Final Project INF5620,
Nonlinear Diffusion Equation

Jens Kristoffer Reitan Markussen

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1 Introduction

Final project report for INF5620 on numerical investigation and analysis of a nonlinear diffusion equation.

2 Nonlinear diffusion model

The project will investigate this nonlinear diffusion equation with known coefficient ϱ and known function coefficient of $\alpha(u)$. Known initial condition in time at 0. Naumann condition on all boundaries.

$$\varrho u_t = \nabla \cdot (\alpha(u)\nabla u) + f, \tag{1}$$

$$u(0) = I, \tag{2}$$

$$\frac{\partial u}{\partial n} = 0 \tag{3}$$

3 Approximation

While we will use the finite elements method to approximate the PDE in space, we use the backwards euler scheme to derive an implicit approximation scheme in time.

$$\left[\varrho D_t^- u = \nabla \cdot (\alpha(u)\nabla u) + f\right]^n \tag{4}$$

$$\varrho \frac{u^n - u^{n-1}}{\Delta t} = \nabla \cdot (\alpha(u^n)\nabla u^n) + f^n \tag{5}$$

Now that we have an expression for u in time, we need to derive an approximation for u in space using the finite elements method. We then need to derive a variational formulation for the spatial problem, $F(u; v)$.

$$\int_\Omega \varrho \frac{u^n - u^{n-1}}{\Delta t} v dx = -\int_\Omega \alpha(u^n)\nabla u^n \cdot \nabla v dx + \int_\Omega f^n v dx + \int_{\partial\Omega} \alpha(u^n)\frac{\partial u}{\partial n} v ds \tag{6}$$

3.1 Picard iterations for nonlinear term $\alpha(u)$

The problem described above is nonlinear because of the coefficient term $a(u)$. Before we can solve the PDE we need to somehow make this term linear. This can be done by replacing the unknown variable in the nonlinear term with something known.

To reduce the amount of indices in our equations we introduce a new notation. Where k is the picard iteration index.

$$\begin{aligned} u^n &: u \\ n^{n,k} &: u_- \\ u^{n-1} &: u_1 \end{aligned}$$

We can then make all the nonlinear terms linear by replacing the u with the result from the previous picard iteration.

$$\int_\Omega \varrho \frac{u - u_1}{\Delta t} v dx = - \int_\Omega \alpha(u_-) \nabla u \cdot \nabla v dx + \int_\Omega f^n v dx + \int_{\partial\Omega} \alpha(u_-) \frac{\partial u}{\partial n} v ds \quad (7)$$

4 Verification

Verification tests are implemented using the python nose unit-test framework. The tests can be ran from the terminal with the command `nosetests -v -s wave2d_module.py`, the runtime option `-v` activates verbose output, and the option `-s` deactivates stdout suppression.

4.1 Convergence rate

Convergence rate tests are done on a test case where $I = \cos(\pi x)$, $\alpha(u) = 1$ and $f = 0$. The exact solution is then $u_e = e^{-\pi^2 t} \cos(\pi x)$. The error in time for the backward euler scheme is then $O(\Delta t)$, which is computed by $E = \sqrt{(u_e - u)^2}$

4.2 Method of manufactured solution

Method of manufactured solution test are implemented by restricting the domain to one dimension and choosing $u_e = tx^2 \left(\frac{1}{2} - \frac{x}{3}\right)$ and $\alpha(u) = 1 + u^2$. The source term is computed to be,

$$f = -\frac{1}{3}\varrho x^3 + \frac{1}{2}\varrho x^2 + \frac{8}{9}t^3 x^7 - \frac{28}{9}t^3 x^6 + \frac{7}{2}t^3 x^5 - \frac{5}{4}t^3 x^4 + 2tx - t.$$

The approximated solution u is then compared to u_e over time

5 Experiment

Two experiments are implemented using the solver module. `ex_01.py` runs the solver with the same parameters as the convergence test, except in one dimension. `ex_02.py` runs the solver in two dimensions simulating nonlinear diffusion of the gaussian function. Both experiments visualize the solution with a graph updated for each timestep