Homework 0 solutions

January 21, 2018

1

(a) Fair coin implies probability is .5

(b) N is sum of 27 Bernoulli r.v. with probability .5. So $N \sim Bin(27, .5)$

(c) Expectation is $27 \times .5$ and variance is $27 \times .5(1 - .5)$. $P(9 \leq N \leq 19) = .964$ approx.

2

(a) Fair coin implies probability is .6

(b) N is sum of 27 Bernoulli r.v. with probability .6. So $N \sim Bin(27, .6)$

(c) Expectation is $27 \times .6$ and variance is $27 \times .6(1 - .6)$. $P(12 \leq N \leq 21) = .951$ approx.

3

Let X denote the coin selected and Y denote the outcome of the coin. X follows a discrete uniform distribution. Let p_x be the probability of Y = "HEAD" when $X = x$. So,

$$E[Y|X] = p_x$$
$$\implies E[Y] = E[E[Y|X]] = \frac{\sum_x p_x}{6} = .35 = P[Y = 1]$$

Similarly, using $V[Y] = V[E[Y|X]] + E[V[Y|X]]$, we get $V[Y] = .35(1 - .35)$ Hence expected number of quiz is $27 \times .35$ and Variance is $27 \times .35(1 - .25)$

4

U follows Uniform(0,1). Given $U = u,\ Y \sim Ber(u)$. Therefore

$$E[Y] = E[E[Y|U]] = E[U] = .5$$

. So expected number of Quizzes is $27 \times .5$ Using $V[Y] = V[E[Y|U]] + E[V[Y|U]]$, we have

$$\begin{aligned} V[Y] = &\quad V[U] + E[U(1-U)] \\ = &\quad E[U^2] - E[U]^2 + E[U] - E[U^2] = \quad E[U] - E[U]^2 = \quad .5(1-.5) \end{aligned}$$

Required variance is $27 \times .5(1-.5)$

5

The following code is an illustration of question 4 through simulation

```
> set.seed(12)
> U = matrix(runif(2700,0,1),100,27)
> X = matrix(0,100,27)
> for(i in 1:100)
+ for(j in 1:27){
+ {
+ X[i,j] = rbinom(1,1,U[i,j])
+ }
+ }
> x = rowSums(X)
> mean(x); var(x)
[1] 13.33
[1] 6.344545
```

6

Log likelihood is proportional to

$$x \log\theta + (n-x)\log(1-\theta)$$

setting derivative equals to 0 we get $\hat{\theta} = \frac{x}{n} = \frac{16}{27} = .59$ approx.
The Fisher's information matrix $\mathbb{E}[-\ell''(\theta)] = \frac{n}{\theta(1-\theta)}$ Using asymptotic properties of MLE and estimate of the Fisher's Information Matrix, the estimated standard error is $\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}$.
The 95% CI is then defined as $\hat{\theta} \pm 1.96 * \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}} = (0.4072, 0.7779)$

For the test under H_0, $Z = \frac{\hat{\theta}-\theta_0}{\sqrt{\frac{\theta_0(1-\theta_0)}{n}}}$ follows standard normal as $n \to \infty$. Here value of test statistic under null is .9623 which is less than 1.96, hence we fail to reject null.