STATS 551 HW0 Zhen Qin

1\.

(a) Because the coin is fair, results are independent, P(having a test) = P(head) = P(tail) = $\frac{1}{2}$. P(having a quiz for each lecture) = $\frac{1}{2^{27}}$

(b) Suppose r.v.s $X_i = 1$ when having a test in the ith class, $X_i = 0$ otherwise. It's clear that $X_i \sim Bernoulli(\frac{1}{2})$, and X_i are iid. So the total number $X = \sum_{i=1}^{27} X_i \sim$ binomial(27, 0.5)

(c) $EX = 27 \times 0.5 = 13.5$, $VarX = 27 \times 0.5 \times (1-0.5) = 6.75$

$P(X \le 8) = 0.0261$, $P(X \le 19) = 0.9904$, $P(X = k) = \binom{27}{k}\frac{1}{2^k}$,

so $P(9 \le X \le 19) = 0.9904 - 0.0261 > 0.95$.

[9, 19] is a 95% CI.

2\.

(a) P(having a test) = P(head) = 0.6

P(having a quiz for each lecture) = 0.6^{27}

(b) Use the notation in 1(b). $X_i \overset{iid}{\sim} Bernoulli(0.6)$,

so $X = \sum_{i=1}^{27} X_i \sim$ binomial(27, 0.6)

(c) $EX = 27 \times 0.6 = 16.2$, $VarX = 27 \times 0.6 \times (1-0.6) = 6.48$

$P(X \le k) = \sum_{i=0}^{k} \binom{27}{i} \times 0.6^i \times 0.4^{27-i}$

$P(X \le 11) = 0.0337$, $P(X \le 21) = 0.9845$

$P(12 \le X \le 21) = 0.9845 - 0.0337 > 0.95$

so [12, 21] is a 95% CI.

3. 'Head'

Suppose θ_i is the ~~ith~~ probability of the ith experiment.

$X_i = 1$ if the result is head, $X_i = 0$ if the result is tail.

The number of quizzes $X = X_1 + X_2 + \cdots + X_{27}$.

$P(\theta_i = 0.1) = \cdots = P(\theta_i = 0.6) = \frac{1}{6}$. ~~$X_i$ are iid $X_i \overset{iid}{\sim} Bernoulli(\theta_i)$~~

$EX = \sum_{i=1}^{27} EX_i = 27 EX_1$ ~~$= 27 \times \frac{1}{6} \times (0.1 + \cdots + 0.6) = 9.45$~~

$Var X = \sum_{i=1}^{27} Var X_i = 27 Var X_1$

$X_i | \theta_i \sim Bernoulli(\theta_i)$, $EX_1 = E(E(X_1|\theta_1)) = E\theta_1 =$

$\frac{1}{6} \times (0.1 + \cdots + 0.6) = 0.35$. So $EX = 27 \times 0.35 = 9.45$

$Var X_1 = E(Var(X_1|\theta_1)) + Var(E(X_1|\theta_1))$

$= E(\theta_1(1-\theta_1)) + Var\theta_1 = E\theta_1 - E\theta_1^2 + E\theta_1^2 - (E\theta_1)^2$

$= 0.35 - 0.35^2 = 0.2275$

So $Var X = 27 \times 0.2275 = 6.1425$

4.

The notation is as above. $\theta_i \sim U[0,1]$, $E\theta_i = \frac{1}{2}$,

$X_i | \theta_i \overset{iid}{\sim} Bernoulli(\theta_i)$

$EX = 27 EX_1 = 27 E\theta_1 = 13.5$

$Var X = 27 Var X_1 = 27 \times (0.5 - 0.5^2) = 6.75$

6.

Likelihood function $p(x;\theta) = \binom{27}{x} \theta^x (1-\theta)^{27-x} 1(0 \le \theta \le 1)$. when $x = 16$,

$\frac{\partial}{\partial\theta} \log p(16;\theta) = \binom{27}{16} \frac{\partial}{\partial\theta}(16 \log\theta + 11 \log(1-\theta)) = \binom{27}{16}\left(\frac{16}{\theta} - \frac{11}{1-\theta}\right)$.

If $\theta < \frac{16}{27}$, $p(x;\theta)$ is increasing with respect to x; if $\theta > \frac{16}{27}$, it's inverse, so the maximum likelihood estimator $\hat{\theta} = \frac{16}{27}$.

$p(\theta | x=16) \propto \binom{27}{16} \theta^{16} (1-\theta)^{11}$, so $\theta | x=16 \sim Beta(17, 12)$. when $x=16$

$P(\theta < 0.41) = 0.025$, $P(\theta \le 0.76) = 0.975$.

So $[0.41, 0.76]$ is a 95% CI.