AM213B Assignment #5

Problem 1 (Theoretical)

Part 1: Carry out von Neumann stability analysis to show that the BTCS method is unconditionally stable

Part 2: Carry out Taylor expansions to show that the local truncation error of the Crank-Nicolson method is

$$e_i^n(\Delta x, \Delta t) = \Delta t\, O\left((\Delta t)^2 + (\Delta x)^2\right)$$

In the final expression, be sure to convert r back to $\Delta t/(\Delta x)^2$.

Problem 2 (Theoretical)

Consider matrix

$$A = \frac{1}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}_{(N-1)\times(N-1)}, \qquad \Delta x = \frac{1}{N}$$

Part 1: Verify that the set below are eigenvalues and eigenvectors of matrix A.

$$\left.\begin{aligned} \lambda^{(k)} &= \frac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x) - 1\right) \\ w^{(k)} &= \left\{\sin(k\pi i \Delta x),\quad i = 1, 2, \ldots, N-1\right\} \end{aligned}\right\}, \quad k = 1, 2, \ldots, N-1$$

Part 2: Verify that

$$\frac{1}{(\Delta x)^2}\left(\cos(k\pi(i-1)\Delta x) - 2\cos(k\pi i\Delta x) + \cos(k\pi(i+1)\Delta x)\right)$$

$$= \frac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x) - 1\right)\cdot\cos(k\pi i\Delta x), \qquad k = 1, 2, \ldots, N-1$$

Part 3: Explain why $u^{(k)} = \left\{\cos(k\pi i\Delta x),\quad i = 1, 2, \ldots, N-1\right\}$ is NOT an eigenvector of A.

Hint: What boundary conditions did we use in defining matrix A?

Problem 3 (Computational)

Consider the IBVP of the heat equation:

$$\begin{cases} u_t = u_{xx}, & x \in (0, 2), \quad t > 0 \\ u(x,0) = f(x), & x \in (0, 2) \\ u(0,t) = g_L(t), & u(2,t) = g_R(t) \end{cases}$$

where $g_L(t) = 1, \quad g_R(t) = 0, \quad f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$.

Implement the FTCS method to solve the IBVP to $T = 0.2$. Use $\Delta x = 0.01$.

- Try $\Delta t = \frac{(\Delta x)^2}{2} \cdot \frac{1}{0.99}$, which is slightly above the stability threshold.
- Try $\Delta t = \frac{(\Delta x)^2}{2} \cdot \frac{1}{1.01}$, which is slightly below the stability threshold.

For each Δt, plot $u(x, t)$ vs x at $t = 0.01, 0.04, 0.09$, and 0.2 in one figure.

Problem 4 (continue with the IBVP in problem 3)

Implement the BTCS method and the Crank-Nicolson method to solve the IBVP to $T = 0.2$. Use $\Delta x = \Delta t = 0.01$ in both methods.

For each method, plot $u(x, t)$ vs x at $t = 0.01, 0.04, 0.09$, and 0.2 in one figure.

Do you see anything unexpected in the results of the Crank-Nicolson?

Hint: First set up the MOL. Then use the backward Euler to solve the MOL. See the sample code on implementing an implicit RK to solve a linear ODE system.

Problem 5 (continue with the IBVP in problem 3)

Now we change the initial and boundary conditions to

$$f(x) = 0.5x, \quad g_L(t) = \cos(2t), \quad g_R(t) = \sin(2t)$$

Implement the 2s-DIRK (with $\alpha = 1 - 1/\sqrt{2}$) on the MOL discretization.

Solve the IBVP to $T = 3$ with $\Delta x = 0.01$ and $\Delta t = 0.01$

Part 1: Plot $u(x, t)$ vs x at $t = 0.02, 0.5, 1$, and 3 in one figure.

Part 2: Repeat the calculation with $\Delta x = 0.01$ and $\Delta t = 0.01/2$. Use the two numerical solutions to estimate the error associated with the time discretization.

Plot the estimated error vs x at $t = 0.5, 1$, and 3 in one figure. Use the algebraic values of errors (do not take absolute values). Use linear scales for both the error and x.

Plot the estimated error vs x at $t = 0.02$ in a separate figure.

Problem 6

Consider the IBVP with a boundary condition modeling the radiation heat loss:

$$\begin{cases} u_t = u_{xx}\,, & x \in (0,\ L)\,,\quad t > 0 \\ u(x,0) = p(x)\,, & x \in (0,\ L) \\ u_x(0,t) - \alpha u(0,t) = 0\,, & u(L,t) = q(t) \end{cases}$$

where $L = 2$, $\alpha = 0.4$, $p(x) = (1-0.5x)^2$, $q(t) = 2\sin^2(t)$.

<u>Numerical grid:</u>

$$\Delta x = \frac{L}{N-0.5}\,, \qquad x_i = (i-0.5)\Delta x\,, \quad i = 0, 1, \ldots, N$$

$$x_0 = -0.5\Delta x, \qquad x_1 = 0.5\Delta x\,, \qquad x_{N-1} = L - \Delta x\,, \qquad x_N = L$$

<u>The FTCS method:</u>

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{(\Delta x)^2}\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right), \qquad i = 1, 2, \ldots, N-1$$

<u>Boundary conditions:</u>

<u>The left end</u>

$$\frac{u_1^n - u_0^n}{\Delta x} - \alpha \frac{u_1^n + u_0^n}{2} = 0 \qquad ==> \qquad (2-\alpha\Delta x)u_1^n = (2+\alpha\Delta x)u_0^n$$

$$==> \qquad \boxed{u_0^n = \frac{(2-\alpha\Delta x)}{(2+\alpha\Delta x)}u_1^n}$$

<u>The right end</u>: $\quad u_N^n = q(n\Delta t)$

Use the FTCS method to solve the IBVP to $T = 3$ with $N = 200$ and $\Delta t = 4\times10^{-5}$.

<u>Plot</u> $u(x, t)$ vs x at $t = 0.02, 0.5, 1$, and 3 in one figure.