STATS 551 HW1 Zhen Qin

Q1.

1.

Prior distribution is $\theta \sim N(60, 20^2)$, and $y|\theta \sim N(\theta, 10^2)$

Posterior distribution is:

$p(\theta|y_{1\cdots n}) \propto P(y_{1\cdots n}|\theta)P(\theta) \propto e^{\frac{\Sigma(y_i-\theta)^2}{200}} \cdot e^{\frac{(\theta-60)^2}{800}}$

In fact, $\theta|y_{1\cdots n}$ is also normally distributed,

$\mu_n = \dfrac{\frac{60}{20^2} + \frac{n}{10^2}\times\bar{y}}{\frac{1}{20^2}+\frac{n}{10^2}}$, $\dfrac{1}{\tau_n^2} = \dfrac{1}{20^2} + \dfrac{n}{10^2}$

so $\mu_n = 70 - \frac{10}{1+4n}$, $\tau_n^2 = \frac{400}{1+4n}$, $\theta|y_{1\cdots n} \sim N(70-\frac{10}{1+4n}, \frac{400}{1+4n})$

2.

$\tilde{y}|y$ is also normally distributed.

$E(\tilde{y}|y_{1\cdots n}) = E(E(\tilde{y}|\theta, y_{1,\cdots,n})|y_{1\cdots n}) = \mu_n = 70 - \frac{10}{1+4n}$

$Var(\tilde{y}|y_{1\cdots n}) = \tau_n^2 + 10^2 = 100 + \frac{400}{1+4n}$

So $\tilde{y}|y \sim N(70-\frac{10}{1+4n}, 100+\frac{400}{1+4n})$

3.

$ucb_\theta = 70 - \frac{10}{1+4n} + 1.96\times\sqrt{\frac{400}{1+4n}}$

$lcb_\theta = 70 - \frac{10}{1+4n} - 1.96\times\sqrt{\frac{400}{1+4n}}$

when $n=9$, $ucb_\theta = 76.18$, $lcb_\theta = 63.28$, a CI is $[63.28, 76.18]$

$ucb_{\tilde{y}} = 70 - \frac{10}{1+4n} + 1.96\times\sqrt{100+\frac{400}{1+4n}}$

$lcb_{\tilde{y}} = 70 - \frac{10}{1+4n} - 1.96\times\sqrt{100+\frac{400}{1+4n}}$

when $n=9$. $ucb_{\tilde{y}} = 90.37$, $lcb_{\tilde{y}} = 49.09$, a CI is $[49.09, 90.37]$

4.

When $n=99$, $ucb_\theta = 71.95$, $lcb_\theta = 68.00$, a CI is $[68.00, 71.95]$

When $n=99$, $ucb_{\tilde{y}} = 89.68$, $lcb_{\tilde{y}} = 50.27$, a CI is $[50.27, 89.68]$

Q2

1.

$P(N) = 0.01 \times (0.99)^{N-1}$, for $N = 1, 2, \cdots$

Suppose a cable car is numbered X.

$P(X|N) = \frac{1}{N}$, $X = 1, 2, \cdots, N$

$P(N|X) \propto \frac{1}{N} \times (0.99)^{N-1}$, $N \geq X$ and $N \in N^+$

$\sum_{n=X}^{\infty} \frac{1}{n}(0.99)^{n-1}$ # converges because $0.99 < 1$

So the posterier distribution is $P(N|X=203) = \frac{(0.99)^{N-1}}{N \sum_{n=203}^{\infty} \frac{1}{n}(0.99)^{n-1}}$, $N \geq 203$

2.

Suppose $T = \sum_{n=203}^{\infty} \frac{1}{n}(0.99)^{n-1}$, the $P(N|X=203) = \frac{(0.99)^{N-1}}{NT}$

$E(N|X=203) = \sum_{n=203}^{\infty} n \times \frac{(0.99)^{n-1}}{nT} = \frac{1}{T} \sum_{n=203}^{\infty} (0.99)^{n-1} = \frac{1}{T} \times 100 \times (0.99)^{202}$

$E(N^2|X=203) = \sum_{n=203}^{\infty} n^2 \times \frac{(0.99)^{n-1}}{nT} = \frac{1}{T} \sum_{n=203} n(0.99)^{n-1} = \frac{1}{T} \times 30200 \times (0.99)^{202}$

$\cancel{T = \sum_{n=203}^{1000} \frac{1}{n}(0.99)^{n-1} + \sum_{n=1001}^{\infty} \frac{1}{n}(0.99)^{n-1}}$

$\cancel{\sum_{n=1001}^{\infty} \frac{1}{n}(0.99)^{n-1} \leq \frac{1}{1000} \sum_{n=1001}^{\infty} 0.99^{n-1} = \frac{1}{10} 0.99^{1000} = 4.3 \times 10^{-6}}$

$\cancel{\sum_{n=203}^{1000} \frac{1}{n}(0.99)^{n-1} = 0.04705, \text{ so } T \approx 0.047}$

$\cancel{\text{Thus } E(N|203) = 279.39}$ 279.1

$\cancel{Sd(N|203) = \sqrt{E(N^2|203) - (E(N|203))^2} = 79.48}$

Approximate them on computer, I get

$E(N|X=203) = 279.1$

Standard deviation is $\cancel{\sqrt{E(N^2|203) - (E(N|203))^2} = 80.0}$

$\cancel{\sqrt{E(N^2|X=203) - (E(N|X=203))^2} = 80.0}$

$\sqrt{E(N^2|X=203) - (E(N|X=203))^2} = 80.0$