

# Exercise 4

Molecular Statistic, Week 4

2014

## 1 Introduction

The goals of this exercise:

1. Extend your MD-program to three dimension using the supplied module.
2. Implement a 3D Velo-Verlet solver in Numpy.
3. Extend your program with functions to calculate the following properties:
  - (a) Kinetic energy
  - (b) Total energy
  - (c) Temperature
  - (d) Pressure
4. Visualize your results in a video.

### 1.1 Lennard-Jones potential and periodic boundary conditions

The potential we are using this week is almost the same Lennard-Jones potential as we saw last week. You may recall from your simulation last week, that the calculation of the particle pair-interactions took a long time to compute, even with only 10-20 particles. To speed up the calculation time we can make the approximation, that only particles closer than a certain minimum distance  $r_{\text{cut}}$  are interacting. This is a justified approximation, since particles that are further apart have comparatively weaker interactions. Here we set  $r_{\text{cut}} = 2.5$ . Using this approximation, we can define the the potential energy between two interacting particles  $i$  and  $j$  is given by:

$$U_{ij} = \begin{cases} 4\epsilon \left[ \left( \frac{\sigma}{r_{ij}} \right)^{12} - \left( \frac{\sigma}{r_{ij}} \right)^6 \right] - E_{\text{cut}} & \text{if } r_{ij} < r_{\text{cut}} \\ 0 & \text{if } r_{ij} \geq r_{\text{cut}} \end{cases} \quad (1.1)$$

The derivatives of the potential energy are used to calculate the forces between particles, i.e.:

$$\mathbf{F} = -\nabla U = - \left( \frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial y_1}, \frac{\partial U}{\partial z_1}, \frac{\partial U}{\partial x_2}, \frac{\partial U}{\partial y_2}, \frac{\partial U}{\partial z_2}, \dots, \frac{\partial U}{\partial x_n}, \frac{\partial U}{\partial y_n}, \frac{\partial U}{\partial z_n} \right) \quad (1.2)$$

The  $x$ -components of the force between two interacting particles  $i$  and  $j$  are:

$$-\frac{\partial}{\partial x_i} U_{ij} = \begin{cases} -48 \frac{x_j - x_i}{r_{ij}^2} \left[ \left( \frac{1}{r_{ij}} \right)^{12} - 0.5 \left( \frac{1}{r_{ij}} \right)^6 \right] & \text{if } r_{ij} < r_{\text{cut}} \\ 0 & \text{if } r_{ij} \geq r_{\text{cut}} \end{cases} \quad (1.3)$$

$$-\frac{\partial}{\partial x_j} U_{ij} = \begin{cases} 48 \frac{x_j - x_i}{r_{ij}^2} \left[ \left( \frac{1}{r_{ij}} \right)^{12} - 0.5 \left( \frac{1}{r_{ij}} \right)^6 \right] & \text{if } r_{ij} < r_{\text{cut}} \\ 0 & \text{if } r_{ij} \geq r_{\text{cut}} \end{cases} \quad (1.4)$$

The  $y$ - and  $z$ -components are, of course, similar.

## 1.2 Velo-Verlet integration and periodic boundary conditions

Last week your setup confined the particles in the simulation inside a box with hard walls. Today's exercise uses periodic boundary conditions in the minimum-image convention (see chapter 6.7 in Leach), but you don't have to implement this code, since it's already implemented in the FORTRAN library you will be using. The Velo-Verlet integrator you have to implement is thus very simple (and identical to last week). The Velo-Verlet equations for integrating the positions and velocities are (in the  $x$ -direction for one particle):

$$x(t + dt) = x(t) + dt v_x(t) + 0.5 dt^2 a_x(t) \quad (1.5)$$

$$v_x(t + dt) = v_x(t) + 0.5 dt [a_x(t) + a_x(t + dt)] \quad (1.6)$$

In the program you will be implementing today, we set the mass of the particles to 1, so the value of the acceleration is equal to the value of the force.

Let's recap: The Velo-Verlet algorithm uses the forces, velocities and positions from the current position to calculate the forces, velocities and positions after the next time-step. In short it is a three-step procedure:

1. Calculate new positions using Eq. 1.5
2. Calculate new forces using Eq. 1.3-1.4
3. Calculate new velocities using Eq. 1.6

The resulting forces, velocities and particle positions are then saved as input for the next Velo-Verlet integration step.

## 1.3 Kinetic energy

The total kinetic energy of the system can be calculated as:

$$E_{\text{kin}} = \sum_i \frac{1}{2} m_i |\mathbf{v}_i|^2 \quad (1.7)$$

In our simulation  $m_i$  is set to 1, so the above reduces to:

$$E_{\text{kin}} = \frac{1}{2} \sum_i |\mathbf{v}_i| \cdot |\mathbf{v}_i| \quad (1.8)$$

**Hint:** If you do this in Numpy, the above summation is simply carried out as (something like):

```
1 e_kin = np.sum(V * V) * 0.5
```

Why this is true? I.e. what does  $V * V$  do and what does  $\text{np.sum}()$  do?

## 1.4 Conservation of energy

If your simulation is successful the total energy will be conserved. That is:

$$E_{\text{total}} = E_{\text{pot}} + E_{\text{kin}} \approx \text{constant} \quad (1.9)$$

Conservation of energy can sometimes fail during an MD simulation. Often this is due to the timestep  $dt$  being too large, which causes the numerical Velo-Verlet integration to fail.

## 1.5 Instantaneous temperature

The first thermodynamic property we will derive from our simulation is the instantaneous temperature, that is the temperature as a function of time,  $T(t)$ . You will sometimes see the temperature defined in terms of the average kinetic energy *per* degree of freedom in a system:

$$\langle E_\alpha \rangle = \frac{1}{2} k_B T \quad (1.10)$$

where  $\alpha$  denotes the degrees of freedom in the system. The above states that each degree of freedom gives a contribution to the temperature which is proportional to the kinetic energy of the individual particles. Instead of this formulation it is more practical for our purpose to write the temperature in its instantaneous form:

$$T(t) = 2 \frac{\langle E_\alpha(t) \rangle}{k_B N_\alpha} \quad (1.11)$$

$$= \sum_i \frac{m_i |\mathbf{v}_i| \cdot |\mathbf{v}_i|}{k_B N_\alpha} \quad (1.12)$$

where  $N_\alpha$  is the number of degrees of freedoms, i.e.  $N_\alpha = N_{\text{dimensions}} \cdot N_{\text{particles}}$ . Notice, that we use fundamental units and simple masses, so  $k_B = 1$  and  $m_i = 1$ .

**Hint:** If you do this in Numpy, this can be written easily as:

```
1 temperature = 2.0 * e_kin / np.size(V)
2
3 # or
4
5 temperature = np.mean(V * V)
```

Why this is true? What does `np.size()` calculate? What does `np.mean()` calculate? Why do both implementations the yield the same result?

## 1.6 Instantaneous Pressure

The instantaneous pressure in a Van der Waal gas simulation can be calculated as:

$$P(t) = \rho T(t) + \frac{1}{3V_{\text{box}}} \sum_{i>j} \mathbf{F}_{ij} \cdot \mathbf{r}_{ij} \quad (1.13)$$

$$= \rho T(t) + \frac{1}{3V_{\text{box}}} W_{\text{vir}} \quad (1.14)$$

where  $V_{\text{box}}$  is the volume of the box,  $\rho$  is the particle-density of the box and  $W_{\text{vir}} = \sum_{i>j} \mathbf{F}_{ij} \cdot \mathbf{r}_{ij}$  is the Clausius virial function (or *virial* for short) of the system.

Since we were not using a standard Lennard-Jones potential (we use a small cut-off distance), we have to add a small correction (the *tail correction*) to Eq. 1.14 in order to account for the interaction we are neglecting:

$$\Delta P_{\text{tail}} = \frac{16}{3} \pi \rho \left[ \frac{2}{3} \left( \frac{\sigma}{r_{\text{cut}}} \right)^9 - \left( \frac{\sigma}{r_{\text{cut}}} \right)^3 \right] \quad (1.15)$$

Here we simply set  $\sigma = 1$  (see last week's exercise). Remember  $r_{\text{cut}} = 2.5$ .

## 2 Numpy Exercises

When working with Python "in the real world" you will most often use Numerical Python (NumPy) and vectorization. This is because it is faster and easier. Firstly we would want to import the module `numpy`.

1. Create a new Python file in which you can carry out the following Numpy exercises. In the first line of the new file, import the Numpy library:

```
1 import numpy as np
```

What does `as np` mean? And why would you write that?

Numpy is a fast library of functions and data types. It is mostly used for substituting the normal Python List for a new data type called `ndarray`, which is a mathematical vector-like data type.

For example generating an empty list of zeros with the two different methods.

```
1 list = [0.0 for i in range(5)] # Normal list
2 vector = np.zeros(5)          # Numpy vector
```

Now for the advantages. A huge advantage is vectorization commands. You can change a variable by using commands like `"+="`, `"-="`, `"*="`, `"/="`. So the code

```
1 a = 45
2 a = a + 5
```

is equivalent too

```
1 a = 45
2 a += 5
```

2. Execute and evaluate the above command. What does it do?

This smart can also be used on Numpy arrays (just like vectors). Note: This will fail for Python Lists.

```
1 V = np.zeros(5)
2 V += 5
```

3. Execute and evaluate the above command. What is the result of V?

**Note:** When you create a Numpy array, you always specify a size. This also represents how much computer memory is allocated by Numpy. For this reason it is not possible to use the `append` method, as with the normal Python list.

If you are not sure how big your array is going to be, you can create a list, append elements, and then after all elements are appended, convert it too a Numpy array as following.

```
1 L = []
2 for i in range(10):
3     L.append(i)
4 L = np.array(L)
```

A very useful command to use is the `np.arange()`, which is similar to the `range()`.

4. Execute the code

```
1 num_vector = np.arange(0, 5)
2 print num_vector
3 num_vector = np.arange(0, 5, 0.1)
4 print num_vector
```

and explain what the method `np.arange()` does.

5. What does `np.sum()` do and what is the result of the follow code?

```
1 np.sum(num_vector)
```

Answer: 122.5

6. What does `np.mean()` do and what is the result of the follow code?

```
1 np.mean(num_vector)
```

Answer: 2.45

7. What is the value of `np.pi`.

8. What does `np.random.uniform(0.0, 1.0)` do?

9. (Optional) How to get a random float from a normal distribution, using Numpy? Create a list of 1000 random numbers and plot the distribution using Pylab.

*Hint:* Use Google.

### 3 Molecular Dynamics simulation

Before you start this week exercise, please make sure you have downloaded the material from the course website.

Make a new Python program, and name it `exercise_4.py`. In the first two lines import Numpy and the three functions from the downloaded module.

```
1 import numpy as np
2 import md_header as md
3 import video3d
```

Last week you wrote your own code to initialize the particle positions, velocities, box\_width etc. The `initialize_particles()` function you imported is a bit more advanced. The new `initialize_particles()` function takes as arguments the number of particles, the given temperature and a desired particle density.

The temperature option is used to scale the particle velocities appropriately (see section 1.5) and the `rho` keyword is used to scale the width of the box according to the number of particles (we need to be able to specify a particle density in order to calculate the pressure correctly, see section 1.6)

1. Initialize your system with the following code:

```
1 R, V, F, box_width = initialize_particles(n_particles, temperature, rho)
```

Suggested values are `temperature = 2.0` and `rho = 0.1`. Don't worry you will get a chance to play around with these values later

You will notice that `initialize_particles()` returns four values. `R` is the coordinate matrix. It is an Nx3 matrix, which holds for each of N particles the X, Y and Z coordinates. Likewise `V` and `F` holds the initial velocities and forces, respectively. `box_width` is the width of the scaled box, which today you cannot specify this yourself, because it is based on the density of the particles.

Now it's time to test the `initialize_particles` function.

2. What comes out of `initialize_particles(n_particles, temperature, rho)` if you set `n_particles = 10`, `temperature = 2.0` and `rho = 0.1`? What do `R`, `V` and `F` contain? Can you explain why?

The second new function you got from the `md` module is `lennard_jones()` - which can be used like this:

```
1 e_pot, F, w_vir = lennard_jones(R, box_width)
```

`lennard_jones` takes as arguments the position matrix `R` and `box_width`. Just like last week, the `lennard_jones` function returns the potential energy `e_pot`, a force matrix `F`, but now also the virial of the system, `w_vir` (see section 1.6).

3. Now it's time to test the `lennard_jones` function. Instead of giving `R` as input to `lennard_jones`, call the function with the following `R_test` matrix:

```
1 R_test = np.array([[0.7, 2.3, 0.7], [0.7, 0.7, 2.3], [0.7, 0.7, 0.7]])
```

and `box_width = 3.10`. That is:

```
1 print lennard_jones(R_test, 3.10)
```

The potential energy is -0.635, the virial -3.732. The forces matrix (rounded values):

```
1 array([[ -1.16    1.24   -0.09]
2         [-1.16   -0.09    1.24]
3         [ 0.0     0.0     0.   ]])
```

The  $z$ -components of the force are correctly 0 (why?).

Now it is time to implement the Velo-Verlet solver from last week, but using the new module and Numpy.

4. Somewhere in your program file put the solver code in:

```
1 for n in range(n_steps):
2     # Step 1: Calculate new positions
3     R = R + ...
4
5     # Step 2: Calculate new forces
6     F_old = copy.copy(F)
7     e_pot, F, w_vir = lennard_jones(R, box_width)
8
9     # Step 3: Calculate new Velocities:
10    V = V + ...
```

**Hint:** Look at the last exercise if you need inspiration on where to put the solver code.

5. Initialize your system with 40 particles, `rho = 1.1` and `temperature = 2.0`. Run your simulation for 1000 steps with  $dt = 0.001$ . Every 10 steps print the potential energy. The potential energy should stabilize around -20 if your solver works properly.
6. Now would be a good time to implement the video function. The new 3D video is used like the other by adding frames like;

```
1 # Add Frame
2 video3d.add_frame(R)
3
4 # Save movie
5 video3d.save(box_width, 'my_video')
```

And add frame every 10'th step in the Velo-Verlet step.

7. Run 500 steps and save a video.

You have now successfully implemented an MD simulation of a Lennard-Jones gas. The goal now is to use the equations from section 1 to calculate interesting physical properties of your system.

8. Use the equations from section 1.3 to calculate the kinetic energy. Print the kinetic energy, potential energy and total energy each 10 step.
9. Make three empty lists called `Ekin_list`, `Epot_list` and `Etot_list`. Append

```
1 Ekin_list = []
2 Epot_list = []
3 Etot_list = []
4
5 for n in range(n_steps):
6
7     """ Your Velo-Verlet solver code
8         is here ...
9     """
10
11    if (n % 10):
12        # Calculate e_kin and e_tot
13        # here!
14
15        Ekin_list.append(e_kin)
16        Epot_list.append(e_pot)
17        Etot_list.append(e_tot)
```

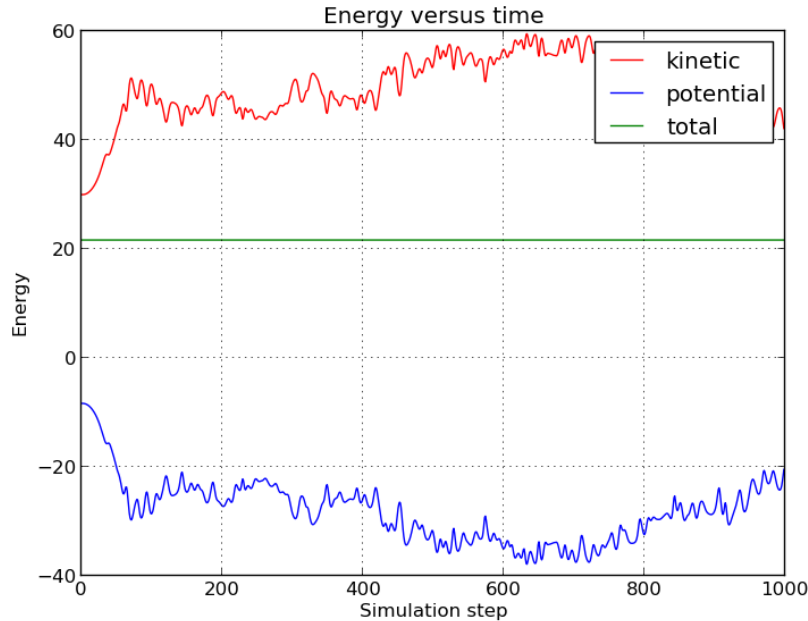
- Run your simulation for 10,000 steps. Do not save a video, since this will take several minutes to compile. Use Pylab to make a plot of the energies vs. time.:

```

1 pylab.clf()
2 pylab.plot(Ekin_list, 'r-', label='Kinetic Energy')
3 pylab.plot(Epot_list, 'b-', label='Potential Energy')
4 pylab.plot(Etot_list, 'g-', label='Total Energy')
5 pylab.legend() # Plot labels in figure
6 pylab.savefig("my_energy.png")

```

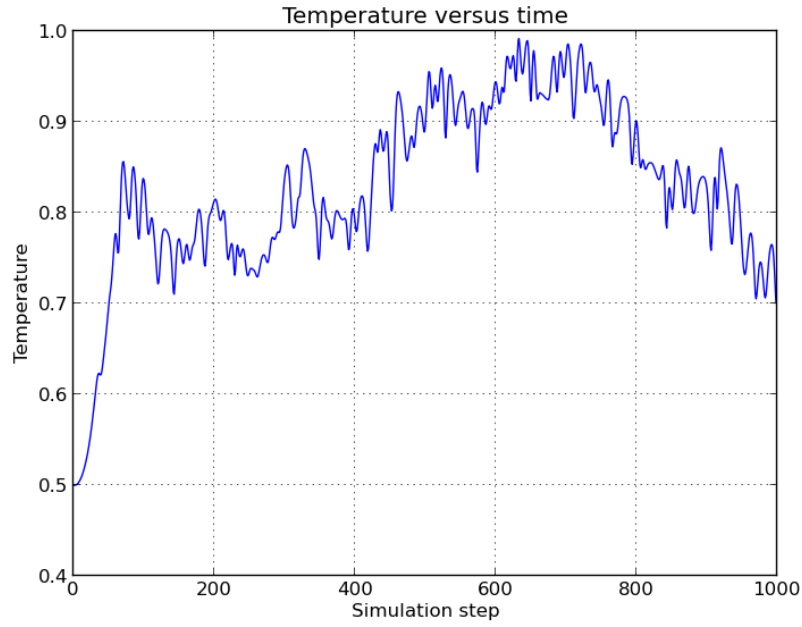
How does the energies evolve if you run your simulation with 40 particles at  $T = 2.0$ ? Now run your simulation at a lower temperature (maybe  $T = 0.5$ ). How does the energy change ?



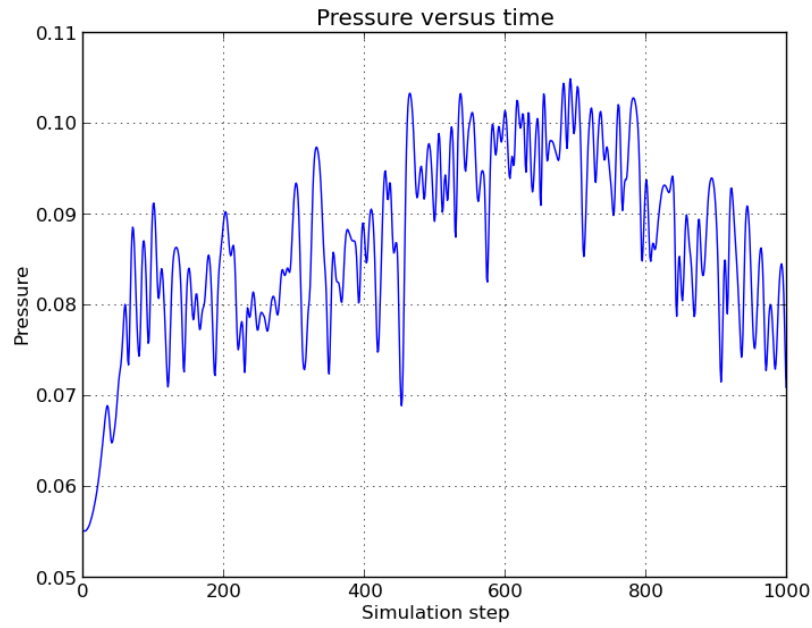
**Figure 3.1:** Evolution of energies during an MD simulation with temperature = 0.5, 40 particles,  $\rho = 0.1$ , for 10,000 steps and saving every 10'th step.

- Use the equations from section 1.5 to calculate the instantaneous temperatures during the simulation.
- Like in the previous question, make an empty list called `T_list` and append the current temperature every 10 steps. Use Pylab to make a plot of this list. See Figure 3.2 for example.  
Does the temperature you calculate correspond to the temperature you specified when you initialized your particles? Run your simulation at  $T = 0.05$ . What could easily go wrong now?
- Answer this question before you proceed: Why is the gas a Van der Waals gas and not an ideal gas? I.e. why isn't the pressure simply  $p = \frac{NT}{V}$ .  
Use the equations from section 1.6 to calculate the instantaneous pressure during the simulation.
- Use Pylab to plot the pressure during the simulation, as done in the previous exercises. How does the temperature and particle density affect the pressure? See Figure 3.3 for example.





**Figure 3.2:** Evolution of temperature during an MD simulation with  $T = 0.5$ , 40 particles,  $\rho = 0.1$ .



**Figure 3.3:** Evolution of pressure during an MD simulation with  $T = 0.5$ , 40 particles,  $\rho = 0.1$ .