HW5 Report

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Problem 1

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Problem 2

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Problem 3

We implement the Forward Time Central Space numerical method (FTCS) to solve the IBVP of the heat equation:

$$\begin{cases} u_t = u_{xx}, \quad x \in (0,2), \ \ t > 0 \\ u(x,0) = f(x), \quad x \in (0,2) \\ u(0,t) = g_L(t), \quad u(2,t) = g_R(t) \end{cases}$$

with $g_L(t) = 1, \ \ g_R(t) = 0$, and

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases} .$$

We solve the IBVP up to $T = 0.2$ with $\Delta x = 0.01$ and two varying timesteps,

$$\Delta t = \frac{(\Delta x)^2}{2(0.99)} \quad \text{and} \quad \Delta t = \frac{(\Delta x)^2}{2(1.01)}$$

The first timestep is slightly above the stability threshold, while the second is slightly below. We can see this in the figures provided below. At each Δt, we plot our numerical solution vs x at $t = 0.01, 0.04, 0.09, 0.2$. Note: I wasn't sure how to make the unstable solution plot look informative.

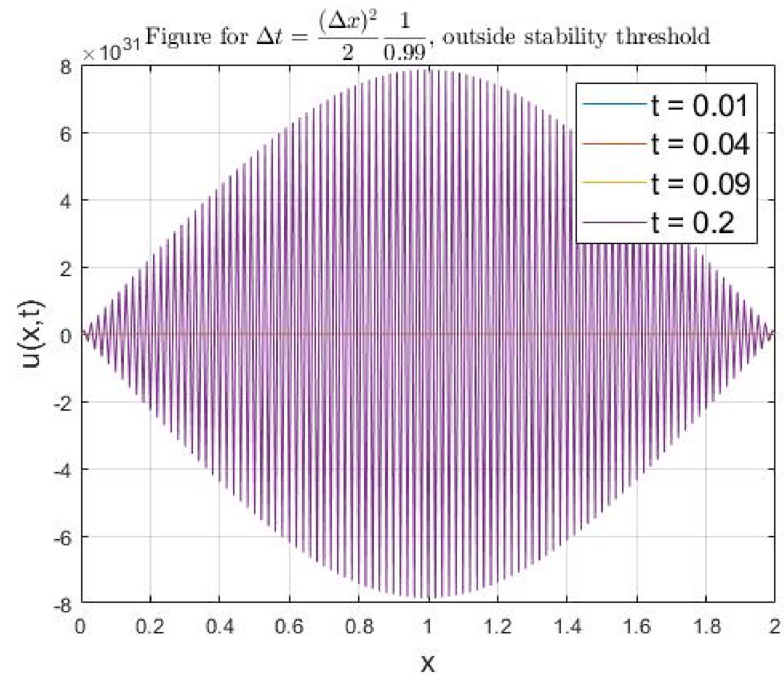


Figure for $\Delta t = \frac{(\Delta x)^2}{2}\frac{1}{0.99}$, outside stability threshold
$\times 10^{31}$
u(x,t)
x
t = 0.01
t = 0.04
t = 0.09
t = 0.2

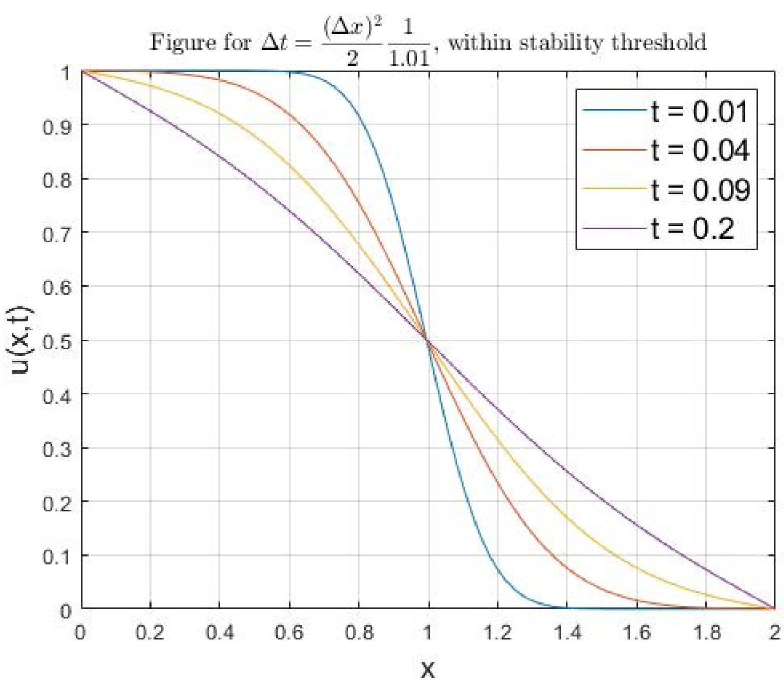
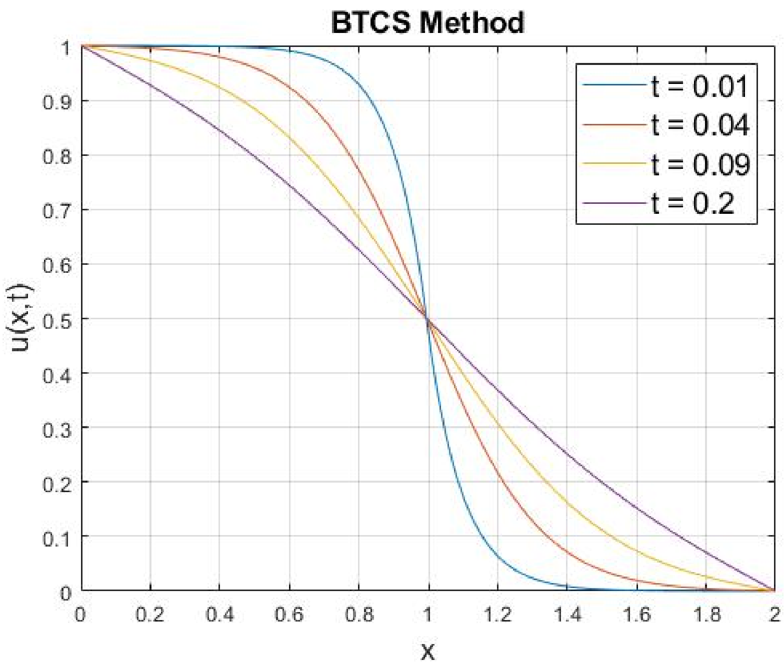
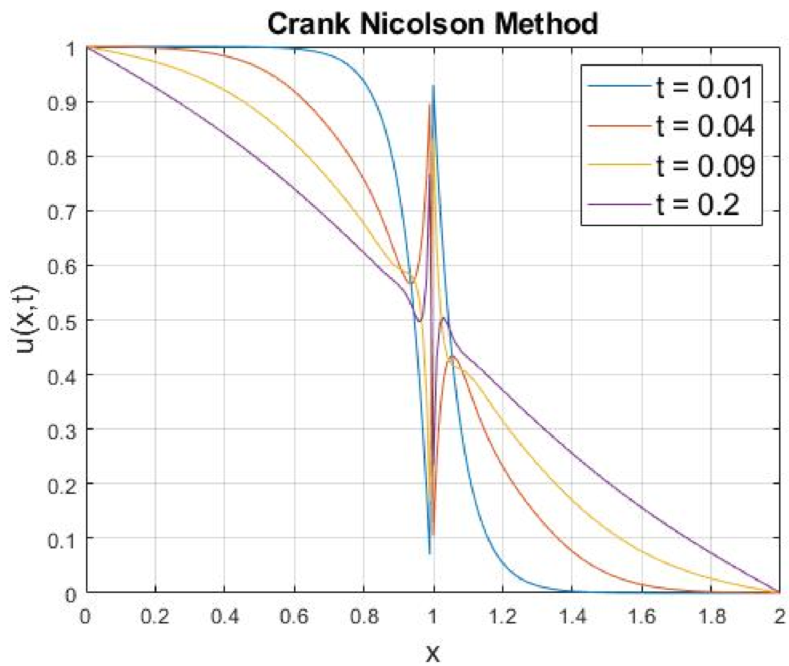


Figure for $\Delta t = \frac{(\Delta x)^2}{2}\frac{1}{1.01}$, within stability threshold
u(x,t)
x
t = 0.01
t = 0.04
t = 0.09
t = 0.2

Problem 4

In this problem, we implement the Backward Time Central Space method (BTCS) as well as the Crank-Nicolson method to solve the same IBVP in problem 3. We solve up to time $T = 0.2$, with $\Delta x = \Delta t = 0.01$. To do this, we implement method of lines (MOL) and then utilize the backward Euler method (resulting in BTCS) and then the trapezoidal method (resulting in Crank-Nicolson). For both methods, we plot our numerical solution $u(x,t)$ vs x at times $t = 0.01, 0.04, 0.09, 0.2.$. Notice that the Crank-Nicolson method is quite unstable around $x = 1$ due to the discontinuity in $f(x)$.



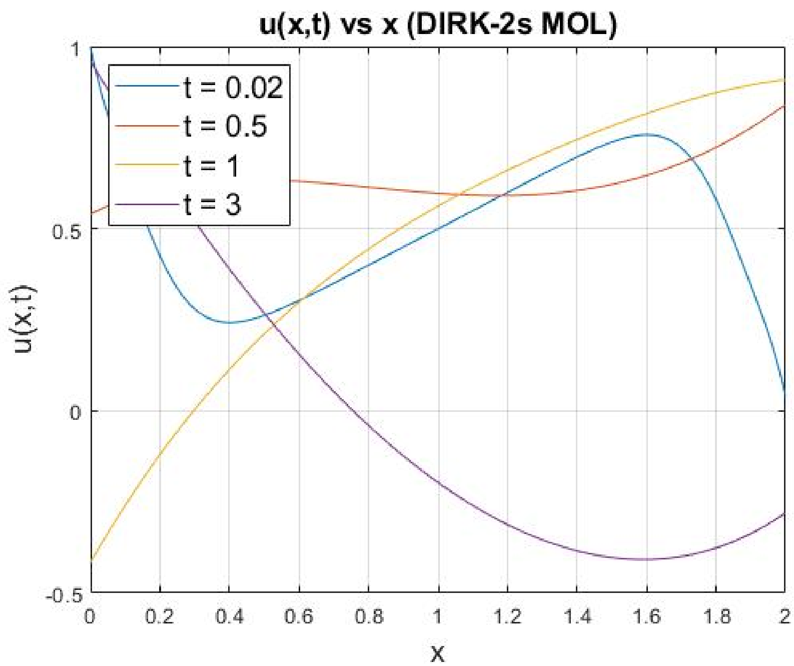


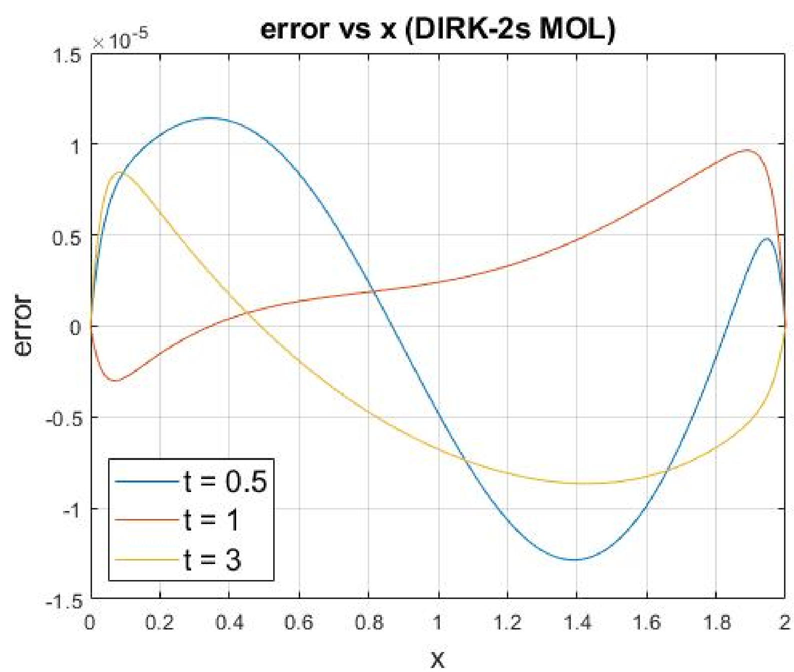
Problem 5

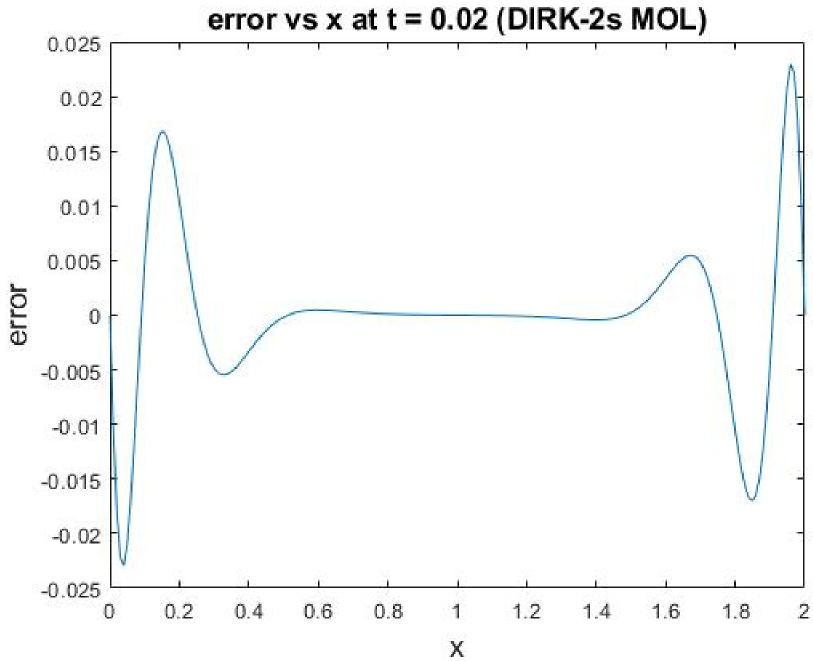
In this problem, we continue further with the IBVP from problem 3. We use new initial and boundary value conditions,

$$f(x) = 0.5x, \quad g_L(t) = \cos(2t), \quad g_R(t) = \sin(2t).$$

Utilizing our MOL discretization, we then implement the 2s-DIRK method (with $\alpha = 1 - 1/\sqrt{2}$) and solve the IBVP to $T = 3$ with $\Delta x = \Delta t = 0.01$. We plot the numerical solution $u(x,t)$ vs x at time $t = 0.02, 0.5, 1, 3$ in a single figure. Then, repeating our calculation with $\Delta x = 0.01$, $\Delta t = 0.01/2$, we use our two numerical solutions to perform error estimation on the time discretization. We then plot our estimated error vs x at time $t = 0.5, 1, 3$ in one figure. We also provide a separate plot of the estimated error specifically at $t = 0.02$.


u(x,t) vs x (DIRK-2s MOL)
t = 0.02
t = 0.5
t = 1
t = 3
u(x,t)
x
1
0.5
0
-0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


error vs x (DIRK-2s MOL)
$\times 10^{-5}$
t = 0.5
t = 1
t = 3
error
x
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



Problem 6

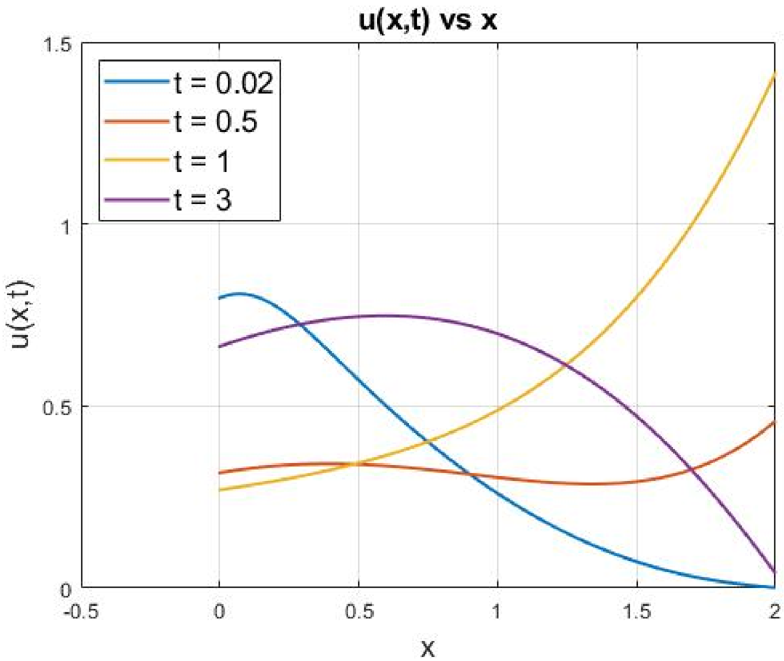
In this problem, we consider a new IBVP with a boundary condition that models radiation heat loss,

$$\begin{cases} u_t = u_{xx}, \quad x \in (0, L), \;\; t > 0 \\ u(x, 0) = p(x), \quad x \in (0, L) \\ u_x(0, t) - \alpha u(0, t) = 0., \quad u(L, t) = q(t) \end{cases}$$

with $L = 2,\ \alpha = 0.4,\ p(x) = (1 - 0.5x)^2,\ q(t) = 2\sin^2(t)$. As provided in the problem description, we discretize the boundary conditions above, finding that

$$u_0^n = \frac{(2 - \alpha\Delta x)}{(2 + \alpha\Delta x)} u_1^n \quad \text{and} \quad u_N^n = q(n\Delta t).$$

We use the FTCS method to solve the IBVP numerically to time $T = 3$, with $N = 200$ and $\Delta t = 4 \times 10^{-5}$. We plot the solution $u(x, t)$ vs x at time $t = 0.02, 0.5, 1, 3$ in a single figure (pictured below).


u(x,t) vs x
t = 0.02
t = 0.5
t = 1
t = 3
u(x,t)
x
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2

Theory

Thursday, April 29, 2021 4:57 PM

Problem 1 (Theoretical)

Part 1: Carry out von Neumann stability analysis to show that the BTCS method is unconditionally stable

P1

Part 2: Carry out Taylor expansions to show that the local truncation error of the Crank-Nicolson method is

$$e_i^n(\Delta x, \Delta t) = \Delta t\, O\left((\Delta t)^2 + (\Delta x)^2\right)$$

In the final expression, be sure to convert r back to $\Delta t/(\Delta x)^2$.

Part 1: The BCTS method is

$$U_i^{n+1} = U_i^n + r\left(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}\right), \quad r = \frac{\Delta t}{(\Delta x)^2}$$

Rearranging to write

$L_{num}(U^{n+1}) = L_{num}(U^n)$, we have

$$U_i^{n+1} - r\left(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}\right) = U_i^n$$

Substitute in $U_i^n = \rho^n \cdot e^{\sqrt{-1}\xi i \Delta x}$,

$$\Rightarrow \rho^{n+1} \cdot e^{\sqrt{-1}\xi i \Delta x} - r\left(\rho^{n+1} \cdot e^{\sqrt{-1}\xi (i+1)\Delta x} - 2\rho^{n+1} \cdot e^{\sqrt{-1}\xi i \Delta x} + \rho^{n+1} \cdot e^{\sqrt{-1}\xi (i-1)\Delta x}\right)$$

$$= \rho^n e^{\sqrt{-1}\xi i \Delta x}$$

Divide both sides by $\rho^n e^{\sqrt{-1}\xi i \Delta x}$:

$$\rho - r\left(\rho \cdot e^{\sqrt{-1}\xi\Delta x} - 2\rho + \rho \cdot e^{-\sqrt{-1}\xi\Delta x}\right) = 1$$

$$\Rightarrow \rho - r\rho\left(e^{\sqrt{-1}\xi\Delta x} + e^{-\sqrt{-1}\xi\Delta x} - 2\right) = 1$$

$$\Rightarrow \rho - r\rho\left(2\cosh(\sqrt{-1}\xi\Delta x) - 2\right) = 1$$

$$\Rightarrow \rho\left[1 - r\left(2\cosh(\sqrt{-1}\xi\Delta x) - 2\right)\right] = 1$$

Hence

$$\rho = \frac{1}{1 - r(2\cosh(\sqrt{-1}\xi\Delta x) - 2)} = \frac{1}{1 - 2r(\cos(\xi\Delta x) - 1)}, \text{ since } \cosh(x) = \cos(\sqrt{-1}x)$$

For unconditional stability, we need that for all $r > 0$,

$|\rho| \leq 1 + C\Delta t$ for all ξ and small Δt.

This is immediate since using

$$\cos(\alpha) - 1 = -2\sin^2\left(\frac{\alpha}{2}\right), \quad \text{we have}$$

$$|\rho| = \left|\frac{1}{1 + 4r\sin^2\left(\frac{\xi\Delta x}{2}\right)}\right| = \boxed{\frac{1}{\underbrace{1 + 4r\sin^2\left(\frac{\xi\Delta x}{2}\right)}_{\text{always positive and } \geq 1 \ \forall\, r>0.}} \leq 1}$$

Thus, BTCS is unconditionally stable.

Part 2:

The Crank-Nicolson method is

$$u_i^{n+1} = u_i^n + \frac{r}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + \frac{r}{2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right), \quad r = \frac{\Delta t}{(\Delta x)^2}$$

WTS $e_i^n(\Delta x, \Delta t) = \Delta t \cdot O\left((\Delta t)^2 + (\Delta x)^2\right)$.

We have

$$e_i^n(\Delta x, \Delta t) = u(x_i, t_{n+1}) - L_{num}\left(u(x_i, t_n)\right)$$

$\Rightarrow$ For Crank-Nicolson:

$$e_i^n(\Delta x, \Delta t) = \underbrace{u(x_i, t_{n+1}) - u(x_i, t_n)}_{1} - \frac{r}{2}\Big[\overbrace{u(x_{i+1}, t_n) - 2u(x_i, t_n) + u(x_{i-1}, t_n)}^{2} + \underbrace{u(x_{i+1}, t_{n+1}) - 2u(x_i, t_{n+1}) + u(x_{i-1}, t_{n+1})}_{3}\Big]$$

Expanding everything around (x_i, t_n) (recall $t_{n+1} = t_n + \Delta t$, $x_{i+1} = x_i + \Delta x$)

For 1, we have

$$\Rightarrow \quad u(x_i, t_{n+1}) - u(x_i, t_n) = u_t \Delta t + \frac{1}{2} u_{tt} (\Delta t)^2 + O\left((\Delta t)^3\right)$$

$$\left\{ u(t_n + h) = u(t_n) + u'(t_n)h + u''(t_n)\frac{h^2}{2!} + \dots \right.$$

And for 2, we have

$$u(x_{i+1}, t_n) - 2u(x_i, t_n) + u(x_{i-1}, t_n) = u(x_i + \Delta x, t_n) - 2u(x_i, t_n) + u(x_i - \Delta x, t_n)$$

$$= +\cancel{u(x_i,t_n)} + \cancel{u_x \Delta x} + \frac{1}{2!}u_{xx}(\Delta x)^2 + \mathcal{O}((\Delta x)^3) - 2\cancel{u(x_i,t_n)}$$

$$+ \cancel{u(x_i,t_n)} - \cancel{u_x \Delta x} + \frac{1}{2!}u_{xx}(\Delta x)^2 + \mathcal{O}((\Delta x)^3)$$

$$= u_{xx}(\Delta x)^2\Big|_{(x_i,t_n)} + \mathcal{O}((\Delta x)^4)$$

For 3, from 2 we have:

$u(x_{i+1},t_{n+1}) - 2u(x_i,t_{n+1}) + u(x_{i-1},t_{n+1}) =$

$$(\Delta x)^2\left[u_{xx}\Big|_{(x_i,t_n)} + \Delta t\, u_{xxt}\Big|_{(x_i,t_n)} + \mathcal{O}(\Delta t)^2\right] + \mathcal{O}((\Delta x)^4)$$

Thus, substituting these expansions into $e_i^n(\Delta x,\Delta t)$:

$$e_i^n(\Delta x,\Delta t) = u_t\Delta t + \frac{1}{2}u_{tt}(\Delta t)^2 + \mathcal{O}((\Delta t)^3) - \frac{\Delta t}{2\cancel{(\Delta x)^2}}\Big[\cancel{(\Delta x)^2}\big(u_{xx} + u_{xx} + \Delta t\, u_{xxt}$$
$$+ \mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta x)^4)\Big]$$

$$= u_t\Delta t + \frac{1}{2}u_{tt}(\Delta t)^2 - u_{xx}\Delta t - \frac{1}{2}u_{xxt}(\Delta t)^2 + \mathcal{O}((\Delta t)^3) + \Delta t\,\mathcal{O}((\Delta x^2))$$

Using $u_t = u_{xx}$:

$$\Rightarrow e_i^n(\Delta x,\Delta t) = \cancel{u_t\Delta t} + \frac{1}{2}\cancel{u_{tt}}(\Delta t)^2 - \cancel{u_t\Delta t} - \frac{1}{2}\cancel{u_{tt}(\Delta t)^2}$$

$$+$$

$$\Rightarrow e_i^n(\Delta x,\Delta t) = \Delta t\left(\mathcal{O}((\Delta t)^2 + (\Delta x)^2)\right).$$

P2

Problem 2 (Theoretical)

Consider matrix

$$A = \frac{1}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}_{(N-1)\times(N-1)}, \qquad \Delta x = \frac{1}{N}$$

Part 1: Verify that the set below are eigenvalues and eigenvectors of matrix A.

$$\left.\begin{aligned} \lambda^{(k)} &= \frac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x)-1\right) \\ w^{(k)} &= \left\{\sin(k\pi i\Delta x),\ \ i=1,2,\ldots,N-1\right\} \end{aligned}\right\},\quad k=1,2,\ldots,N-1$$

Part 2: Verify that

$$\frac{1}{(\Delta x)^2}\left(\cos(k\pi(i-1)\Delta x)-2\cos(k\pi i\Delta x)+\cos(k\pi(i+1)\Delta x)\right)$$
$$= \frac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x)-1\right)\cdot\cos(k\pi i\Delta x), \qquad k=1,2,\ldots,N-1$$

Part 3: Explain why $u^{(k)} = \left\{\cos(k\pi i\Delta x),\ \ i=1,2,\ldots,N-1\right\}$ is NOT an eigenvector of A.

Hint: What boundary conditions did we use in defining matrix A?

Part 1:

We want to show that

$$Aw^{(k)} = \lambda^{(k)} w^{(k)} \quad \text{for } k=1,\ldots,N-1.$$

Since $D_x^2\{u\}_i = \dfrac{u_{i+1}-2u_i+u_{i-1}}{(\Delta x)^2}$ corresponds to the i^{th} component of the resulting vector Au, we can simply show

$$\frac{w_{i+1}^{(k)} - 2w_i^{(k)} + w_{i-1}^{(k)}}{(\Delta x)^2} = \lambda^{(k)} w_i^{(k)}, \qquad w_0^{(k)}=0,\ w_N^{(k)}=0$$

$w^{(k)} = \left\{\sin(k\pi i\Delta x),\ \ i=1,2,\ldots,N-1\right\}$ $\quad \lambda^{(k)} = \dfrac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x)-1\right)$

$$\Rightarrow \quad \frac{\sin(k\pi(i+1)\Delta x) - 2\sin(k\pi i\Delta x) + \sin(k\pi(i-1)\Delta x)}{\Delta x^2} \overset{?}{=} \frac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x) - 1\right) \cdot \sin(k\pi i\Delta x)$$

Using angle sum formulas:

$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$, the numerator becomes:

$$\sin(k\pi i\Delta x)\cos(k\pi\Delta x) + \cancel{\sin(k\pi\Delta x)\cos(k\pi i\Delta x)} - 2\sin(k\pi i\Delta x)$$
$$+ \sin(k\pi i\Delta x)\cos(k\pi\Delta x) - \cancel{\sin(k\pi\Delta x)\cos(k\pi i\Delta x)}$$

$\Rightarrow$ The entire expression is

$$\frac{2\sin(k\pi i\Delta x)\cos(k\pi\Delta x) - 2\sin(k\pi i\Delta x)}{\Delta x^2}$$

$$= \frac{2}{(\Delta x)^2}\left[\cos(k\pi\Delta x) - 1\right]\sin(k\pi i\Delta x)$$

$\Delta x = \frac{1}{N}$

At $i=0$: $\quad \sin(0) = 0 \Rightarrow w_0^{(k)} = 0$

$i=N$: $\quad \sin(k\pi N \cdot \frac{1}{N}) = 0 \Rightarrow w_N^{(k)} = 0$

Hence,

$$\frac{w_{i+1}^{(k)} - 2w_i^{(k)} + w_{i-1}^{(k)}}{(\Delta x)^2} = \lambda^{(k)} w_i^{(k)}, \qquad w_0^{(k)} = 0,\ w_N^{(k)} = 0$$

$$\Rightarrow \qquad A w^{(k)} = \lambda^{(k)} w_i^{(k)} \qquad \text{for } k = 1, \dots, N-1.$$

Part 2: We're asked to verify

$$\frac{1}{(\Delta x)^2}\Big(\cos(k\pi(i-1)\Delta x) - 2\cos(k\pi i \Delta x) + \cos(k\pi(i+1)\Delta x)\Big)$$

$$= \frac{2}{(\Delta x)^2}\left(\cos(k\pi \Delta x) - 1\right)\cdot \cos(k\pi i \Delta x), \quad k = 1, 2, \dots, N-1.$$

The LHS is:

$$\frac{1}{(\Delta x)^2}\Big(\cos(k\pi i\Delta x - k\pi\Delta x) - 2\cos(k\pi i \Delta x) + \cos(k\pi i\Delta x + k\pi \Delta x)\Big)$$

using $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$,

$$\Rightarrow$$

$$\begin{aligned} LHS = \frac{1}{(\Delta x)^2}\Big(&\cos(k\pi i \Delta x)\cos(k\pi\Delta x) + \cancel{\sin(k\pi i\Delta x)\sin(k\pi\Delta x)} \\ &+ \cos(k\pi i\Delta x)\cos(k\pi\Delta x) - \cancel{\sin(k\pi i \Delta x)\sin(k\pi\Delta x)} \\ &- 2\cos(k\pi i \Delta x)\Big) \end{aligned}$$

$$= \frac{2}{(\Delta x)^2}\Big[\cos(k\pi\Delta x) - 1\Big]\cos(k\pi i\Delta x) = RHS \ \checkmark.$$

Part 3

This satisfies the 2^{nd} order difference. however it

This satisfies the 2^{nd} order difference, however it fails to satisfy the values $w_0^{(k)} = 0 = w_N^{(k)}$;

$$w_0^{(k)} = \cos(0) = 1 \neq 0$$

$$w_N^{(k)} = \cos(k\pi) = \pm 1 \neq 0$$