INF5620 - Obligatory Exercise 2

Jens Kristoffer Reitan Markussen

October 12, 2013

1 Introduction

2 Equations

2.1 Partial Differential Equation

Two-dimensional, linear wave equation, with damping,

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t)$$

with boundary condition,

$$\frac{\partial u}{\partial n} = 0$$

in a rectangular spatial domain with these conditions

$$\Omega = [0, L_x] \times [0, L_y]$$

$$u(x,y,0) = I(x,y)$$

$$u_t(x,y,0) = V(x,y)$$

2.2 Numerics

PDE in compact finite difference notation

$$[D_t D_t u]^n_{i,j} + b[D_{2t} u]^n_{i,j} = [D_x \bar{q}^x D_x u]^n_{i,j} + [D_y \bar{q}^y D_y u]^n_{i,j}$$

Central difference approximations

$$\frac{\partial^2 u}{\partial t^2} \approx [D_t D_t u]^n_{i,j} = \frac{u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j}}{\Delta t^2}$$

$$b\frac{\partial u}{\partial t} \approx b[D_{2t} u]^n_{i,j} = b\frac{u^{n+1}_{i,j} - 2u^{n-1}_{i,j}}{2\Delta t}$$

$$\begin{aligned}\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) \approx [D_x \bar{q}^x D_x u]^n_{i,j} &= \frac{1}{\Delta x}\left[q_{i+\frac{1}{2},j}\frac{u^n_{i+1,j} - u^n_{i,j}}{\Delta x} - q_{i-\frac{1}{2},j}\frac{u^n_{i,j} - u^n_{i-1,j}}{\Delta x}\right] \\ &= \frac{1}{\Delta x^2}\left(q_{i+\frac{1}{2},j}(u^n_{i+1,j} - u^n_{i,j}) - q_{i-\frac{1}{2},j}(u^n_{i,j} - u^n_{i-1,j})\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) \approx [D_y \bar{q}^y D_y u]^n_{i,j} &= \frac{1}{\Delta y}\left[q_{i,j+\frac{1}{2}}\frac{u^n_{i,j+1} - u^n_{i,j}}{\Delta y} - q_{i,j-\frac{1}{2}}\frac{u^n_{i,j} - u^n_{i,j-1}}{\Delta y}\right] \\ &= \frac{1}{\Delta y^2}\left(q_{i,j+\frac{1}{2}}(u^n_{i,j+1} - u^n_{i,j}) - q_{i,j-\frac{1}{2}}(u^n_{i,j} - u^n_{i,j-1})\right)\end{aligned}$$

Relation for creating initial scheme

$$\frac{\partial u}{\partial n} \approx \frac{u^{n+1}_{i,j} - 2u^{n-1}_{i,j}}{2\Delta t}1 = 0$$

$$u^{-1}_{i,j} = u^1_{i,j}$$

Discretization

Make q only valid at the grid points using the arithmetic mean

$$\begin{aligned}q_{i,j+\frac{1}{2}} &= \frac{1}{2}(q_{i,j+1} + q_{i,j}) \\ q_{i,j-\frac{1}{2}} &= \frac{1}{2}(q_{i,j} + q_{i,j-1}) \\ q_{i+\frac{1}{2},j} &= \frac{1}{2}(q_{i+1,j} + q_{i,j}) \\ q_{i-\frac{1}{2},j} &= \frac{1}{2}(q_{i,j} + q_{i-1,j})\end{aligned}$$

Full approximation scheme for $u_{i,j}^{n+1}$

$$u_{i,j}^{n+1} = \left(1 + b\frac{\Delta t}{2}\right)^{-1} \Bigg[2u_{i,j}^{n} + u_{i,j}^{n-1}\left(b\frac{\Delta t}{2} - 1\right) + \frac{\Delta t^2}{2\Delta x^2}((q_{i+1,j} + q_{i,j})(u_{i+1,j}^{n} - u_{i,j}^{n}) - (q_{i,j} + q_{i-1,j})(u_{i,j}^{n} - u_{i-1,j}^{n})) + \frac{\Delta t^2}{2\Delta y^2}((q_{i,j+1} + q_{i,j})(u_{i,j+1}^{n} - u_{i,j}^{n}) - (q_{i,j} + q_{i,j-1})(u_{i,j}^{n} - u_{i,j-1}^{n}))\Bigg]$$

Full approximation scheme at boundary ghost cells $u_{0,j}^{n+1}$, $u_{i,0}^{n+1}$, $u_{L_x,j}^{n+1}$ and u_{i,L_y}^{n+1}, and corner cells $u_{0,0}^{n+1}$, $u_{L_x,0}^{n+1}$,u_{L_x,L_y}^{n+1} and u_{0,L_y}^{n+1}

$$u_{0,j}^{n+1} = \left(1 + b\frac{\Delta t}{2}\right)^{-1} \Bigg[2u_{i,j}^{n} + u_{i,j}^{n-1}\left(b\frac{\Delta t}{2} - 1\right) + \frac{\Delta t^2}{2\Delta x^2}((q_{1,j} + q_{0,j})(u_{1,j}^{n} - u_{0,j}^{n}) - (q_{0,j} + q_{1,j})(u_{0,j}^{n} - u_{1,j}^{n})) + \frac{\Delta t^2}{2\Delta y^2}((q_{0,j+1} + q_{0,j})(u_{0,j+1}^{n} - u_{0,j}^{n}) - (q_{0,j} + q_{0,j-1})(u_{0,j}^{n} - u_{0,j-1}^{n}))\Bigg]$$

Equivalent scheme can be made for the other cells.
When computing u^{n+1} we required knowledge of the mesh points u^n and u^{n-1}, which means that when computing u^1 we require knowledge of u^0 and u^{-1}. u^0 is defined by the initial conditions $I(x, y)$ and $V(x, y)$, however we need a modified scheme for u^{-1}.

$$u_{i,j}^{-1} = u_{i,j}^{0} + \frac{\Delta t^2}{4\Delta x^2}((q_{i+1,j} + q_{i,j})(u_{i+1,j}^{0} - u_{i,j}^{0}) - (q_{i,j} + q_{i-1,j})(u_{i,j}^{0} - u_{i-1,j}^{0})) + \frac{\Delta t^2}{4\Delta y^2}((q_{i,j+1} + q_{i,j})(u_{i,j+1}^{0} - u_{i,j}^{0}) - (q_{i,j} + q_{i,j-1})(u_{i,j}^{0} - u_{i,j-1}^{0}))$$

Scheme can also be used as initial condition scheme if modified in the same way we modified the inner scheme.

3 Implementation

4 Conclusion