INF5620 Exam, Problem 2

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2D/3D wave equation with finite differences

a) 2D wave equation with variable wave velocity

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right), \tag{1}$$

$$\frac{\partial t}{\partial n} = 0, \tag{2}$$

$$u(x,y,0) = I, \tag{3}$$

$$\frac{\partial}{\partial t}u(x,y,0) = V \tag{4}$$

Model can be used to simulate waves on a lake or an ocean, where the function $q(x,y)$ describes sub-sea hill which would cause variable wave velocity

b) Compact differentiation notation

$$[D_t D_t u = D_x \overline{q}^x D_x u + D_y \overline{q}^y D_y u]^n_{i,j} \tag{5}$$

Replace derivatives with differences. Apply a centered difference in time,

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j}}{\Delta t^2}, \tag{6}$$

$$\tag{7}$$

then discretize the outer derivative of the variable coefficients,

$$\emptyset = q(x,y)\frac{\partial u}{\partial x}, \tag{8}$$

$$[D_x \emptyset]^n_{i,j} = \frac{\emptyset_{i+\frac{1}{2},j} - \emptyset_{i-\frac{1}{2},j}}{\Delta x}, \tag{9}$$

$$\emptyset_{i+\frac{1}{2},j} = q_{i+\frac{1}{2},j}\frac{u^n_{i+1,j} - u^n_{i,j}}{\Delta x} \tag{10}$$

$$\emptyset_{i-\frac{1}{2},j} = q_{i-\frac{1}{2},j}\frac{u^n_{i,j} - u^n_{i-1,j}}{\Delta x} \tag{11}$$

then combine the results,

$$\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) \approx \frac{1}{\Delta x^2}\left(q_{i+\frac{1}{2},j}(u^n_{i+1,j}-u^n_{i,j}) - q_{i-\frac{1}{2},j}(u^n_{i,j}-u^n_{i-1,j})\right). \tag{12}$$

We then need an approximation for the variable coefficient between mesh points, we can get that through by averaging by the arithmetic mean

$$q_{i+\frac{1}{2},j} = \frac{1}{2}(q_{i,j}+q_{i+1,j})$$
$$q_{i-\frac{1}{2},j} = \frac{1}{2}(q_{i,j}+q_{i-1,j})$$

Then we need to do the same for the last term, and get this,

$$\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) \approx \frac{1}{\Delta y^2}\left(q_{i,j+\frac{1}{2}}(u^n_{i,j+1}-u^n_{i,j}) - q_{i,j-\frac{1}{2}}(u^n_{i,j}-u^n_{i,j-1})\right). \tag{13}$$

In the end we combine everything and rearrange so we get $u^{n+1}_{i,j}$ on the left-hand side

$$u^{n+1}_{i,j} = -u^{n-1}_{i,j} + 2u^n_{i,j} + \tag{14}$$
$$\left(\frac{\Delta x}{\Delta t}\right)^2\left(q_{i+\frac{1}{2},j}(u^n_{i+1,j}-u^n_{i,j}) - q_{i-\frac{1}{2},j}(u^n_{i,j}-u^n_{i-1,j})\right) + \tag{15}$$
$$\left(\frac{\Delta y}{\Delta t}\right)^2\left(q_{i,j+\frac{1}{2}}(u^n_{i,j+1}-u^n_{i,j}) - q_{i,j-\frac{1}{2}}(u^n_{i,j}-u^n_{i,j-1})\right) \tag{16}$$

The initial condition are implemented by special schemes for $n = 0, 1$,

$$u^0_{i,j} = I_{i,j} \tag{17}$$
$$u^1_{i,j} = -u^0_{i,j} + \Delta t V_{i,j} + \tag{18}$$
$$\left(\frac{\Delta x}{\Delta t}\right)^2\left(q_{i+\frac{1}{2},j}(u^n_{i+1,j}-u^n_{i,j}) - q_{i-\frac{1}{2},j}(u^n_{i,j}-u^n_{i-1,j})\right) + \tag{19}$$
$$\left(\frac{\Delta y}{\Delta t}\right)^2\left(q_{i,j+\frac{1}{2}}(u^n_{i,j+1}-u^n_{i,j}) - q_{i,j-\frac{1}{2}}(u^n_{i,j}-u^n_{i,j-1})\right) \tag{20}$$

Now $u^n_{i,j}$ and $u^{n-1}_{i,j}$ are known, so we can compute $u^{n+1}_{i,j}$. Since its a PDE, we also need to handle boundary conditions

c) In 3 dimensions we get an addition term,

$$\frac{\partial}{\partial z}\left(q(x,y,z)\frac{\partial u}{\partial z}\right)$$

that can be descretized in the same way as the two other spacial terms.

d) Stability limit for Δt

$$\Delta t \leqslant \beta \frac{1}{\sqrt{\max q(x,y,z)}} \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2} \right)^{-\frac{1}{2}} \tag{21}$$

Accuracy measurement

$$\widetilde{w} = \tag{22}$$

e) Explain how we can verify the implementation of the scheme

1. MMS
2. Convergance rate
3. ..

f) When looking at the schemes we notice that non of the spacial indices (i, j, k) are dependant on any neighbouring indices being already computed, because of this we can for each time step, compute all the spacial nodes in parallel