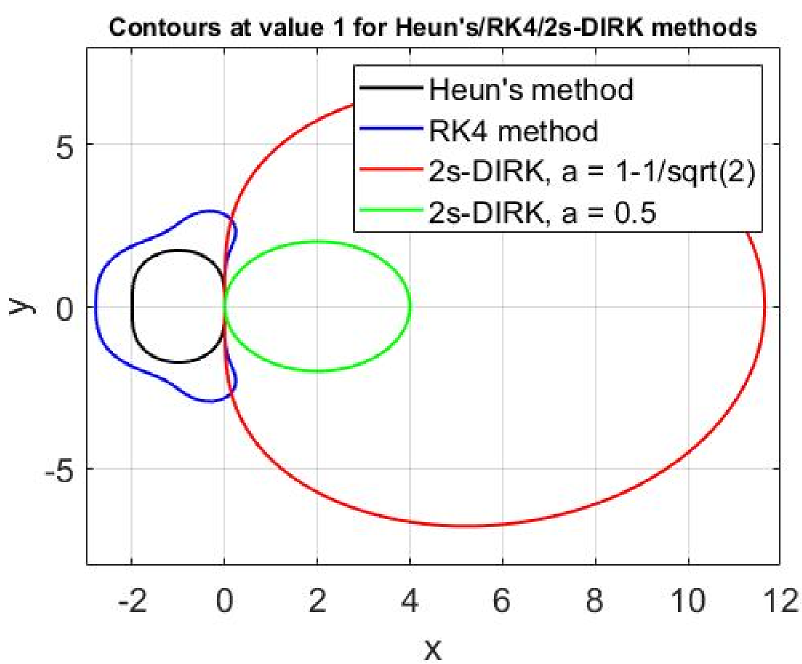
Problem (1-3)

Problems 1-3 are appended to the bottom of this PDF in a handwritten format.

Problem 4

In this problem, we are asked to plot the region of absolute stability for Heun's method, RK4, 2s-DIRK with $\alpha = 1 - 1/\sqrt{2}$, and 2s-DIRK with $\alpha = 0.5$. To do this, we used the provided sample code as guidance to learn how to plot contours of a function of x and y. Below are the contours of each method, at the constant value of 1.

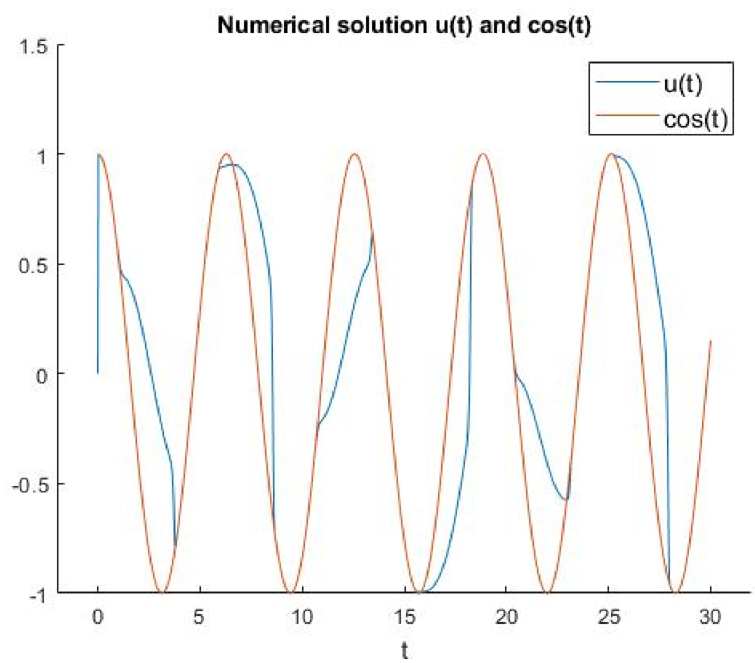


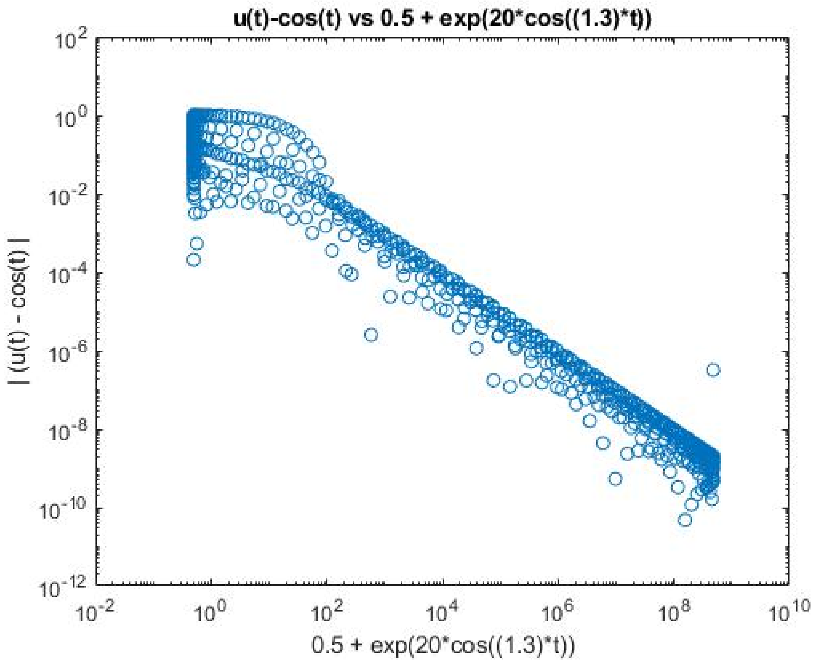
Problem 5

This problem requires the implementation of the 2s-DIRK method for $\alpha = 1 - 1/\sqrt{2}$. To correctly implement this, the provided 3s-DIRK method was followed as guidance. We solve the following IVP to $T = 30$,

$$\begin{cases} u' = -\Big(0.5 + \exp\big(20\cos(1.3t)\big)\Big)\sinh\big(u - \cos(t)\big) \\ u(0) = 0 \end{cases}.$$

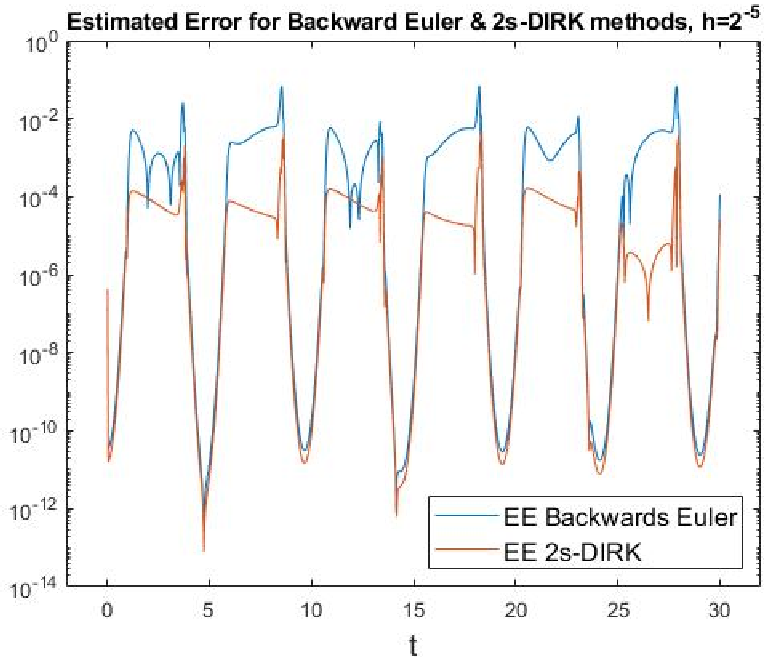
Once we have obtained the numerical solution for $u(t)$, we plot $u(t)$ vs t for time-step size $h = 2^{-5}$. On the same plot, we compare $u(t)$ to $\cos(t)$, noticing they share many of the same values, but don't always follow each other closely. We also provide a log-scale plot of $|u(t) - \cos(t)|$ vs $-\Big(0.5 + \exp\big(20\cos(1.3t)\big)\Big)$, in which we can see that for values along the horizontal axis greater than 10^6, we have $|u(t) - \cos(t)| < 10^{-6}$. In other words, $u(t)$ follows $\cos(t)$ fairly closely.



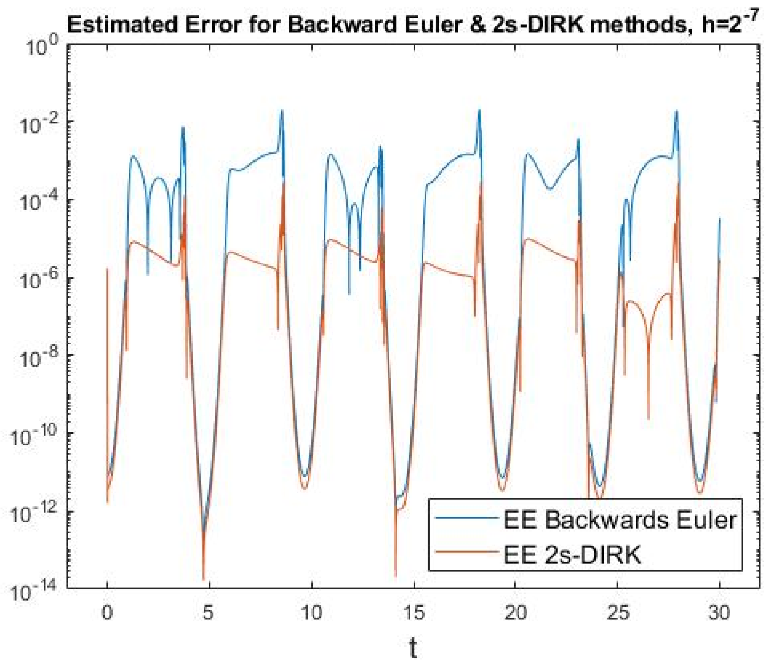


Problem 6

This problem is a continuation of the IVP from Problem 5. We implement the backward Euler to solve the above IVP along side the 2s-DIRK method for $\alpha = 1 - 1/\sqrt{2}$. We solve the IVP with both methods up to $T = 30$, using time step sizes of $h = \frac{1}{2^3}, \frac{1}{2^4}, \ldots, \frac{1}{2^8}$. Then, for each numerical method, we carry out numerical error estimation. Below, we plot figures of the estimated error vs t, for time step sizes of $h = 2^{-5}$, and $h = 2^{-7}$ respectively.



Estimated Error for Backward Euler & 2s-DIRK methods, h=2^-5
EE Backwards Euler
EE 2s-DIRK
t



Estimated Error for Backward Euler & 2s-DIRK methods, h=2^-7
EE Backwards Euler
EE 2s-DIRK
t

Theory

Wednesday, April 14, 2021 2:01 PM

P1)

Problem 1 (Theoretical)

Part 1: Derive the stability function $\phi(z)$ for each of the two RK methods below

- Predictor-corrector method (Heun's method)
- Classic 4-th order Runge-Kutta method (RK4)

Hint: Check your expression of $\phi(z)$ with the theorem we studied.

Theorem: If an RK method is p-th order accurate, then it must satisfies

$\phi(z) = e^z + O(|z|^{p+1})$ for any complex z with small $|z|$

Part 2: Study the zero-stability for each of the two LMMs below

- $u_{n+2} - 2u_{n+1} + u_n = hf(u_{n+1}, t_{n+1}) - hf(u_n, t_n)$
- $u_{n+2} - u_n = h\left[\frac{1}{3}f(u_{n+2}, t_{n+2}) + \frac{4}{3}f(u_{n+1}, t_{n+1}) + \frac{1}{3}f(u_n, t_n)\right]$

Part 1

We apply Heun's method and RK4 to the first order ODE $u' = \gamma u$.

- Heun's Method stability function

Heun's method has Butcher Tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \xRightarrow{\text{RK form}} \begin{cases} k_1 = hf(u_n, t_n) \\ k_2 = hf(u_n + k_1, t_n + h) \\ u_{n+1} = u_n + \frac{1}{2}k_1 + \frac{1}{2}k_2 \end{cases} \quad \star$$

Our goal is to find $\phi(z)$ such that

$u_{n+1} = \phi(z) \cdot u_n$, $z = h\gamma$.

We have $f(u) = \gamma u$, so $\star$ gives

$k_1 = h \cdot f(u_n, t_n) = h\gamma \cdot u_n = z u_n$

$k_2 = hf(u_n + k_1, t_n + h) = h\gamma(u_n + zu_n)$

$$K_1 = hf(u_n, t_n) = \dots$$

$$K_2 = h \cdot f(u_n + z \cdot u_n,\ t_n + h) = h\gamma(u_n + zu_n)$$
$$= z(u_n + zu_n).$$

Thus,

$$u_{n+1} = u_n + \frac{z}{2}u_n + \frac{z}{2}(u_n + zu_n)$$
$$= u_n + \frac{z}{2}u_n + \frac{z}{2}u_n + \frac{z^2}{2}u_n$$
$$= \frac{z^2}{2}u_n + zu_n + u_n$$
$$u_{n+1} = \left(\frac{z^2}{2} + z + 1\right)u_n \Rightarrow \boxed{\phi(z) = \frac{z^2}{2} + z + 1 \quad \text{for Heun's.}}$$

- Runge-Kutta 4 stability function

The Butcher Tableau for RK4 is

$$\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}$$

$$\Rightarrow \quad k_1 = h\cdot f(u_n, t_n) = h\gamma u_n = z\cdot u_n$$

$$k_2 = h\cdot f(u_n + \tfrac{1}{2}k_1, t_n + \tfrac{h}{2}) = h\gamma\left(u_n + \tfrac{z}{2}\cdot u_n\right) = z\left(u_n + \tfrac{z}{2}\cdot u_n\right)$$

$$k_3 = hf\left(u_n + \tfrac{1}{2}k_2, t_n + \tfrac{h}{2}\right) = z\left(u_n + \tfrac{z}{2}\left(u_n + \tfrac{z}{2}\cdot u_n\right)\right)$$

$$k_4 = hf(u_n + k_3, t_n + h) = z\left(u_n + \left(z\left(u_n + \tfrac{z}{2}\left(u_n + \tfrac{z}{2}\cdot u_n\right)\right)\right)\right)$$

$$\Rightarrow \quad k_1 = z\cdot u_n$$

$$k_2 = \left(\frac{z^2}{2} + z\right)u_n$$

$$k_3 = \left(\frac{z^3}{4} + \frac{z^2}{2} + z\right)u_n$$

$$k_4 = \left(\frac{z^4}{4} + \frac{z^3}{2} + z^2 + z\right)u_n$$

and

$$u_{n+1} = u_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4$$

$$\Rightarrow u_{n+1} = \left(\frac{z^4}{24} + \frac{z^3}{12} + \frac{z^2}{6} + \frac{z}{6} + \frac{z^3}{12} + \frac{z^2}{6} + \frac{z}{3} + \frac{z^2}{6} + \frac{z}{3} + \frac{z}{6} + 1\right)u_n$$

$$= \left(\frac{z^4}{24} + \frac{z^3}{6} + \frac{z^2}{2} + z + 1\right)u_n$$

Hence,

$$\Phi(z) = \frac{z^4}{24} + \frac{z^3}{6} + \frac{z^2}{2} + z + 1 \quad \text{for RK4.}$$

Part 2

We have

- $U_{n+2} - 2U_{n+1} + U_n = h \cdot f(U_{n+1}, t_{n+1}) - hf(U_n, t_n)$

$$\Rightarrow \rho(\xi) = \xi^2 - 2\xi + 1, \quad \sigma(\xi) = \xi - 1.$$

Factoring $\rho(\xi) = 0$, we have

$$(\xi - 1)(\xi - 1) = 0 \Rightarrow \xi = 1, \text{ multiplicity } 2.$$

Hence, this is $\boxed{\text{not zero-stable,}}$ since an LMM is stable iff $\rho(\xi)$ satisfies the root condition, which it fails since

$|\xi| = 1$ (to pass, it would need $|\xi| < 1$).

- The second method is

$$U_{n+2} - U_n = h\left[\frac{1}{3} f(U_{n+2}, t_{n+2}) + \frac{4}{3} f(U_{n+1}, t_{n+1}) + \frac{1}{3} f(U_n, t_n)\right]$$

$$\Rightarrow \rho(\xi) = \xi^2 - 1, \quad \sigma(\xi) = \frac{1}{3}\xi^2 + \frac{4}{3}\xi + \frac{1}{3}$$

Roots are $\xi_1 = 1$, and $\xi_2 = -1$.

since the simple roots

roots are $\xi_1 = 1$, and $\xi_2 = $ ...

Hence, the method is zero-stable since the simple roots $|\xi_1| = 1$ and $|\xi_2| = 1$, satisfying the root condition for simple roots.

P2)

Problem 2 (Theoretical)

Consider the Runge-Kutta method described by Butcher tableau

Butcher tableau:
$$\begin{array}{c|cc} \alpha & \alpha & 0 \\ 1 & 1-\alpha & \alpha \\ \hline & 1-\alpha & \alpha \end{array}$$

where $\alpha > 0$. This is called a 2s-DIRK (2-Stage Diagonally Implicit Runge-Kutta) method.

The matrix A of a DIRK method is lower triangular so $\{k_1, k_2, k_3, \ldots\}$ can be solved sequentially. The first row of A gives an equation on k_1 without involving $\{k_2, k_3, \ldots\}$. The second row of A gives an equation on k_2 without involving $\{k_3, \ldots\}$ where k_1 is already known. This is in contrast to a fully implicit Runge-Kutta where $\{k_1, k_2, k_3, \ldots\}$ has to be solved simultaneously in a joint system.

Part 1: Show that method is second order for $\alpha = 1 - 1/\sqrt{2}$.

Hint: check the internal consistency condition, the condition for first order and the additional condition for second order.

Part 2: Apply the 2s-DIRK to solving $u' = \gamma u$.

Derive the expressions for k_1, k_2 and the stability function $\phi(z)$.

$$k_1 = \frac{z}{1-\alpha z} u_n, \qquad k_2 = \frac{(1-\alpha z)z + (1-\alpha)z^2}{(1-\alpha z)^2} u_n$$

$$\phi(z) = \frac{1+(1-2\alpha)z}{(1-\alpha z)^2}$$

Part 3: Suppose the 2s-DIRK is A-stable for $\alpha = 1 - 1/\sqrt{2}$ (see Problem 4 below).

Show that it satisfies the second condition of L-stability.

Part 1 We have Butcher Tableau

$$\begin{array}{c|c} c^T & A \\ \hline & b \end{array} = \begin{array}{c|cc} \alpha & \alpha & 0 \\ 1 & 1-\alpha & \alpha \\ \hline & 1-\alpha & \alpha \end{array}.$$

for $\alpha = 1 - \frac{1}{\sqrt{2}}$

$1-\alpha \quad \alpha$ $\sqrt{2}$

To show the method above is second order, we can simply check the internal consistency condition, plus the first and second order consistency conditions.

Internal Consistency Condition:

$$c_i \stackrel{?}{=} \sum_{j=1}^{p} a_{ij} \rightsquigarrow \begin{array}{l} \alpha = \alpha + 0 \quad \checkmark \\ 1 = 1-\alpha+\alpha \quad \checkmark \end{array}$$

"Does row sum of i^{th} row add to i^{th} element in c?"

$\boxed{\checkmark}$

First Order Condition:

$$\sum_{i=1}^{p} b_i \stackrel{?}{=} 1 \rightsquigarrow 1-\alpha+\alpha = 1 \quad \checkmark$$

$\boxed{\checkmark}$

Second Order Condition (with $c_i = \sum_{j=1}^{p} a_{ij}$):

$$\sum_{i=1}^{p} b_i c_i \stackrel{?}{=} \frac{1}{2} \rightsquigarrow (1-\alpha)\alpha + \alpha$$

For $\alpha = 1 - \frac{1}{\sqrt{2}}$, we have

$$\left(1-\left(1-\frac{1}{\sqrt{2}}\right)\right)\left(1-\frac{1}{\sqrt{2}}\right) + \left(1-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{2} + 1 - \frac{1}{\sqrt{2}} = \frac{1}{2}. \qquad \square$$

Thus, the method described by the above Butcher Tableau is 2^{nd} order when $\alpha = 1 - \frac{1}{\sqrt{2}}$.

<u>Part 2</u> The 2s-DIRK method is

$$\begin{cases} K_1 = hf(u_n + \alpha K_1, t_n + \alpha h) \\ K_2 = hf(u_n + (1-\alpha)K_1 + \alpha K_2, t_n + h) \\ u_{n+1} = u_n + (1-\alpha)K_1 + \alpha K_2 \end{cases}$$

Applying the method to $u' = \gamma u$, we have

$$\begin{cases} K_1 = h\gamma(u_n + \alpha K_1) = z u_n + \alpha z K_1 \\ K_2 = h\gamma(u_n + (1-\alpha)K_1 + \alpha K_2) = z u_n + (1-\alpha) z K_1 + \alpha z K_2 \end{cases}$$

Thus, $K_1 - \alpha z K_1 = z u_n$

$$\Rightarrow \quad K_1 = \frac{z}{\quad} u_n$$

$$\Rightarrow \quad k_1 = \frac{z}{1-\alpha z} u_n$$

and

$$k_2 = z u_n + (1-\alpha) z \left(\frac{z}{1-\alpha z}\right) u_n + \alpha z k_2$$

$$\Rightarrow k_2 = \frac{z + (1-\alpha) z \left(\frac{z}{1-\alpha z}\right)}{1-\alpha z} u_n$$

$$= \frac{z(1-\alpha z) + (1-\alpha) z^2}{(1-\alpha z)^2} u_n \qquad \checkmark$$

$$u_{n+1} = u_n + (1-\alpha) k_1 + \alpha k_2$$

$$= u_n + \frac{(1-\alpha) z}{1-\alpha z} u_n + \alpha \left(\frac{z(1-\alpha z) + (1-\alpha) z^2}{(1-\alpha z)^2} \right) u_n$$

$$= \left(1 + \frac{(1-\alpha) z}{1-\alpha z} + \alpha \left(\frac{z(1-\alpha) + (1-\alpha) z^2}{(1-\alpha z)^2} \right) \right) u_n$$

$$\Rightarrow \phi(z) = 1 + \frac{(1-\alpha) z}{1-\alpha z} + \alpha \left(\frac{z(1-\alpha) + (1-\alpha) z^2}{(1-\alpha z)^2} \right)$$

$$= \frac{(1-\alpha z)^2 + (1-\alpha)(1-\alpha z) z + z\alpha(1-\alpha z) + z^2 \alpha (1-\alpha)}{(1-\alpha z)^2}$$

$z - \alpha z^2 - \alpha z + \alpha^2 z^2$

$$= \frac{1 - 2\alpha z + \cancel{\alpha^2 z^2} + z - \cancel{\alpha z^2} - \cancel{\alpha z} + \cancel{\alpha^2 z^2} + \cancel{\alpha z} - \cancel{\alpha^2 z^2} + \cancel{z^2 \alpha} - \cancel{z^2 \alpha^2}}{(1-\alpha z)^2}$$

$$= \frac{1 - 2\alpha z + z}{(1-\alpha z)^2}$$

$$\boxed{\phi(z) = \frac{1 + (1-2\alpha) z}{(1-\alpha z)^2}}$$

<u>Part 3</u>: Here, we suppose that the 2s-DIRK method is A-stable for $\alpha = 1 - \frac{1}{\sqrt{2}}$.

To be L-stable, we need

$$\lim_{z \to \infty} \phi(z) = 0.$$

$$\phi(z) = \frac{1 + (1-2\alpha) z}{(1-\alpha z)^2} = \frac{1}{(1-\alpha z)^2} + \frac{(1-2\alpha) z}{(1-\alpha z)^2}$$

$\Rightarrow$

$\Rightarrow$ Indeed $\lim_{z \to \infty} \phi(z) = 0$, since

$$\lim_{z\to\infty} \frac{1 \to 0}{\cancel{(1-\alpha z)^2}} + \lim_{z\to\infty} \cancel{\frac{\overbrace{(1-2\alpha)z}^{0}}{\underbrace{(1-\alpha z)^2}_{\text{deg } 2}}} \Big\} \text{deg } 1.$$

Problem 3 (Theoretical)

Consider the implicit 2-step method below

$$u_{n+2} - u_n = h\left[\frac{1}{3} f(u_{n+2}, t_{n+2}) + \frac{4}{3} f(u_{n+1}, t_{n+1}) + \frac{1}{3} f(u_n, t_n)\right]$$

P3)

Part 1: Use Taylor expansion to show $e_n(h) = O(h^5)$.

Hint: Expand everything around t_{n+1}.

Part 2: The stability polynomial is

$$\pi(\xi, z) = (\xi^2 - 1) - z\left(\frac{1}{3}\xi^2 + \frac{4}{3}\xi + \frac{1}{3}\right)$$

Consider $z = -\varepsilon$ with small $\varepsilon > 0$. We examine the two roots of $\pi(\xi, -\varepsilon)$.

Show that the two roots $\xi_1(\varepsilon)$ and $\xi_2(\varepsilon)$ have the expansions

$$\xi_1(\varepsilon) = 1 - \varepsilon + O(\varepsilon^2), \qquad \xi_2(\varepsilon) = -\left(1 + \frac{\varepsilon}{3}\right) + O(\varepsilon^2)$$

Therefore, $z = -\varepsilon$ is NOT in the region of absolute stability.

Part 1:

We have

$$U_{n+2} - U_n = h\left[\frac{1}{3} f(U_{n+2}, t_{n+2}) + \frac{4}{3} f(U_{n+1}, t_{n+1}) + \frac{1}{3} f(U_n, t_n)\right].$$

$\Rightarrow$

7/

$$e_n(h) = u(t_n) - u(t_{n+1}+h) + \frac{h}{3} f(u(t_{n+1}+h), t_{n+1}+h) + \frac{4}{3} h f(u(t_{n+1}), t_{n+1})$$
$$+ \frac{h}{3} f(u(t_n), t_n)$$

Then, since $f(u(t_n)) = u'(t_n)$, we can write

$$e_n(h) = u(t_{n+1}-h) - u(t_{n+1}+h) + \frac{h}{3} u'(t_{n+1}+h) + \frac{4}{3} h u'(t_{n+1})$$
$$+ \frac{h}{3} u'(t_{n+1}-h),$$ where $t_{n+2} = t_{n+1}+h$, $t_n = t_{n+1}-h$.

Expanding everything around t_{n+1}, we have

$$u(t_{n+1}-h) = u(t_{n+1}) - h u'(t_{n+1}) + \frac{h^2}{2!} u''(t_{n+1}) - \frac{h^3}{3!} u^{(3)}(t_{n+1})$$
$$+ \frac{h^4}{4!} u^{(4)}(t_{n+1}) + O(h^5)$$

$$u(t_{n+1}+h) = u(t_{n+1}) + h u'(t_{n+1}) + \frac{h^2}{2!} u''(t_{n+1}) + \frac{h^3}{3!} u^{(3)}(t_{n+1})$$
$$+ \frac{h^4}{4!} u^{(4)}(t_{n+1}) + O(h^5)$$

$$u'(t_{n+1}+h) = u'(t_{n+1}) + h u''(t_{n+1}) + \frac{h^2}{2!} u^{(3)}(t_{n+1})$$
$$+ \frac{h^3}{3!} u^{(4)}(t_{n+1}) + O(h^4)$$

$$u'(t_{n+1}-h) = u'(t_{n+1}) - h u''(t_{n+1}) + \frac{h^2}{2!} u^{(3)}(t_{n+1})$$
$$- \frac{h^3}{3!} u^{(4)}(t_{n+1}) + O(h^4)$$

Substituting in to $e_n(h)$,

$$e_n(h) = -2hu'(t_{n+1}) - \frac{2h^3}{3!}u^{(3)}(t_{n+1}) + \frac{h}{3}\Big[2u'(t_{n+1})$$
$$+ h^2 u^{(3)}(t_{n+1}) + O(h^4)\Big] + \frac{4}{3}hu'(t_{n+1})$$

$$= -2hu'(t_{n+1}) - \frac{h^3}{3}\cancel{u^{(3)}}(t_{n+1}) + \frac{2}{3}hu'(t_{n+1})$$
$$+ \frac{h^3}{\cancel{3}}\cancel{u^{(3)}}(t_{n+1}) + \frac{4}{3}hu'(t_{n+1}) + O(h^5)$$

$$= -2h\cancel{u'(t_{n+1})} + \frac{6}{3}\cancel{u'(t_{n+1})} + O(h^5)$$

$$\Rightarrow \quad e_n(h) = O(h^5).$$

Part 2: We're given that the stability polynomial is

$$\pi(\xi, z) = (\xi^2 - 1) - z\left(\frac{1}{3}\xi^2 + \frac{4}{3}\xi + \frac{1}{3}\right)$$

$\Rightarrow$ for small $\varepsilon > 0$,

$$\pi(\xi, -\varepsilon) = \xi^2 - 1 + \frac{\varepsilon}{3}\xi^2 + \frac{4}{3}\varepsilon\xi + \frac{\varepsilon}{3}$$
$$= \left(1 + \frac{\varepsilon}{3}\right)\xi^2 + \frac{4}{3}\varepsilon\xi + \left(\frac{\varepsilon}{3} - 1\right)$$

We apply the quadratic formula to find the roots of $\pi(\xi, -\varepsilon)$:

$$\xi = \frac{-\frac{4}{3}\varepsilon \pm \sqrt{\frac{16}{9}\varepsilon^2 - 4\left(1+\frac{\varepsilon}{3}\right)\left(\frac{\varepsilon}{3}-1\right)}}{2\left(1+\frac{\varepsilon}{3}\right)}$$

$$= \frac{-\frac{4}{3}\varepsilon \pm \sqrt{\frac{16}{9}\varepsilon^2 - \frac{4\varepsilon^2}{9} + 4}}{2\left(1+\frac{\varepsilon}{3}\right)}$$

$$= \frac{-4\varepsilon \pm \sqrt{16\varepsilon^2 - 4\varepsilon^2 + 36}}{6+2\varepsilon}$$

$$= \frac{-4\varepsilon \pm 2\sqrt{3(\varepsilon^2+3)}}{6+2\varepsilon}$$

$$\Rightarrow \xi_1 = \frac{-4\varepsilon + 6\sqrt{1+\frac{\varepsilon^2}{3}}}{6\left(1+\frac{\varepsilon}{3}\right)}$$

$$(1+\eta)^\alpha = 1 + \alpha\eta + \frac{\alpha(\alpha-1)}{2!}\eta^2 + \ldots$$

$$\xi_2 = \frac{-4\varepsilon - 6\sqrt{1+\frac{\varepsilon^2}{3}}}{6\left(1+\frac{\varepsilon}{3}\right)}$$

denom: $1 + \alpha\left(\frac{\varepsilon}{3}\right) + \frac{\alpha(\alpha-1)}{2!}\left(\frac{\varepsilon}{3}\right)^2$

numer: $1 + \alpha\left(\frac{\varepsilon^2}{3}\right) + \ldots$

Expanding the denominator around $\varepsilon = 0$,

Using $(1+\eta)^\alpha = 1 + \alpha\eta + \frac{\alpha(\alpha-1)}{2!}\eta^2 + \ldots$, we have

Denominator

$$1 + \frac{\varepsilon}{3} = 1 - \frac{\varepsilon}{3} + \frac{\varepsilon^2}{9} + \ldots$$

Numerator

$$\left(1+\frac{\varepsilon^2}{3}\right)^{1/2} = 1 + \frac{\varepsilon^2}{6} + \ldots$$

Thus, $\dfrac{-4\varepsilon + 6\sqrt{1+\frac{\varepsilon^2}{3}}}{6\left(1+\frac{\varepsilon}{3}\right)}$

$$\xi_1 = -\frac{2}{3}\varepsilon\left(1-\frac{\varepsilon}{3}+\frac{\varepsilon^2}{9}+\ldots\right) + \left(1+\frac{\varepsilon^2}{6}+\ldots\right)\left(1-\frac{\varepsilon}{3}+\frac{\varepsilon^2}{9}+\ldots\right)$$

$$= -\frac{2}{3}\varepsilon + O(\varepsilon^2) + 1 - \frac{\varepsilon}{3} + O(\varepsilon^2)$$

$$= 1 - \varepsilon + O(\varepsilon^2) \quad \checkmark$$

$$\xi_2 = -\frac{2}{3}\varepsilon\left(1-\frac{\varepsilon}{3}+\frac{\varepsilon^2}{9}+\ldots\right) - \left(1+\frac{\varepsilon^2}{6}+\ldots\right)\left(1-\frac{\varepsilon}{3}+\frac{\varepsilon^2}{9}+\ldots\right)$$

$$= -\frac{2}{3}\varepsilon + O(\varepsilon^2) - 1 + \frac{\varepsilon}{3} + O(\varepsilon^2)$$

$$= -\left(1+\frac{\varepsilon}{3}\right) + O(\varepsilon^2) \quad \checkmark$$