Minimization of a Function in the Domain of Complex Numbers

Ryan Jensen

June 14, 2016

1 Problem

Given:

$$g(m) = \frac{1}{m} - m \tag{1}$$

Minimize $|g(m)|$ where m is a complex number.

2 Solution

2.1 Expressing the Magnitude

Choose to represent m in vector notation where $m = M\underline{/\theta}$ where M and θ are real numbers.

$$g(M\underline{/\theta}) = \frac{1}{M}\underline{/-\theta} - M\underline{/\theta} \tag{2}$$

Manipulate Equation 2 into rectangular form:

$$\begin{aligned} g(M\underline{/\theta}) &= \frac{1}{M}[\cos(-\theta) + j\sin(-\theta)] - M[\cos(\theta) + j\sin(\theta)] \\ &= \frac{1}{M}\cos\theta - j\frac{1}{M}\sin\theta - M\cos\theta - jM\sin\theta \\ &= \left(\frac{1}{M} - M\right)\cos\theta - j\left(\frac{1}{M} + M\right)\sin\theta \end{aligned}$$

Express the magnitude of the complex number $g(M\underline{/\theta})$:

$$|g(M\underline{/\theta})| = \sqrt{\left(\left(\frac{1}{M} - M\right)\cos\theta\right)^2 + \left(\left(\frac{1}{M} + M\right)\sin\theta\right)^2} \tag{3}$$

Minimizing the square of the magnitude is the same as minimizing the magnitude itself, and it is more convenient:

$$
\begin{aligned}
|g(M\angle\theta)|^2 &= \left(\left(\frac{1}{M} - M\right)\cos\theta\right)^2 + \left(\left(\frac{1}{M} + M\right)\sin\theta\right)^2 \\
&= \left(\frac{1}{M} - M\right)^2 \cos^2\theta + \left(\frac{1}{M} + M\right)^2 \sin^2\theta \\
&= \left(\frac{1}{M^2} - 2 + M^2\right)\cos^2\theta + \left(\frac{1}{M^2} + 2 + M^2\right)\sin^2\theta \\
&= \frac{\cos^2\theta}{M^2} - 2\cos^2\theta + M^2\cos^2\theta + \frac{\sin^2\theta}{M^2} + 2\sin^2\theta + M^2\sin^2\theta \\
&= \frac{\left(\cos^2\theta + \sin^2\theta\right)}{M^2} + 2\left(\sin^2\theta - \cos^2\theta\right) + M^2\left(\cos^2\theta + \sin^2\theta\right) \\
&= \frac{1}{M^2} + 2\left(\sin^2\theta - \left(1 - \sin^2\theta\right)\right) + M^2 \\
&= \frac{1}{M^2} + M^2 + 2\left(2\sin^2\theta - 1\right) \\
&= \frac{1}{M^2} + M^2 + 4\sin^2\theta - 2 \qquad (4)
\end{aligned}
$$

In order to minimize the magnitude of $g(m)$, the above line should be made to be as small as possible. Because both terms $\left(\frac{1}{M^2} + M^2\right)$ and $\left(4\sin^2\theta\right)$ are independent, they can be minimized separately.

2.2 Minimizing $\left(\frac{1}{M^2} + M^2\right)$

To find the local maxima and minima of $\left(\frac{1}{M^2} + M^2\right)$, take the derivative of the function and set it equal to zero:

$$
\begin{aligned}
\frac{d}{dM}\left(\frac{1}{M^2} + M^2\right) &= 0 \\
-2\frac{1}{M^3} + 2M &= 0 \\
\frac{1}{M^3} - M &= 0 \\
1 - M^4 &= 0 \\
M^4 &= 1 \\
M &= \sqrt[4]{1} \\
M &= 1, j, -1, -j
\end{aligned}
$$

There are four solutions, but only 1 and -1 are relevant to the original problem statement. This is because M, being the magnitude of the vector m, is a real number.

To determine if these values of M are local minima or maxima, the function $\left(\frac{1}{M^2} + M^2\right)$ is differentiated twice, and the sign of the function is analyzed.

$$\frac{d^2}{dM^2}\left(\frac{1}{M^2}+M^2\right)=\frac{d}{dM}\left(-2\frac{1}{M^3}+2M\right)=6\frac{1}{M^4}+2 \tag{5}$$

Equation 5 is always positive for real values of M. Therefore, all local extrema of the $\left(\frac{1}{M^2}+M^2\right)$ are local minima.

Considering the "endpoints": when $M \to -\infty$ or $M \to +\infty$, the expression $\left(\frac{1}{M^2}+M^2\right) \to \infty$. These two cases can be ignored in the search for the minimum. Hence, we are left with:

$$M = \pm 1 \tag{6}$$

2.3 Minimizing $\left(4\sin^2\theta\right)$

To minimize $\left(4\sin^2\theta\right)$, take the derivative of the function and set it equal to zero to find the local extrema:

$$\begin{aligned}
\frac{d}{d\theta}\left(4\sin^2\theta\right) &= 0\\
\frac{d}{d\theta}\sin^2\theta &= 0\\
\frac{d}{d\theta}\sin\theta\sin\theta &= 0\\
2\sin\theta\cos\theta &= 0\\
\sin\theta\cos\theta &= 0\\
\theta &= \frac{k}{2}\pi \qquad \text{where k is any integer}
\end{aligned}$$

$4\sin^2\left(\frac{k}{2}\pi\right)$ equals 4 for any odd integer k.

$4\sin^2\left(\frac{k}{2}\pi\right)$ equals 0 for any even integer k.

Therefore, even values of k will minimize $\left(4\sin^2\theta\right)$.

$$\begin{aligned}
\theta &= k\pi \qquad \text{where k is any integer} \\
\theta &= \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots
\end{aligned} \tag{7}$$

3 Conclusion

The conditions which yield minimum $|g(M\underline{/\theta})|$ are: $M = \pm 1$ while simultaneously θ is an integer multiple of π.

Expressing this in terms of the original complex number, m, yields:

$$m = \pm 1 \tag{8}$$

Evaluating $|g(m)|$ at these values of m yields:

$$|g(1)| = \left| \frac{1}{(1)} - (1) \right| = |1 - 1| = |0| = 0 \tag{9}$$

$$|g(-1)| = \left| \frac{1}{(-1)} - (-1) \right| = |-1 + 1| = |0| = 0 \tag{10}$$

$|g(m)|$ doesn't minimize more than that.

4 Alternative Solution

1. Assume there are values of m which make $|g(m)| = 0$
2. Recognize that $|g(m)| = 0$ only when $g(m) = 0$
3. Solve for $g(m) = 0$

$$\begin{aligned} g(m) = \frac{1}{m} - m &= 0 \qquad (m \neq 0) \\ 1 - m^2 &= 0 \\ m^2 &= 1 \\ m &= \pm 1 \end{aligned} \tag{11}$$