AM213B Assignment #1

Problem 1 (Theoretical)

Suppose E_n satisfies the recursive inequality

$$E_{n+1} \le (1+Ch)E_n + h^2 \qquad \text{for } n \ge 0$$
$$E_0 = 0$$

where $C > 0$ is a constant independent of h and n.

Derive that $\quad E_N \le \dfrac{e^{CT}-1}{C} h \quad \text{for } Nh \le T$

Problem 2

Use the composite trapezoidal rule and the composite Simpson's rule, respectively, to approximate the integral

$$I \equiv \int_1^3 \sqrt{2+\cos^3(x)}\, \exp\big(\sin(x)\big) dx$$

For each method, carry out simulations at a sequence of numerical resolutions:

$N = 2^2,\ 2^3,\ 2^4,\ \ldots,\ 2^{10}$.

State and compare the numerical solutions of the two methods at $N = 2^{10}$.

For each method, use the numerical results to do numerical error estimations.

For each method, plot the estimated error (absolute value) as a function of h. Use log-log plot to accommodate the wide ranges of h and error.

Plot the two curves in ONE figure to compare the performance of the two methods.

Problem 3

Implement Newton's method to solve the non-linear equation of x given below.

$$x - \alpha + \beta \sinh\big(x - \cos(s-1)\big) = 0$$

where sinh() is the hyperbolic sine function: $\sinh(z) = \frac{1}{2}\left(e^{z} - e^{-z}\right)$.

Solve the equation for each value of s = [0:0.1:20]. Use $\alpha = 0.9$ and $\beta = 50000$.

Plot x vs s, and compare with $\cos(s-1)$ vs s.

Hint: Look at the sample code on how to implement Newton's method.

Problem 4

Implement the Euler method and the backward Euler method to solve the IVP below.

$$\begin{cases} u' = -\lambda \sinh\left(u - \cos(t-1)\right), \qquad \lambda = 10^6 \\ u(0) = 0 \end{cases}$$

Part 1: For the Euler method, solve the IVP to $T = 2^{-10}$. Try $h = 2^{-18}, 2^{-19}, 2^{-20}, \ldots$

At what time step size, the numerical solution remains bounded?

Plot one representative figure showing the behavior of numerical solution when the time step is not small enough.

Plot another representative figure when the time step is small enough.

Part 2: For the backward Euler method, solve the IVP to $T = 10$. Use Newton's method to solve the non-linear equation in each time step. Use $h = 0.1$ in your simulations.

Plot the numerical solution vs t for the backward Euler method.

Problem 5:

Implement the trapezoidal method to solve the IVP in Problem 4.

Use Newton's method to solve the non-linear equation in each time step.

Part1: Solve the IVP to $T = 10$. Use $h = 0.1$ in your simulations.

Plot the numerical solution vs t. Is the numerical solution bounded?

Do you observe any oscillation in the numerical solution with $h = 0.1$?

Part2: Reduce the time step to $h = 2^{-7}, 2^{-8}, 2^{-9}, \ldots$.

What happens to the oscillation, as the time step is refined?

Problem 6:

Use the Euler method and the 2-step midpoint method, respectively, to solve the IVP

$$\begin{cases} u' = -u \\ u(0) = 1 \end{cases}$$

The exact solution of the IVP is $u_{\text{exact}}(t) = \exp(-t)$. In the midpoint method, use the exact solution $u_1 = \exp(-h)$ to get started. Use $h = 0.2$ for both methods.

Part 1: Solve the IVP to $T = 2$. Compare the numerical results of the two methods and the exact solution in ONE figure.

Is the midpoint method more accurate than the Euler in this time period?

Part 2: Solve the IVP to $T = 20$. Compare the numerical results of the two methods and the exact solution in ONE figure.

Is the result of midpoint method well behaved over this longer period?

Part 3: With $T = 20$, reduce the time step to $h = 0.2/32$, $0.2/64$, $0.2/128$. Does that reduce the growth of error in the midpoint method?