AM213B Assignment #4

Problem 1 (Theoretical)

Consider the one-stage implicit RK method described by Butcher tableau

$$\begin{array}{c|c} c^T & A \\ \hline & b \end{array} = \begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

Part 1: Show that it has second order accuracy.

Hint: Check the conditions on second order accuracy.

Part 2: Derive its stability function $\phi(z)$.

Part 3: Show that it is A-stable, but not L-stable.

Problem 2 (computational)

Plot the region of absolute stability (S) for each of the methods below.

- 3-step Adams-Moulton,
- the 2-step 4th-order LMM in Assignment 3

$$u_{n+2} - u_n = h\left[\frac{1}{3} f(u_{n+2}, t_{n+2}) + \frac{4}{3} f(u_{n+1}, t_{n+1}) + \frac{1}{3} f(u_n, t_n)\right]$$

- 2-step BDF (BDF2), and
- 3-step BDF (BDF3).

Hint: You don't need to shade the region if it is tricky to do so. But you do need to describe clearly the region in the context of the plotted candidate boundary curve.

Remark: You will see

- 3-step Adams-Moulton is not A-stable
- The 2-step 4th-order LMM is not practically useful.
- BDF2 is A-stable.
- BDF3 is almost A-stable.

Problem 3 (Computational)

Implement the shooting method to solve the two-point BVP.

$$\begin{cases} u'' - \left(1 + 0.5 u'^2\right) u = \sin x \\ u(0) = 1, \quad u(2) = 0.5 \end{cases}$$

You first need to convert the 2nd order ODE into a 1st order ODE system.

In the shooting method, use RK4 with h = 0.002 as the ODE solver; use Newton's method as the non-linear equation solver. Start the shooting method with $v_0 = -1$ (where v denotes $u'(0)$).

Report the value of v found in the shooting method.

Plot $u(x)$ vs x.

Hint: Write a Matlab function for evaluating $G(v) \equiv u(2)\Big|_{u'(0)=v} - 0.5$. Then solve $G(v) = 0$.

Remark: This problem demonstrates the advantage of shooting method. If we apply the finite difference method, the resulting non-linear system is difficult to solve.

Problem 4 (Computational)

Use the finite difference method (FDM) to solve the two-point BVP.

$$\begin{cases} u'' - 625u = -625x \\ u(0) = 1, \quad u(2) = 1 \end{cases}$$

The exact solution of the BVP is given by

$$u_{exact}(x) = x + \frac{1+e^{-50}}{1-e^{-100}}\left(e^{-25x} - e^{25(x-2)}\right)$$

Solve the BVP numerically using the FDM with N = 1000 (h = 0.002).

Part 1: Plot the numerical $u(x)$ vs x and the exact $u(x)$ vs x in one figure.

Part 2: Estimate the error in numerical $u(x)$ using the results of N=1000 and N=2000. Calculate the exact error in numerical $u(x)$ using the exact $u(x)$ given above.

Plot the estimated error vs x and the exact error vs x in one figure. Use linear scales for both x and the errors.

Hint: See the sample code on implementing the finite difference method.

Remark: This problem demonstrates the advantage of FDM. If we apply the shooting method, $G(v)$ will be either +large or –large. We will never be able to make $G(v)$ = 0 no matter what value we use for v.

Problem 5 (Theoretical and computational)

Design the numerical grid and the discretization of the finite difference method (FDM) for solving the two-point BVP below (similar to type 2 BVP in lectures).

$$\begin{cases} u'' - \left(1+\exp(-\sin x)\right)u = -5 - (\sin x)^2 \\ u'(0) = 2.5, \quad u(2) = 0.5 \end{cases} \tag{P5}$$

We study the general two-point BVP of the same type.

$$\begin{cases} u''+p(x)u'+q(x)u=r(x) \\ u'(a)=\alpha\,, \quad u(b)=\beta \end{cases}$$

We follow the approach used in lectures on type 2 BVP.

To accommodate $u'(a) = \alpha$, we need to put the left end a between two grid points.

<u>Numerical grid and notation:</u>

$h=\dfrac{b-a}{N-0.5}$, $\qquad x_i = a + (i-0.5)h$, $\qquad i = 0, 1, 2, \ldots, N$

$x_0 = a-0.5h$, $\qquad x_{0.5} = a$, $\qquad x_1 = a+0.5h$, $\qquad x_N = a + (N-0.5)h = b$

Internal points = $\{x_i,\ i = 1, 2, \ldots, N-1\}$

u_i = numerical approximation of $u(x_i)$

Combining the finite difference discretization and the boundary conditions (BCs), we write out a <u>linear system</u> of $\{u_1, u_2, \ldots, u_{N-1}\}$:

$$\begin{aligned} &\left(\frac{1}{h^2}+\frac{p_i}{2h}\right)u_{i+1}+\left(-\frac{2}{h^2}+q_i\right)u_i+\left(\frac{1}{h^2}-\frac{p_i}{2h}\right)u_{i-1}=r_i\,, \quad 1\le i\le N-1 \\ &\boxed{\frac{u_1-u_0}{h}=\alpha \longrightarrow u_0=u_1-h\alpha}\,, \qquad u_N=\beta \end{aligned} \tag{E02B}$$

We write the linear system as $Au = g$.

Only the first row of (E02B) is different from type 1 (affected by the new BC).

<u>$i = 1$:</u>

$$\left(\frac{1}{h^2}+\frac{p_1}{2h}\right)u_2+\left(-\frac{2}{h^2}+q_1\right)u_1+\left(\frac{1}{h^2}-\frac{p_1}{2h}\right)\underbrace{(u_1-h\alpha)}_{u_0}=r_1$$

==> $$\left(\frac{1}{h^2}+\frac{p_1}{2h}\right)u_2+\left(-\frac{1}{h^2}-\frac{p_1}{2h}+q_1\right)u_1=r_1+\left(\frac{1}{h}-\frac{p_1}{2}\right)\alpha$$

==> $$\boxed{a_{1,1}=-\frac{1}{h^2}-\frac{p_1}{2h}+q_1}\,, \qquad a_{1,2}=\frac{1}{h^2}+\frac{p_1}{2h}$$

$$\boxed{g_1=r_1+\left(\frac{1}{h}-\frac{p_1}{2}\right)\alpha}$$

<u>Implement</u> (E02B) in Matlab. Solve the BVP (P5) with $N = 1000$.

<u>Plot</u> the numerical $u(x)$ vs x.

Hint: See the discussion in lectures on constructing the linear system for type 2 BVP.

See the sample code on implementing the FDM on type 2 BVP.

Note in particular that the numerical grid is different from both type 1 and type 2.

Once you obtain the numerical solution $\{u_j\}$ of the FDM, you can solve the IVP using RK4 with $u(0) = u_1 - \alpha h/2$ and $u'(0) = \alpha$. You can then compare the solution of the BVP and the solution of the IVP. The two solutions should match each other. This is a good approach of validating your numerical discretization/solution of the FDM.

Problem 6 (Theoretical and computational)

Consider a slightly different version of the BVP in Problem 5.

$$\begin{cases} u'' - \left(1+\exp(-\sin x)\right)u = -5-(\sin x)^2 \\ u(0)-u'(0)=1.5, \quad u(2)=0.5 \end{cases} \tag{P6}$$

Design the discretization of the finite difference method (FDM). Implement it in Matlab. Solve the BVP (P6) with $N = 1000$.

Plot the numerical $u(x)$ vs x.

Hint: Discretize the left BC as

$$u(0)-u'(0)=\alpha$$

$$\Longrightarrow \quad \frac{u_1+u_0}{2} - \frac{u_1-u_0}{h} = \alpha \qquad \Longrightarrow \quad (2+h)u_0 - (2-h)u_1 = 2\alpha h$$

$$\Longrightarrow \quad u_0 = \frac{2\alpha h + (2-h)u_1}{2+h}$$

Only the first row of the linear system is different from type 1.

Write out the first row as we did in lectures and in Problem 5 above.

Again once you obtain the numerical solution $\{u_j\}$, you can solve the IVP using RK4 with $u(0) = (2u_1+\alpha h)/(2+h)$ and $u'(0) = 2(u_1-\alpha)/(2+h)$. You can then compare the solution of the BVP and the solution of the IVP. The two solutions should match each other. This is a good approach of validating your numerical discretization/solution of the FDM.