HW8 Report

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Problem 1

In this problem, we solve the IVP of Burgers' equation,

$$\begin{cases} u_t + (\frac{1}{2}u^2)_x = 0, \quad t > 0 \\ u(x,0) = \begin{cases} \frac{-1}{2}, & x \le 0 \\ 1, & 0 < x \le 1 \\ 0, & x > 1 \end{cases} \end{cases}.$$

When $t \le 2$, the exact solution for Burgers' equation is

$$u_{ext}(x,t) = \begin{cases} \frac{-1}{2}, & x \le \frac{-1}{2}t \\ \frac{x}{t}, & \frac{-1}{2}t < x \le t \\ 1, & t < x \le 1 + \frac{1}{2}t \\ 0, & x > 1 + \frac{1}{2}t \end{cases}.$$

We solve the IVP numerically with three methods: an upwind method with no entropy fix, and upwind method with the LeVeque entropy fix, and the Lax-Wendroff method. Our computation domain is $[L_1, L_2] = [-1, 2]$, where $\Delta x = \frac{L_2 - L_1}{N}, \quad x_i = L_1 + (i - 0.5)\Delta x, \quad i = 0, 1, \ldots, N+1$. We utilize artificial boundary conditions, $u_0^n = u_1^n, \quad u_{N+1}^n = u_N^n$. In our computations, we have $N = 300$ and $r = 0.5$. Below are differing plots for the problem, with appropriate figure captions.

Part 1)

As seen in Figure 1, both the Lax-Wendroff and Upwind 1 method (no entropy fix) produce fake shocks at $x = 0$, while the Upwind 2 method (LeVeque entropy fix) is quite accurate in comparison.

Part 2)

In this subsection, we plot (Figure 2) the numerical solutions of the Upwind 2 method at various times.

Problem 2

This problem is a continuation of the IVP for Burgers' equation. We use the same computational domain and grid resolution as before.

Part 1)

We change the value of r to be greater than the CFL condition ($r \leq 1$) by setting $r = 10/8$. As seen in Figure 3, when the CFL condition is surpassed, there are massive oscillations in the numerical solution.

Part 2)

We again change the value of r, this time to $r = 10/8.5$. As seen in Figure 4, the behavior isn't the same massive numerical oscillations in Figure 3, but a fake shock is produced close to $x \approx 1.355$.

Problem 3

We consider an IVP of conservation law

$$\begin{cases} u_t + (\frac{1}{4}u^4)_x = 0, \quad t > 0 \\ u(x, 0) = \sin(\pi x) \end{cases}$$

and implement the Upwind 2 method to numerically solve it. We choose the computational domain of $[L_1, L_2] = [0, 4]$, with $N = 400$, $r = 0.5$. The solution is periodic, hence we have the periodic boundary conditions $u_0^n = u_N^n, \quad u_{N+1}^n = u_1^n$.

Part 1)

In this part, we plot $u(x, t)$ vs x for various times, as shown in Figure 5 below.

Part 2)

In this part, we create Figure 6, where we plot

$$\chi := \Big(u(x, t)/ \max_x u(x, t)\Big)$$

vs x for $t = 0, 1, 3, 10, 40$.

Problem 4

Consider the 2D IVP

$$\begin{cases} \dfrac{\partial u(x, y, t)}{\partial t} + \nabla \cdot (\vec{a}(x, y)u(x, y, t)) = 0 \\ u(x, y, 0) = u_0(x, y) \equiv \sin^2(x + y) \end{cases}$$

where $\vec{a}(x, y)$ is defined as

$$\vec{a}(x, y) = \begin{bmatrix} a_1(x, y) \\ a_2(x, y) \end{bmatrix} \equiv \begin{bmatrix} \sin(x)\sin(y) \\ 1 - \exp(\sin(x + y)) \end{bmatrix}.$$

We apply the method of characteristics to solve this problem. After computing the divergence and rearranging terms, we have that

$$u_t + a_1(x, y)u_x + a_2(x, y)u_y = b(x, y)u.$$

Hence, using the definition of $\vec{a}$, we have

$$b(x,y) = -\frac{\partial a_1}{\partial x} - \frac{\partial a_2}{\partial y} = -\cos(x)\sin(y) + \exp(\sin(x+y))\cos(x+y).$$

To apply the method of characteristics to an arbitrary point (ξ, η, T), we first perform a back-tracing step by solving the ODE system

$$\begin{aligned} \frac{dX}{dt} &= a_1(X,Y) \\ \frac{dY}{dt} &= a_2(X,Y) \\ X(T) = \xi,& \quad Y(T) = \eta \end{aligned}$$

from time $t = T$ to $t = 0$ via an RK4 solver. Then, $x_0 = X(0)$ and $y_0 = Y(0)$. We can then solve the forward evolution of the system

$$\begin{aligned} \frac{dx}{dt} &= a_1(x,y) \\ \frac{dy}{dt} &= a_2(x,y) \\ \frac{dv}{dt} &= b(x,y)v \\ x(0) = x_0,& \ \ y(0) = y_0, \ \ v(0) = u_0(x_0, y_0) \end{aligned}$$

from $t = 0$ to $t = T$ using an RK4 solver. Then, we have that $u(\xi, \eta, T) = v(T)$. In computations, we set our timestep $h = 0.01$.

Part 1)

In this part, we create Figure 7. First, we set the x-coordinate of our arbitrary point to be fixed at $x = 3.9$. We then solve the IVP and plot $u(x, y, T)$ as a function of y, where $y \in [0, 2\pi]$. We do this for the times $T = 0.75, 1.0, 1.25$.

Part 2)

Nearly identical to part 1 in approach, we set the x-coordinate to be fixed at $x = 2.5$, and plot in the same manner, creating Figure 8.

Problem 5

This problem is a continuation of the IVP in problem 4. Previously, we fixed our x value and plotted $u(x, y, t)$ as a function of y. Now, we fix the pair (x, y) and plot $u(x, y, t)$ as a function of t, where $t \in [0, 1.25]$ (Figure 9).

Problem 6

This problem is another continuation of the IVP in problem 4. We plot contours, (Figures 10, 11, 12, 13, and 14) of for $u(x, y, T)$ at $T = 0$, 0.5, 1.0, 1.25. In our computations, we use the grid

$$x_i = i\Delta x, \quad y_j = j\Delta y, \quad 0 \leq i, j \leq N,$$

where $\Delta x = \Delta y = \frac{2\pi}{N}$.

Figure 1: The three numerical solutions against the exact, plotted for $t = 1$.

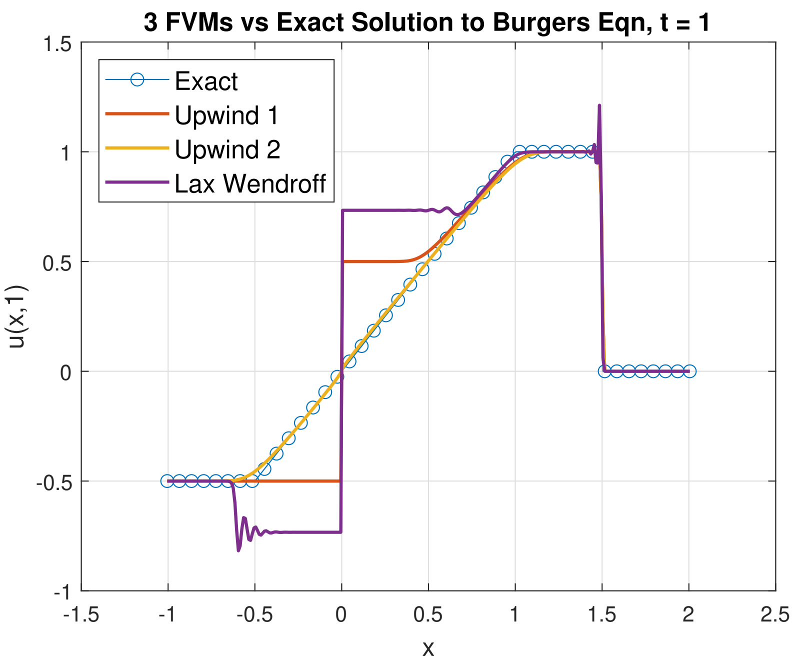

Figure 2: The Upwind 2 numerical solution for various times.

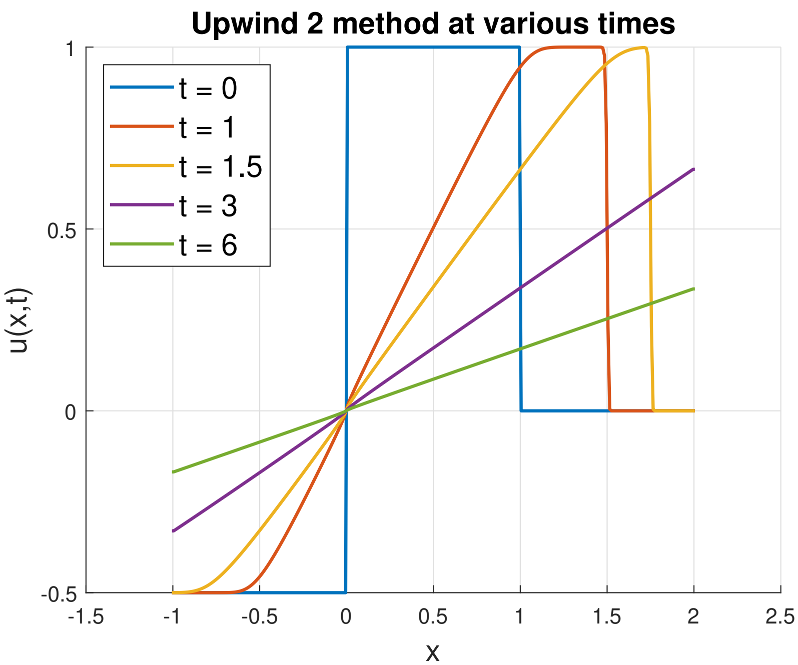

Figure 3: Upwind 2 numerical solution for $r = 10/8$.

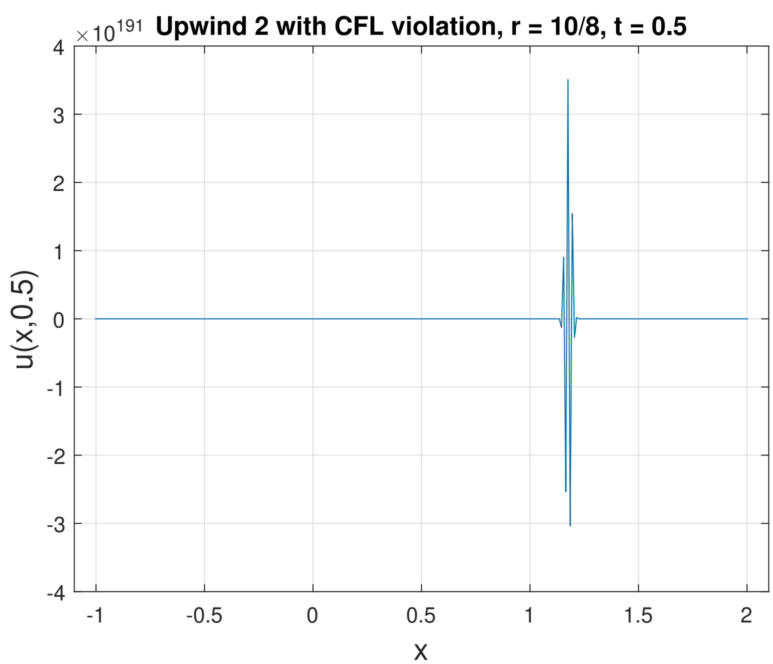

Figure 4: Upwind 2 numerical solution for $r = 10/8.5$.

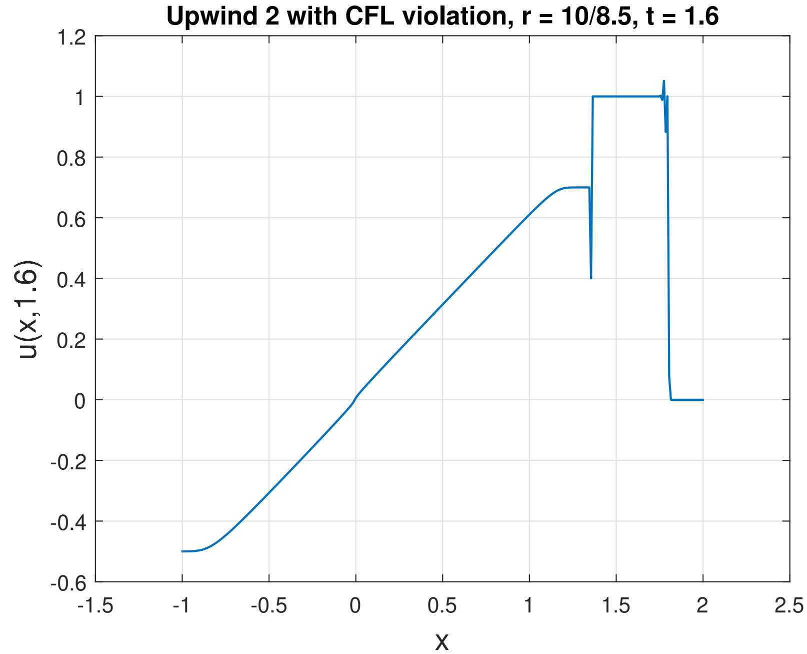

Figure 5: $u(x,t)$ for various time levels.

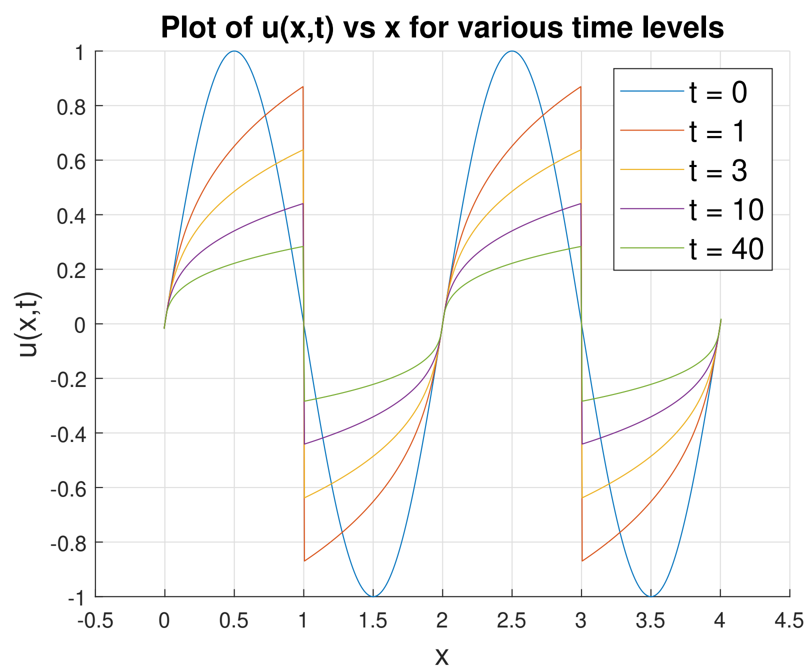

Figure 6: χ vs x for various time levels.

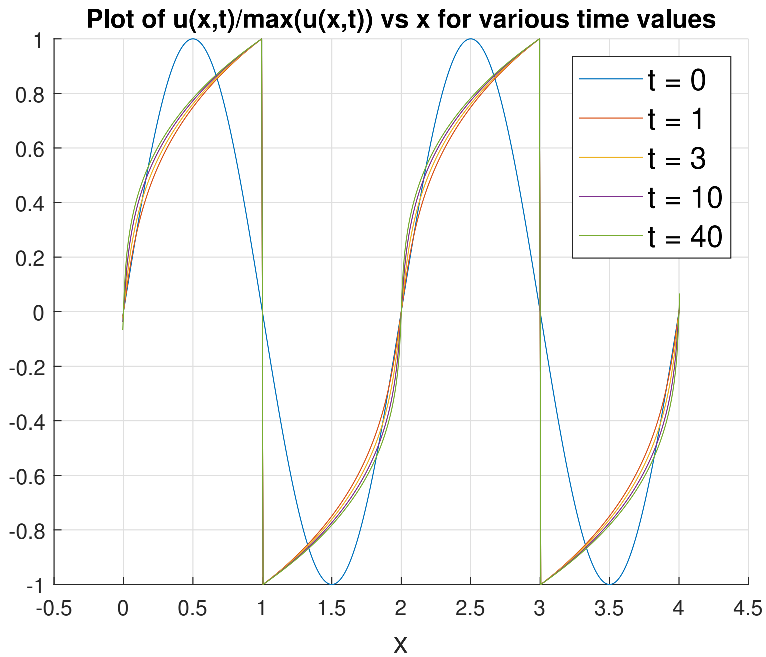

Figure 7: $u(x, y, T)$ vs y for $x = 3.9$

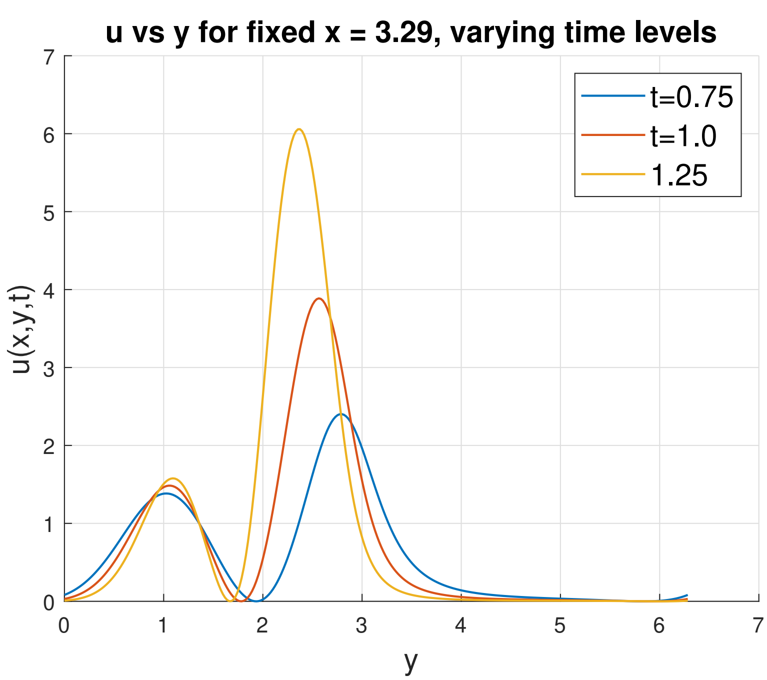

Figure 8: $u(x, y, T)$ vs y for $x = 2.5$

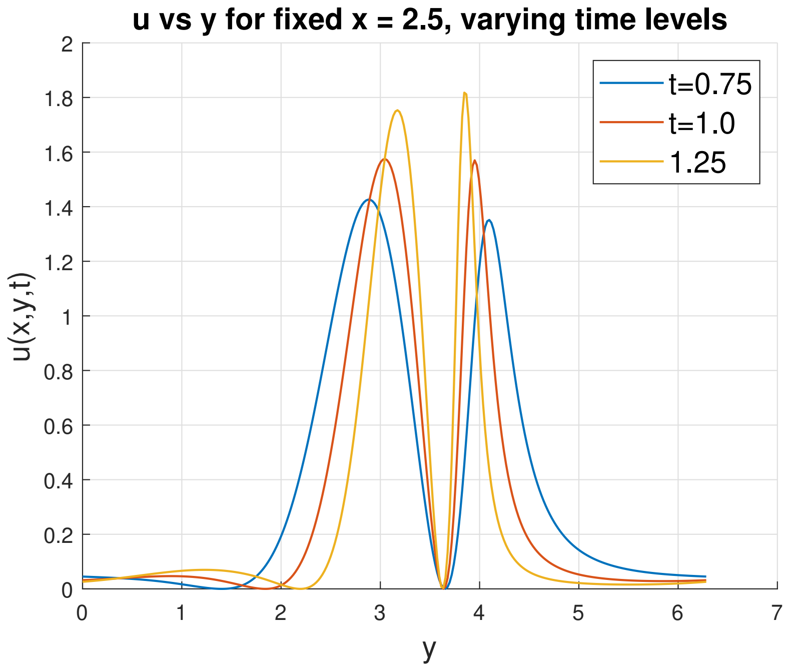

Figure 9: Numerical solution for various fixed (x, y) vs t.

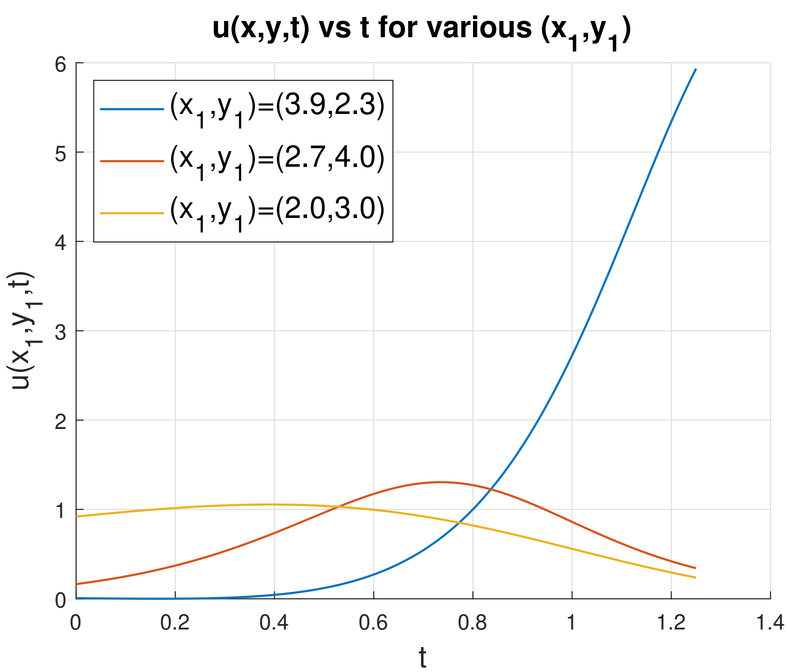

Figure 10: Numerical solution $u(x, y, t)$ at $t = 0$.

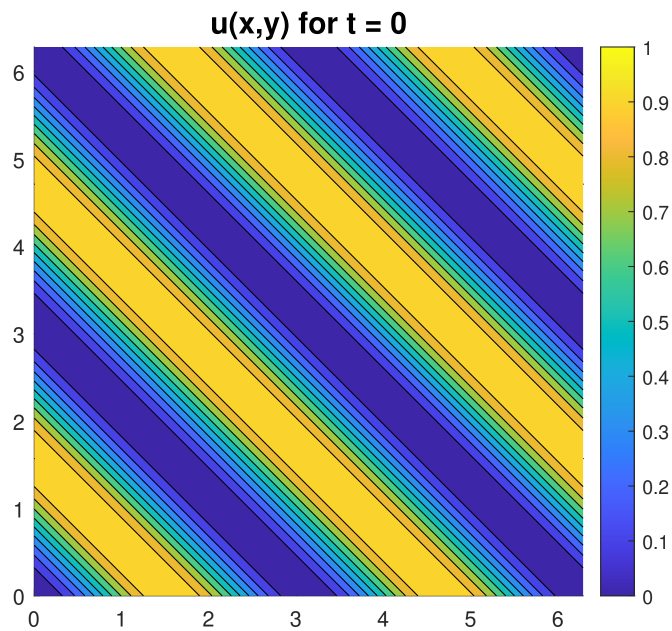

Figure 11: Numerical solution $u(x, y, t)$ at $t = 0.5$.

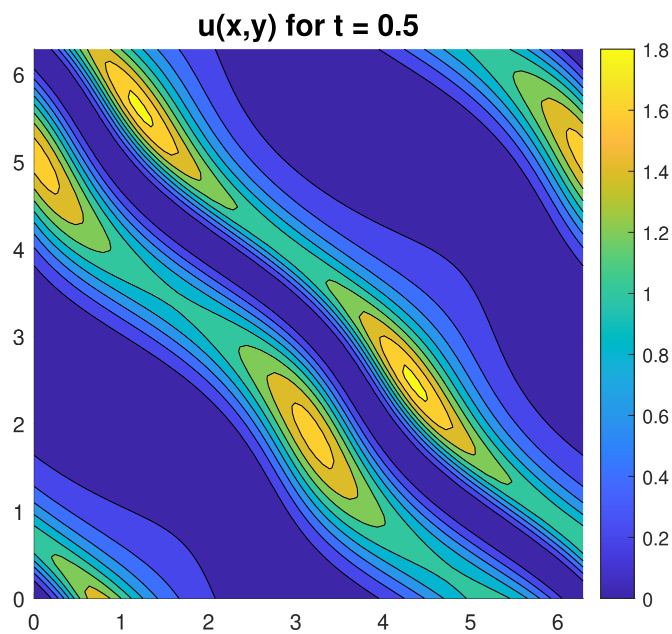

Figure 12: Numerical solution $u(x, y, t)$ at $t = 1.0$.

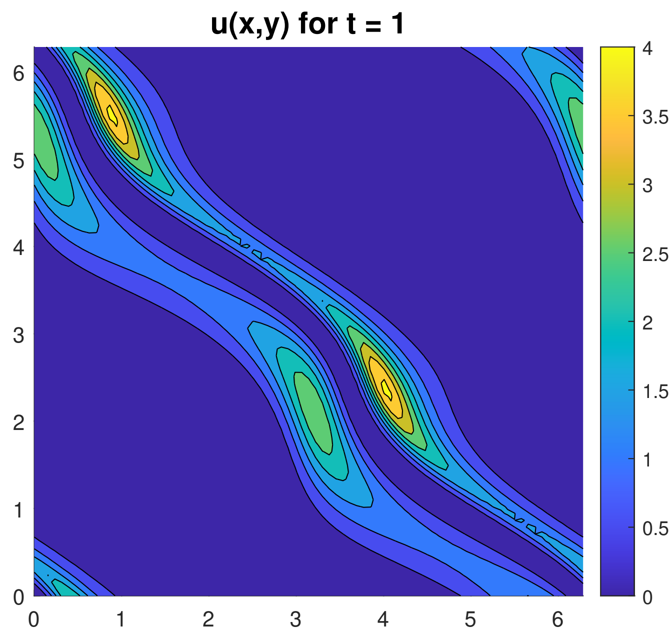

Figure 13: Numerical solution $u(x, y, t)$ at $t = 1.25$.