

Problem 1 (Theoretical)

Part 1: Carry out von Neumann stability analysis to show that the BTCS method is unconditionally stable

Part 2: Carry out Taylor expansions to show that the local truncation error of the Crank-Nicolson method is

$$e_i^n(\Delta x, \Delta t) = \Delta t O((\Delta t)^2 + (\Delta x)^2)$$

In the final expression, be sure to convert r back to $\Delta t/(\Delta x)^2$.

P1

Part 1: The BTCS method is

$$U_i^{n+1} = U_i^n + r(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}), \quad r = \frac{\Delta t}{(\Delta x)^2}$$

Rearranging to write

$$L_{num}(U^{n+1}) = L_{num}(U^n), \quad \text{we have}$$

$$U_i^{n+1} - r(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) = U_i^n$$

$$\text{Substitute in } U_i^n = \rho \cdot e^{\sqrt{-1}\xi_i \Delta x},$$

$$\Rightarrow \rho \cdot e^{\sqrt{-1}\xi_i \Delta x} - r \left(\rho \cdot e^{\sqrt{-1}\xi_{(i+1)} \Delta x} - 2\rho \cdot e^{\sqrt{-1}\xi_i \Delta x} + \rho \cdot e^{\sqrt{-1}\xi_{(i-1)} \Delta x} \right)$$

$$= \rho \cdot e^{\sqrt{-1}\xi_i \Delta x}$$

Divide both sides by $\rho \cdot e^{\sqrt{-1}\xi_i \Delta x}$:

$$\rho - r \left(\rho \cdot e^{\sqrt{F_1} \xi \Delta x} - 2\rho + \rho \cdot e^{-\sqrt{F_1} \xi \Delta x} \right) = 1$$

$$\Rightarrow \rho - r\rho \left(e^{\sqrt{F_1} \xi \Delta x} + e^{-\sqrt{F_1} \xi \Delta x} - 2 \right) = 1$$

$$\Rightarrow \rho - r\rho (2\cosh(\sqrt{F_1} \xi \Delta x) - 2) = 1$$

$$\Rightarrow \rho [1 - r(2\cosh(\sqrt{F_1} \xi \Delta x) - 2)] = 1$$

Hence

$$\rho = \frac{1}{1 - r(2\cosh(\sqrt{F_1} \xi \Delta x) - 2)} = \frac{1}{1 - 2r(\cos(\xi \Delta x) - 1)}, \text{ since } \cosh(x) = \cos(\sqrt{F_1} x)$$

For unconditional stability, we need that for all $r > 0$,

$$|\rho| \leq 1 + C\Delta t \quad \text{for all } \xi \text{ and small } \Delta t.$$

This is immediate since using

$$\cos(\alpha) - 1 = -2\sin^2\left(\frac{\alpha}{2}\right), \quad \text{we have}$$

$$|\rho| = \left| \frac{1}{1 + 4r\sin^2\left(\frac{\xi \Delta x}{2}\right)} \right| = \boxed{\frac{1}{1 + 4r\sin^2\left(\frac{\xi \Delta x}{2}\right)}} \leq 1$$

always positive
and $\geq 1 \forall r > 0$.

Thus, BTCS is unconditionally stable.

Part 2:

The Crank-Nicolson method is

$$U_i^{n+1} = U_i^n + \frac{r}{2} (U_{i+1}^n - 2U_i^n + U_{i-1}^n) + \frac{r}{2} (U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}), \quad r = \frac{\Delta t}{(\Delta x)^2}$$

WTS $e_i^n(\Delta x, \Delta t) = \Delta t \cdot O((\Delta t)^2 + (\Delta x)^2)$.

We have

$$e_i^n(\Delta x, \Delta t) = U(x_i, t_{n+1}) - L_{num}(U(x_i, t_n))$$

\Rightarrow For Crank-Nicolson:

$$\begin{aligned} e_i^n(\Delta x, \Delta t) &= U(x_i, t_{n+1}) - U(x_i, t_n) - \frac{r}{2} \left[\underbrace{U(x_{i+1}, t_n) - 2U(x_i, t_n) + U(x_{i-1}, t_n)}_{(1)} \right. \\ &\quad \left. + \underbrace{U(x_{i+1}, t_{n+1}) - 2U(x_i, t_{n+1}) + U(x_{i-1}, t_{n+1})}_{(3)} \right] \end{aligned}$$

(2)

Expanding everything around (x_i, t_n) (recall $t_{n+1} = t_n + \Delta t$)
 $x_{i+1} = x_i + \Delta x$

For (1), we have

$$\Rightarrow U(x_i, t_{n+1}) - U(x_i, t_n) = U_t \Delta t + \frac{1}{2} U_{tt}(\Delta t)^2 + \Theta((\Delta t)^3)$$

$(U(t_n+h) = U(t_n) + U'(t_n)h + U''(t_n)\frac{h^2}{2!} + \dots)$

And for (2), we have

$$U(x_{i+1}, t_n) - 2U(x_i, t_n) + U(x_{i-1}, t_n) = U(x_i + \Delta x, t_n) - 2U(x_i, t_n) + U(x_i - \Delta x, t_n)$$

$$\begin{aligned}
&= +U(x_i, t_n) + U_x \Delta x + \frac{1}{2!} U_{xx}(\Delta x)^2 + \mathcal{O}((\Delta x)^3) - 2U(x_i, t_n) \\
&\quad + U(x_i, t_n) - U_x \Delta x + \frac{1}{2!} U_{xx}(\Delta x)^2 + \mathcal{O}((\Delta x)^3) \\
&= U_{xx}(\Delta x)^2 \Big|_{(x_i, t_n)} + \mathcal{O}((\Delta x)^4)
\end{aligned}$$

For ③, from ② we have:

$$\begin{aligned}
U(x_{i+1}, t_{n+1}) - 2U(x_i, t_{n+1}) + U(x_{i-1}, t_{n+1}) &= \\
(\Delta x)^2 \left[U_{xx} \Big|_{(x_i, t_n)} + \Delta t U_{xxt} \Big|_{(x_i, t_n)} + \mathcal{O}((\Delta t)^2) \right] + \mathcal{O}((\Delta x)^4)
\end{aligned}$$

Thus, substituting these expansions into $e_i^n(\Delta x, \Delta t)$:

$$\begin{aligned}
e_i^n(\Delta x, \Delta t) &= U_+ \Delta t + \frac{1}{2} U_{++} (\Delta t)^2 + \mathcal{O}((\Delta t)^3) - \frac{\Delta t}{2(\Delta x)^2} \left[(\Delta x)^2 (U_{xx} + U_{xx} + \Delta t U_{xxt} \right. \\
&\quad \left. + \mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta x)^4)) \right]
\end{aligned}$$

$$= U_+ \Delta t + \frac{1}{2} U_{++} (\Delta t)^2 - U_{xx} \Delta t - \frac{1}{2} U_{xxt} (\Delta t)^2 + \mathcal{O}((\Delta t)^3) + \Delta t \mathcal{O}((\Delta x^2))$$

Using $U_+ = U_{xx}$:

$$\begin{aligned}
\Rightarrow e_i^n(\Delta x, \Delta t) &= U_x \Delta t + \frac{1}{2} U_{++} (\Delta t)^2 - U_x \Delta t - \frac{1}{2} U_{++} (\Delta t)^2 \\
&\quad +
\end{aligned}$$

$$\Rightarrow e_i^n(\Delta x, \Delta t) = \Delta t (\mathcal{O}((\Delta t)^2 + (\Delta x)^2)).$$

Problem 2 (Theoretical)

Consider matrix

$$A = \frac{1}{(\Delta x)^2} \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}_{(N-1) \times (N-1)}, \quad \Delta x = \frac{1}{N}$$

P2

Part 1: Verify that the set below are eigenvalues and eigenvectors of matrix A.

$$\left. \begin{aligned} \lambda^{(k)} &= \frac{2}{(\Delta x)^2} (\cos(k\pi\Delta x) - 1) \\ w^{(k)} &= \{\sin(k\pi i \Delta x), \quad i=1,2,\dots,N-1\} \end{aligned} \right\}, \quad k=1,2,\dots,N-1$$

Part 2: Verify that

$$\begin{aligned} &\frac{1}{(\Delta x)^2} (\cos(k\pi(i-1)\Delta x) - 2\cos(k\pi i \Delta x) + \cos(k\pi(i+1)\Delta x)) \\ &= \frac{2}{(\Delta x)^2} (\cos(k\pi\Delta x) - 1) \cdot \cos(k\pi i \Delta x), \quad k=1,2,\dots,N-1 \end{aligned}$$

Part 3: Explain why $u^{(k)} = \{\cos(k\pi i \Delta x), \quad i=1,2,\dots,N-1\}$ is NOT an eigenvector of A.

Hint: What boundary conditions did we use in defining matrix A?

Part 1:

We want to show that

$$Aw^{(k)} = \lambda^{(k)} w^{(k)} \quad \text{for } k=1,\dots,N-1.$$

Since $D_x^2 \{u\}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$ corresponds to the

i^{th} component of the resulting vector Au , we can simply show

$$\underbrace{\frac{w_{i+1}^{(k)} - 2w_i^{(k)} + w_{i-1}^{(k)}}{(\Delta x)^2}}_{= \lambda^{(k)} w_i^{(k)}} = \lambda^{(k)} w_i^{(k)}, \quad w_0^{(k)} = 0, \quad w_N^{(k)} = 0$$

$$w^{(k)} = \{\sin(k\pi i \Delta x), \quad i=1,2,\dots,N-1\} \quad \lambda^{(k)} = \frac{2}{(\Delta x)^2} (\cos(k\pi\Delta x) - 1)$$

$$\Rightarrow \frac{\sin(k\pi(i+1)\Delta x) - 2\sin(k\pi i \Delta x) + \sin(k\pi(i-1)\Delta x)}{\Delta x^2} \stackrel{?}{=} \frac{2}{(\Delta x)^2} (\cos(k\pi \Delta x) - 1) \cdot \sin(k\pi i \Delta x)$$

Using angle sum formulas:

$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$, the numerator becomes:

$$\begin{aligned} & \sin(k\pi i \Delta x) \cos(k\pi \Delta x) + \sin(k\pi \Delta x) \cos(k\pi i \Delta x) - 2\sin(k\pi i \Delta x) \\ & + \sin(k\pi i \Delta x) \cos(k\pi \Delta x) - \sin(k\pi \Delta x) \cos(k\pi i \Delta x) \end{aligned}$$

\Rightarrow The entire expression is

$$\frac{2\sin(k\pi i \Delta x) \cos(k\pi \Delta x) - 2\sin(k\pi i \Delta x)}{\Delta x^2}$$

$$= \frac{2}{(\Delta x)^2} [\cos(k\pi \Delta x) - 1] \sin(k\pi i \Delta x) \quad \text{with } \Delta x = \frac{1}{N}$$

$$\text{At } i=0: \quad \sin(0) = 0 \Rightarrow w_0^{(k)} = 0$$

$$\text{At } i=N: \quad \sin(k\pi N \cdot \frac{1}{N}) = 0 \Rightarrow w_N^{(k)} = 0$$

Hence,

$$\frac{w_{i+1}^{(k)} - 2w_i^{(k)} + w_{i-1}^{(k)}}{(\Delta x)^2} = \lambda^{(k)} w_i^{(k)}, \quad w_0^{(k)} = 0, \quad w_N^{(k)} = 0$$

$$\Rightarrow A w^{(k)} = \lambda^{(k)} w_i^{(k)} \quad \text{for } k=1, \dots, N-1.$$

Part 2: We're asked to verify

$$\begin{aligned} & \frac{1}{(\Delta x)^2} (\cos(k\pi(i-1)\Delta x) - 2\cos(k\pi i \Delta x) + \cos(k\pi(i+1)\Delta x)) \\ &= \frac{2}{(\Delta x)^2} (\cos(k\pi \Delta x) - 1) \cdot \cos(k\pi i \Delta x), \quad k=1, 2, \dots, N-1. \end{aligned}$$

The LHS is:

$$\frac{1}{(\Delta x)^2} (\cos(k\pi i \Delta x - k\pi \Delta x) - 2\cos(k\pi i \Delta x) + \cos(k\pi i \Delta x + k\pi \Delta x))$$

using $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$,

$$\begin{aligned} \Rightarrow \text{LHS} &= \frac{1}{(\Delta x)^2} (\cos(k\pi i \Delta x)\cos(k\pi \Delta x) + \sin(k\pi i \Delta x)\cancel{\sin(k\pi \Delta x)} \\ &\quad + \cos(k\pi i \Delta x)\cos(k\pi \Delta x) - \cancel{\sin(k\pi i \Delta x)}\sin(k\pi \Delta x) \\ &\quad - 2\cos(k\pi i \Delta x)) \\ &= \frac{2}{(\Delta x)^2} [\cos(k\pi \Delta x) - 1] \cos(k\pi i \Delta x) = \text{RHS} \quad \checkmark. \end{aligned}$$

Part 3

This satisfies the 2nd order difference. however it

This satisfies the 2nd order difference, however it fails to satisfy the values $w_0^{(k)} = 0 = w_n^{(k)}$;

$$w_0^{(k)} = \cos(0) = 1 \neq 0$$

$$w_n^{(k)} = \cos(k\pi) = \pm 1 \neq 0$$